

p -ADIC BROWNIAN MOTION IS A SCALING LIMIT

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ABSTRACT. A p -adic Brownian motion is a continuous time stochastic process in a p -adic state space that has a Vladimirov operator as its infinitesimal generator. The current work shows that any such process is the scaling limit of a discrete time random walk on a discrete group. Earlier work required the exponent of the Vladimirov operator to be in $(1, \infty)$, and the convergence was the weak convergence of probability measures on the Skorohod space of paths on a compact time interval. The current approach simplifies the earlier approach, allows for any positive exponent, eliminates the restriction to compact time intervals, and establishes some moment estimates for the discrete time processes that are of independent interest.

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1. INTRODUCTION

The fundamental solution to the diffusion equation in the real setting induces the Wiener measure W on $C([0, \infty): \mathbb{R})$, the space of continuous real-valued functions on $[0, \infty)$. The probability space for Brownian motion is the pair $(C([0, \infty): \mathbb{R}), W)$. The continuous paths form a closed subset of $D([0, \infty): \mathbb{R})$, the Skorohod space of real-valued càdlàg functions on $[0, \infty)$ endowed with the Skorohod metric. There is a sequence of probability measures (P^m) on $D([0, \infty): \mathbb{R})$ so that each P^m is the probability measure for a discrete time random walk on a grid in $\mathbb{R}$, and (P^m) converges weakly to W . The random walks associated to (P^m) arise from the scaling of a single discrete time process on a discrete group. In this precise sense, Brownian motion in $\mathbb{R}$ is a scaling limit. There are direct analogs of Brownian motion for paths in $\mathbb{Q}_p$, the p -adic numbers, that are functions of a real time parameter. Do these real time p -adic Brownian motions also arise as scaling limits? An earlier study partially answered this question [7], but only treated Brownian motions with Vladimirov operators with exponents greater than 1 as their infinitesimal generators. Furthermore,

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the convergence that this study established is, for any positive T , the weak convergence of probability measures on the Skorohod space $D([0, T]: \mathbb{Q}_p)$ of $\mathbb{Q}_p$ -valued paths.

The current work improves the earlier study [7], simplifies the approach by reducing all calculations that involve scaled processes to calculations that involve a single discrete time stochastic process that has a discrete quotient group of $\mathbb{Q}_p$ as its state space, and establishes moment estimates for the discrete time processes that are of independent interest. It treats any Vladimirov operator with a positive exponent, and the convergence it establishes is the weak convergence of probability measures on $D([0, \infty): \mathbb{Q}_p)$. The current approach should be robust enough to permit an extension to the very general type of non-Archimedean analogs of Brownian motion that Varadarajan introduced [26].

Authors have traditionally referred to the type of processes that the present article studies as p -adic ultrametric diffusion processes [4, 9]. However, such processes may potentially be more general than the processes that the present work studies. Reference to the *real time* p -adic Brownian motion as a p -adic Brownian motion agrees with the general practice of referring to a *real time* analog of Brownian motion in a space other than $\mathbb{R}$ as a Brownian motion in the space, but it introduces some ambiguity with respect to the parameter space for those familiar with p -adic mathematical physics. In particular, Bikulov and Volovich introduced in their seminal work [8] a very different stochastic process, parameterized by a p -adic time variable, that they also referred to as p -adic Brownian motion. In some respects, such a process is the true analog of real Brownian motion. To clarify, the p -adic Brownian motion that the current paper studies is a stochastic process with a real time parameter and a p -adic state space.

The two principle ideas that motivate the study of p -adic mathematical physics are the idea that non-Archimedean physical models could describe the observed ultrametricity in certain complex systems and the idea that spacetime could have a non-Archimedean structure at extremely small distance and time scales. Ultrametric structures in spin glasses were already implicit in Parisi's early works [19, 20]. Avetisov, Bikulov, and Kozyrev [3] and Parisi and Sourlas [21] independently proposed p -adic models to describe replica symmetry breaking. The understanding that there are fundamental limitations on physical measurements below the Planck scale motivated Volovich to introduce the *Volovich hypothesis* [27, Chapter 6], the idea that physical theories that deal with such ultramicroscopic measurements should involve non-Archimedean local fields [33]. The review of Dragovich, Khrennikov, Kozyrev, and Volovich [13] and the more recent work of Khrennikov, Kozyrev, and Zúñiga-Galindo [16] both give excellent accounts of the development of p -adic mathematical physics and list a variety of areas of potential application for the results of the current paper. A significant application of p -adic Brownian motion to physics that has experimental confirmation is the utilization by Avetisov, Bikulov, and Zubarev [5] and by Bikulov and Zubarev [10] of the Vladimirov equation in the description of protein molecule dynamics. The current work could also permit a study from a discrete perspective of the work of Khrennikov and Kochubei [15] on a p -adic analog of the porous medium equation, as well as its generalization by the previous two authors with Antoniouk [2]. It would also be of interest to extend the finite approximation of non-Archimedean quantum systems [6] to a discrete time setting.

Taibleson introduced the idea of a pseudo-differential operator in the context of a local field [25]. Saloff-Coste studied such operators [23] and generalized them to the setting of local groups [24]. The seminal works of Vladimirov and Volovich on p -adic quantum systems [30, 31] and Vladimirov's works in which he studied the *Vladimirov operator* and computed

its spectrum [28, 29] dramatically increased interest in analogs of diffusion equations in the p -adic setting. Following the work of Kochubei [17] and Albeverio and Karwowski [1], Varadarajan introduced the idea of a Brownian motion in a state space that is a finite dimensional vector space over a division ring which is finite dimensional over a local field of arbitrary characteristic [26]. The current paper takes the perspective of Varadarajan as its starting point, but restricts to the setting of p -adic state spaces.

Section 2 presents some necessary background for the current work and introduces to a general readership a framework for studying scaling limits in both the real and p -adic settings. Section 3 introduces the state space of the primitive process and some properties of the primitive process and its refinements. It establishes in Theorem 3.1 upper bounds for low order moments of the primitive process. Section 4 introduces a family of scaled processes in $\mathbb{Q}_p$ and uses Theorem 3.1 to determine uniform upper bounds for the scaled processes that imply that the family of measures associated to the scaled processes is uniformly tight. Uniform tightness of the measures and the convergence of the finite dimensional distributions of the scaled processes imply the main result of the paper, Theorem 4.1.

2. BACKGROUND AND FRAMEWORK

2.1. Path spaces. Take I to be either $[0, \infty)$ or $\mathbb{N}_0$, the natural numbers with 0. For any Polish space $\mathcal{S}$, denote by $F(I: \mathcal{S})$ the set of all paths in $\mathcal{S}$ with domain I . Two spaces of paths are of particular importance, the set $C([0, \infty): \mathcal{S})$ of continuous paths in $\mathcal{S}$ with domain $[0, \infty)$ equipped with the topology of uniform convergence on compacta, and the *Skorohod space* $D([0, \infty): \mathcal{S})$ of càdlàg functions from $[0, \infty)$ to $\mathcal{S}$ equipped with the Skorohod metric [11]. Probability measures on the second space are the central objects of the current study. Denote by $\Omega(I)$ any of the spaces $F(I: \mathcal{S})$, $C([0, \infty): \mathcal{S})$, or $D([0, \infty): \mathcal{S})$ and by $\mathcal{B}(\mathcal{S})$ the set of Borel subsets of $\mathcal{S}$.

