A Unified Approach to Evaluation and Routing in Public Transport Systems

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May 3, 2024

Abstract

Both evaluating the service quality of a public transport system and understanding how passengers choose between modes or routes is imperative for public transport operators, providers of competing mobility services and policy makers. However, the literature does not offer consensus on how either of these tasks should be performed, which can lead to inconsistent or counter-intuitive results. This paper provides a formal treatment on how common manifestations of public transport (route sets, timetables and line plans) can be evaluated consistently, and how passengers distribute over routes. Our main insight is that evaluation and routing are two sides of the same coin: by solving an appropriate optimization model one obtains both the quality of the route set, timetable or line plan (the optimal objective value), and the distribution of the travelers over the routes (the optimal solution itself). We further demonstrate that the framework developed in this paper enables planners to (i) improve service by taking better decisions and (ii) assess to what degree of accuracy traveler behavior should be modeled on their network, potentially avoiding investing in complicated methods that may not be necessary.

Keywords: Passenger Mobility, Public Transport, Choice Modeling, Route Choice, Line Planning, Transit Network Design, Timetabling.

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1 Introduction

In passenger mobility research, it is common to define some measure for the service quality of a public transport system (e.g., average travel time) and to assume some choice model that distributes travelers over routes and/or modes (e.g., the multinomial logit model). Despite their ubiquity, there is currently no consensus on *how* the service quality of public transport should be measured and *what* choice model is appropriate for what context. Instead, the literature offers a variety of approaches, which makes it difficult to compare and validate obtained results. Furthermore, it is generally accepted that public transport systems can improve service quality by adding routes, making routes faster or increasing the frequency at which routes are operated (at least, in the absence of congestion effects). However, this is not always reflected in existing approaches, leading to counter-intuitive and inconsistent results.

To illustrate how seemingly sensible route choice models and service quality measures can lead to unexpected outcomes and suboptimal decisions, suppose that travelers choose routes according to the logit model and that average travel time is used as a measure of service quality. For the system depicted in Figure 1a, where travelers can choose between route 1 with a duration of 15 minutes and route 2 with duration of l_2 minutes, Figure 1b shows the average travel time as a function of l_2 . After some point, making route 2 slower actually *improves* the measure, because it increases the likelihood that travelers switch to route 1, which is better in terms of travel time. Public transport planners using such a measure are insufficiently incentivized to speed up or add routes. In fact, they may even decide to slow down or remove routes, as it could occur that their measure suggests that doing so improves service quality. Note that this phenomenon is fundamentally different than Braess' paradox, because in our setting travel times are fixed, whereas in Braess' paradox travel times depend on the number of travelers that choose a route (Braess, 1968).



(a) Routes. (b) Travel Time as function of l_2 . (c) Perceived Travel Time as function of l_2 .

Figure 1: Illustration of two route set measures, where travelers are distributed over the routes using the logit model with $\beta = 0.22$. The duration of route *i* is denoted by l_i .

In this paper, we present a formal treatment of route choice models and service quality measures for three models of public transport: route sets, (periodic) timetables and line plans, concepts illustrated in Figure 2. Throughout the paper, we consider a single origin destination (OD) pair, but this can be generalized to multiple OD pairs by taking a weighted average. We analyze the two predominant route choice models in the literature: *shortest path* routing,



Figure 2: Illustration of the differences between a route set, timetable and line plan. l_i and θ_i represent the route duration and departure time of route *i*, respectively. *T* is the cycle time.

where all travelers choose the shortest route, and *logit* routing, where travelers distribute over the routes according to the logit model. We then define desirable properties of measures, show which measures fail to meet these properties and develop measures that do. To ensure that the derived measures are consistent and interpretable, our approach is hierarchical: the line plan measures build upon the timetable measures, which again build upon the route set measures. Ultimately, this results in a ready-to-use framework for routing and evaluation in public transport, as well as multiple managerial insights.

1.1 Framework

This section provides a high-level overview of our approach and results. A route set is simply a set of routes with given durations (see Figure 2a). A measure for route set quality can be constructed by pairing a routing model, e.g. logit, with an evaluation function, e.g. travel time. The desirable properties that we define are *monotonicity* and *consistency*. A measure is monotonic if increasing route durations or removing a route cannot improve the measure. Figure 1b illustrates that the combination of logit routing and travel time evaluation induces a measure that fails to be monotonic. We say a measure is consistent if the routing model minimizes the evaluation function. Again, travel time under logit routing fails to satisfy this property, since a shortest path routing will always lead to a shorter travel time compared to logit routing.

The observation that logit routing does not minimize travel time raises the question if there exists an alternative evaluation function that is minimized by logit. We answer this question affirmatively, by showing that logit routing optimizes the evaluation function we refer to as *perceived travel time*. This results in a measure that is both consistent and monotonic. The monotonicity can also be observed in Figure 1c, which depicts perceived travel time under logit routing. Moreover, we provide a motivation for the terminology "perceived travel time" through the random utility model (RUM) interpretation of logit. In addition to evaluating perceived travel time under logit routing, combining regular travel time evaluation with shortest path routing also induces a consistent and monotonic measure.

To also support practitioners in other stages of the public transport planning process, this paper subsequently extends the analysis from route sets to periodic timetables. A periodic timetable is a route set that is operated periodically with a given cycle time or period and with given departure times (see Figure 2b). The time until a traveler reaches its destination now not only depends on the route durations, but also on the waiting time that the traveler experiences.

As service quality should be measured with respect to the actual (unobserved) demand rather than the induced demand, we assume that travel demand is distributed uniformly over the period, similar to for example Kaspi and Raviv (2013) and Polinder et al. (2021). Under this assumption, we show how to efficiently compute the average travel time under shortest path routing, and the average perceived travel time under logit routing, resulting in two consistent and monotonic measures for the quality of a periodic timetable.

Finally, we consider line plans, which are route sets that will be performed periodically, but with unknown departure times (see Figure 2c). Line plans are commonly used in strategic planning, when the timetable is not yet known. There, one needs measures for service quality and route choice models that are independent of the timetable that will be operated. To deal with this additional source of uncertainty, we construct line plan measures by optimizing the respective timetable measures over all possible timetables. Surprisingly, we are able to prove that the solution to this optimization problem can be interpreted as a routing, such that we obtain the route choice model *for free:* one simultaneously finds an estimate of service quality (the optimal value) and a routing (the optimal solution itself). We additionally develop algorithms that efficiently solve this optimization problem both for shortest path and for logit routing.

1.2 Practical Insights

The developed framework reveals a deep parallel between evaluation and routing: by solving an appropriate optimization model one obtains both the quality of the route set, timetable or line plan (the optimal objective value), and the distribution of the travelers over the routes (the optimal solution itself). Hence, every evaluation function *implies* some route choice model, and vice versa. The importance of this connection for public transport planners is evident for route sets from the example in Figure 1, but this behavior extends all the way to line plans: situations where the evaluation function and routing do not correspond lead to missed opportunities for improving services and taking wrong decisions that can harm the experience of public transport users.

Further numerical experiments disclose more important insights for practitioners. Route choice and service quality can differ substantially between route sets, timetables, and line plans, highlighting that it is crucial to select the right model at the right stage of planning. Furthermore, while logit is the model of choice for accurately capturing traveler behavior, the benefit of logit over shortest path diminishes as we move from route sets to timetables to line plans. This observation has major managerial implications, as it implies that *it is not always necessary for public transport planners to accurately model travel behavior*. Especially for long-term or strategic planning such as line planning, the shortest path model may be sufficiently accurate to make good decisions.

