

Linked orbits of homeomorphisms of the plane and Gambaudo-Kolev Theorem

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J. P. Boroński¹

*Dedicated to the memory of my dad, Wojtek Boroński (1947-2021),
who taught me how to solve linear equations in one variable.*

Abstract

Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism of the plane. For any bounded orbit $\mathcal{O}(x) = \{h^n(x) : n \in \mathbb{Z}\}$ there exists a fixed point $p \in \mathbb{R}^2$ of h linked to $\mathcal{O}(x)$ in the sense of Gambaudo: one cannot find a Jordan curve $C \subseteq \mathbb{R}^2$ around $\mathcal{O}(x)$, separating it from p , that is isotopic to $h(C)$ in $\mathbb{R}^2 \setminus (\mathcal{O}(x) \cup \{p\})$.

1 Introduction

Given a set D by $\text{cl}(D)$, $\text{int}(D)$, and ∂D we shall denote the closure, interior and boundary of D respectively. \mathbb{S}^n denotes the n -dimensional sphere, and \mathbb{R}^2 denotes the plane. Given an orientation preserving homeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the *orbit* of x is given by $\mathcal{O}(x) = \{h^n(x) : n \in \mathbb{Z}\}$. The present paper is concerned with the existence of fixed points of orientation preserving homeomorphisms of \mathbb{R}^2 linked to bounded orbits, in the sense introduced by J.-M. Gambaudo in [11]. Let \mathcal{O}_1 and \mathcal{O}_2 be two sets invariant for a homeomorphism h . Following [11] we say that these two sets are *unlinked* if there exist two discs $D_1, D_2 \subset \mathbb{R}^2$ with the following properties:

- $\mathcal{O}_i \subset \text{int}(D_i)$ for $i=1,2$;
- $D_1 \cap D_2 = \emptyset$;
- $h(\partial D_i)$ is isotopic to ∂D_i in $\mathbb{R}^2 \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$, for $i = 1, 2$.

Note that if $\text{cl}(\mathcal{O}_1) \cap \mathcal{O}_2 \neq \emptyset$ then \mathcal{O}_1 and \mathcal{O}_2 are linked in a trivial way, as there is no Jordan curve around \mathcal{O}_2 separating the two sets. Gambaudo showed that for any C^1 -embedding f of a disk, and any periodic orbit O of f , there exists a fixed point p linked to O . In consequence, for the torus flow ϕ_t suspending f , the sets $\{\phi_t(O)\}_{t \geq 0}$ and $\{\phi_t(p)\}_{t \geq 0}$ are linked as knots in \mathbb{S}^3 . A similar result was obtained simultaneously by B. Kolev in [14], who showed linking of a periodic orbit to a fixed point for orientation preserving C^1 -diffeomorphisms of \mathbb{R}^2 . The result of Gambaudo and Kolev was generalized to orientation reversing homeomorphisms of \mathbb{S}^2 by M. Bonino [1], who showed linking of periodic orbits of period at least 3, to periodic orbits of least period 2. Bonino also pointed out that Kolev's proof, in the orientation preserving case, works in C^0 as well, as it is enough to perturb a given homeomorphism slightly, by smoothing it out in a small neighborhood of the periodic orbit and then apply the same proof. In the present paper we improve on the results of Gambaudo and Kolev, by proving that *any* bounded orbit is linked to a fixed point.

Theorem 1.1. *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism. For any bounded orbit $\mathcal{O}(x)$ there exists a fixed point $p \in \mathbb{R}^2$ linked to $\mathcal{O}(x)$.*

Outline of Proof.

¹Faculty of Mathematics and Computer Science, Jagiellonian University in Kraków, ul. Łojasiewicza 6, 30-348 Kraków, Poland – and – National Supercomputing Centre IT4Innovations, Division of the University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 70103 Ostrava, Czech Republic, e-mail: jan.boronski@osu.cz

- Suppose no point in $\text{Fix}(h)$ is linked to the compact $K = \text{cl } \mathcal{O}(x)$.
- Then there exists a lift \tilde{h} of h to the universal cover (τ, \tilde{U}) of an open surface U (which is the component of $\mathbb{R}^2 \setminus \text{Fix}(h)$ containing K), and a (compact) sheet \tilde{K} over K , such that $\tilde{h}(\tilde{K}) = \tilde{K}$. This step is possible since $h(U) = U$, as a consequence of a result of Brown and Kister [6].
- Since \tilde{h} is orientation preserving and fixed point free, and \tilde{K} is compact, we obtain a contradiction with Brouwer Translation Theorem.
- To show that \tilde{K} is invariant under \tilde{h} , one projects \tilde{K} from the universal cover onto an infinite cyclic covering space, to realize that \tilde{K} is \tilde{h} -invariant unless a fixed point of h linked to K already exists.

