ON THE NUMBER OF PFISTER FORMS AND THE SYMBOL LENGTH OF FIELDS WITH FINITE SQUARE CLASS NUMBER

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ABSTRACT. Let F be a field of characteristic not 2 with finitely many square classes. Using combinatorial arguments applied to objects related to vector spaces over finite fields, we deduce an upper bound for the number of Pfister forms over F. Moreover, we compute upper bounds for the n-symbol length of F ($n \in \mathbb{N}$), i.e., the smallest integer $\mathrm{sl}_n(F) \geq 0$ such that to each quadratic form $\phi \in \mathrm{I}^n(F)$ there exists some $0 \leq k \leq \mathrm{sl}_n(F)$ and Pfister forms $\pi_1, \dots, \pi_k$ such that $\varphi \equiv \pi_1 + \dots + \pi_k \mod \mathrm{I}^{n+1}(F)$. In particular, we rediscover a bound that can also be deduced from a result by Bruno Kahn that he stated without proof.

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1. Introduction

Throughout this paper, let F be a field of characteristic different from 2. By a *quadratic form* or just *form* for short, we will always mean a finite dimensional non-degenerate quadratic form over F.

An *n-fold Pfister form* π for some $n \in \mathbb{N}$ is an n-fold tensor product of binary forms that represent 1, i.e., π is a form of the shape $\langle\langle a_1, \dots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$ with $a_1, \dots, a_n \in F^* = F \setminus \{0\}$. The set of n-fold Pfister forms over F is denoted by $\mathsf{P}_n(F)$; the set of forms that are similar to n-fold Pfister forms is denoted by $\mathsf{GP}_n(F)$. Forms in $\mathsf{GP}_n(F)$ are called *general Pfister forms*.

Both $\mathsf{P}_n(F)$ and $\mathsf{GP}_n(F)$ generate additively the n-th power $\mathrm{I}^n(F)$ of the *fundamental ideal* $\mathrm{I}(F)$ of Witt classes of even-dimensional forms in the Witt ring $\mathsf{W}(F)$, i.e., any Witt class $\varphi \in \mathrm{I}^n(F)$ can be written as a sum $\varphi = \pi_1 + \dots + \pi_m$ for some m and with $\pi_i \in \mathsf{GP}_n(F)$ (resp. with π_i or $-\pi_i \in \mathsf{P}_n F$). By convention, we denote by $\langle 1\rangle$ the unique 0-fold Pfister form and set $\mathrm{I}^0(F) = \mathsf{W}(F)$.

For a form $\varphi \in \mathrm{I}^n(F)$, we are interested in the *n-symbol length* of φ (or *symbol length* for short if the integer is clear from the context). The n-symbol length is defined as

$$\mathrm{sl}_n(\varphi) := \min\{k \in \mathbb{N} \mid \exists \pi_1, \dots, \pi_k \in \mathsf{GP}_n(F) : \varphi \equiv \pi_1 + \dots + \pi_k \mod \mathrm{I}^{n+1}(F)\}.$$

Note that one may have used $\mathsf{P}_n(F)$ in this definition instead since for all $c \in F^*$ and all $\pi \in \mathsf{P}_n(F)$, one has $\pi \equiv c\pi \mod \mathrm{I}^{n+1}(F)$.

We further define

$$\mathrm{sl}_n(F) := \sup_{\varphi \in \mathrm{I}^n(F)} \{\mathrm{sl}_n(\varphi)\} \in \mathbb{N} \cup \{\infty\}.$$

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(In the literature, one also finds the notation $\lambda_n(F)$ or $\lambda^n(F)$ to denote the symbol length.)

Due to the groundbreaking results by V. Voevodsky et al. in [OVV07] and [Voe03], the symbol length has connections to Milnor K-theory and Galois cohomology. In characteristic not 2, the Milnor K-groups $K_n(F)/2$, the Galois cohomology groups $H^n(F, \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{I}^n(F)/\mathrm{I}^{n+1}(F)$ are all mutually isomorphic. Under these isomorphisms, generators of the form $\{a_1, \ldots, a_n\}$ in $K_n(F)/2$ correspond to n-fold cup products $(a_1) \cup \ldots \cup (a_n)$ in $H^n(F, \mathbb{Z}/2\mathbb{Z})$ and to n-fold Pfister forms $\langle\langle a_1, \ldots, a_n\rangle\rangle$ modulo $\mathrm{I}^{n+1}(F)$, respectively. So the analogous notions of symbol lengths can be translated from any one of these three groups to any other.

The determination of the symbol length is an active field of research, see, e.g., [Sal15], [Mat16] or [Cha20] for some recent progress in the field. The 3-symbol length (and the related concept of 3-Pfister number, i.e., the minimal length of a representation of forms in $I^3(F)$ as sums of general 3-fold Pfistter forms) has, for example, been studied in [BRV10], [PST09], [Rac13], [Lor23].

The aim of this article is to find upper bounds for $\mathrm{sl}_n(\varphi)$ and $\mathrm{sl}_n(F)$ in the case where the square class group F^*/F^{*2} is finite, i.e., $q(F) = |F^*/F^{*2}| = 2^d < \infty$. As this problem is rather trivial for $n = 1$ and in order to avoid subtleties in our calculations, we assume throughout that n be an integer ≥ 2. Note that K. Becher and the first author obtained best possible general upper bounds in the case $n = 2$ in terms of the cardinality $q(F)$, see [BH04].

Our estimates depend on some field invariants such as the level, the Pythagoras number and the size of certain quotients of subgroups of the multiplicative group of the field related to sums of squares.

In Section 2, we provide a counting argument for the number of Pfister forms of a certain type over a field with finitely many square classes, and provide upper bounds for the size of certain quotient groups in a filtration of the square class group of a field with finitely many square classes. Using these results, we use more refined counting methods to obtain bounds for the n-symbol length (Theorem 3.8). The bound is of a rather technical nature and can be weakened to yield a polynomial bound in d of degree $n-1$ (Corollary 3.10). Such a polynomial can be computed explicitly (see Example 3.12 where such polynomials are explicitly computed in the case $n = 3$). A polynomial bound of that type can also be deduced from a result that was stated by B. Kahn but without proof, [Kah05, Proposition 2.3(h)].

In the remainder of this introduction, we recall some further basic definitions and facts from the algebraic theory of quadratic forms and refer to [Lam05] for any undefined terminology or additional facts that we state without further reference.