1.3 Contributions and Structure of the Paper

Summarizing, the main contribution of this paper is twofold. Firstly, we develop and analyze a framework for traveler route choice and service quality evaluation for route sets, timetables and line plans. Compared to existing work, the framework introduces a new evaluation function that is consistent with the logit model, and routing models and measures tailored to the specific nature of line plans. Secondly, based on the framework we derive and discuss several managerial implications. Among other insights, we demonstrate that the framework developed in this paper enables planners to (i) improve service by taking better decisions and to (ii) assess the difference between shortest path and logit measures for their network, avoiding investments in complicated methods when they are unnecessary.

The remainder of this paper is structured as follows. Section 2 describes related literature. Sections 3.1, 3.2 and 3.3 consider the measures for route sets, timetables and line plans, respectively. Section 4 presents numerical experiments and managerial implications. Section 5 concludes the paper and describes directions for future research.

2 Literature Review

Existing measures of the service quality of public transport can be divided into two categories. On the one hand, there are travel time-based approaches that capture the quality of public transport through a travel time equivalent. On the other hand, there are utility-based approaches that aim to quantify the total utility experienced by users of the public transport system, which may depend on other factors in addition to travel time. Analogously (although this connection is not always made explicit), when it comes to route choice models in public transport we can distinguish between the shortest path model and the multinomial logit model and variants thereof.

Travel time-based measures are most prominent in the Operations Research literature, as their linear nature lends itself for use in optimization models. However, because routing travelers endogenously heavily increases the complexity of the model, passenger travel time has traditionally been modeled using fixed routes, both in line planning (also referred to as transit network design) (Schöbel, 2012) and timetabling (Borndörfer et al., 2017). Since actual shortest paths (and hence travel times) strongly depend on line planning or timetabling decisions, the literature has been moving towards the routing of travelers on shortest paths during the optimization, see e.g. Borndörfer et al. (2007) and Bertsimas et al. (2021) for line planning with integrated routing and Schiewe and Schöbel (2020) and Polinder et al. (2021) for timetabling with integrated routing.

Another relatively recent development in timetabling models is to expand the origin-destination matrix with a time dimension that indicates *when* people want to travel and including waiting or adaptation time at the start or end of trips in the total travel time (Kaspi and Raviv, 2013; Polinder et al., 2021; Robenek et al., 2016). In periodic timetabling settings, it is typically assumed that demand is distributed uniformly over the period, which then rewards timetables that distribute routes more evenly over the period (a timetable repeating every hour with departures at xx:05 and xx:35 results in a lower expected waiting time than one with departures at xx:05 and xx:20). In line planning models, including waiting time is non-trivial, since it is dependent on the timetable. We are unaware of papers that address this issue. The line planning measures and route choice models that we develop in Section 3.3 do account for waiting time, and can efficiently be incorporated in line planning models.

A significant disadvantage of travel time-based approaches is the implicit assumption that

all travelers care about is travel time, and therefore always take the shortest route. Even when aspects such as the number of transfers, transfer time and waiting time are factored into generalized measures of travel time, this assumption is known to prove false in practice (Van der Hurk et al., 2015). While it is true that travelers favor shorter routes, they tend to distribute more or less evenly between routes with similar (generalized) travel times. For this reason, studies that do not depend on solvers that require linear formulations typically assess service quality of public transport by analyzing the offered travel options through the lens of discrete choice theory (Ben-Akiva and Lerman, 1985; McFadden, 1974). In this context, service quality is also referred to as accessibility, user benefits or consumer surplus. The predominant model in these approaches is the logit model, which naturally provides a quality measure for a set of travel options known as the *logsum* (Williams, 1977). The logsum is a dimensionless quantity that reflects the expected (dis)utility perceived by travelers. However, since it is not possible to attribute any meaning to absolute utility, the absolute value of the logsum cannot be interpreted directly. Therefore, one is required to define a baseline scenario and assess the logsum change (Sweet, 1997). For cost-benefit analyses or analyzing the benefits of the introduction of new transport modes, the logsum change can be converted into monetary or temporal units (Cats et al., 2022; De Jong et al., 2007; Standen et al., 2019). In addition to the dependence on a (potentially arbitrary) baseline scenario, another shortcoming of existing measures is that timetables and line plans are treated similarly as route sets, neglecting that decreasing headway variance or increasing line frequencies improves service quality and also impacts the routing decisions of travelers. The measures and route choice models that we propose in this paper are both easily interpretable, and tailored to the distinctive characteristics of route sets, timetables and line plans.

Finally, there are recent contributions mixing travel time-based and utility-based evaluation, where travelers are routed according to the logit model, but the objective is to minimize travel time (Bertsimas et al., 2020; Hartleb and Schmidt, 2022). Since not all travelers take the shortest route in the logit model, this implies a discrepancy between the objective of the model and the preferences of travelers. In this paper, we formally show that this approach corresponds to optimizing a measure that lacks consistency and monotonicity, and propose an alternative objective function that is consistent and monotonic with logit routing.

3 Framework for Routing and Evaluation

This section introduces a novel and ready-to-use framework for routing and evaluation in public transport. New routing models and evaluation measures are built up hierarchically from route sets to periodic timetables and all the way up to line plans. Along the way, we define desirable properties, show which measures fail to meet these properties, and develop measures that do. We introduce optimization models to efficient evaluate the new measures, and show that these optimization models can be interpreted as routing models themselves, proving that evaluation and routing are two sides of the same coin.

3.1 Route Sets

We start building the new framework from the most basic object: a set of routes $\mathcal{R} = \{1, 2, ..., n\}$ that travelers can choose from. Each route $i \in \mathcal{R}$ has a duration $l_i \in \mathbb{R}$, which may include more general quality aspects (all converted to time) such as the number of transfers, transfer time, and waiting time. Travelers are assumed to make probabilistic decisions, and assign probabilities to the routes according to a *route choice model*. When the probabilities are set, the expected quality of the route set can be evaluated with an *evaluation function*. We refer to the combination of a route choice model and an evaluation function as a *measure* of route set quality.

Route Choice Models This paper considers the two predominant route choice models in the literature: shortest path routing and logit routing. Let a route choice model, or a *routing* for short, be a function that assigns probabilities to the routes. That is, a routing is a function p that maps the routes \mathcal{R} to $p(\mathcal{R}) \in \mathcal{P}$, where \mathcal{P} is the set of probability vectors. The two routing models are defined as follows:

• Shortest Path Routing:

$$p_i^{sp} = 1$$
 if $i = \underset{j \in \mathcal{R}}{\operatorname{argmin}} \{l_j\}$ and $p_i^{sp} = 0$ else, for all $i \in \mathcal{R}$

• Logit Routing:

$$p_i^{logit} = \frac{e^{-\beta l_i}}{\sum_{j \in \mathcal{R}} e^{-\beta l_j}}, \text{ for all } i \in \mathcal{R}, \text{ with parameter } \beta > 0.$$

Shortest path routing assigns probability one to the shortest path and probability zero otherwise, where ties are broken arbitrarily (e.g., by lowest index). Logit routing assigns probabilities accordingly to the multinomial logit model (McFadden, 1974). The parameter β controls the sensitivity of travelers to the route durations. For smaller β , travelers distribute more evenly, while for larger β they are more likely to choose the shortest route. In fact, it can be seen that shortest path routing is the limiting case of logit routing for $\beta \to \infty$. Figure 3 provides an example for $\mathcal{R} = \{1, 2\}$ and $l_1 = 15$ minutes. For shortest path routing, travelers choose route 2 if the duration is shorter than 15 minutes, and route 1 otherwise. For logit routing, the probability to choose route 2 is a smooth function of l_2 , the steepness of which depends on β .