For any natural number N , an *epoch* of length N in I is a strictly increasing finite sequence in I with domain $\{0, \dots, N\}$ such that $e(0)$ is equal to 0. A *route*, U , of length N in $\mathcal{S}$ is a finite sequence from $\{0, \dots, N\}$ to $\mathcal{B}(\mathcal{S})$. A *history*, h , of length N for $\mathcal{S}$ -valued paths with domain I is a finite sequence of ordered pairs in $I \times \mathcal{B}(\mathcal{S})$ with $N + 1$ places and with the property that if

$$h = ((t_i, U_i))_{i \in \{0, \dots, N\}},$$

then (t_i) is an epoch in I and (U_i) is a route in $\mathcal{B}(\mathcal{S})$. For any history h , denote by e_h , $\ell(h)$, and U_h the associated epoch, length, and route of h , respectively. Take $H(I, \mathcal{S})$ to be the set of all histories for $\mathcal{S}$ -valued paths with domain I . On specifying a path space $\Omega(I)$, take $\mathcal{C}$ to be defined for any h in $H(I, \mathcal{S})$ by

$$\mathcal{C}(h) = \{\omega \in \Omega(I) : \forall i \in \mathbb{N}_0 \cap [0, \ell(h)], \omega(e_h(i)) \in U_h(i)\}.$$

For any history h , $\mathcal{C}(h)$ is a *simple cylinder set*. The set $\mathcal{C}(H(I, \mathcal{S}))$ is the *set of all simple cylinder sets of $\Omega(I)$* and is a π -system, but not an algebra. The *set of cylinder sets* is the σ -algebra generated by $\mathcal{C}(H(I, \mathcal{S}))$ —its elements are the *cylinder sets*.

For any t in I , take Y_t to be the function that acts on any path ω in $\Omega(I)$ by

$$(1) \quad Y_t(\omega) = \omega(t).$$

Take Y to be the function

$$Y: I \times \Omega(I) \rightarrow \mathcal{S} \quad \text{by} \quad Y(t, \omega) = \omega(t).$$

For any probability measure P on the σ -algebra of cylinder sets of $\Omega(I)$, the pair $(\Omega(I), P)$ is a *path space with probability measure* P . For any t in I , $(\Omega(I), P, Y_t)$ is a random variable and $(\Omega(I), P, Y)$ is a stochastic process. The *finite dimensional distributions* for $(\Omega(I), P, Y)$ are the probabilities for the simple cylinder sets of $\Omega(I)$. The convergence of stochastic processes that this paper investigates involves processes that have the same state and parameter spaces, but different probability measures. The level of precision in referring to stochastic processes as triples facilitates reframing the study of the convergence of a sequence of stochastic processes as the study of the convergence of a sequence of probability measures on the same Polish space.

2.2. Abstract processes and models. The construction of a probability measure on a space of paths typically involves for each N in $\mathbb{N}_0$ and each epoch e of length N the construction of a premeasure on the π -system of Borel subsets of $\mathcal{S}^N$ that are products of Borel sets in each component. If this family of premeasures satisfies the Kolmogorov consistency conditions, then the Kolmogorov extension theorem guarantees that there is a unique probability measure P on the cylinder sets of $F(I: \mathcal{S})$ that agrees with the given premeasures [11]. It is helpful to initially view the premeasures as being the finite dimensional distributions of a stochastic process, even before it is known that there is a stochastic process with the given finite dimensional distributions. After determining that there is such a stochastic process, verification of certain moment estimates for the process will guarantee that the process has a version in either $C(I: \mathcal{S})$ or $D(I: \mathcal{S})$ [11, 12].

Refer to the construction of the finite dimensional distributions for a stochastic process as the construction of the *abstract stochastic process* $\tilde{Y}$. Refer to the construction of the law for a random variable without actually specifying its domain as the construction of an *abstract random variable* $\tilde{X}$. A model for the abstract random variable $\tilde{X}$ is a random variable with the same law as $\tilde{X}$. A model for the abstract stochastic process $\tilde{Y}$ is a stochastic process with the same finite dimensional distributions as $\tilde{Y}$. To distinguish the probabilities associated with abstract random variables and stochastic processes from the probabilities associated with their models, denote by $\text{Prob}(\tilde{X} \in A)$ the probability that $\tilde{X}$ takes a value in some Borel set A , and use an analogous notation for $\tilde{Y}$.

2.3. Brownian motion in $\mathbb{R}$. For any D in $(0, \infty)$, the equation

$$(2) \quad \frac{\partial u}{\partial t} = \frac{D}{2} \frac{\partial^2 u}{\partial x^2}$$

is the diffusion equation with diffusion constant D . Its fundamental solution is the function ρ that is given for any pair (t, x) in $(0, \infty) \times \mathbb{R}$ by

$$(3) \quad \rho(t, x) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{x^2}{2Dt}\right).$$

The set $\{\rho(t, \cdot): t \in \mathbb{R}_+\}$ forms a semigroup under the convolution operation, a *convolution semigroup of probability density functions*. There is a unique probability measure W on $C([0, \infty): \mathbb{R})$ so that for any history h , and with x_0 taken to be equal to 0,

$$(4) \quad W(\mathcal{C}(h)) = \int_{U_h(1)} \cdots \int_{U_h(\ell(h))} \prod_{i \in \{1, \dots, \ell(h)\}} \rho(e_h(i) - e_h(i-1), x_i - x_{i-1}) dx_1 \cdots dx_{\ell(h)}$$

if 0 is in $U_h(0)$ and is equal to 0 otherwise. In studying questions about the approximation of Brownian motion by discrete time random walks, it is helpful to view the measure W as being a measure on $D([0, \infty): \mathbb{R})$ that gives full measure to the closed subset $C([0, \infty): \mathbb{R})$.

2.4. The additive group $\mathbb{Q}_p$. See the books by Gouvêa [14] and by Ramakrishnan and Valenza [22] for further background on the p -adic numbers that this subsection summarizes. Refer to the book by Vladimirov, Volovich, and Zelenov [32] for both an introduction to p -adic mathematical physics and for many helpful examples of integration over $\mathbb{Q}_p$.

For any prime p , denote by $|\cdot|_p$ the p -adic absolute value on $\mathbb{Q}$ and by $\mathbb{Q}_p$ the field of p -adic numbers, the analytic completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_p$. Denote by $\mathbb{B}_k(x)$ and $\mathbb{S}_k(x)$ the sets

$$\mathbb{B}_k(x) = \{z \in \mathbb{Q}_p: |x - z|_p \leq p^k\} \quad \text{and} \quad \mathbb{S}_k(x) = \{z \in \mathbb{Q}_p: |x - z|_p = p^k\},$$

and by $\mathbb{Z}_p$ the *ring of integers*, the set $\mathbb{B}_0(0)$. Take μ to be the unique Haar measure on the locally compact Hausdorff abelian group $(\mathbb{Q}_p, +)$ that gives $\mathbb{Z}_p$ unit measure. Compress notation by henceforth suppressing p in the notation for $|\cdot|_p$, and wherever else it is unambiguous to do so.

For any x in $\mathbb{Q}_p$, there is a unique function a_x with

$$(5) \quad a_x: \mathbb{Z} \rightarrow \{0, 1, 2, \dots, p-1\} \quad \text{and} \quad x = \sum_{k \in \mathbb{Z}} a_x(k) p^k.$$

Denote respectively by $\{x\}$ and χ the rational number and the additive character given by

$$\{x\} = \sum_{k < 0} a_x(k) p^k \quad \text{and} \quad \chi(x) = e^{2\pi\sqrt{-1}\{x\}}.$$

The Fourier transform $\mathcal{F}$ and inverse Fourier transform $\mathcal{F}^{-1}$ are unitary operators on $L^2(\mathbb{Q}_p)$ that are given for any f in $L^2(\mathbb{Q}_p)$ by

$$(\mathcal{F}f)(x) = \int_{\mathbb{Q}_p} \chi(-xy) f(y) d\mu(y) \quad \text{and} \quad (\mathcal{F}^{-1}f)(y) = \int_{\mathbb{Q}_p} \chi(xy) f(x) d\mu(x),$$

where the integrals are improper integrals for functions that are in $L^2(\mathbb{Q}_p) \setminus L^1(\mathbb{Q}_p)$.

2.5. Brownian motion in $\mathbb{Q}_p$. For further background on parabolic equations over non-Archimedean fields and the theory of p -adic diffusion, refer to the books by Kochubei [18] and Zúñiga-Galindo [35]. The latter reference discusses diffusion processes of a more general type than those to be presently studied.