Isometry of two forms φ_1, φ_2 will be denoted by $\varphi_1 \cong \varphi_2$, their orthogonal sum by $\phi_1 \perp \phi_2$, and their tensor product by $\phi_1 \otimes \phi_2$. The orthogonal sum of m copies of ϕ will be written $m \times \phi$. By abuse of notation, we will denote the *Witt class* of a quadratic form φ in the *Witt ring* $\mathsf{W}(F)$ again by φ.

For a quadratic form φ defined over a vector space V, we denote by $\mathsf{D}_F(\varphi) = \{a \in F^* \mid \exists v \in V : \varphi(v) = a\}$ the set of nonzero elements represented by φ and by $\mathsf{G}_F(\varphi) = \{a \in F^* \mid a\varphi \cong \varphi\}$ the multiplicative group of similarity factors of φ. We define $\mathsf{D}_F(m) = \mathsf{D}_F(m \times \langle 1\rangle)$, the nonzero sums of m squares in F.

A form ϕ is called *round* if $\mathsf{D}_F(\phi) = \mathsf{G}_F(\phi)$. Pfister forms are always round. In particular, if φ is isometric to the m-fold Pfister form $\langle\langle -1, \ldots, -1\rangle\rangle$, we obtain

$\mathsf{G}_F(\varphi) = \mathsf{D}_F(\varphi) = \mathsf{D}_F(2^m)$, the group of nonzero sums of 2^m squares in F. The set of nonzero sums of squares in F will be denoted by $\mathsf{D}_F(\infty) = \bigcup_{m=1}^{\infty} \mathsf{D}_F(m)$.

The *level* of F, denoted by $s(F)$, is the least integer k such that -1 is a sum of k squares in F, or ∞ if no such k exists. A famous result by A. Pfister [Pfi65] states that $s(F)$ is a power of 2 if finite, and that each power of 2 occurs as a level of a suitable field. A field is called *formally real* (or *real* for short), if -1 is not a sum of squares, and *nonreal* otherwise. The *Pythagoras number* of F, denoted by $p(F)$ is the least integer k such that any sum of squares in F is a sum of k squares, or ∞ if no such integer exists. For nonreal fields, we have $s(F) \leq p(F) \leq s(F) + 1$, see [Lam05, Chapter XI. Theorem 5.6]. For real fields, it was shown by the first author in [Hof99, Theorem 1] that every positive integer can be realized as the Pythagoras number of a suitable field.

Since we will focus on fields with finite square class number, it should be noted that there is no known example of a field with $q(F) < \infty$ and finite level $s(F) \geq 8$. In fact, if the so-called *Elementary Type Conjecture* were true, then any field with $q(F) < \infty$ would have level $s(F) \in \{1, 2, 4, \infty\}$, see also [Lam05, Chapter XIII. Question 6.2].

We will use several combinatorial arguments in the sequel. We need the *binomial coefficient* $\binom{n}{k}$ for integers $n, k \in \mathbb{Z}$, i.e., the number of subsets with k elements of a set with n elements. For $0 \leq k \leq n$, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and in all other cases, we have $\binom{n}{k} = 0$. Finally, for a set S, we denote the *power set* of S, i.e., the set consisting of all subsets of S by $\mathcal{P}(S)$. More generally, for $m \in \mathbb{N}$, we denote by $\mathcal{P}_m(S)$ the set of all subsets $A \subseteq S$ with $|A| = m$. In particular, if S is finite, then $|\mathcal{P}_m(S)| = \binom{|S|}{m}$.

2. The Number of Pfister Forms

In this section, we obtain an upper bound for the number of isometry classes of Pfister forms over fields with finite square class number. Since we will use a crucial relation between Pfister forms and certain vector spaces, we collect some basic facts from linear algebra.

Let $\mathbb{F}_q$ be the finite field with q elements, let $0 \leq m \leq d < \infty$ and denote by $V(q, d, m)$ the number of m-dimensional subspaces of a d-dimensional vector space over $\mathbb{F}_q$ This number is well known (or an easy exercise in linear algebra) and given in the following result.

Proposition 2.1 *We have*

$$V(q, d, m) = \prod_{\ell=0}^{m-1} \frac{q^{d-\ell} - 1}{q^{m-\ell} - 1}.$$

We will apply these results in particular to the groups $F^*/F^{*2}, F^*/\mathsf{D}_F(2^m)$ and $F^*/\pm\mathsf{D}_F(2^m)$ for some nonnegative integer m. Since these groups all have exponent ≤ 2, they can be interpreted as vector spaces over $\mathbb{F}_2$. Thus, if finite, these groups have order 2^k and thus $\mathbb{F}_2$-dimension k for some nonnegative integer k.

We therefore give an upper bound in the special case of $\mathbb{F}_2$-vector spaces for later reference.

Corollary 2.2

$$V(2,d,m) \leq 2^{m(d-m)}\alpha_m \quad \textit{where} \quad \alpha_m = \prod_{\ell=1}^{m}\left(1+\frac{1}{2^\ell-1}\right).$$

Proof. Using Proposition 2.1 we have

$$\begin{aligned} V(2,d,m) &= \prod_{\ell=0}^{m-1}\frac{2^{d-\ell}-1}{2^{m-\ell}-1} \\ &= \prod_{\ell=0}^{m-1}\frac{2^{d-m}\left(2^{m-\ell}-1\right)+2^{d-m}-1}{2^{m-\ell}-1} \\ &< \prod_{\ell=0}^{m-1}\frac{2^{d-m}\left(2^{m-\ell}-1\right)+2^{d-m}}{2^{m-\ell}-1} \\ &= \prod_{\ell=0}^{m-1}2^{d-m}\cdot\left(1+\frac{1}{2^{m-\ell}-1}\right) \\ &= 2^{m(d-m)}\cdot\prod_{\ell=1}^{m}\left(1+\frac{1}{2^\ell-1}\right) \end{aligned}$$

□

Remark 2.3 One readily obtains

$$\alpha_m = \begin{cases} 1 & \text{if } m=0; \\ 2 & \text{if } m=1; \\ \frac{8}{3} & \text{if } m=2. \end{cases}$$

For $m \geq 3$, we write

$$\alpha_m = \frac{8}{3}\prod_{\ell=3}^{m}\left(1+\frac{1}{2^\ell-1}\right)$$

and note that for $\ell \geq 3$, we have $3 \cdot 2^{\ell-2} \leq 2^\ell - 1$, so

$$\alpha_m \leq \frac{8}{3}\prod_{\ell=1}^{m-2}\left(1+\frac{1}{3\cdot 2^\ell}\right)$$

Using $1+x \leq e^x$ we get for all $k \geq 1$,

$$\prod_{\ell=1}^{k}\left(1+\frac{1}{3\cdot 2^\ell}\right) \leq \exp\left(\frac{1}{3}\sum_{\ell=1}^{k}2^{-\ell}\right) < e^{1/3}$$

and thus, $\alpha_m < \frac{8}{3}e^{1/3}$.