Evaluation Functions After the route probabilities $p \in \mathcal{P}$ have been determined, the quality of the route set is evaluated with an evaluation function $\mathcal{E}(\mathcal{R}, p) \in \mathbb{R}$ that assigns lower scores to better outcomes. We consider two different evaluation functions: the expected *travel time* and a new *perceived travel time*.

• Travel Time:

$$\mathcal{E}_{tt}(\mathcal{R},p) = \sum_{i \in \mathcal{R}} l_i p_i.$$



Figure 3: Routing models for route set $\mathcal{R} = \{1, 2\}$ with $l_1 = 15$ minutes and varying l_2 .

	Shortest Path Routing			Logit Routing		
Evaluation	Measure	Monot.	Consist.	Measure	Monot.	Consist.
Travel Time	l_{\min}	1	✓ ✓	$\sum_{i \in \mathcal{R}} p_i^{logit} l_i$	X	X
Perceived Travel Time	l_{\min}	v	×	$-\frac{1}{\beta}\log\left(\sum_{i\in\mathcal{R}}e^{-\beta n_i}\right)$	<i>√ √</i>	v

Table 1: Overview of route set measures and properties (double checkmark for strict monotonic).

• Perceived Travel Time:

$$\mathcal{E}_{ptt}(\mathcal{R}, p) = \sum_{i \in \mathcal{R}} l_i p_i + \frac{1}{\beta} \sum_{i \in \mathcal{R}} p_i \log(p_i), \text{ with parameter } \beta > 0.$$

Travel time evaluation is a straightforward calculation of the expected travel time. More interestingly, perceived travel time evaluation combines travel time with a *dispersion* term that is weighted by the parameter $\beta > 0$. This evaluation function was inspired by Anderson et al. (1988), who introduce a similar function and show that it has favorable properties in the logit context. The dispersion term captures a distributional property of the probability vector $p \in \mathcal{P}$. As such, perceived travel time favors probability vectors that are spread out over the routes, which creates robustness against uncertainty.

Route Set Measures Finally, we obtain a measure of route set quality by combining a routing (a model for traveler behavior) with an evaluation function (a scoring function for given behavior). That is, for routing $p(\mathcal{R})$ and evaluation function $\mathcal{E}(\mathcal{R}, p)$, we define the measure $\mathcal{M}(\mathcal{R}) = \mathcal{E}(\mathcal{R}, p(\mathcal{R}))$. Table 1 summarizes the four measures that can be obtained by substituting one of the two routings into one of the two evaluation functions. For shortest path routing with travel time evaluation, the measure is simply the length l_{\min} of the shortest route. The same measure is obtained when perceived travel time is used, because the dispersion term evaluates to zero when the full probability is assigned to a single route.¹ The measure for logit routing with travel time evaluation corresponds to calculating the expected length with logit probabilities. Logit routing with perceived travel time simplifies to a logsum that is scaled by a factor $-\frac{1}{\alpha}$.

While logit routing with travel time evaluation has been used in the literature (e.g., Bertsi-

¹Using the convention that $p_i \log(p_i) = 0$ for $p_i = 0$.



Figure 4: Example measures (y-axis) for $\mathcal{R} = \{1, 2\}, l_1 = 15, \beta = 0.22$, and varying l_2 (x-axis).

mas et al. (2020); Hartleb and Schmidt (2022)), the example in Figure 4 reveals some unexpected behavior: When the length of route 2 is increased from 20 minutes to 30 minutes, *the measure improves while the route set has become worse*. This shortcoming of logit routing with travel time evaluation is formalized in the next section. The other route set measures are assigned a symbol, and will be used later as building blocks for the evaluation of timetables and line plans:

• Route Set Measure for Shortest Path Routing with Travel Time Evaluation:

$$\mathcal{M}_{sp}^{RS}(\mathcal{R}) = l_{\min}.$$
 (1)

• Route Set Measure for Logit Routing with Perceived Travel Time Evaluation:

$$\mathcal{M}_{logit}^{RS}(\mathcal{R}) = -\frac{1}{\beta} \log \left(\sum_{i \in \mathcal{R}} e^{-\beta l_i} \right).$$
⁽²⁾

Monotonicity and Consistency This section introduces two desirable properties of route set measures: monotonicity and consistency. First, we define a partial order on route sets to capture the intuition that a route set is better when the routes are shorter and more options are available. In particular, consider two route sets \mathcal{R} and \mathcal{R}' and corresponding route length vectors l and l'. Now sort l and l' and append routes with infinite length to make the vectors the same size. We say that $\mathcal{R} \leq \mathcal{R}'$ if $l \leq l'$ elementwise, and $\mathcal{R} \prec \mathcal{R}'$ if additionally $l \neq l'$. Informally, route set \mathcal{R} is better when each route in \mathcal{R}' can be matched to a better route in \mathcal{R} .

The first desirable property is monotonicity. We say that a route set measure is *monotonic* if better route sets (according to the partial order defined above) receive better scores. The second desirable property is consistency. A route set measure is *consistent* when the travelers behave in such a way that they optimize the evaluation function. This suggests that the route set is evaluated in a way that matches what is important to the travelers. When a measure is inconsistent, the performance is measured according to a metric that the travelers do not agree with, as revealed through their routing choices. These properties are formalized as follows:

• Monotonicity: A route set measure is monotonic if, for all route sets \mathcal{R} and \mathcal{R}' ,

$$\mathcal{R} \preceq \mathcal{R}' \Longrightarrow \mathcal{M}(\mathcal{R}) \leq \mathcal{M}(\mathcal{R}'),$$

and strict monotonic if additionally

$$\mathcal{R} \prec \mathcal{R}' \Longrightarrow \mathcal{M}(\mathcal{R}) < \mathcal{M}(\mathcal{R}').$$

• Consistency: A route set measure that combines routing $p(\mathcal{R})$ and evaluation function $\mathcal{E}(\mathcal{R}, p)$ is consistent if, for any route set \mathcal{R} ,

 $p(\mathcal{R})$ is an optimal solution to the minimization problem $\min_{p \in \mathcal{P}} \mathcal{E}(\mathcal{R}, p)$.

Table 1 summarizes the monotonicity and consistency properties of the four route set measures, with proofs provided in Appendix A. The table shows that shortest path routing with travel time evaluation and logit routing with perceived travel time evaluation both exhibit monotonicity and consistency. The counter-intuitive results for logit routing with travel time evaluation are explained by a lack of monotonicity, and the measure lacks consistency as well. Finally, we remark that monotonicity alone is not sufficient to obtain a good measure: for example, $\mathcal{M}(\mathcal{R}) = 0$ is monotonic but practically useless. Consistency is also useful to motivate the assumptions that may appear in other parts of the analysis. For example, Table 1 shows that shortest path routing with travel time evaluation and shortest path routing with perceived travel time evaluation have the same measure, but using travel time yields a more consistent theory.

Random Utility Model Interpretation The logit model is often motivated by formulating it as a random utility model (Train, 2009). This perspective is also useful in the current context, and we introduce this model here as the *random travel time model*. In the random travel time model, the *perceived duration* χ_i of a route is given by

$$\chi_i = l_i - \frac{1}{\beta} \varepsilon_i, \quad \forall i \in \mathcal{R},$$

with independent error terms ε_i that are distributed according to a Gumbel distribution. Each traveler chooses the route that minimizes perceived duration for them. This choice is motivated by the route lengths l_i , but also by a personal random component ε_i for each route. Again, for $\beta \to \infty$ the randomness disappears and travelers will pick the shortest path.