Fix b to be in $(0, \infty)$ and denote by $SB(\mathbb{Q}_p)$ the Schwarz-Bruhat space of compactly supported, locally constant, $\mathbb{C}$ -valued functions. Take $\mathcal{M}$ to be the multiplication operator that acts on $SB(\mathbb{Q}_p)$ by

$$(\mathcal{M}f)(x) = |x|^b f(x).$$

Denote by Δ'_b the unique self-adjoint extension of the essentially self-adjoint operator that acts on any f in $SB(\mathbb{Q}_p)$ by

$$(\Delta'_b f)(x) = (\mathcal{F}^{-1} \mathcal{M} \mathcal{F} f)(x).$$

For any $\mathbb{C}$ -valued function g with domain $(0, \infty) \times \mathbb{Q}_p$ and any t in $(0, \infty)$, take g_t to be the function that is defined for every x in $\mathbb{Q}_p$ by

$$g_t(x) = g(t, x).$$

Denote by $\mathcal{D}(\Delta_b)$ the set of all such g so that for all t in $(0, \infty)$, g_t is in the domain of Δ'_b . Take Δ_b to be the *Vladimirov operator with exponent b* that acts on any g in $\mathcal{D}(\Delta_b)$ by

$$(\Delta_b g)(t, x) = (\Delta_b g_t)(x).$$

Similarly extend the Fourier transform to act on $\mathbb{C}$ -valued functions on $(0, \infty) \times \mathbb{Q}_p$ that are square integrable for each positive t and the operator $\frac{d}{dt}$ to act on $\mathbb{C}$ -valued functions on $(0, \infty) \times \mathbb{Q}_p$ that for any x in $\mathbb{Q}_p$ are differentiable in the t variable.

Fix σ to be in $(0, \infty)$. The function ρ that is for any (t, x) in $(0, \infty) \times \mathbb{Q}_p$ given by

$$(6) \quad \rho(t, x) = \left(\mathcal{F}^{-1} e^{-\sigma t |\cdot|^b} \right)(x)$$

is the fundamental solution to the pseudo-differential equation

$$(7) \quad \frac{du}{dt} = -\sigma \Delta_b u.$$

A minor modification of Varadarajan's arguments [26] shows that

$$(8) \quad \rho(t, x) = \sum_{k \in \mathbb{Z}} p^k \left(e^{-\sigma t p^{kb}} - e^{-\sigma t p^{(k+1)b}} \right) \mathbf{1}_{\mathbb{B}_{-k}}(x).$$

The set $\{\rho(t, \cdot) : t \in (0, \infty)\}$ is a convolution semigroup of probability density functions that determines the probabilities of simple cylinder sets in the same way as in the real case, by (4). These probabilities determine a probability measure P on $D([0, \infty) : \mathbb{Q}_p)$ that is concentrated on the paths that are initially at 0. The triple $(D([0, \infty) : \mathbb{Q}_p), P, Y)$ is a *p-adic Brownian motion*.

2.6. Construction of approximants. Henceforth, take m to be an index that varies in $\mathbb{N}_0$ and F to be either $\mathbb{R}$ or $\mathbb{Q}_p$. For any discrete, infinite, additive abelian group G , take $\tilde{X}$ to be an abstract G -valued random variable and $(\tilde{X}_i)$ to be a sequence of independent abstract random variables with the same distribution as $\tilde{X}$. Denote by $\tilde{S}_n$ the abstract random variable

$$(9) \quad \tilde{S}_n = \tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_n,$$

where $\tilde{X}_0$ almost surely takes on the value 0, the identity element of the group. The abstract stochastic process $\tilde{S}$ is a *primitive process* with *generator* $\tilde{X}$. For any history $((0, U_0), (n_1, U_1), \dots, (n_N, U_N))$ of length N with epoch in $\mathbb{N}_0$ and route in G , define

$$(10) \quad \begin{aligned} & \text{Prob} \left((\tilde{S}_0 \in U_0) \cap \cdots \cap (\tilde{S}_{n_N} \in U_N) \right) \\ &= \sum_{x_0 \in U_0} \cdots \sum_{x_N \in U_N} \text{Prob} \left(\tilde{S}_0 = x_0 \right) \prod_{1 \leq i \leq N} \text{Prob} \left(\tilde{S}_{n_i - n_{i-1}} = x_i - x_{i-1} \right). \end{aligned}$$

For each n , take S_n to be the function

$$S_n : F(\mathbb{N}_0 : G) \rightarrow G \quad \text{by} \quad S_n(\omega) = \omega(n).$$

The Kolmogorov Extension theorem guarantees the existence of a measure P^* on the cylinder sets of $F(\mathbb{N}_0 : G)$ so that the stochastic process $(F(\mathbb{N}_0 : G), P^*, S)$, the *primitive process*, is a model for $\tilde{S}$.

Definition 2.1. For any positive null sequence (δ_m) , a sequence (Γ_m) of *spatial embeddings* of G into $\mathbb{F}$ with *spatial scale* δ is a sequence of injective functions from G to $\mathbb{F}$ so that for any f in $\mathbb{F}$, there is a g in G so that

$$0 < |\Gamma_m(g) - f| \leq \delta_m,$$

and for any g_1 and g_2 in G ,

$$|\Gamma_m(g_1) - \Gamma_m(g_2)| < \delta_m \quad \text{implies that} \quad g_1 = g_2.$$

Definition 2.2. For any positive null sequence (τ_m) , a *sequence of spatiotemporal embeddings* of $\mathbb{N}_0 \times G$ into $[0, \infty) \times \mathbb{F}$ with time scale τ and sequence of spatial embeddings Γ is a sequence ι with

$$\iota_m: \mathbb{N}_0 \times G \rightarrow [0, \infty) \times \mathbb{F} \quad \text{by} \quad \iota_m(n, g) = (n\tau_m, \Gamma_m(g)).$$

For any sequence ι of spatiotemporal embeddings of $\mathbb{N}_0 \times G$ into $[0, \infty) \times \mathbb{F}$ and for any t in $[0, \infty)$, take $\tilde{Y}^m$ to be the abstract stochastic process that is given by

$$\tilde{Y}_t^m = \Gamma_m(\tilde{S}_{\lfloor \frac{t}{\tau_m} \rfloor}).$$

In the cases to be investigated, bounds on moments guarantee that $\tilde{Y}^m$ has a model with sample paths in $D([0, \infty): \mathbb{F})$ that almost surely take values in $\Gamma_m(G)$. Denote by P^m the measure on the cylinder sets of $D([0, \infty): \mathbb{F})$ that gives rise to the finite dimensional distributions of $\tilde{Y}^m$. Define $\iota_m S$ to be the process $(D([0, \infty): \mathbb{F}), P^m, Y)$. For appropriate choices of the primitive process and the sequence of spatiotemporal embeddings, the sequence of measures (P^m) converges weakly to the probability measure P for a Brownian motion in $\mathbb{F}$. Such a sequence of approximating measures exists for any Brownian motion in $\mathbb{F}$. In this precise sense, Brownian motion in $\mathbb{F}$ is a scaling limit of a primitive discrete time random walk on G .

2.7. Classical example: Real Brownian motion. The abstract random variable $\tilde{X}$ that is given by the law

$$\begin{cases} \text{Prob}(\tilde{X} = -1) = \frac{1}{2} \\ \text{Prob}(\tilde{X} = 1) = \frac{1}{2} \end{cases}$$

generates the abstract $\mathbb{Z}$ -valued stochastic process $\tilde{S}$, by (9). Calculate the mean of $\tilde{S}_n$, $\mathbb{E}[\tilde{S}_n]$, and the variance of $\tilde{S}_n$, $\text{Var}[\tilde{S}_n]$, to verify that

$$\mathbb{E}[\tilde{S}_n] = 0 \quad \text{and} \quad \text{Var}[\tilde{S}_n] = n.$$

For any space scale (δ_m) and time scale (τ_m) , define Γ and ι for any (n, z) in $N_0 \times \mathbb{Z}$ by

$$\Gamma_m(z) = \delta_m z \quad \text{and} \quad \iota_m(n, z) = (n\tau_m, \Gamma_m(z)).$$

The stochastic process $(F([0, \infty): \mathbb{R}), P^m, Y)$ is a model for $\tilde{Y}^m$. Denote by $\mathbb{E}^m$ the expected value with respect to P^m . Independence of the $\tilde{X}_i$ imply that for any positive t ,

$$\mathbb{E}^m \left[|Y_{t_2} - Y_{t_1}|^2 \right] = \mathbb{E} \left[\left| \delta_m \tilde{S}_{\lfloor \frac{t_2}{\tau_m} \rfloor} - \delta_m \tilde{S}_{\lfloor \frac{t_1}{\tau_m} \rfloor} \right|^2 \right] = \delta_m^2 \left(\lfloor \frac{t_2}{\tau_m} \rfloor - \lfloor \frac{t_1}{\tau_m} \rfloor \right).$$

Independence of the increments of Y imply Proposition 2.1.