Let $\mathsf{P}_{n,m}(F)$ be the set of isometry classes of *anisotropic* n-fold Pfister forms over F that can be written with m slots that are -1, but such that there is no representation with $m+1$ slots that are -1. Of course, this set may be empty.

Lemma 2.4 *Let $\varphi \in \mathsf{P}_{n,m}(F)$ and $x_1,\dots,x_{n-m} \in F^*$ such that we have*

$$\varphi \cong \langle\langle -1,\dots,-1,-x_1,\dots,-x_{n-m}\rangle\rangle.$$

Then $x_1, \dots, x_{n-m}$ are linearly independent when considered as elements in the $\mathbb{F}_2$-vector space $F^*/\pm \mathsf{D}_F(2^m)$ and thus also in $F^*/\mathsf{D}_F(2^m)$.

Proof. If $x_1, \dots, x_{n-m}$ are linearly dependent in $F^*/\pm \mathsf{D}_F(2^m)$, after renumbering, there is some $i \in \{0, \dots, n-m-1\}$ and some $c \in \pm\mathsf{D}_F(2^m)$ with

$$x_{i+1} = c\prod_{k=1}^{i} x_k.$$

We claim that we can find $a \in \pm\mathsf{D}_F(2^m)$ and $n-m-1$ elements $y_1, \dots, y_{n-m-1}$ in $\{x_1, \dots, x_{n-m}\}$ such that we have

$$\varphi \cong \langle\langle -1, \dots, -1, a, -y_1, \dots, -y_{n-m-1}\rangle\rangle$$

This is clear for $i = 0$, so let now $i \geq 1$. Recall that for all $a, b \in F^*$, we have an isometry $\langle\langle -a, -b\rangle\rangle \cong \langle\langle -a, -ab\rangle\rangle$. Using this equality i times, we obtain

$$\begin{aligned}\langle\langle -x_1, \dots, -x_i, -x_{i+1}\rangle\rangle &\cong \langle\langle -x_1, \dots, -x_i, -cx_1 \cdot \dots \cdot x_i\rangle\rangle \\ &\cong \langle\langle -x_1, \dots, -x_i, -c\rangle\rangle \cong \langle\langle -c, -x_1, \dots, -x_i\rangle\rangle,\end{aligned}$$

i.e. for $a = -c \in \pm\mathsf{D}_F(2^m)$ we have

$$\varphi \cong \langle\langle -1, \dots, -1, a, -x_1, \dots, -x_i, \widehat{-x_{i+1}}, -x_{i+2}, \dots, -x_{n-m-1}\rangle\rangle$$

(here, the hat means to omit $-x_{i+1}$). This establishes the claim. We now distinguish whether $a \in \mathsf{D}_F(2^m)$ or $a \in -\mathsf{D}_F(2^m)$. For $a \in \mathsf{D}_F(2^m) = \mathsf{G}_F(2^m)$ we have

$$\begin{aligned}\langle\langle -1, \dots, -1, a\rangle\rangle &= \langle\langle -1, \dots, -1\rangle\rangle \perp -a\langle\langle -1, \dots, -1\rangle\rangle \\ &\cong \langle\langle -1, \dots, -1\rangle\rangle \perp -\langle\langle -1, \dots, -1\rangle\rangle,\end{aligned}$$

which contradicts the anisotropy of φ. If we have $a \in -\mathsf{D}_F(2^m) = -\mathsf{G}_F(2^m)$, we have

$$\begin{aligned}\langle\langle -1, \dots, -1, a\rangle\rangle &= \langle\langle -1, \dots, -1\rangle\rangle \perp -a\langle\langle -1, \dots, -1\rangle\rangle \\ &\cong \langle\langle -1, \dots, -1\rangle\rangle \perp \langle\langle -1, \dots, -1\rangle\rangle = \langle\langle \underbrace{-1, \dots, -1}_{m+1}\rangle\rangle,\end{aligned}$$

contradicting the maximality of m. $\square$

By abuse of notation, we will henceforth often identify an element $x \in F^*$ with its class in $F^*/\mathsf{D}_F(2^m)$ for a given $m \in \mathbb{N}$, and denote by $U_{n,m}$ the set of $(n-m)$-dimensional subspaces of $F^*/\mathsf{D}_F(2^m)$.

Corollary 2.5 *(a) We have a well defined map*

$$\begin{aligned}\varphi_{n,m} : U_{n,m} &\to \mathsf{P}_n(F) \\ U &\mapsto \langle\langle -1, \dots, -1, -x_1, \dots, -x_{n-m}\rangle\rangle\end{aligned}$$

where $\{x_1, \dots, x_{n-m}\}$ is an arbitrary $\mathbb{F}_2$-basis of U. We then write

$$\pi_{U,m} = \varphi_{n,m}(U).$$

(b) Every form in $\mathsf{P}_{n,m}(F)$ is in the image of $\varphi_{n,m}$. In particular, we have $|\mathsf{P}_{n,m}(F)| \leq |U_{n,m}|$.

Proof. Let $x, y \in F^*$ be such that there is a $c \in \mathsf{D}_F(2^m) = \mathsf{G}_F(2^m)$ with $y = cx$. We then have

$$\langle\langle -1, \dots, -1, -y \rangle\rangle = \langle\langle -1, \dots, -1, -cx \rangle\rangle \cong \langle\langle -1, \dots, -1, -x \rangle\rangle.$$

Furthermore, any basis from a subspace U can be transformed into any other basis of U by successively replacing a subset $\{x, y\}$ by $\{y, x\}$ or by $\{x, xy\}$. Also, we have isometries $\langle\langle -x, -xy \rangle\rangle \cong \langle\langle -x, -y \rangle\rangle \cong \langle\langle -y, -x \rangle\rangle$ for all $x, y \in F^*$.