Analogous to the random utility model, for the random travel time model, it can be shown that

$$\mathbb{E}\left(\min_{i\in\mathcal{R}}\chi_i\right) = -\frac{1}{\beta}\log\left(\sum_{i\in\mathcal{R}}e^{-\beta l_i}\right) + C,$$

where C is an unknown constant that reflects the base utility of the travelers (Train, 2009). Note that (up to a constant) this quantity is precisely \mathcal{M}_{logit}^{RS} . This shows that the measure that combines perceived travel time and logit routing can be interpreted as the expected perceived travel time of a random traveler.

The term $\log \left(\sum_{i \in \mathcal{R}} e^{-\beta l_i}\right)$ is referred to as the *logsum* in the literature, and has been used extensively to assess the user benefits of public transport. However, as discussed in Section 2, existing approaches compute the logsum change compared to a baseline scenario, which can subsequently be converted to units of time. Our approach is to directly convert the logsum to units of time, which eases its interpretation and omits the need for a baseline scenario.

3.2 Timetables

We now shift attention to evaluating timetables for systems that provide periodic service. Such a system is defined by a route set $\mathcal{R} = \{1, \ldots, n\}$, a period T > 0, and a timetable $\boldsymbol{\theta} \in [0, T)^n$ of departure times (see Figure 2). The routes are performed periodically, such that route $i \in \mathcal{R}$ departs at times θ_i , $\theta_i + T$, $\theta_i + 2T$, etc. Rather than a static route set, travelers now face an *observed route set* with added waiting times that depend on the arrival time. We will evaluate the experience for a particular traveler by evaluating the observed route set with the measures developed in the previous section. To evaluate the full timetable, we propose to calculate the *expected observed route set measure* for a random arrival, reflecting the average experience of travelers facing this timetable.

This section first introduces the observed route set measure, and studies how it changes as a function of arrival time. We show how to calculate characteristic values that describe this function, and how to use them to calculate timetable measures under uniform arrivals. These measures are further specialized to shortest path routing and logit routing, respectively, to obtain measures that are completely separable and convex in the characteristic values, paving the way for extensions to line planning.

The following notation is used to work with cyclic timetables. The timetable $\boldsymbol{\theta} \in [0, T)^n$ defines a natural cyclic ordering of the routes by departure time. For a route $i \in \mathcal{R}$, let $\pi(i)$ be its predecessor and let $\sigma(i)$ be its successor. Routes that depart at the same time are ordered according to their index. For example, if subsequent routes i and $\sigma(i)$ depart at the same time $\theta_i = \theta_{\sigma(i)}$, then it must be that $i < \sigma(i)$. Time intervals [t, t'] are also considered to be cyclical. That is, [t, t'] equals [t, t'] when $t \leq t'$ and $[t, T) \cup [0, t']$ otherwise.

Observed Route Set Measures In the periodic setting, a traveler entering the system at time t has to take waiting into account. The waiting time for route $i \in \mathcal{R}$ is given by $\theta_i + zT - t$ for the smallest value of $z \in \mathbb{Z}$ such that the waiting time is non-negative. For convenience, we define the function $[x]_T$ to be the modulo-like operation that maps x onto [0,T) by adding or subtracting a multiple of T. Waiting time for route $i \in \mathcal{R}$ is then simply $[\theta_i - t]_T$. We define the *observed route set* $\mathcal{R}_{\theta}(t)$ to be the route set as experienced by a traveler arriving at time t. That is, the observed route lengths are given by $l_i + [\theta_i - t]_T$ for every route $i \in \mathcal{R}$.

We are interested in studying the properties of the *observed route set measure* $\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t))$ as a function of time, where \mathcal{M}^{RS} represents either the route set measure for shortest path (1)



Figure 5: Example timetable and observed shortest path route set measure for route set $\mathcal{R} = \{1, 2, 3, 4\}$, route lengths l = (20, 30, 15, 10), period T = 60, and schedule $\boldsymbol{\theta} = (5, 10, 20, 50)$.

or for logit (2). That is, we consider the following two functions:

$$\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(t)) = \min_{i \in \mathcal{R}} \left\{ l_i + [\theta_i - t]_T \right\},\tag{3}$$

$$\mathcal{M}_{logit}^{RS}(\mathcal{R}_{\theta}(t)) = -\frac{1}{\beta} \log \left(\sum_{i \in \mathcal{R}} e^{-\beta (l_i + [\theta_i - t]_T)} \right).$$
(4)

The reason to study these functions is to later derive simple expressions for the expected observed route set measure $\mathbb{E}_t \left(\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t)) \right)$, which will be used as a measure for timetable quality.

Figure 5 visualizes the observed route set measure for shortest path routing. Figure 5a shows the timetable, where circled numbers indicate departures of a given length that are located according to the schedule. For example, the number 10 indicates that a route of length 10 departs at time 50. Figure 5b plots the function $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(t))$. At $t = \theta_1 = 5$ the shortest path is to use route 1, which departs immediately. The corresponding length is $l_1 = 20$ and the waiting time is zero, which results in an observed route set measure of $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(5)) = 20$. The choice for route 1 is optimal everywhere in the interval (50,5]. At t = 55, for example, it is optimal to wait for 10 minutes and then take route 1, which results in an observed route set measure of $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(55)) = 30$. Route 2 ($l_2 = 30$) is sub-optimal: It is always better to wait an additional 10 minutes to take route 3 ($l_3 = 15$). This is reflected in the plot by the lack of a jump at $\theta_2 = 10$, as the route choice does not change.

The example also displays two useful properties that are true in general, as proven in Appendix B.1-2:

- Constant Routing between Departures: For shortest path routing and logit routing the route choice does not change between departures. That is, for route $i \in \mathcal{R}$ and successor $\sigma(i) \in \mathcal{R}$ it holds that $p(\mathcal{R}_{\theta}(t))$ is constant for $t \in (\theta_i, \theta_{\sigma(i)}]$.
- Translation Invariance: Both \mathcal{M}_{sp}^{RS} and \mathcal{M}_{logit}^{RS} are translation invariant between depar-



Figure 6: Two representations for the observed route set measure $\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t))$.

tures. That is, for route $i \in \mathcal{R}$ and successor $\sigma(i) \in \mathcal{R}$ it holds that

$$\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t)) = [\theta_{\sigma(i)} - t]_T + \mathcal{M}^{RS}(\mathcal{R}_{\theta}(\theta_{\sigma(i)})), \quad \forall t \in (\theta_i, \theta_{\sigma(i)}].$$

The first property states that travelers do not make routing decisions upon arrival, but postpone their decisions until the first route departs, essentially treating the waiting time until the first departure as sunk cost. As a direct consequence, the second property states that the observed route set measure can be split into two components: the waiting time until the first departure and the observed route set measure at the first departure. During the waiting part, the slope of the observed route set measure is -1, as can also be seen in Figure 5b.

Characteristic Values Given that the observed route set measure has a fixed slope between departures, we can represent $\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t))$ with a finite number of values. We define the following characteristic values:

- δ_i : Waiting time between predecessor route $\pi(i)$ and route $i \in \mathcal{R}$.
- τ_i : Observed route set measure at time θ_i when $i \in \mathcal{R}$ is the next departure.
- Δ_i : Change in observed route set measure due to missing route $i \in \mathcal{R}$.

The characteristic values are visualized by Figure 6. There are two ways to fully describe $\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t))$ with the characteristic values, up to shifts of the complete schedule. The δ -representation (Figure 6a) defines the function based on the gaps δ_i between departure times, and the observed route set measures τ_i at these departures times. Using instead the vertical differences, the Δ -representation (Figure 6b) describes the function with the observed route set measures τ_i at each departure time, and the change Δ_i in value that would result from missing this route.