Proposition 2.1. For any strictly increasing sequence (t_1, t_2, t_3) in $[0, \infty)$,

$$\mathbb{E}^m \left[|Y_{t_2} - Y_{t_1}|^2 |Y_{t_3} - Y_{t_2}|^2 \right] \leq \frac{\delta_m^4}{\tau_m^2} (t_3 - t_1)^2.$$

Proposition 2.1 implies that the process has a version with sample paths in $D([0, \infty): \mathbb{R})$ [11, 12]. Denote once again by P^m the probability measure for this version and take $\iota_m S$ to be the process $(D([0, \infty): \mathbb{R}), P^m, Y)$. If there is a positive constant D with

$$(11) \quad \frac{\delta_m^2}{\tau_m} \rightarrow D,$$

then the estimate given by Proposition 2.1 is uniform in m , which implies the uniform tightness of (P^m) . Uniform tightness and the convergence on the simple cylinder sets of (P^m) to the Wiener measure W with diffusion constant D together imply the weak convergence of (P^m) to W [11, 12]. Although each P^m gives full measure to the $\delta_m \mathbb{Z}$ -valued step functions whose jumps occur almost surely in $\tau_m \mathbb{N}$, the measure W is concentrated on the continuous functions.

3. THE PRIMITIVE PROCESS AND ITS GENERATOR

3.1. The primitive p -adic state space and its refinements. Take G_m to be the quotient group $\mathbb{Q}_p/p^m \mathbb{Z}_p$ and $[\cdot]_m$ to be the quotient map from $\mathbb{Q}_p$ onto G_m that is given for each x in $\mathbb{Q}_p$ by

$$[x]_m = x + p^m \mathbb{Z}_p.$$

Define the absolute value $|\cdot|_m$ on G_m by

$$|[x]_m|_m = \begin{cases} |x| & \text{if } [x]_m \neq [0]_m \\ 0 & \text{if } [x]_m = [0]_m. \end{cases}$$

For any g in G_m , denote by $\mathbb{B}_k^m(g)$ and $\mathbb{S}_k^m(g)$ the sets

$$\mathbb{B}_k^m(g) = \{h \in G_m : |h - g|_m \leq p^k\} \quad \text{and} \quad \mathbb{S}_k^m(g) = \{h \in G_m : |h - g|_m = p^k\}.$$

Suppress g in the notation when g is equal to $[0]$.

Denote by G the group G_0 and by $[\cdot]$ the quotient map $[\cdot]_0$. Take μ_m to be the counting measure on G_m scaled by a factor of p^{-m} , so that for any g the volume of $\mathbb{B}_k^m(g)$ and $\mathbb{S}_k^m(g)$ are respectively given by

$$\text{Vol}(\mathbb{B}_k^m(g)) = p^{-m} p^k \quad \text{and} \quad \text{Vol}(\mathbb{S}_k^m(g)) = p^{-m} p^k \left(1 - \frac{1}{p}\right).$$

For any integrable function f on G_m ,

$$(12) \quad \int_{G_m} f([x]_m) d\mu_m([x]_m) = \sum_{[x]_m \in G_m} f([x]_m) p^{-m}.$$

Proposition 3.1 follows from the observation that any function that is defined on $\mathbb{Q}_p$ by

$$x \mapsto f([x]_m)$$

is locally constant with radius of local constancy equal to p^{-m} , and the fact that any ball in $\mathbb{Q}_p$ with radius at least p^{-m} is a disjoint union of balls of radius p^{-m} .

Proposition 3.1. *For any $\mathbb{C}$ -valued integrable function f on G_m and any ball B of radius at least p^{-m} in $\mathbb{Q}_p$,*

$$\int_{[B]_m} f([x]_m) d\mu_m([x]_m) = \int_B f([x]_m) d\mu(x).$$

Take α_m to be the isomorphism from G to G_m that is defined for any $[x]$ in G by

$$\alpha_m(x + \mathbb{Z}_p) = p^m x + p^m \mathbb{Z}_p.$$

The group $p^{-m}\mathbb{Z}_p$ is the Pontryagin dual of the group G_m and any element of G_m is $[x]_m$ for some x in $\mathbb{Q}_p$. Use the canonical inclusion map that takes $p^{-m}\mathbb{Z}_p$ into $\mathbb{Q}_p$ to define the dual pairing $\langle \cdot, \cdot \rangle_m$, the function

$$\langle \cdot, \cdot \rangle_m : p^{-m}\mathbb{Z}_p \times G_m \rightarrow \mathbb{S}^1 \quad \text{by} \quad \langle [x]_m, y \rangle_m = \chi(xy).$$

For any $([x]_m, w, y)$ in $G_m \times p^m\mathbb{Z}_p \times p^{-m}\mathbb{Z}_p$, the product wy is in $\mathbb{Z}_p$. The equality

$$\chi((x+w)y) = \chi(xy)\chi(wy) = \chi(xy)$$

follows from the fact that χ is an additive rank 0 character on $\mathbb{Q}_p$, and it implies that the definition of the dual pairing is independent of the choice of the representative. The equalities

$$\langle \alpha_m([x]), y \rangle_m = \langle [p^m x]_m, y \rangle_m = \chi(p^m xy) = \langle [x], p^m y \rangle$$

imply Proposition 3.2.

Proposition 3.2. *For any (x, y) in $\mathbb{Q}_p \times p^{-m}\mathbb{Z}_p$,*

$$\langle [x]_m, y \rangle_m = \langle [x], p^m y \rangle.$$

Define the Fourier transform $\mathcal{F}_m$ that takes $L^2(G_m)$ to $L^2(p^{-m}\mathbb{Z}_p)$ and the inverse Fourier transform $\mathcal{F}_m^{-1}$ that takes $L^2(p^{-m}\mathbb{Z}_p)$ to $L^2(G_m)$ to be given for any f in $L^2(G_m)$ and any $\tilde{f}$ in $L^2(p^{-m}\mathbb{Z}_p)$ by

$$\begin{aligned} (\mathcal{F}_m f)(y) &= \int_{G_m} \langle -[x]_m, y \rangle_m f([x]_m) d\mu_m([x]_m) \\ \text{and } (\mathcal{F}_m^{-1} \tilde{f})([x]_m) &= \int_{p^{-m}\mathbb{Z}_p} \langle [x]_m, y \rangle_m \tilde{f}(y) d\mu(y). \end{aligned}$$

3.2. The generator of the primitive process. Take $\tilde{X}$ to be the G_0 -valued abstract random variable with probability mass function $\rho_{\tilde{X}}$ given by

$$(13) \quad \begin{cases} \rho_{\tilde{X}}([x]) = \frac{p^b - 1}{p^{ib}} \frac{1}{p^i - p^{i-1}} & \text{if } |[x]| = p^i \\ \rho_{\tilde{X}}([0]) = 0, \end{cases} \quad \text{so that} \quad \begin{cases} \text{Prob}(\tilde{X} \in \mathbb{S}_i^0) = \frac{p^b - 1}{p^{ib}} \\ \text{Prob}(\tilde{X} = [0]) = 0. \end{cases}$$