The map is thus well-defined and we have proved (a). Finally, for (b), every form in $\mathsf{P}_{n,m}(F)$ has a preimage due to Lemma 2.4. □

Example 2.6 In [Cor73], Cordes introduced so-called $\overline{C}$*-fields*. These are fields with finite square class group, i.e., $q(F) < \infty$, and such that $|\mathsf{W}(F)| = 2^{q(F)}$. Such fields will always have level $s(F) = 1$ or $s(F) = 2$.

Over such fields, every subset of F^*/F^{*2} is the value set of a unique anisotropic quadratic form over F by [Lam05, Chapter XI. Theorem 7.19 (4)]. In particular, we have $\mathsf{D}_F(\pi_{U,0}) = U$ and $\mathsf{D}_F(\pi_{U,1}) = U \cup -U$. One now readily sees that each form in $\mathsf{P}_{n,m}(F)$ has exactly one preimage under $\varphi_{n,m}$.

Note that $\overline{C}$-fields are examples of so-called *rigid fields* whose *Pfister number*, an invariant related to the symbol length, was studied by the second author in [Lor23].

Definition 2.7 Let m be a nonnegative integer. For $m \geq 1$, we define

$$q_m = \dim_{\mathbb{F}_2} \left(D_F(2^m)/D_F(2^{m-1}) \right).$$

If $q_m < \infty$, we have

$$2^{q_m} = \left[\mathsf{D}_F\left(2^m\right) : \mathsf{D}_F\left(2^{m-1}\right) \right].$$

For $m \geq 0$, we define

$$s_m = \dim_{\mathbb{F}_2} \left(D_F(\pm 2^m)/D_F(2^m) \right) = \log_2 \left(\left[\mathsf{D}_F\left(\pm 2^m\right) : \mathsf{D}_F\left(2^m\right) \right] \right) \in \{0, 1\}$$

and

$$d_m = \dim_{\mathbb{F}_2} \left(F^* / \pm \mathsf{D}_F\left(2^m\right) \right).$$

If $d_m < \infty$, we have

$$2^{d_m} = \left[F^* : \pm \mathsf{D}_F\left(2^m\right) \right].$$

Note that $p(F)$ is finite if some d_m is finite. It is easy to calculate s_m in terms of the level $s(F)$.

Remark 2.8 (a) Let F be a field of finite level $s(F) = 2^s$ and let π be the m-fold Pfister form $\langle\langle -1, \dots, -1 \rangle\rangle$. Using the roundness of π, we see that for $m < s$, we have $\mathsf{D}_F(\pi) \cap -\mathsf{D}_F(\pi) = \emptyset$ and for $m \geq s$, we have $\mathsf{D}_F(\pi) = -\mathsf{D}_F(\pi)$. In particular we have $s_m = 1$ if $m < s$ and $s_m = 0$ otherwise.

(b) If F is a real field, we clearly have $s_m = 1$ for all $m \in \mathbb{N}$.

(c) Note that $p(F) \leq 2^m$ iff $q_{m+1} = 0$ iff $q_k = 0$ for all $k \geq m+1$. Thus, one readily sees that if $d_m < \infty$, then $p(F) \leq 2^{m+d_m+s_m}$.

With this notation, we can formulate the following:

Corollary 2.9 *Let $0 \leq m \leq n$ be integers and let F be a field with $d_m < \infty$. Then*

$$|\mathsf{P}_{n,m}(F)| \leq 2^{(n-m)(d_m+s_m-n+m)} \cdot \alpha_{n-m},$$

where α_{n-m} is defined as in Corollary 2.2.

Proof. This follows immediately from Corollary 2.5 and Corollary 2.2. □

The rest of the section is now devoted to finding upper bounds for the d_m in order to optimize the above bound for the number of Pfister forms. We will see in the next result that we can express d_m in terms of s_m and the q_k for $k \leq m$.

Proposition 2.10 *Let F be a field with $q(F) = 2^d < \infty$. We then have*

$$d_m = d - s_m - \sum_{k=1}^{m} q_k$$

for all $m \in \mathbb{N}$.

Proof. For all $m \in \mathbb{N}$, we have

$$\begin{aligned} 2^d &= [F^* : F^{*2}] \\ &= [F^* : \pm \mathsf{D}_F(2^m)] \cdot [\pm \mathsf{D}_F(2^m) : \mathsf{D}_F(2^m)] \cdot \prod_{k=1}^{m} [\mathsf{D}_F(2^k) : \mathsf{D}_F(2^{k-1})] \\ &= 2^{d_m} \cdot 2^{s_m} \cdot \prod_{k=1}^{m} 2^{q_k} = 2^{d_m + s_m + \sum_{k=1}^{m} q_k}. \end{aligned}$$

The claim follows readily. □

We now want to collect estimates for the invariants q_m. To do so we will use the following extension of a classical result from I. Kaplansky.

Theorem 2.11 *Let F be a field with finite $p(F)$, i.e., $2^s \leq p(F) < 2^{s+1}$ for some integer $s \geq 0$. For $k \in \{1, \ldots, s\}$, we have*

$$q_k \geq s + 1 - k.$$

For $k = s + 1$, we have $q_k \geq 1$ if $p(F) > 2^s$ and $q_k = 0$ if $p(F) = 2^s$. For $k > s + 1$, we have $q_k = 0$.

Proof. For $k \in \{1, \ldots, s\}$ the proof can be found in [Lam05, Chapter XI. Kaplansky's Lemma 7.1]. For $k = s + 1$, we have $q_k = 0$ if and only if $D_F(2^s) = D_F(2^{s+1})$ if and only if $p(F) = 2^s$. For $k > s + 1$, we clearly have $D_F(2^k) = D_F(2^{k+1})$ and the assertion is clear. □

Remark 2.12 (i) If F in the above theorem is nonreal, then $2^s \leq p(F) < 2^{s+1}$ is equivalent to $s(F) = 2^s$, in which case $p(F) \in \{2^s, 2^s + 1\}$.

(ii) In the case $k = s + 1$ and $p(F) > 2^s$, q_k can be arbitrary large. To see this, consider the field $F_n = \mathbb{F}_3(\!(t_1)\!) \ldots (\!(t_n)\!)$. For all $n \geq 1$, we have $s(F_n) = 2 = 2^1$ and $p(F_n) = 3$, $\mathsf{D}_{F_n}(2^1) = \{1, 2\}$ but $\mathsf{D}_{F_n}(2^2) = F_n^*$, so $q_2 = n$.