Algorithm 1 provides pseudocode to calculate the characteristic values. The calculation of δ (Line 13) and τ (Lines 15 and 17) follows immediately from the definitions, except for a waiting time correction in the case of simultaneous departures (Line 1). This technical detail is justified in Appendix B.3. The value of the measure after missing route $i \in \mathcal{R}$ is conveniently expressed

Algorithm 1 Calculating the Characteristic Values

1: function w(i, j)▷ Waiting time correction for simultaneous departures if $\theta_i = \theta_j$ and j < i then 2: return T \triangleright Add full period T if j is processed before i 3: 4: else return $[\theta_i - \theta_i]_T$ \triangleright Regular wait time 5:6: procedure CHARACTERISTICVALUES(\mathcal{R}, T, θ) 7:8: if $|\mathcal{R}| = 1$ then $\delta_1 \leftarrow T$ 9: $\tau_1 \leftarrow l_1$ 10: 11: else for $i \in \mathcal{R}$ do 12: $\delta_i \leftarrow w(\pi(i), i)$ ▷ From definition 13:if shortest path routing then 14: \triangleright From route set measure (3) $\tau_i \leftarrow \min_{i \in \mathcal{R}} \{ l_i + w(i, j) \}$ 15:else if logit routing then 16: $\tau_i \leftarrow -\frac{1}{\beta} \log \left(\sum_{j \in \mathcal{R}} e^{-\beta(l_j + w(i,j))} \right)$ \triangleright From route set measure (4) 17:for $i \in \mathcal{R}$ do 18: $\Delta_i \leftarrow \delta_{\sigma(i)} + \tau_{\sigma(i)} - \tau_i$ \triangleright From translation invariance 19:20: return δ, τ, Δ

as $\delta_{\sigma(i)} + \tau_{\sigma(i)}$ by virtue of translation invariance (visualized in Figure 6a for $\delta_4 + \tau_4$). It follows that Δ_i can be calculated as $\delta_{\sigma(i)} + \tau_{\sigma(i)} - \tau_i$ (Line 19).

Timetable Measures To measure the quality of timetables, we propose to use the *expected* observed route set measure, $\mathbb{E}_t \left(\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t)) \right)$. This measure can be interpreted as the expected (perceived) travel time experienced by a random traveler, or as a weighted average over all travelers. As a weighted average of route set measures, the timetable measure inherits (strict) monotonicity, and improves when the underlying routes are reduced in length or when new routes are added. The measure also remains consistent, in the sense that the timetable is evaluated in a way that now matches what is important to the *average* traveler.

• Timetable Measure for Shortest Path Routing or Logit Routing (general arrivals):

$$\mathcal{M}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \mathbb{E}_t \left(\mathcal{M}^{RS}(\mathcal{R}_{\boldsymbol{\theta}}(t)) \right).$$
(5)

Note that the function $\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t))$ has been completely specified previously. As a result, the timetable measure can easily be estimated through sampling or numerical integration.

Furthermore, the measure can be expressed in closed form if we assume that traveler demand is distributed uniformly over the period. This is a common assumption in periodic timetabling (e.g., see Kaspi and Raviv (2013); Polinder et al. (2021)) that reflects the idea that service quality should be measured with respect to the actual (unobserved) demand rather than the induced demand. Using the characteristic values calculated with Algorithm 1, it is proven in Appendix B.5 that the timetable measure can be calculated in two distinct ways that correspond to the δ - and Δ -representation, respectively:

• Timetable Measure for Shortest Path Routing or Logit Routing (uniform arrivals):

$$\mathcal{M}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \delta_i^2 + \tau_i \delta_i \right) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \Delta_i^2 + \tau_i \Delta_i \right).$$
(6)

Finally, we provide expressions for the timetable measures that are completely separable and convex. These properties are extremely important when timetable measures are used as a building block, and they enable the efficient calculation of line plan measures in the next section. For shortest path routing, note the following: if for route $i \in \mathcal{R}$ we have that $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(\theta_i)) < l_i$, then it means that route *i* is not even the best option when it is about to depart, and the route can safely be removed without affecting the measure. After removing the suboptimal routes, we have $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(\theta_i)) = l_i$ for all routes, such that τ_i in (6) may be substituted for l_i . For logit routing, τ_i may be replaced by an expression in Δ_i that is proven in Appendix B.6. We then obtain the following two expressions:

• Timetable Measure for Shortest Path Routing after preprocessing (uniform arrivals):

$$\mathcal{M}_{sp}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \delta_i^2 + l_i \delta_i \right).$$
(7)

• Timetable Measure for Logit Routing (uniform arrivals):

$$\mathcal{M}_{logit}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \Delta_i^2 + l_i \Delta_i + \frac{\Delta_i}{\beta} \log\left(\frac{1 - e^{-\beta \Delta_i}}{1 - e^{-\beta T}}\right) \right).$$
(8)

The convexity of (7) is trivial, and the convexity of (8) is proven in Appendix B.7. The fact that (8) is convex demonstrates the importance of viewing the observed route set measure through the Δ -representation. In fact, it can be shown that reformulating τ_i using the δ -representation leads to an expression that is *not* convex.

Route Choice Models for Timetables The proposed measures naturally correspond to route choice models for which travelers distribute according to the expected route choice of a random traveler. Formally, the routing $p^{TT}(\mathcal{R}, T, \theta)$ for a timetable can be defined as follows:

• Route Choice Model for Shortest Path Routing or Logit Routing (general arrivals):

$$\boldsymbol{p}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \mathbb{E}_t \left(\boldsymbol{p}(\mathcal{R}_{\boldsymbol{\theta}}(t)) \right).$$
(9)

For shortest path routing, it holds that after removing the suboptimal routes, random travelers always choose the first departing route after they enter the system. This gives a simple expression in δ for the route choice under uniform arrivals:

• Route Choice Model for Shortest Path Routing after preprocessing (uniform arrivals):

$$\boldsymbol{p}_{sp}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \left(\delta_1/T, ..., \delta_n/T\right).$$
(10)

It is not obvious how to derive a similar expressions for logit routing when travelers face an arbitrary timetable. But surprisingly, the next section will show that when random travelers face an *optimal* timetable, they use route $i \in \mathcal{R}$ with probability Δ_i/T . This again stresses the importance of the Δ -representation.

3.3 Line Plans

Finally, we consider measures and route choice models for line plans. A line plan is defined by a set of routes $\mathcal{R} = \{1, \ldots, n\}$ that are performed every period T > 0. The exact timetable $\boldsymbol{\theta} \in [0, T)^n$ has not yet been decided, and will be optimized at a later stage (see Figure 2). Therefore, we propose to evaluate the quality of a line plan by calculating the minimum value of the timetable measure over all possible timetables, and to base the routing on the optimal timetable. When multiple OD pairs are involved, timetabling is known to be hard in theory and in practice (Liebchen, 2008; Lindner and Reisch, 2022), and there is a need for line plan measures that can be evaluated efficiently. The approach taken here is to not solve the full timetabling problem at once, but rather to optimize the timetable for each OD pair independently. This section analyzes a single OD pair, but measures for multiple OD pairs can easily be combined, e.g., by taking a weighted average. The resulting measure can be seen as the *potential* of a line plan.

To derive new measures and route choice models for line plans (for a single OD pair), we first discuss how to construct optimal timetables. This requires solving different optimization problems, and it is shown how to do so efficiently. The optimal solutions are then used to define the line plan measures and route choice models.