The law for $\tilde{X}$ is different from the one previously introduced [7]. Denote by $\phi_{\tilde{X}}$ the Fourier transform of $\rho_{\tilde{X}}$, the characteristic function of $\tilde{X}$ that has domain equal to $\mathbb{Z}_p$. To compress notation, write

$$\rho(i) = \frac{p^b - 1}{p^{ib}} \cdot \frac{1}{p^i - p^{i-1}}.$$

For any y in $\mathbb{Z}_p$, there is a k in $\mathbb{N}_0$ so that

$$(14) \quad |y| = p^{-k},$$

and so

$$\begin{aligned} \phi_{\tilde{X}}(y) &= \int_G \langle -[x], y \rangle \rho_X([x]) d\mu_0([x]) \\ &= \sum_{i \in \mathbb{N}} \rho(i) \int_{\mathbb{S}_i^0} \langle -[x], y \rangle d\mu_0([x]) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in \mathbb{N}} \rho(i) \left\{ \int_{\mathbb{B}_i^0} \langle -[x], y \rangle d\mu_0([x]) - \int_{\mathbb{B}_{i-1}^0} \langle -[x], y \rangle d\mu_0([x]) \right\} \\
&= -\rho(1) \int_{\mathbb{B}_0^0} \langle -[x], y \rangle d\mu_0([x]) + \sum_{i \in \mathbb{N}} (\rho(i) - \rho(i+1)) \int_{\mathbb{B}_i^0} \langle -[x], y \rangle d\mu_0([x]) \\
&= -\frac{p^b - 1}{p^b} \cdot \frac{1}{p-1} \mathbb{1}_{\mathbb{Z}_p}(y) + (p^b - 1) \left(\frac{p}{p-1} \right) \left(1 - \frac{1}{p^{b+1}} \right) \sum_{i \in \mathbb{N}} \frac{1}{p^{ib}} \mathbb{1}_{p^{-i}}(y) \\
(15) \quad &= -\frac{p^b - 1}{p^b} \cdot \frac{1}{p-1} + (p^b - 1) \left(\frac{p}{p-1} \right) \left(1 - \frac{1}{p^{b+1}} \right) \sum_{i=1}^k \frac{1}{p^{ib}}.
\end{aligned}$$

Use (14) to simplify (15) and obtain Proposition 3.3.

Proposition 3.3. *For any y in $\mathbb{Z}_p$,*

$$\phi_{\tilde{X}}(y) = 1 - \alpha|y|^b \quad \text{where} \quad \alpha = \frac{p^{b+1} - 1}{p^{b+1} - p^b}.$$

Proposition 3.4. *The values $\frac{\alpha}{p^b}$ and $\alpha - 1$ are both in $(0, 1)$.*

Proof. The Vladimirov exponent b is positive and p is at least 2, so

$$\alpha - 1 = \frac{1}{p-1} \cdot \frac{p^b - 1}{p^b} < 1.$$

Multiply $2p^{2b+1} \ln(p)$ by both sides of the inequality

$$\frac{p-1}{p} > \frac{1}{2p^b}$$

and write both sides of the resulting inequality as derivatives to obtain the inequality

$$\frac{d}{db}(p^{2b+1} - p^{2b}) > \frac{d}{db}(p^{b+1} - 1).$$

The equality of $p^{b+1} - 1$ and $p^{2b+1} - p^{2b}$ when b is equal to 0 implies that for any positive b ,

$$\frac{\alpha}{p^b} = \frac{p^{b+1} - 1}{p^{2b+1} - p^{2b}} < 1.$$

□

3.3. The primitive process. Take $\tilde{S}$ to be the abstract stochastic process defined by (9) and, for any n in $\mathbb{N}$, take $\rho^*(n, \cdot)$ to be the probability mass function for $\tilde{S}_n$.

Proposition 3.5. *For any natural number n and $[x]$ in G ,*

$$(16) \quad \rho^*(n, [x]) = (1 - \alpha)^n \mathbb{1}_{\mathbb{B}_0^0}([x]) + \sum_{i \in \mathbb{N}} \left(\left(1 - \frac{\alpha}{p^{ib}} \right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}} \right)^n \right) p^{-i} \mathbb{1}_{\mathbb{B}_i^0}([x]).$$

Proof. The Fourier transform takes the convolution of functions to their product, so compute the inverse Fourier transform of the n^{th} power of $\phi_{\tilde{X}}$ to obtain the equalities

$$\rho^*(n, [x]) = \sum_{i \in \mathbb{N}_0} \int_{|y|=p^{-i}} \langle [x], y \rangle (1 - \alpha|y|^b)^n dy$$

$$\begin{aligned}
&= \sum_{i \in \mathbb{N}_0} \left(1 - \frac{\alpha}{p^{ib}}\right)^n \int_{|y|=p^{-i}} \langle [x], y \rangle dy \\
&= (1 - \alpha)^n \left(\int_{|y| \leq 1} \langle [x], y \rangle dy - \int_{|y| \leq p^{-1}} \langle [x], y \rangle dy \right) \\
&\quad + \sum_{i \in \mathbb{N}} \left(1 - \frac{\alpha}{p^{ib}}\right)^n \left(\int_{|y| \leq p^{-i}} \langle [x], y \rangle dy - \int_{|y| \leq p^{-(i+1)}} \langle [x], y \rangle dy \right) \\
&= (1 - \alpha)^n \int_{|y| \leq 1} \langle [x], y \rangle dy \\
&\quad + \sum_{i \in \mathbb{N}} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right) \int_{|y| \leq p^{-i}} \langle [x], y \rangle dy \\
&= (1 - \alpha)^n \mathbf{1}_{\mathbb{B}_0^0}([x]) + \sum_{i \in \mathbb{N}} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right) p^{-i} \mathbf{1}_{\mathbb{B}_i^0}([x]).
\end{aligned}$$

□

The Kolmogorov extension theorem guarantees that $\tilde{S}$ has a model with sample paths in $F(\mathbb{N}_0: G)$, a stochastic process $(F(\mathbb{N}_0: G), P^*, S)$ with finite dimensional distributions given by (10). Take $\mathbb{E}_*[\cdot]$ to be the expected value with respect to P^* .

3.4. Bounds for low order moments of the primitive process. In the real setting, moment estimates come from a straightforward calculation of variance. Determining sufficient estimates in the $\mathbb{Q}_p$ setting, irrespective of the value of b , is more involved.

Theorem 3.1. *There is a K in $\mathbb{R}$ so that for any r in $(0, b)$ and any n in $\mathbb{N}$,*

$$(17) \quad \mathbb{E}_* [|S_n|^r] < K n^{\frac{r}{b}}.$$

Proof. Use the explicit expression for $\rho^*(n, x)$ to obtain the equalities

$$\begin{aligned}
(18) \quad \mathbb{E}_* [|S_n|^r] &= \int_G |[x]|^r \rho^*(n, [x]) d\mu_0([x]) \\
&= \sum_{i \in \mathbb{N}} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right) \int_G |[x]|^r p^{-i} \mathbf{1}_{\mathbb{B}_i^0}([x]) d\mu_0([x]).
\end{aligned}$$

An integral over G is a countable sum of integrals over the $\mathbb{S}_i^0$, namely

$$\begin{aligned}
(19) \quad \int_G |[x]|^r p^{-i} \mathbf{1}_{\mathbb{B}_i^0}([x]) d\mu_0([x]) &= \left(\int_{\mathbb{S}_i^0} |[x]|^r d\mu_0([x]) + \cdots + \int_{\mathbb{S}_1^0} |[x]|^r d\mu_0([x]) + 0 \right) p^{-i} \\
&= \left(p^r (p^1 - 1) + \cdots + p^{ir} (p^i - p^{i-1}) \right) p^{-i} \\
&= \frac{p^{r+1} - p^r}{p^{r+1} - 1} (p^{ir} - p^{-i}) < (p^{ir} - p^{-i}).
\end{aligned}$$

Equalities (18) and (19) together imply that

$$\begin{aligned}
\mathbb{E}_* [|S_n|^r] &< \sum_{i \in \mathbb{N}} (p^{ir} - p^{-i}) \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right) \\
&= \sum_{i \in \mathbb{N}} p^{ir} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right) - \sum_{i \in \mathbb{N}} p^{-i} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right).
\end{aligned}$$

For any natural number i that is greater than 1,

$$(20) \quad \left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n = n \int_{1 - \frac{\alpha}{p^{(i-1)b}}}^{1 - \frac{\alpha}{p^{ib}}} s^{n-1} ds.$$

The integrand in (20) is increasing, and so

$$(21) \quad \left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n < n \left(1 - \frac{\alpha}{p^{ib}}\right)^{n-1} \frac{\alpha}{p^{(i-1)b}} \left(1 - \frac{1}{p^b}\right).$$