We are finally in a good position to give upper bounds for d_m that only depend on m, on the Pythagoras number of F and on whether F is real or not.

Corollary 2.13 *Let F be a field with $q(F) = 2^d < \infty$ and let $s \geq 0$ be the integer with $2^s \leq p(F) < 2^{s+1}$.*

(a) For $m < s$, we have

$$d_m \leq d - m \cdot \frac{2s - m + 1}{2} - 1.$$

(b) For $m = s$, we have

$$d_m \leq \begin{cases} d - s \cdot \frac{s+1}{2}, & \text{if } F \text{ is nonreal} \\ d - s \cdot \frac{s+1}{2} - 1, & \text{if } F \text{ is real.} \end{cases}$$

(c) For $m = s + 1$:

(i) if F is nonreal, then $d_m = 0$.

(ii) if F is real with $p(F) = 2^s$, we have

$$d_m \leq d - s \cdot \frac{s+1}{2} - 1.$$

(iii) if F is real with $p(F) > 2^s$, we have

$$d_m \leq d - s \cdot \frac{s+1}{2} - 2.$$

(d) For $m > s + 1$ we have

$$d_m = 0 \text{ if } F \text{ is nonreal and } d_m \leq d - s \cdot \frac{s+1}{2} - 1, \text{ if } F \text{ is real.}$$

Proof. (a), (b): We use Proposition 2.10 and plug in the values obtained in Remark 2.8 and Theorem 2.11. In fact, we have

$$\begin{aligned} d_m &= d - s_m - \sum_{k=1}^{m} q_k \\ &\leq d - s_m - \sum_{k=1}^{m} (s + 1 - k) \\ &= d - s_m - m \cdot s - m + \frac{m(m+1)}{2} \\ &= d - m \cdot \frac{2s - m + 1}{2} - s_m \end{aligned}$$

and the result follows.

The nonreal case of (c) follows since in this case, the m-fold Pfister form $\langle\langle -1, \ldots, -1 \rangle\rangle$ is clearly universal. The other parts of (c) and (d) now follow using the same arguments as above. The details are omitted and left to the reader □

As we already mentioned, $\pi \equiv a\pi \mod \mathrm{I}^{n+1}(F)$ for all $\pi \in \mathsf{P}_n(F), a \in F^*$, we immediately get the following result.

Lemma 2.14 *For any field F, we have*

$$\sup_{\varphi \in \mathrm{I}^n(F)} \{\mathrm{sl}_n(\varphi)\} \leq |\mathsf{P}_n(F)|.$$

Thus we could deduce upper bounds for the symbol length from our above calculations. Nevertheless we do not work out the details since these bounds will obviously grow exponentially, while the upper bounds we will obtain in the next section grow polynomially.

3. Bounding the Symbol Length

Proposition 3.1 *There is a basis $\mathcal{B}$ of F^*/F^{*2} with a filtration*

$$\mathcal{B} = \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \dots$$

such that for each $m \in \mathbb{N}$, the classes of the elements in $\mathcal{B}_m$ form a basis of $F^*/\mathsf{D}_F(2^m)$.

Proof. By basic linear algebra, we can find sets $\mathcal{A}_m$ for any $m \in \mathbb{N}$ such that $\mathcal{A}_m$ is a basis of $\mathsf{D}_F(2^m)/F^{*2}$ and such that we have $\mathcal{A}_m \subseteq \mathcal{A}_{m+1}$. We can then extend $\bigcup_{m\in\mathbb{N}} \mathcal{A}_m$ to a basis $\mathcal{B}$ of the square class group F^*/F^{*2}. The result now follows since we can identify $\mathcal{B}_m := \mathcal{B} \setminus \mathcal{A}_m$ with a basis of

$$(F^*/F^{*2})/(\mathsf{D}_F(2^m)/F^{*2}) \cong F^*/\mathsf{D}_F(2^m).$$

□

Remark 3.2 (a) The union $\bigcup_{m\in\mathbb{N}} \mathcal{A}_m$ in the proof of Proposition 3.1 is a basis itself of $\mathsf{D}_F(\infty)/F^{*2}$ and thus equal to $\mathcal{B}$ if and only if every element is a sum of squares, i.e., if and only if F is nonreal.
(b) With our notation from Definition 2.7, we have $|\mathcal{B}_m| = d_m + s_m$.

The following easy lemma will be used frequently in the sequel.

Lemma 3.3 *For all $x, y, x_2, \dots, x_k \in F^*$, we have*

$$\langle\langle xy, x_2, \dots, x_k\rangle\rangle \equiv \langle\langle x, x_2, \dots, x_k\rangle\rangle + \langle\langle y, x_2, \dots, x_k\rangle\rangle \mod \mathrm{I}^{k+1}(F).$$

Proof. From the Witt equivalence $\langle\langle xy\rangle\rangle = \langle\langle x\rangle\rangle + x\langle\langle y\rangle\rangle$, we obtain

$$\begin{aligned}\langle\langle xy, x_2, \dots, x_k\rangle\rangle &= \langle\langle x, x_2, \dots, x_k\rangle\rangle + x\langle\langle y, x_2, \dots, x_k\rangle\rangle \\ &\equiv \langle\langle x, x_2, \dots, x_k\rangle\rangle + \langle\langle y, x_2, \dots, x_k\rangle\rangle \mod \mathrm{I}^{k+1}(F)\end{aligned}$$

□

Proposition 3.4 *Let $\mathcal{B}$ be a basis of F^*/F^{*2} as in Proposition 3.1. For $\varphi \in \mathrm{I}^n(F)$, there are finite subsets $C_m \subseteq \mathcal{P}_m(\mathcal{B}_{n-m})$ for $m \in \{0, \dots, n\}$ such that we have*

$$\varphi \equiv \sum_{m=0}^{n} \sum_{U\in C_m} \pi_{\mathrm{span}(U),n-m} \mod \mathrm{I}^{n+1}(F).$$

Proof. We consider a representation

$$\varphi \equiv \pi_1 + \ldots + \pi_k \mod \mathrm{I}^{n+1}(F)$$

such that each π_j is written with as many slots equal to -1 as possible, i.e. if $\pi_j \in \mathsf{P}_{n,m}(F)$ for some $m \in \mathbb{N}$, then exactly m slots of π_j are equal to -1 and the other $n-m$ slots can be chosen to be of shape $-\prod_{i=1}^{\ell} b_i$ with $b_i \in \mathcal{B}_m$ and suitable ℓ, see Corollary 2.5.