Constructing Optimal Timetables The easiest way to construct an optimal timetable is to directly minimize the timetable measure for shortest path routing (7) or logit routing (8). Figure 7 introduces two optimization models for this purpose. The decision variables $x \ge 0$ represent the values of δ in (7), and the decision variables $y \ge 0$ represent Δ in (8). The objectives (11a) and (12a) are chosen to match the corresponding timetable measure. Finally, normalization constraints (11b) and (12b) are added to exclude solutions that are not meaningful. These constraints are justified by the fact that $\sum_{i\in\mathcal{R}} \delta_i = T$ and $\sum_{i\in\mathcal{R}} \Delta_i = T$ for any feasible timetable (Appendix B.4).

Given an optimal solution to Problem (11) or Problem (12), Algorithm 2 provides methods to extend this information about δ or Δ into a complete timetable θ that is feasible, and for which the measure matches the optimal objective value. This is achieved by calculating desired values for the characteristic values, and then constructing the timetable accordingly. It is not trivial that this is always possible, but Appendix C.9-13 proves that this is indeed the case. Finally, Appendix C.14-15 proves that the output of Algorithm 2 is indeed an optimal timetable. This is a consequence of the fact that the measure of the constructed timetable matches the optimal

$$\min \quad \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} x_i^2 + l_i x_i \right), \quad (11a) \quad \min \quad \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} y_i^2 + l_i y_i + \frac{y_i}{\beta} \log \left(\frac{1 - e^{-\beta y_i}}{1 - e^{-\beta T}} \right) \right), \quad (12a)$$

s.t.
$$\sum_{i \in \mathcal{R}} x_i = T$$
, (11b) s.t. $\sum_{i \in \mathcal{R}} y_i = T$, (12b)

$$x_i \ge 0 \quad \forall i \in \mathcal{R}.$$
 (11c) $y_i \ge 0 \quad \forall i \in \mathcal{R}.$ (12c)

(a) Shortest Path Routing.

(b) Logit Routing.



Alg	Algorithm 2 Construct an Optimal Timetable					
1:	procedure TIMETABLESHORTESTP	PATH (x) > Input: optimal solution to Problem (11)				
2:	$\theta_1 \leftarrow 0$	\triangleright Start timetable at arbitrary time 0				
3:	for $i \in \{2, \ldots, n\}$ do					
4:	$\hat{\delta}_i \leftarrow x_i$	\triangleright Desired value for δ_i (from definition x)				
5:	$\theta_i \leftarrow \theta_{\pi(i)} + \hat{\delta}_i$	\triangleright Progress time table with desired gap between routes				
6:	$\mathbf{return} \theta$					
7:						
8:	procedure TIMETABLELOGIT (y)	\triangleright Input: optimal solution to Problem (12)				
9:	$\theta_1 \leftarrow 0$	\triangleright Start time table at arbitrary time 0				
10:	for $i \in \mathcal{R}$ do					
11:	$\hat{\Delta}_i \leftarrow y_i$	\triangleright Desired value for Δ_i (from definition y)				
12:	$\hat{\tau}_i \leftarrow l_i + \frac{1}{\beta} \log \left(\frac{1 - e^{-\beta \Delta_i}}{1 - e^{-\beta T}} \right)$	\triangleright Desired value for τ_i (from Appendix B.6)				
13:	for $i \in \{2, \ldots, n\}$ do					
14:	$\hat{\delta}_i \leftarrow \hat{\tau}_{\pi(i)} + \hat{\Delta}_{\pi(i)} - \hat{\tau}_i$	\triangleright Desired value for δ_i (from translation invariance)				
15:	$\theta_i \leftarrow \theta_{\pi(i)} + \hat{\delta}_i$	\triangleright Progress time table with desired gap between routes				
16:	$\mathbf{return}\;\theta$					

objective value.

An interesting property of optimal timetables is that for any permutation of the routes, there exists an optimal timetable with departures in that order. This follows from the fact that objectives (11a) and (12a) are separable, and relabeling the routes does not affect the optimal solution. Furthermore, the specific ordering of the routes plays no role in Algorithm 2. Figure 8 demonstrates this property with two optimal timetables. The timetable at the top alternates slower and faster routes, while the timetable at the bottom follows two slow routes by two fast routes. We obtain the managerial insight that both cyclical orderings can be optimal if the spacing between departures is optimized accordingly.

Solving Problems (11) and (12) Next, we focus on solving Problems (11) and (12). We have already seen that the solutions can be used by Algorithm 2 to construct an optimal timetable, and the optimal objective values will be used as line plan measures. A crucial observation is that (11) and (12) are convex optimization problems that can be solved efficiently. The convexity stems from the convex timetable measures (7) and (8), and the fact that all constraints are linear.



(a) Alternating timetable with observed route set measure $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(t))$.



(b) Non-alternating timetable with observed route set measure $\mathcal{M}_{sp}^{RS}(\mathcal{R}_{\theta}(t))$.

Figure 8: Optimal timetables for different route orderings.

The Δ -representation has been crucial to achieve this result, as the optimization problem for logit routing is not convex in the standard δ -representation.

There are a number of tools and techniques to solve Problems (11) and (12). Using a classical cutting plane method (Kelley Jr., 1960), the convex objective can be enforced through linear constraints. This method is compatible with most mixed-integer linear programming solvers such as CPLEX and Gurobi, making it easy to integrate the new line plan measures with other problems in the same framework. In fact, the quadratic objective of Problem (11) is supported directly by both solvers. Another option is to use general non-linear optimization solvers such as Ipopt that converge to a local optimum, which is guaranteed to be globally optimal for convex problems.

When considered in isolation, the two problems can be seen as continuous non-linear resource allocation problems (Patriksson, 2008). For shortest path routing, the resource is the period T that is allocated to the gaps between departures. For logit routing, the same resource is instead allocated to the jumps in the observed route set measure function. Patriksson (2008) surveys non-linear resource allocation problems, and identifies Lagrange multiplier methods as the most common solution technique. These methods introduce a multiplier $\mu \in \mathbb{R}$ to penalize the budget constraint (11b) or (12b) in the objective. For a given μ , each allocation $i \in \mathcal{R}$ is calculated independently. The optimal value of μ can be found with a simple line search to find the point where the independent allocations together satisfy the budget constraint.

For shortest path routing in particular, the Lagrange multiplier method reveals that every optimal solution to Problem (11) satisfies $x_i(\mu) = \max\{0, \mu - l_i\}$ for some Lagrange multiplier $\mu \in \mathbb{R}$ (Appendix C.16). This structure can also be observed in Figure 8. The optimal timetable

only contains routes $i \in \mathcal{R}$ for which $x_i > 0$, and thus $\delta_i = x_i = \mu - l_i$. Furthermore, we have that $\tau_i = l_i$. It follows that when the function jumps, it always jumps to $\delta_i + \tau_i = \mu$, regardless of the route. The optimal multiplier μ is the value for which the function $\sum_{i \in \mathcal{R}} x_i(\mu) = \sum_{i \in \mathcal{R}} \max\{0, \mu - l_i\}$ equals T. This is a piecewise-linear function with O(n) breakpoints, and bisection search can be used to find the intersection with T. The corresponding complexity is $O(n \log n)$ in general, or O(n) when the lengths l_i are already sorted (Patriksson, 2008). Alternatively, Brucker (1984) presents a more complicated algorithm that solves Problem (11) in O(n) even when the routes are in arbitrary order.

Line Plan Measures We propose to evaluate the quality of a line plan by calculating the measure of the optimal timetable. This is motivated by how public transport is planned in practice, where timetables are optimized for a given line plan (Liebchen, 2008):

• Line Plan Measure for Shortest Path Routing or Logit Routing (general arrivals):

$$\mathcal{M}^{LP}(\mathcal{R},T) = \min_{\boldsymbol{\theta} \in [0,T)^n} \mathcal{M}^{TT}(\mathcal{R},T,\boldsymbol{\theta}).$$
(13)

This measure inherits (strict) monotonicity from the timetable measures. After all, when routes are reduced in length or when new routes are added, all timetable measures improve, including the minimum.