Our proof is inspired by M. Brown's paper [5], the writing of [2], and motivated by [15] where Brown's approach was used to show existence of periodic points in neighborhoods of adding machines in the plane. Unlike the proofs of results in [1], [11] and [14], our proof does not employ any elements of the Nielsen-Thurston theory, but purely topological arguments concerning covering spaces of open surfaces. We believe that this proof reveals a deeper explanation of the phenomenon described by the result of Gambaudo and Kolev. Our result is applicable not only to periodic orbits, but also to non-periodic bounded orbits, and can be used, for example, to guarantee fixed points linked to an invariant Cantor set $C = \text{cl}(\mathcal{O}(x))$. This is particularly useful when $h|C$ is minimal², or more generally aperiodic³, as then C does not contain any fixed points. It is easy to find examples of local dynamics of h for which h may be such that the result of Kolev and Gambaudo does not guarantee a linked fixed point, but the above theorem does. In particular this may occur for a disk $D \subseteq \mathbb{R}^2$ such that $h|_{\partial D}$ is conjugate to a circle homeomorphism with irrational rotation number (minimal or Denjoy-type), or $h|C$ is an odometer with $C \subseteq D$. However, the set of minimal homeomorphisms of the Cantor set is very rich, and contains homeomorphisms with any prescribed topological entropy, even topologically mixing [18], and so potential applications go far beyond these two examples⁴. Moreover, our result also applies to orbits whose closures might be connected sets, even separating plane continua. Finally, our theorem can also be used to detect linking of one-dimensional invariant sets of suspension flows in \mathbb{S}^3 , more general than just knots considered in [11], such as 1-dimensional *matchbox manifolds*; i.e. the class of compact connected metrizable spaces in which every point has a neighborhood homeomorphic to the product $[0, 1] \times C$. This class of spaces includes many familiar examples from the dynamics literature, such as Denjoy exceptional minimal sets of flows [21], solenoids [23], and DA-attractors [13]; see e.g. [9] and [10] to learn more.

2 Proof of Theorem 1.1

Suppose $\mathcal{O}_1 = \{h^n(x) : n \in \mathbb{Z}\}$ is an orbit and $\mathcal{O}_2 = \{p\}$ is a fixed point of h , such that $p \notin \text{cl}(\mathcal{O}_1)$. If D_2 is a sufficiently small disk containing \mathcal{O}_2 then ∂D_2 is always isotopic to $h(\partial D_2)$ ⁵. Therefore for linking of orbits to fixed points, Gambaudo's definition reduces to the following condition.

- \mathcal{O}_1 and p are *linked* if there does not exist a closed disk D_1 with $\mathcal{O}_1 \subset \text{int}(D_1)$, such that $p \notin D_1$ and ∂D_1 is isotopic to $h(\partial D_1)$ in $\mathbb{R}^2 \setminus (\mathcal{O}_1 \cup \{p\})$.

If such a Jordan curve ∂D_1 does exist, and consequently \mathcal{O}_1 and p are unlinked, then we shall call ∂D_1 an *unlinking* of \mathcal{O}_1 and p .

Recall that if U is an open surface and (\tilde{U}, τ) is its universal covering space then given a homeomorphism $h : U \rightarrow U$ there exists a *lift* homeomorphism $\tilde{h} : \tilde{U} \rightarrow \tilde{U}$ such that the following

² $h|C$ is said to be minimal if the orbit of every point is dense in C

³ $h|C$ is aperiodic if $h|C$ has no periodic orbits

⁴Note that any Cantor set homeomorphism extends to a homeomorphism of \mathbb{R}^2 [19, Chapter 13, Theorem 7, p. 93].