If we have $\pi_j = \langle\langle x_1, \dots, x_{n-1}, x_n\rangle\rangle$, where $x_n = -b_1 \cdot \ldots \cdot b_\ell = (-1)^{\ell-1} \prod_{i=1}^{\ell}(-b_i)$ with $b_1, \dots, b_\ell \in \mathcal{B}_m$ and $\ell \geq 2$, for some $j \in \{1, \dots, k\}$, then by Lemma 3.3, we can replace π_j by the sum

$$\sum_{i=1}^{\ell}\langle\langle x_1,\dots,x_{n-1},-b_i\rangle\rangle + \langle\langle x_1,\dots,x_{n-1},(-1)^{\ell-1}\rangle\rangle.$$

If we have $\pi_j \in \mathsf{P}_{n,m}(F)$ for some $m \in \mathbb{N}$, the resulting Pfister forms are hyperbolic or will lie in $\mathsf{P}_{n,m'}(F)$ for some $m' \geq m$. We can clearly omit the hyperbolic Pfister forms in our representation and thus now consider any of the resulting Pfister forms lying in $\mathsf{P}_{n,m'}(F)$ for some $m' \geq m$

If we have $m' = m$, we can repeat the above until the considered slot in any newly introduced Pfister form lies in $\mathcal{B}_m$. If we have $m' > m$, we first substitute the representation of this form with an isometric Pfister form with m' slots equal to -1 and all other slots being products of -1 and of elements lying in $\mathcal{B}_{m'}$. We can then apply the above procedure to this new form.

By repeating this process for all of the $\pi_j, j \in \{1,\dots,k\}$ and all slots that neither lie in the appropriate $\mathcal{B}_m$ (up to a factor -1) nor are equal to -1, and possibly rearranging the terms, we will eventually obtain a representation as desired. $\square$

Remark 3.5 If F is nonreal of level $s(F) = 2^s$, it is obviously enough to restrict the outer sum occuring in Proposition 3.4 to $m \geq \max\{0, n-s\}$.

We can directly deduce the following upper bound for the symbol length from Proposition 3.4.

Corollary 3.6 *Let F be a field with $q(F) = 2^d < \infty$, and let $n \in \mathbb{N}$ be an integer.*
(a) If F is nonreal with $s(F) = 2^s$, we have

$$\mathrm{sl}_n(F) \leq \sum_{m=0}^{\min\{s,n\}} \binom{d_m + s_m}{n-m}$$

(b) If F is real, we have

$$\mathrm{sl}_n(F) \leq \sum_{m=0}^{n} \binom{d_m + s_m}{n-m}$$

We will now present two strategies for adding certain Pfister forms that appear in Proposition 3.4 in order to obtain better bounds for the symbol length.

Corollary 3.7 *Let F be a field with $q(F) = 2^d < \infty$, and let $n \in \mathbb{N}$.*
(a) If F is nonreal with $s(F) = 2^s$, we have

$$\mathrm{sl}_n(F) \leq \sum_{m=0}^{\left\lfloor \frac{\min\{s,n\}}{2} \right\rfloor} \binom{d_{2m} + s_{2m}}{n-2m}.$$

(b) If F is real, we have

$$\mathrm{sl}_n(F) \leq \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{d_{2m} + s_{2m}}{n-2m}.$$

Proof. We write φ as in Proposition 3.4 and use the notation of the statement of this proposition. Fix a subset $A = \{x_1, \dots, x_{n-2m}\} \in \mathcal{P}_{n-2m}(\mathcal{B}_{2m})$ for some $m \geq 1$. We now consider the set $C \subseteq C_{n-2m+1} \subseteq \mathcal{P}_{n-2m+1}(\mathcal{B}_{2m-1})$ consisting of all $U \in C_{n-2m+1}$ with $A \subseteq U$. Let $a_1, \dots, a_k \in \mathcal{B}_{2m-1}$ be pairwise different such that

$$C = \{A \cup \{a_j\} \mid j = 1, \dots, k\}.$$

The Pfister forms

$$\langle\langle -1 \rangle\rangle^{\otimes 2m-1} \otimes \langle\langle -x_1, \dots, -x_{n-2m} \rangle\rangle \otimes \langle\langle -a_j \rangle\rangle, \text{ for } j \in \{1, \dots, k\}$$

then all occur in the given representation of $\varphi \mod \mathrm{I}^{n+1}(F)$. It may further happen that the Pfister form

$$\langle\langle -1 \rangle\rangle^{\otimes 2m} \otimes \langle\langle -x_1, \dots, -x_{n-2m} \rangle\rangle$$

(corresponding to the set A from the beginning of the proof) occurs in the representation. By Lemma 3.3 the sum (modulo $\mathrm{I}^{n+1}(F)$) of these Pfister forms is equivalent to

$$\langle\langle -1 \rangle\rangle^{\otimes 2m-1} \otimes \langle\langle -x_1, \dots, -x_{n-2m} \rangle\rangle \otimes \langle\langle \epsilon a_1 \cdot \ldots \cdot a_k \rangle\rangle$$

where $\epsilon \in \{\pm 1\}$ depends on the parity of k and on whether $\langle\langle -1 \rangle\rangle^{\otimes 2m} \otimes \langle\langle x_1, \dots, x_{n-2m} \rangle\rangle$ has to be included or not. We will sum up all Pfister forms in the given representation except those in $\mathsf{P}_{n,0}(F)$ if we apply this procedure for all possible A for all $m \geq 1$. There are at most $\binom{d_0+s_0}{n}$ elements in $\mathsf{P}_{n,0}(F)$, and for $m \geq 1$, there are at most $\binom{d_{2m}+s_{2m}}{n-2m}$ possible choices of a subset A as above. Thus the claim follows. $\square$

Theorem 3.8 *Let F be a field with $q(F) = 2^d < \infty$ and let $n \in \mathbb{N}$ be an integer.*
(a) If F is nonreal with $s(F) = 2^s$, we have

$$\mathrm{sl}_n(F) \leq \sum_{m=0}^{\min\{s,n-1\}} \sum_{r=0}^{\lfloor \frac{n-m-1}{2} \rfloor} \binom{\lfloor \frac{d_m+s_m}{2} \rfloor}{2r} \binom{\lfloor \frac{d_m+s_m+1}{2} \rfloor}{n-m-1-2r}$$