The general measures are again specialized to uniform arrivals. In this case, the measures can be calculated efficiently by solving Problem (11) or Problem (12), respectively, as discussed in the previous section. We obtain:

• Line Plan Measure for Shortest Path Routing (uniform arrivals):

$$\mathcal{M}_{sp}^{LP}(\mathcal{R},T) = \min_{\substack{\sum x_i = T \\ x \ge 0}} \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} x_i^2 + l_i x_i \right).$$
(14)

• Line Plan Measure for Logit Routing (uniform arrivals):

$$\mathcal{M}_{logit}^{LP}(\mathcal{R},T) = \min_{\substack{\sum y_i = T \\ y \ge 0}} \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} y_i^2 + l_i y_i + \frac{y_i}{\beta} \log\left(\frac{1 - e^{-\beta y_i}}{1 - e^{-\beta T}}\right) \right).$$
(15)

There are major benefits to *optimizing* the timetable, rather than evaluating the line plan with a particular timetable. Fixing the timetable to $\theta = 0$ makes evaluation easy, but it ignores all periodic aspects. Another approach could be to use an equidistant timetable that depends on the number of routes. However, the resulting line plan measure may not be monotonic, even when the underlying timetable measure is. For example, consider two short routes that are equally spaced. Adding a very long third route changes the spacing between the short routes, potentially making the measure worse despite the fact that a route has been added. Optimizing the timetable as part of the measure avoids these problems, and it can be done in an efficient manner.

Route Choice Models for Line Plans We again observe a natural parallel between measures and route choice models for line plans. In the case of shortest path routing, it was observed in Section 3.2, (10), that travelers choose route $i \in \mathcal{R}$ with probability δ_i/T . In the optimal timetable, the values of δ are obtained from the *x*-variables in Problem (11). The substitution $p_i = x_i/T$ then results in the following route choice model:

• Route Choice Model for Shortest Path Routing (uniform arrivals):

$$\boldsymbol{p}_{sp}^{LP} = \operatorname*{argmin}_{p \in \mathcal{P}} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} T p_i^2 + l_i p_i \right).$$
(16)

Note that constraint (11b) is captured by $p \in \mathcal{P}$, which requires the probabilities to sum to one.

The route choice model expresses how travelers distribute over the routes when the timetable is optimized but the arrival time is uncertain. These travelers do not only care about the expected travel time $\sum_{i \in \mathcal{R}} l_i p_i$, but also appreciate a small Simpson index $\sum_{i \in \mathcal{R}} p_i^2$, which indicates a high diversity in route choices (Simpson, 1949). This diversity term is given more weight when the period T increases, as not diversifying the routes comes with larger consequences in terms of waiting time. When the period decreases $T \to 0$, the route choice model simply selects the shortest path, matching the route choice model of the underlying route set.

Finally, and most surprisingly, travelers under logit routing use route $i \in \mathcal{R}$ with probability $p_i = \Delta_i/T$, but only when facing the optimal timetable (Appendix C.17-19). This simple expression suggests that the Δ -representation is in some sense natural for logit routing. As the line plan indeed uses the optimal timetable, a similar substitution of $p_i = y_i/T$ can be made to obtain:

• Route Choice Model for Logit Routing (uniform arrivals):

$$\boldsymbol{p}_{logit}^{LP} = \operatorname*{argmin}_{p \in \mathcal{P}} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} T p_i^2 + l_i p_i + \frac{p_i}{\beta} \log\left(\frac{1 - e^{-\beta T p_i}}{1 - e^{-\beta T}}\right) \right).$$
(17)

Compared to shortest path routing, this route choice model includes an additional term that puts extra emphasis on avoiding probabilities that are close to zero or one. This encourages multiple good route options to be available, regardless of when the traveler arrives. For a decreasing period $T \rightarrow 0$, the new term tends to $\frac{1}{\beta}p_i \log(p_i)$ in the limit. The route choice model then minimizes perceived travel time, and matches the route choice model of the underlying route set. Finally, we note the deep connection between the line plan measures and the route choice models: they correspond to the objective value and the optimal solution to the same optimization problem.

4 Practical Insights

After introducing the new framework and proving its theoretical properties, this section presents numerical examples to also offer practical insights. We start from the baseline route set, timetable, and line plan that are visualized in Figure 9. The baseline considers two routes with durations of 20 and 30 minutes. The timetable and line plan are periodic and repeat every 30 minutes. When the timetable is fixed, departures are set to be equidistant. Starting from this baseline, the following figures show how the routings and measures change when varying the route duration l_2 (Figure 10), the period T (Figure 11), and the departure time θ_2 (Figure 12).

The figures reveal that route choice and service quality can differ considerably between route sets, timetables, and line plans, demonstrating that it is important to use the right model at the right stage of planning. In Figure 10, for example, travelers under shortest path routing who are faced with a static route set will all choose route 2 if the duration is less than 20 minutes. In the timetable setting, this only happens when route 2 is so short that it is the best option regardless of the arrival time (duration less than 5 minutes). In general, Figure 10 shows clear differences between route sets (which ignore periodicity), and timetables and line plans (which include periodicity), both in terms of route choice and service quality. The difference between timetable measures and line plan measures is less obvious in Figure 10, but this strongly depends on the quality of the timetable. For the baseline, the equidistant timetable is relatively close to optimal, and therefore the timetable measure is relatively close to the line plan measure (recall that the line plan measure is defined as the timetable measure of the optimal timetable). The situation is different in Figure 12b, which changes the timetable by varying the departure time of route 2. Each curve shows the timetable measure for different timetables, and the minimum value is equal to the line plan measure (e.g., $31\frac{2}{3}$ at $\theta_2 = 10$ for shortest path). As the timetable moves further away from optimal, the difference between the timetable measure and the line plan measure becomes apparent, again emphasizing that it is important to use routings and measures tailored to the form of public transport that one considers.

In line with the literature on travel demand analysis, the smooth curves provided by the logit model suggest that it is a much more reasonable model for travel behavior than the shortest path model. Route choice based on shortest path yields large jumps in routing probabilities in Figures 10a, 10c, 11a and 12a, which translate to kinks in the corresponding service quality measures. In practice, routings and service quality are expected to vary smoothly when the system's parameters change, which better matches the logit model. Another strength of the logit model is that it provides better guidance for the local improvement of existing timetables. Consider the timetables in Figure 12, for example. For $\theta_2 > 20$ minutes, the timetable is such that route 2 is never the shortest path. This results in a flat region of the shortest path curve in Figure 12b, which provides no guidance on how to improve the timetable. For logit the curve is smooth, and the gradient suggests moving route 2 up in the schedule, towards the optimal departure time.

While logit is the model of choice for accurately capturing traveler behavior, the benefit of logit over shortest path seems to diminish as we move from route sets to timetables to line plans. This can be seen in Figure 10, for example, where the curves for shortest path and logit get closer to eachother as we move down the plots, both for routings and for measures. We conjecture that this behavior is due to the following two effects. First, when there is a clear shortest path, the shortest path model and the logit model give similar results. This can be seen in Figure 10a for example, where all models assign a high probability to route 2 when it is short and a low probability when it is long, but the models disagree when there is no clear



Figure 9: Baseline route set, timetable, and line plan for sensitivity analysis.



Figure 10: Sensitivity analysis for varying route duration l_2 in Figure 9.