⁵Note that this shows that this form of linking is not well suited for homeomorphisms of \mathbb{S}^2 , since there any two Jordan curves separating \mathcal{O}_1 and x' are isotopic in $\mathbb{S}^2 \setminus (\mathcal{O}_1 \cup \{x'\})$. This contrasts with the case discussed in [1], when \mathcal{O}_2 is of least period 2, in which case this form of linking is meaningful.

diagram commutes.

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{h}} & \tilde{U} \\ \tau \downarrow & & \tau \downarrow \\ U & \xrightarrow{h} & U \end{array}$$

Additionally if $h(x) = y$ then \tilde{h} is uniquely determined by the choice of two points $\tilde{x} \in \tau^{-1}(x)$, $\tilde{y} \in \tau^{-1}(y)$ and setting $\tilde{h}(\tilde{x}) = \tilde{y}$. If $D \subset U$ is a disk (or its subset), then $\tau^{-1}(D)$ consists of pairwise disjoint homeomorphic copies of D in \tilde{U} , called *sheets*. We shall need the following celebrated result of Brouwer. Brouwer's theorem with its subsequent generalizations is much stronger, but we shall only need the weaker version stated below. The reader is referred to [22] for a historical account of the proof of Brouwer's result.

Theorem 2.1 (Brouwer Translation Theorem). [7] *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving planar homeomorphism. If there exists an $x \in \mathbb{R}^2$ such that $\mathcal{O}(x)$ is bounded, then there exists a fixed point $p \in \mathbb{R}^2$ of h .*

Now we are ready to prove Theorem 1.1.

Proof. (of Theorem 1.1) Let $\mathcal{O}(x)$ be a bounded orbit and $p \in \text{Fix}(h)$. If $p \in \text{cl}(\mathcal{O}(x)) \cap \text{Fix}(h)$ then $\mathcal{O}(x)$ and p are linked, and so from now on we shall assume that $\text{cl}(\mathcal{O}(x)) \cap \text{Fix}(h) = \emptyset$.

Standing Assumption: $\text{cl}(\mathcal{O}(x)) \cap \text{Fix}(h) = \emptyset$.

We start with the case when $K = \text{cl}(\mathcal{O}(x))$ does not separate \mathbb{R}^2 , as the case when K separates is easier.

CASE I: $\text{cl}(\mathcal{O}(x))$ does not separate \mathbb{R}^2 .

By contradiction, suppose that no point in $\text{Fix}(h)$ is linked to $\mathcal{O}(x)$. Let U be the component of $\mathbb{R}^2 \setminus \text{Fix}(h)$ that contains x . Recall that $h(U) = U$ by [6], and so $K \subset U$. Note that U cannot be simply connected, as otherwise we obtain a contradiction with Brouwer Translation Theorem, since U is then homeomorphic to \mathbb{R}^2 and contains a bounded orbit, but no fixed point. Let $x_n = h^n(x)$ for each \mathbb{N} . Let (\tilde{U}, τ) be the universal cover of U . Note that \tilde{U} is homeomorphic to \mathbb{R}^2 and, since K is contained in a disk disjoint from $\text{Fix}(h)$, K lifts to pairwise disjoint homeomorphic copies of itself (sheets) in \tilde{U} , each of which maps homeomorphically onto K by τ . Let \tilde{K} be one such a sheet. We have $\tau(\tilde{K}) = K$. Let $\tilde{x} = \tau^{-1}(x) \cap \tilde{K}$ and $\tilde{x}_1 = \tau^{-1}(x_1) \cap \tilde{K}$. The homeomorphism h lifts to a unique homeomorphism \tilde{h} such that $\tilde{h}(\tilde{x}) = \tilde{x}_1$.

Claim 2.2. $\tilde{h}^n(\tilde{x}) \in \tilde{K}$ for every $n \in \mathbb{N}$.

Proof. (of Claim 2.2) By contradiction, suppose that there exists an N such that $\tilde{h}^N(\tilde{x}) \notin \tilde{K}$. Without loss of generality we assume that $N = 2$. Let $\tilde{x}'_2 = \tau^{-1}(x_2) \cap \tilde{K}$ and $\tilde{x}_2 = \tilde{h}(\tilde{x}_1)$. We have $\tilde{x}_2 \in \tilde{h}(\tilde{K}) \neq \tilde{K}$.