(b) If F is real, we have

$$\mathrm{sl}_n(F) \leq \sum_{m=0}^{n-1} \sum_{r=0}^{\lfloor \frac{n-m-1}{2} \rfloor} \binom{\lfloor \frac{d_m+s_m}{2} \rfloor}{2r} \binom{\lfloor \frac{d_m+s_m+1}{2} \rfloor}{n-m-1-2r}$$

Proof. For all $m \in \{0, \dots, n\}$, we decompose $\mathcal{B}_m$ into $\mathcal{B}_m = \mathcal{A}_m \cup \mathcal{A}'_m$ with $\mathcal{A}_m \cap \mathcal{A}'_m = \emptyset$ and $|\mathcal{A}_m| = \lfloor \frac{d_m+s_m}{2} \rfloor$ for some suitably chosen subsets $\mathcal{A}_m \subseteq \mathcal{B}_m$. We then have (see also Remark 3.2(b))

$$|A'_m| = d_m + s_m - \left\lfloor \frac{d_m + s_m}{2} \right\rfloor = \left\lfloor \frac{d_m + s_m + 1}{2} \right\rfloor.$$

We further consider

$$\mathcal{P}_{m,r} = \{X \subseteq \mathcal{A}_m \mid |X| = 2r\} \text{ and } \mathcal{P}'_{m,r} = \{Y \subseteq \mathcal{A}'_m \mid |Y| = n - m - 1 - 2r\}.$$

We clearly have

$$|\mathcal{P}_{m,r}| = \binom{\lfloor \frac{d_m+s_m}{2} \rfloor}{2r} \text{ and } |\mathcal{P}'_{m,r}| = \binom{\lfloor \frac{d_m+s_m+1}{2} \rfloor}{n-m-1-2r}.$$

Let now $\varphi \in \mathrm{I}^n(F)$ and consider a representation

$$\varphi \equiv \sum_{m=0}^{n} \sum_{U \in C_m} \pi_{U,n-m} \mod \mathrm{I}^{n+1}(F) \tag{1}$$

as in Proposition 3.4. Let $X \in \mathcal{P}_{m,r}$ and $Y \in \mathcal{P}'_{m,r}$. Then all forms $\pi \in \mathsf{P}_{n,m}(F) \cup \mathsf{P}_{n,m+1}(F)$ that occur in this representation and that have the elements of $X \cup Y$ as slots (up to a factor -1) can be replaced by a single Pfister form using Lemma 3.3 just as in the proof of Corollary 3.7.

Conversely, let some $\pi = \pi_{U,m}$ from this representation be given and let $B = \{-b_1, \ldots, -b_{n-m}\}$ be the set of slots $\neq -1$ of π. Let r be maximal such that there is an $X \in \mathcal{P}_{m,r}$ with $X \subseteq -B$. We clearly have

$$|(-B \setminus X) \cap \mathcal{A}'_m| \in \{n-m-2r-1, n-m-2r\}.$$

There is thus a subset $Y \subseteq \mathcal{A}'_m$ with cardinality $|Y| = n-m-2r-1$ such that $X \cup Y \subseteq -B$.

Hence, for all Pfister forms of type $\pi_{U,m}$ for a suitable U, there is an $r \in \mathbb{N}$, $X \in \mathcal{P}_{m,r}$ and $Y \in \mathcal{P}'_{m,r}$ such that all slots except one are either -1 or lie (up to a factor -1) in $X \cup Y$. Therefore, our above strategy to replace certain Pfister forms by a single one can be applied to all forms on the right hand side in (1).

The symbol length $\mathrm{sl}_n(\varphi)$ can thus be bounded by the cardinality of

$$\{X \times Y \mid \exists m \in \mathbb{N}, r \in \mathbb{N} : X \in \mathcal{P}_{m,r}, Y \in \mathcal{P}'_{m,r}\}$$

which is given by the respective sums of the theorem. □

Lemma 3.9 *Let $k, m \in \mathbb{N}$ be nonnegative integers with $m \geq 2$. Then*

$$\sum_{r=0}^{\lfloor \frac{n-m-1}{2} \rfloor} \binom{k}{2r}\binom{k}{n-m-1-2r} \quad \text{and} \quad \sum_{r=0}^{\lfloor \frac{n-m-1}{2} \rfloor} \binom{k}{2r}\binom{k+1}{n-m-1-2r}$$

are polynomials in k of degree $n-m-1$ with leading coefficient $\frac{2^{n-m-1}}{2(n-m-1)!}$.

Proof. For all r, the expression $\binom{k}{2r}$ is zero or a polynomial in k of degree $2r$ with leading coefficient $\frac{1}{(2r)!}$. Similarly $\binom{k}{n-m-1-2r}$ and $\binom{k+1}{n-m-1-2r}$ are zero or polynomials in k of degree $n-m-1-2r$ with leading coefficient $\frac{1}{(n-m-1-2r)!}$. Thus, in both cases, the products are zero or polynomials in k of degree

$$2r + n - m - 1 - 2r = n - m - 1$$

with leading coefficient

$$\frac{1}{(2r)!(n-m-1-2r)!}.$$

The sum then is a polynomial of degree $n-m-1$ with leading coefficient

$$\begin{aligned}\sum_{r=0}^{\lfloor\frac{n-m-1}{2}\rfloor} \frac{1}{(2r)!(n-m-1-2r)!} &= \frac{1}{(n-m-1)!}\sum_{r=0}^{\lfloor\frac{n-m-1}{2}\rfloor} \frac{(n-m-1)!}{(2r)!(n-m-1-2r)!}\\ &= \frac{1}{(n-m-1)!}\sum_{r=0}^{\lfloor\frac{n-m-1}{2}\rfloor}\binom{n-m-1}{2r}\\ &= \frac{1}{(n-m-1)!}2^{n-m-2}\\ &= \frac{2^{n-m-1}}{2(n-m-1)!}.\end{aligned}$$

□

Corollary 3.10 *Let F be a field with $q(F)=2^d<\infty$ and let $n\geq 1$ be an integer. Then $\mathrm{sl}_n(F)$ is bounded from above by*

$$\frac{d^{n-1}}{2(n-1)!}+f_n(d)$$

where f_n is a polynomial of degree at most $n-2$.