Take $I(n)$ to be the quantity

$$I(n) = \sum_{i>1} p^{ir} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right).$$

For any natural number i that is greater than 1,

$$p^{-i} \left(\left(1 - \frac{\alpha}{p^{ib}}\right)^n - \left(1 - \frac{\alpha}{p^{(i-1)b}}\right)^n \right) > 0,$$

and so

$$(22) \quad \mathbb{E}_* [|S_n|^r] < I(n) + (p^r - p^{-1}) \left(\left(1 - \frac{\alpha}{p^b}\right)^n - (1 - \alpha)^n \right).$$

The inequality (21) implies that

$$(23) \quad \begin{aligned} I(n) &< \sum_{i>1} p^{ir} n \left(1 - \frac{\alpha}{p^{ib}}\right)^{n-1} \frac{\alpha}{p^{(i-1)b}} \left(1 - \frac{1}{p^b}\right) \\ &= n \alpha^{\frac{r}{b}} \sum_{i>1} \left(\frac{\alpha}{p^{ib}}\right)^{-\frac{r}{b}} \left(1 - \frac{\alpha}{p^{ib}}\right)^{n-1} \frac{\alpha}{p^{(i-1)b}} \left(1 - \frac{1}{p^b}\right). \end{aligned}$$

Write

$$x_i = \frac{\alpha}{p^{ib}}, \quad I_i = [x_i, x_{i-1}], \quad \text{and} \quad \Delta I_i = x_{i-1} - x_i$$

to obtain the equality

$$(24) \quad n \alpha^{\frac{r}{b}} \sum_{i>1} \left(\frac{\alpha}{p^{ib}}\right)^{-\frac{r}{b}} \left(1 - \frac{\alpha}{p^{ib}}\right)^{n-1} \frac{\alpha}{p^{(i-1)b}} \left(1 - \frac{1}{p^b}\right) = n \alpha^{\frac{r}{b}} \sum_{i>1} x_i^{-\frac{r}{b}} (1 - x_i)^{n-1} \Delta I_i.$$

Reindex the sum in (23) and use the fact that ΔI_i is equal to $p^b \Delta I_{i+1}$ together with (24) to see that

$$(25) \quad I(n) < n \alpha^{\frac{r}{b}} p^b \sum_{i>2} x_{i-1}^{-\frac{r}{b}} (1 - x_{i-1})^{n-1} \Delta I_i,$$

and so

$$(26) \quad I(n) < n \alpha^{\frac{r}{b}} p^b \int_0^{\frac{\alpha}{p^{2b}}} x^{-\frac{r}{b}} (1 - x)^{n-1} \Delta I_i$$

since the sum in (25) is a lower Riemann sum approximation of the integral in (26). The positivity on $(0, 1)$ of the integrand in (26) implies that

$$(27) \quad \begin{aligned} I(n) &< n \alpha^{\frac{r}{b}} p^b \int_0^1 x^{-\frac{r}{b}} (1 - x)^{n-1} dx \\ &= n \alpha^{\frac{r}{b}} p^b B\left(\frac{b-r}{b}, n\right) = n \alpha^{\frac{r}{b}} p^b \frac{\Gamma\left(\frac{b-r}{b}\right) \Gamma(n)}{\Gamma\left(\frac{b-r}{b} + n\right)}, \end{aligned}$$

where B is the beta function and Γ is the gamma function.

Wendel proved [34] that for any positive x and a in $[0, 1]$,

$$1 \leq \frac{x^a \Gamma(x)}{\Gamma(x+a)} \leq \left(\frac{x}{x+a} \right)^{1-a},$$

and so

$$(28) \quad \frac{\Gamma(n)}{\Gamma\left(\frac{b-r}{b} + n\right)} \leq n^{-\frac{b-r}{b}} \left(\frac{n + \frac{b-r}{b}}{n} \right)^{1-\frac{b-r}{b}} = n^{-\frac{b-r}{b}} \left(\frac{n + \frac{b-r}{b}}{n} \right)^{\frac{r}{b}} = e_1(n) n^{-\frac{b-r}{b}}$$

where for all n ,

$$e_1(n) < 2 \quad \text{and} \quad \lim_{n \rightarrow \infty} e_1(n) = 1.$$

The inequalities (27) and (28) together imply that

$$(29) \quad I(n) < 2\alpha^{\frac{r}{b}} p^b \Gamma\left(\frac{b-r}{b}\right) n^{\frac{r}{b}}.$$

To bound from above the second summand in (22), take

$$\delta = \max \left\{ \left| 1 - \frac{\alpha}{p^b} \right|, |1 - \alpha| \right\} < 1$$

so that

$$(30) \quad 0 < \left| \left(1 - \frac{\alpha}{p^b} \right)^n - (1 - \alpha)^n \right| \leq 2\delta^n = e_2(n) n^{\frac{r}{b}} \quad \text{where} \quad e_2(n) = (2\delta^n n^{-\frac{r}{b}}).$$

Since $e_2(n) n^{\frac{r}{b}}$ tends to zero as n tends to infinity and $e_2(n)$ is bounded above by 2, the inequalities (22), (29), and (30) together imply that

$$\mathbb{E}[|S_n|^r] < 2\alpha^{\frac{r}{b}} p^b \Gamma\left(\frac{b-r}{b}\right) n^{\frac{r}{b}} + 2p^r n^{\frac{r}{b}} = K n^{\frac{r}{b}}, \quad \text{where} \quad K = 2\left(p^r + \alpha^{\frac{r}{b}} p^b \Gamma\left(\frac{b-r}{b}\right)\right).$$

□

4. CONVERGENCE OF THE DISCRETE TIME PROCESSES

4.1. Embeddings into the continuous state space. Take q_m to be an injection from G_m to $\mathbb{Q}_p$ that has the property that for any g in G_m , $[q_m(g)]_m$ is equal to g . Denote by $\mathbf{1}_m$ the indicator function on the set $(-\infty, m] \cap \mathbb{Z}$. Use q_m and (5) to define for any g in G_m the injection j_m from G_m to $\mathbb{Q}$ by

$$j_m(g) = \sum_{k \in \mathbb{Z}} a_{q_m(g)}(k) \mathbf{1}_m(k) p^k,$$

which is independent of the choice of injection q_m . The sequence of functions

$$(31) \quad \Gamma_m: G \rightarrow \mathbb{Q}_p \quad \text{by} \quad \Gamma_m(g) = j_m(\alpha_m(g))$$

is a sequence of spatial embeddings of G into $\mathbb{Q}_p$ with spatial scale (p^{-m}) . For any g in G ,

$$(32) \quad j_m(\alpha_m(g)) = p^m j_0(g).$$

The sequence of discrete sets $(\Gamma_m(G))$ is a sequence of refinements of grids that approximates $\mathbb{Q}_p$ in the sense that, for any m_1 and m_2 in $\mathbb{N}_0$, if m_1 is less than m_2 , then $\Gamma_{m_1}(G)$ is a subset of $\Gamma_{m_2}(G)$, and the union over all m of the $\Gamma_m(G)$ is dense in $\mathbb{Q}_p$. Equation (32) implies Proposition 4.1.

Proposition 4.1. *For any finite sequence $(g_i)_{i \in \{1, \dots, k\}}$ in G ,*

$$\left| \Gamma_m \left(\sum_{i=1}^k g_i \right) \right| = p^{-m} \left| \sum_{i=1}^k g_i \right|.$$

4.2. Spatiotemporal embeddings of the primitive process. Denote by $\tilde{X}^m$ the abstract random variable $\alpha_m \circ \tilde{X}$. The probability density function for $\tilde{X}^m$, $\rho_{\tilde{X}^m}$, is given for any x in $\mathbb{Q}_p$ by

$$\rho_{\tilde{X}^m}([x]_m) = \rho_{\tilde{X}}([x]).$$

The group $p^{-m}\mathbb{Z}_p$ is the Pontryagin dual of G_m and is the domain of $\phi_{\tilde{X}^m}$, the characteristic function of $\tilde{X}^m$.