Let $\sigma : \tilde{U} \rightarrow \tilde{U}$ be the deck transformation such that $\sigma(\tilde{x}'_2) = \tilde{x}_2$. The deck transformation group is isomorphic to the fundamental group $\pi_1(U)$ of U , and one sees σ as an element α of $\pi_1(U)$. There exists a point $p \in \text{Fix}(h)$, such that α is a nontrivial loop in the surface $W = \mathbb{R}^2 \setminus \{p\}$. Let (\tilde{W}, κ) be an infinite cyclic covering space of W . Then (\tilde{U}, τ) is also a universal covering space of \tilde{U} , the component of $\tilde{W} \setminus \kappa^{-1}(\text{Fix}(h) \setminus \{p\})$ that contains $\kappa^{-1}(K)$, and so there exists a covering map $\phi : \tilde{U} \rightarrow \tilde{U}$ such that $\tau = \kappa|_{\tilde{U}} \circ \phi$. Let $\bar{x} = \phi(\tilde{x})$, $\bar{x}_1 = \phi(\tilde{x}_1)$, $\bar{x}_2 = \phi(\tilde{x}_2)$ and $\bar{x}'_2 = \phi(\tilde{x}'_2)$. By the choice of α we have that the element of the deck transformation group (with respect to \tilde{U}) $\bar{\sigma}$ satisfying $\bar{\sigma} \circ \phi = \phi \circ \sigma$ is nontrivial, and $\bar{\sigma}(\bar{x}'_2) = \bar{x}_2$, so $\bar{x}_2 \neq \bar{x}'_2$. Let L be an unlinking of $\text{cl}(\mathcal{O}(x))$ and $\{p\}$. Set $\bar{K} = \phi(\tilde{K})$, and let \bar{L} be a sheet over L that bounds a disk containing \bar{K} , and \bar{h} be the lift of h to \tilde{W} given by $\bar{h}(\bar{x}) = \bar{x}_1$. We have $\phi \circ \tilde{h} = (\bar{h}|_{\tilde{U}}) \circ \phi$, and $\bar{h}(\bar{x}_1) = \bar{x}_2$.

Consider an isotopy $\{i_t : \mathbb{S}^1 \rightarrow \mathbb{R}^2 \setminus (\text{cl}(\mathcal{O}(x)) \cup \{p\}) : 0 \leq t \leq 1\}$ from $i_0(\mathbb{S}^1) = L$ to $i_1(\mathbb{S}^1) = h(L)$. By isotopy lifting property, this isotopy lifts to an isotopy $\tilde{i}_t : \mathbb{S}^1 \rightarrow \tilde{W} \setminus \kappa^{-1}(\text{cl}(\mathcal{O}(x)))$ taking \bar{L} to $\bar{h}(\bar{L})$, both of which are loops in \tilde{W} (since L and $h(L)$ are inessential in $\mathbb{R}^2 \setminus \{p\}$). This leads to a contradiction, since \bar{L} bounds a disk containing \bar{x}_2 , but $\bar{h}(\bar{L})$ does not, so \tilde{i}_t cannot take \bar{L} to $\bar{h}(\bar{L})$ in $\tilde{W} \setminus \kappa^{-1}(\text{cl}(\mathcal{O}(x)))$. This completes the proof of Claim 2.2. \square