Proof. This follows from the above by plugging in $\lfloor\frac{d_0+s_0}{2}\rfloor$ for k in Lemma 3.9 and noting that the sums in Theorem 3.8 can be bounded from above by replacing any d_m+s_m by $d_0+s_0=d$. □

Remark 3.11 In the above corollary, we rediscovered [Kah05, Proposition 2.3h)] for fields with finite square class number 2^d in a way that allows us to determine the polynomial explicitly. Of course, sticking with d_m+s_m in Theorem 3.8 rather than replacing them by d will yield better bounds in general but also makes their computation more difficult.

Example 3.12 We now close the article with a comparison of these bounds for level $s(F)=2^s$ with $s\in\{0,1,2\}$. (Recall that conjecturally, if $s(F)<\infty$ then $q(F)<\infty$ implies $s(F)\leq 4$.) Of interest in this context are certain fields of iterated Laurent series that realize a given value $q(F)$ for the respective value $s(F)$, such as

- $s=1$, $q=2^d$ $(d\geq 0)$: $\mathbb{C}((t_1))\dots((t_d))$;
- $s=2$, $q=2^d$ $(d\geq 1)$: $\mathbb{F}_3((t_1))\dots((t_{d-1}))$;
- $s=4$, $q=2^d$ $(d\geq 3)$: $\mathbb{Q}_2((t_1))\dots((t_{d-3}))$.

(Note that for any field F with $s(F)=4$, one has $q(F)\geq 8$, see [EL73, Theorem 2.7].)

Before giving some explicit estimates, we want to recall the upper bounds that we found above.

First of all, we want to collect upper bounds for d_m for some cases. In the following table, we have summarized the upper bounds we know for d_m obtained in Corollary 2.13:

$m \backslash s$	0	1	2
0	d	$d-1$	$d-1$
1	0	$d-1$	$d-3$
2	0	0	$d-3$

For s_m we have the following values according to Remark 2.8 (a):

$m \backslash s$	0	1	2
0	0	1	1
1	0	0	1
2	0	0	0

Now we plug in the upper bounds for d_m and s_m in all the bounds for the symbol length that we have found so far. It is obvious that this will also yield upper bounds for the symbol length.

- Corollary 3.6 yields

$$\mathrm{sl}_n(F) \leq \begin{cases} \binom{d}{n}, & \text{if } s=0 \\ \binom{d}{n} + \binom{d-1}{n-1}, & \text{if } s=1 \\ \binom{d}{n} + \binom{d-2}{n-1} + \binom{d-3}{n-2}, & \text{if } s=2. \end{cases}$$

- Corollary 3.7 yields

$$\mathrm{sl}_n(F) \leq \begin{cases} \binom{d}{n}, & \text{if } s=0 \\ \binom{d}{n}, & \text{if } s=1 \\ \binom{d}{n} + \binom{d-3}{n-2}, & \text{if } s=2. \end{cases}$$

- Since we do not have a closed formula for the upper bound obtained in Theorem 3.8, we will only write down the case $n=3$ explicitly. We obtain the following upper bounds:

s	0	1	2
bound	$\binom{\lfloor \frac{d+1}{2} \rfloor}{2} + \binom{\lfloor \frac{d}{2} \rfloor}{2}$	$\binom{\lfloor \frac{d+1}{2} \rfloor}{2} + \binom{\lfloor \frac{d}{2} \rfloor}{2} + \lfloor \frac{d}{2} \rfloor$	$\binom{\lfloor \frac{d+1}{2} \rfloor}{2} + \binom{\lfloor \frac{d}{2} \rfloor}{2} + \lfloor \frac{d-1}{2} \rfloor + 1$
if $d=2k$	k^2-k	k^2	k^2
if $d=2k+1$	k^2	k^2+k	k^2+k+1

Let us illustrate these upper bounds for the 3-symbol lengths for certain values of d:

- For $s=0$, i.e. $s(F)=1$:

	$d=4$	$d=5$	$d=7$	$d=10$
Corollary 3.6	4	10	35	120
Corollary 3.7	4	10	35	120
Theorem 3.8	2	4	9	20

- For $s=1$, i.e. $s(F)=2$:

	$d=4$	$d=5$	$d=7$	$d=10$
Corollary 3.6	7	16	50	156
Corollary 3.7	4	10	35	120
Theorem 3.8	4	6	12	25

- For $s = 2$, i.e. $s(F) = 4$:

	$d=4$	$d=5$	$d=7$	$d=10$
Corollary 3.6	6	15	49	155
Corollary 3.7	5	12	39	127
Theorem 3.8	4	7	13	25

It comes as no suprise that the bound obtained in Theorem 3.8 yields the best values.

We now return to the case in which F is an iterated Laurent extension of $\mathbb{C}, \mathbb{F}_3, \mathbb{Q}_2$ with $d = 4$ as above. It is straightforward to see that F is 3-linked in these cases, i.e. for each pair of 3-fold Pfister forms π_1, π_2 over F, there are $a, b, c_1, c_2 \in F^*$ with $\pi_1 \cong \langle\langle a, b, c_1\rangle\rangle, \pi_2 \cong \langle\langle a, b, c_2\rangle\rangle$. In particular, we have $\mathrm{sl}_3(F) = 1$, strictly subceeding the upper bounds above.

Example 3.13 Let us consider the special case of a field F with level $s(F) = 1 = 2^0$. In Theorem 3.8 (a), we then have $m = 0$ and $d_0 + s_0 = d$, so the upper bound simplifies to

$$\mathrm{sl}_n(F) \leq \sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{\left\lfloor\frac{d}{2}\right\rfloor}{2r}\binom{\left\lfloor\frac{d+1}{2}\right\rfloor}{n-1-2r}. \tag{2}$$

We now have a closer look at the case $n > d$. By [Lam05, Chapter XI. Kneser's Lemma 6.5] there are no anisotropic forms of dimension $> 2^d$. In particular, we have $\mathrm{I}^n(F) = 0$ and thus $\mathrm{sl}_n(F) = 0$. For $n \geq d + 2$, one readily sees that the right hand side of (2) also equals 0. For $n = d + 1$, the right hand side of (2) equals 1 if $d \equiv 0, 1 \mod 4$ and equals 0 if $d \equiv 2, 3 \mod 4$. The details of these straightforward calculations are left to the reader.

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