Proposition 4.2. *For any y in $p^{-m}\mathbb{Z}_p$,*

$$\phi_{\tilde{X}^m}(y) = 1 - \alpha \frac{|y|^b}{p^{mb}}.$$

Proof. Proposition 3.1 implies that for any y in $p^{-m}\mathbb{Z}_p$,

$$\begin{aligned} \phi_{\tilde{X}^m}(y) &= \int_{G_m} \langle -[x]_m, y \rangle_m \rho_{\tilde{X}^m}([x]_m) d\mu_m([x]_m) \\ (33) \quad &= \int_{\mathbb{Q}_p} \langle -[x]_m, y \rangle_m \rho_{\tilde{X}^m}([x]_m) d\mu(x). \end{aligned}$$

Proposition 3.2 and (33) together imply that

$$\begin{aligned} \phi_{\tilde{X}^m}(y) &= \int_{\mathbb{Q}_p} \langle -[x], p^m y \rangle \rho_{\tilde{X}^m}([x]_m) d\mu(x) \\ (34) \quad &= \int_{\mathbb{Q}_p} \langle -[x], p^m y \rangle \rho_{\tilde{X}}([x]) d\mu(x). \end{aligned}$$

Proposition 3.1 permits the integral in (34) to be rewritten as an integral over G , and so

$$\begin{aligned} \phi_{\tilde{X}^m}(y) &= \int_G \langle -[x], p^m y \rangle \rho_{\tilde{X}}([x]) d\mu_0([x]) \\ &= (\mathcal{F}\rho_{\tilde{X}})(p^m y) = 1 - \alpha |p^m y|^b = 1 - \frac{\alpha |y|^b}{p^{mb}}, \end{aligned}$$

where Proposition 3.3 implies the penultimate equality. □

Take τ to be the sequence of time scalings $(\frac{\sigma}{\alpha} p^{mb})$ and ι to be the sequence of spatiotemporal embeddings of G into $\mathbb{Q}_p$ with time scaling τ and spatial embeddings given by the Γ_m that (31) defines. As discussed in the previous work [7], the relationship between the space and time scalings that guarantee the convergence of the discrete time processes to the Brownian motion in $\mathbb{Q}_p$ is similar in the p -adic and real settings, namely,

$$\frac{\delta_m^b}{\tau_m} \rightarrow \frac{\sigma}{\alpha}.$$

As should be expected, this relationship continues to hold for more general b . Take $\lambda(m)$ to be the reciprocal of $\tau(m)$ and take $\tilde{Y}^m$ and $\tilde{Z}^m$ to be the abstract processes that are for any

t in $[0, \infty)$ given by

$$\tilde{Y}_t^m = \Gamma_m(\tilde{S}_{\lfloor t\lambda(m) \rfloor}) \quad \text{and} \quad \tilde{Z}_t^m = \alpha_m(\tilde{S}_{\lfloor t\lambda(m) \rfloor}).$$

All probabilities for $\tilde{Y}^m$ may be recovered from the probabilities for $\tilde{Z}^m$, since

$$\tilde{Y}^m = j_m \circ \alpha_m \circ \tilde{S}.$$

For all t in $(0, \infty)$, denote by $\rho_{\tilde{Z}^m}(t, \cdot)$ and $\phi_{\tilde{Z}^m}(t, \cdot)$ the probability density function and characteristic function for $\tilde{Z}_t^m$, respectively. Since the Fourier transform takes convolutions to products, Proposition 4.2 implies Proposition 4.3.

Proposition 4.3. *For any y in $p^{-m}\mathbb{Z}_p$ and t in $[0, \infty)$,*

$$\phi_{\tilde{Z}^m}(t, y) = \left(1 - \frac{\alpha|y|^b}{p^{mb}}\right)^{\lfloor t\lambda(m) \rfloor}.$$

Take E_m to be the function that is defined for any (t, y) in $[0, \infty) \times \mathbb{Q}_p$ by

$$E_m(t, y) = \begin{cases} \left(1 - \frac{\alpha|y|^b}{p^{mb}}\right)^{\lfloor t\lambda(m) \rfloor} & \text{if } |y| \leq p^m \\ 0 & \text{if } |y| > p^m. \end{cases}$$

For any fixed b in $(0, \infty)$, Proposition 3.4 implies Proposition 4.4.

Proposition 4.4. *For any t in $[0, \infty)$, the sequence of functions $(E_m(t, \cdot))$ converges uniformly to $e^{-t\sigma|\cdot|^b}$.*

4.3. Moment estimates for the embedded processes. The Kolmogorov Extension Theorem guarantees the existence of a model $(F([0, \infty): \mathbb{Q}_p), \mathbb{Q}^m, Y)$ for $\tilde{Y}^m$. Denote by $\mathbb{E}_m[\cdot]$ the expected value with respect to $\mathbb{Q}^m$.

Proposition 4.5. *There is a constant C such that for any t in $[0, \infty)$,*

$$\mathbb{E}_m[|Y_t|^r] < Ct^{\frac{r}{b}}.$$

Proof. For any t small enough so that $t\lambda(m)$ is in $[0, 1)$, $\mathbb{E}_m[|Y_t|^r]$ is equal to 0, which verifies the inequality. Denote by K the constant appearing in Proposition 3.1. If $t\lambda(m)$ is in $[1, \infty)$, then (32) implies that

$$\begin{aligned} \mathbb{E}_m[|Y_t|^r] &= \sum_{k \in \mathbb{N}_0} p^{kr} \text{Prob}\left(|\alpha_m \circ \tilde{S}_{\lfloor t\lambda(m) \rfloor}| = p^k\right) \\ &= \sum_{k \in \mathbb{N}_0} p^{(k-m)r} \text{Prob}\left(|\tilde{S}_{\lfloor t\lambda(m) \rfloor}| = p^k\right) \\ &= p^{-rm} \mathbb{E}_* \left[|\tilde{S}_{\lfloor t\lambda(m) \rfloor}|^r \right] \\ &< K p^{-rm} [t\lambda(m)]^{\frac{r}{b}} \leq K \left(\frac{\sigma}{\alpha}\right)^{\frac{r}{b}} t^{\frac{r}{b}}, \end{aligned}$$

where Proposition 3.1 implies the penultimate inequality. $\square$

Proposition 4.6. *There is a sequence of measures (P^m) on $D([0, \infty): \mathbb{Q}_p)$ so that for each m in $\mathbb{N}_0$, the triple $(D([0, \infty): \mathbb{Q}_p), P^m, Y)$ is a model for $\tilde{Y}^m$. Furthermore, the sequence (P^m) is a sequence of uniformly tight probability measures.*

Proof. For any strictly increasing finite sequence (t_1, t_2, t_3) in $[0, \infty)$, the independence of the increments of $\tilde{Y}^m$ implies that

$$\begin{aligned} \mathbb{E}_m[|Y_{t_3} - Y_{t_2}|^r | Y_{t_2} - Y_{t_1}|^r] &= \mathbb{E}_m[|Y_{t_3} - Y_{t_2}|^r] \mathbb{E}_m[|Y_{t_2} - Y_{t_1}|^r] \\ &\leq C(t_3 - t_2)^{\frac{r}{b}} C(t_2 - t_1)^{\frac{r}{b}} \leq C^2(t_3 - t_1)^{\frac{2r}{b}}. \end{aligned}$$

Take r to be in $(\frac{b}{2}, b)$ to verify that $(F([0, \infty): \mathbb{Q}_p), \mathbb{Q}^m, Y)$ satisfies Chentsov's criterion with constant C independent of both m and the choice of (t_1, t_2, t_3) [12]. $\square$

4.4. Convergence of the processes. The sequence $(D([0, \infty): \mathbb{Q}_p), \mathbb{P}^m, Y)$ of stochastic processes describe random walks on the embedded grids $\Gamma_m(G)$ in $\mathbb{Q}_p$, with jumps that occur only at time points that are positive multiples of τ_m . Proposition 4.7 makes this idea precise.