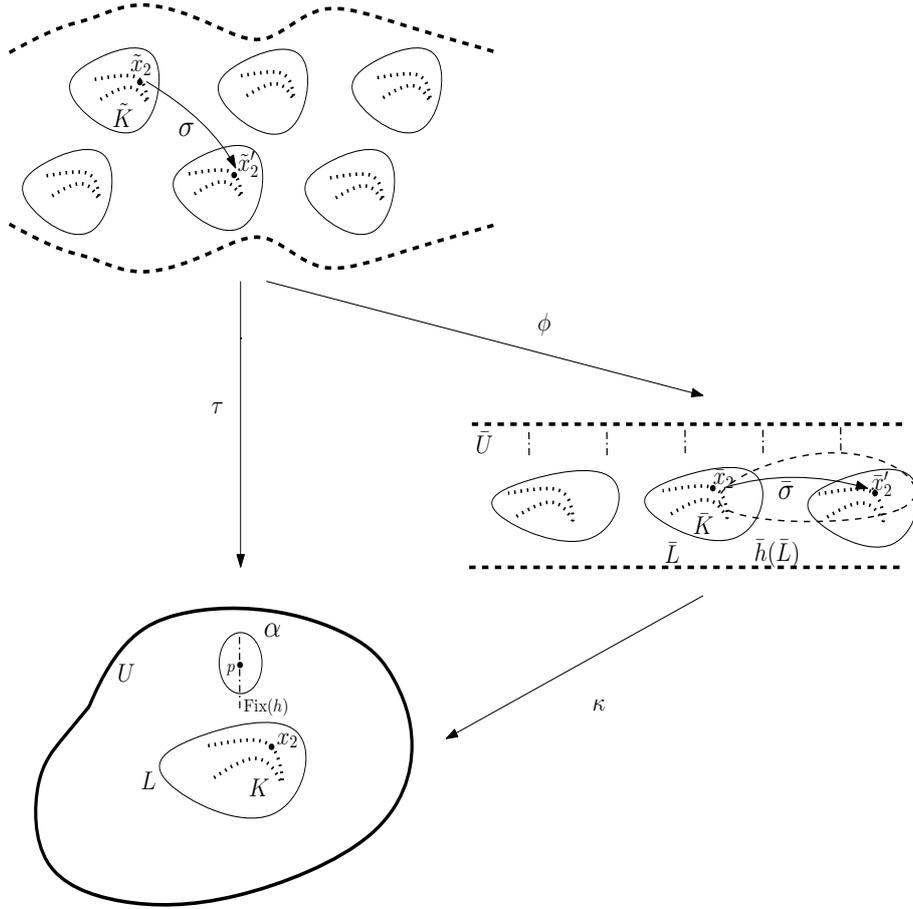


Figure 1: Proof of Theorem 1.1, CASE I: The component U of $\mathbb{R}^2 \setminus \text{Fix}(h)$ containing $\text{cl}(\mathcal{O}(x))$ and the covering spaces \tilde{U} and \bar{U} .

To conclude the proof of CASE I it is now enough to observe that \tilde{h} is an orientation preserving homeomorphism of the plane \tilde{U} , with a compact invariant set \tilde{K} , but no fixed points, contradicting Brouwer Translation Theorem.

CASE II: $\text{cl}(\mathcal{O}(x))$ separates \mathbb{R}^2 .

If no bounded component V of $\mathbb{R}^2 \setminus \text{cl}(\mathcal{O}(x))$ contains a fixed point then we add all such components to $\text{cl}(\mathcal{O}(x))$ and we are back to CASE I.

Otherwise, there exists a bounded component V_o of $\mathbb{R}^2 \setminus \text{cl}(\mathcal{O}(x))$ such that $h(V_o) = V_o$ and V_o contains a fixed point p' of h . But since ∂V_o separates the plane into at least two components, one of which contains p' , and none of which contains $\text{cl}(\mathcal{O}(x))$, there does not exist a disk D_1 such that $\text{cl}(\mathcal{O}(x)) \subset \text{int}(D_1)$ and $p' \notin D_1$. Consequently there is no unlinking of $\text{cl}(\mathcal{O}(x))$ and p' , and these two orbits must be linked. This concludes the proof of CASE II.

The proof of Theorem 1.1 is complete. □

Remark 2.3. *The proof of Theorem 1.1 remains valid if the full orbit $\mathcal{O}(x)$ is replaced with one of the half orbits $\mathcal{O}^+(x) = \{h^n(x) : n \in \mathbb{N}\}$ or $\mathcal{O}^-(x) = \{h^{-n}(x) : n \in \mathbb{N}\}$.*

3 Final Remarks

In 1988 J. Franks defined what seems to be a deeper form of linking, for which one requires from a periodic orbit to have a non-zero rotation number around a fixed point. Franks asked whether every periodic orbit of an orientation preserving homeomorphism is linked in that sense to a fixed point [3]. This difficult open problem was resolved in the affirmative by P. Le Calvez in 2006 [17]. It seems that Le Calvez's result cannot be extended to bounded orbits. A result that seems related to

both forms of linking is proven in [12], where a way of locating fixed points in proximity of recurrent orbits is given, by the means of topological hulls of unions of arcs, connecting a finite number of points of an ϵ -periodic orbit; see also [16] and [20]. Sufficient conditions for the nonremovability of collections of periodic points under isotopy relative to a general compact invariant set can be found in [4].

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