Proposition 4.7. *The measure $\mathbb{P}^m$ is concentrated on the subset of paths in $D([0, \infty): \mathbb{Q}_p)$ that are valued in $\Gamma_m(G)$ and are, for each natural number n , constant on the intervals $[(n-1)\tau_m, n\tau_m)$.*

Proof. For any strictly increasing finite sequence $(t_1, \dots, t_k)$ in $[0, \infty)$, paths in $D([0, \infty): \mathbb{Q}_p)$ are $\mathbb{P}^m$ -almost surely in $\Gamma_m(G)$ at any given place of $(t_1, \dots, t_k)$. Finite intersections of almost sure events are almost sure and so

$$\mathbb{P}^m(\{\omega \in D([0, \infty): \mathbb{Q}_p): \omega(t_i) \in \Gamma_m(G), 1 \leq i \leq k\}) = 1.$$

For any sequence (A_i) of strictly increasing nested finite subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$, continuity from above of $\mathbb{P}^m$ implies that

$$\begin{aligned} \mathbb{P}^m(\{\omega \in D([0, \infty): \mathbb{Q}_p): \omega(s) \in \Gamma_m(G), \forall s \in \mathbb{Q}\}) \\ = \lim_{i \rightarrow \infty} \mathbb{P}^m(\{\omega \in D([0, \infty): \mathbb{R}): \omega(s) \in \Gamma_m(G), \forall s \in A_i\}) = 1. \end{aligned}$$

The right continuity of the paths implies that the set of $\Gamma_m(G)$ -valued paths has full measure with respect to the measure $\mathbb{P}^m$.

For any interval I in $[0, \infty)$ that does not intersect the set $\tau_m \mathbb{N}$, take (V_i) to be a nested sequence of finite subsets of I whose union is $I \cap \mathbb{Q}$. For any pair of real numbers s_1 and s_2 in I , $\lfloor s_1 \lambda(m) \rfloor$ is equal to $\lfloor s_2 \lambda(m) \rfloor$ and so

$$\begin{aligned} \mathbb{P}^m(Y_{s_1} - Y_{s_2} = 0) &= \text{Prob}(\tilde{Y}_{s_1}^m - \tilde{Y}_{s_2}^m = 0) \\ &= \text{Prob}(\tilde{S}_{\lfloor s_1 \lambda(m) \rfloor} - \tilde{S}_{\lfloor s_2 \lambda(m) \rfloor} = 0) = 1. \end{aligned}$$

Continuity from above of the measure $\mathbb{P}^m$ implies that for any t in I ,

$$\begin{aligned} \mathbb{P}^m(\{\omega \in D([0, \infty): \mathbb{Q}_p): |\omega(t) - \omega(s)| = 0, \quad \forall s \in I \cap \mathbb{Q}\}) \\ = \lim_{i \rightarrow \infty} \mathbb{P}^m(\{\omega \in D([0, \infty): \mathbb{Q}_p): |\omega(t) - \omega(s)| = 0, \quad \forall s \in I \cap V_i\}) = 1. \end{aligned}$$

The right continuity of the paths implies that

$$\mathbb{P}^m(\{\omega \in D([0, \infty): \mathbb{Q}_p): |\omega(t) - \omega(s)| = 0, \quad \forall s, t \in I\}) = 1.$$

Take (I_n) to be a sequence of disjoint intervals that do not intersect $\tau_m \mathbb{N}$ and whose union is $\mathbb{Q} \setminus \tau_m \mathbb{N}$. Continuity from above of the measure $\mathbb{P}^m$ implies that

$$\mathbb{P}^m\left(\bigcap_{n \in \mathbb{N}} \{\omega \in D([0, \infty): \mathbb{Q}_p): |\omega(t) - \omega(s)| = 0, \quad \forall s, t \in I_n\}\right) = 1.$$

$\square$

Definition 4.1. The set H_R of all *restricted histories* for paths in $\mathbb{Q}_p$ is the set of all histories in $\mathbb{Q}_p$ whose route is a finite sequence of balls.

Proposition 4.8. For any restricted history h in H_R ,

$$P^m(\mathcal{C}(h)) \rightarrow P(\mathcal{C}(h)).$$

Proof. For any (t, x) in $(0, \infty) \times \mathbb{Q}_p$,

$$\begin{aligned} \rho_{\tilde{Z}^m}(t, [x]_m) &= \int_{p^{-m}\mathbb{Z}_p} \langle [x]_m, y \rangle \left(1 - \frac{\alpha|y|^b}{p^{mb}}\right)^{\lfloor t\ell(m) \rfloor} d\mu(y) \\ &= \int_{\mathbb{Q}_p} \chi(xy) E_m(t, y) d\mu(y). \end{aligned}$$

Denote by F_m the function that takes any x in $\mathbb{Q}_p$ to $E_m(t, x) \mathbb{1}_{\mathbb{B}_{m-1}(0)}(x)$, and denote by A_m the integral of $|E_m(t, \cdot)|$ over $\mathbb{S}_m(0)$. The bounds given by Proposition 3.4 imply that (A_m) is a positive null sequence and that the sequence of functions (F_m) increases to $e^{-t\sigma|\cdot|^b}$. There is, therefore, a positive null sequence (ε_k) so that for any m ,

$$(35) \quad \int_{\mathbb{B}_k(0)^c} |E_m(t, \cdot)| d\mu(x) < \varepsilon_k.$$

Proposition 4.4 and (35) together imply that for any positive t ,

$$\begin{aligned} |\rho_{\tilde{Z}^m}(t, [x]_m) - \rho_Y(t, x)| &\leq \int_{\mathbb{Q}_p} \left| \chi(xy) E_m(t, y) - \chi(xy) e^{-\sigma t|y|^b} \right| d\mu(y) \\ &= \int_{\mathbb{Q}_p} \left| E_m(t, y) - e^{-\sigma t|y|^b} \right| d\mu(y) \rightarrow 0, \end{aligned}$$

and so $(\rho_{\tilde{Z}^m}(t, [\cdot]_m))$ converges uniformly on $\mathbb{Q}_p$ to $\rho_Y(t, \cdot)$.

It is enough to suppose that $U_h(0)$ is the set $\{0\}$. Simplify the notation by writing

$$e_h = (t_0, \dots, t_k) \quad \text{and} \quad U_h = (\{0\}, U_1, \dots, U_k).$$

As long as m is large enough so that p^{-m} is less than the radius of any ball in U_h ,

$$\begin{aligned} P^m(\mathcal{C}(h)) &= P^m(Y_{t_1} \in U_1, Y_{t_2} \in U_2, \dots, Y_{t_n} \in U_n) \\ &= \int_{U_1} \cdots \int_{U_k} \prod_{i \in \{1, \dots, k\}} \rho_{\tilde{Z}^m}(t_i - t_{i-1}, [x_i] - [x_{i-1}]) d\mu(x_k) \cdots d\mu(x_1) \\ &\rightarrow \int_{U_1} \cdots \int_{U_k} \prod_{i \in \{1, \dots, k\}} \rho_Y(t_i - t_{i-1}, x_i - x_{i-1}) d\mu(x_k) \cdots d\mu(x_1) = P(\mathcal{C}(h)), \end{aligned}$$

since the integrand that is indexed by m is uniformly convergent as a product of bounded, uniformly convergent functions. $\square$

Theorem 4.1. The sequence of measures (P^m) converges weakly to P .

Proof. Proposition 4.6 implies that the family measures $\{P^m : m \in \mathbb{N}_0\}$ is uniformly tight. Since the intersection of any two balls in $\mathbb{Q}_p$ is again a ball in $\mathbb{Q}_p$, $\mathcal{C}(H_R)$ is a π -system that generates the σ -algebra of cylinder sets. Proposition 4.8 implies that for any A in $\mathcal{C}(H_R)$, $(P^m(A))$ converges to $P(A)$. The uniform tightness together with the convergence of the measures on a π -system that generates the cylinder sets implies the weak convergence of the P^m to P . $\square$

The convergence demonstrated here is, for any b in $(0, \infty)$, the weak convergence of probability measures on $D([0, \infty): \mathbb{Q}_p)$. In contrast, the convergence established earlier [7] was, for any fixed positive T and b in $(1, \infty)$, the weak convergence of probability measures on $D([0, T]: \mathbb{Q}_p)$. The current proof is quite robust and it should extend to the full generality of the non-Archimedean Brownian motion that Varadarajan introduced [26].

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