Figure 9: Baseline route set, timetable, and line plan for sensitivity analysis. (repeated)



Figure 11: Sensitivity analysis for varying period T in Figure 9. The fixed timetable maintains equidistant departures $\theta_1 = 0$, $\theta_2 = T/2$.



Figure 12: Sensitivity analysis for varying departure time θ_2 in Figure 9.

preference around $l_2 = 20$. Second, when travelers arrive at random, it is likely that many of them have a clearly preferred route based on their arrival time, especially if the departures are spaced out well. As such, the *average* traveler is less affected by the difference between shortest path and logit. Timetable routings and measures are defined based on the average traveler, which explains why the curves get closer together. Line plans further strengthen this effect by optimizing the timetable to intentionally give travelers a clearly preferred choice, resulting in the smallest difference between shortest path and logit, as seen in Figures 10e and 10f.

The observations above suggest that it is not always necessary for public transport planners to accurately model travel behavior. Especially for high-level planning such as line planning, the shortest path model may be sufficiently accurate to make good decisions. This is of major managerial importance, as the shortest path model is easier to integrate in existing systems, and does not require an estimation of the β parameter. The framework in this paper allows practitioners to get a sense of how much shortest path and logit measures differ on their network, before investing in more complicated optimization methods that may not be necessary.

4.1 Consequences of Using Inconsistent Models

Even if there is no significant difference between the shortest path and logit models for line planning, combining the models in an inconsistent way can have significant negative consequences. This behavior will be demonstrated with an example below. It was already shown in the introduction that inconsistent models can lead to counter-intuitive results for route sets, but it is surprising to see that this behavior extends all the way to line plans, which are much less affected by the choice of model. The main purpose of this section is to serve as a warning that being consistent may be much more important than being correct.

The example starts from an original timetable that is depicted in Figure 13a. The timetable uses a period of one hour, and features two routes with a duration of 15 minutes that are spread out evenly. Now suppose that the opportunity arises to add an additional route with a duration of 35 minutes. There are a number of reasonable ways to do so:

- Option 1: Place the new route directly in-between the other departures (Figure 13b);
- Option 2: Use an equidistant timetable with three departures (Figure 13c);
- Option 3: Use the optimal timetable (Figure 13d).

The optimal timetable is calculated for the shortest path model with the methods developed in this paper. As expected, the logit-optimal timetable is almost identical to the shortest pathoptimal timetable, to the extent that there is no benefit in treating these options separately.

Table 2 summarizes how three line planners evaluate the different options. The first planner uses the shortest path measure, the second planner uses the logit measure with $\beta = 0.1$, and the third planner uses an inconsistent measure that combines logit routing with travel time evaluation (instead of the consistent *perceived* travel time evaluation).

When the shortest path measure is used, adding in a slow route does not necessarily improve the service quality. Option 1 does not affect the measure, as travelers can simply ignore the new route and wait for the faster routes they prefer. Option 2 worsens the measure, as it ruins the



Figure 13: Original timetable and three options to include a new route.

	Measure			
Timetable	Shortest Path	Logit ($\beta = 0.1$)	Inconsistent	
Original (two routes)	30.00	29.51	31.42	
Option 1: In-between	30.00	28.23	32.24	
Option 2: Equidistant	31.67	28.59	33.40	
Option 3: Optimal	29.44	28.20	31.87	

Table 2: Three ways to evaluate the options in Figure 13.

spacing of the fast routes without providing any benefit. However, the measure indicates that a strict benefit can be obtained when the routes are spaced out appropriately, as in Option 3. In the logit case, all options are better than the original timetable. This is due to the very low value of β , which assumes that travelers really care about having more options. But even if this assumption were incorrect, the relative ranking between Options 1-3 is the same as for shortest path. Planners are still encouraged to implement Option 1 over Option 2, which would not hurt in the shortest path case, or add the new route and optimize the timetable, which would result in a similar timetable and better service quality in both cases.

This is in stark contrast with the inconsistent measure: adding the new route always makes the measure worse. The inconsistent planner cannot justify adding the new route, even when the timetable is optimized. We conclude that a consistent model can lead to good decisions even when the assumptions are wrong (shortest path versus logit), while an inconsistent model can lead to universally bad decisions. It is remarkable that we already see this behavior in small examples, and we expect that the negative consequences of inconsistency may be worse in integrated models that allow bad decisions to propagate.

5 Conclusions and Further Research

We presented a new framework for passenger routing and evaluating the service quality for route sets, timetables and line plans. We showed that the developed route choice models and measures are easy to interpret and can be computed efficiently, enabling practitioners to use the framework to better plan their networks. Numerical experiments support our claim that the framework enhances decision-making, and also reveal multiple managerial implications. Most notably, we observe that as one moves from immediate route choice to timetabling to line planning, the benefit of logit over shortest path diminishes, implying that the shortest path model, despite being slightly less accurate, may suffice for tactical or strategic planning purposes.

This paper lays the groundwork for a plethora of further research. The developed measures can be incorporated in virtually every public transport problem that concerns passengers such as line planning, timetabling, providing travel advice or determining tariff schemes. For problems with fixed route durations (e.g. line planning), the measures naturally lend themselves to decomposition schemes such as Benders decomposition due to their convex nature. Problems where the route durations depend on decision variables (e.g. timetabling) are more challenging, as this introduces non-convexity. This research also paves the way for new ways to evaluate and optimize the next generation of passenger transport systems, such as on-demand transport or shared mobility services. This requires non-trivial extensions to the framework, since these transport modes are not operated periodically. The application of more advanced choice models such as mixed logit or nested logit also seems highly relevant in this context, as these allow for individual or correlated preferences for modes and other heterogeneous factors. Finally, one could include aspects such as pricing, crowding or robustness. It would be interesting to study whether the insights found in this paper generalize to such settings.

Acknowledgements This research is partly supported by NSF LEAP-HI proposal NSF-1854684.

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A Proofs Section 3.1 (Route Sets)

	Shortest Path Routing			Logit Routing		
Evaluation	Measure	Monot.	Consist.	Measure	Monot.	Consist.
Travel Time Perceived Travel Time	$l_{ m min} \ l_{ m min}$	\ \	✓ ×	$\frac{\sum_{i\in\mathcal{R}} p_i^{logit} l_i}{-\frac{1}{\beta} \log\left(\sum_{i\in\mathcal{R}} e^{-\beta l_i}\right)}$	× √ √	× ✓

This appendix provides the proofs for the properties in Table 1.

Table 1: Overview of route set measures and properties (double checkmark for strict monotonic).

Monotonicity With shortest path routing, increasing route lengths and removing routes can only increase l_{\min} , but is not required to do so strictly. This implies (weak) monotonicity. For logit routing with travel time evaluation, Figure 4 provides a counterexample for monotonicity. For logit routing with perceived travel time evaluation, strictly increasing a route length or removing a route strictly decreases the argument of the logarithm. As $-\frac{1}{\beta} \log(.)$ is strictly decreasing, this strictly increases the measure as required.

Consistency Shortest path routing minimizes travel time by definition, so this combination is consistent. Logit routing assigns a positive probability to every route, and therefore does not minimize the travel time, which makes this combination inconsistent. Next, we prove that logit routing minimizes perceived travel time, i.e., p^{logit} minimizes $\mathcal{E}_{ptt}(\mathcal{R}, .)$. A similar proof is provided by Anderson et al. (1988). We use the Lagrange multiplier rule to minimize $\mathcal{E}_{ptt}(\mathcal{R}, p)$ over $p \in \mathcal{P}$. The Lagrangian is given by

$$\mathcal{L}(\mathcal{R}, p, \lambda) = \sum_{i \in \mathcal{R}} l_i p_i + \frac{1}{\beta} \sum_{i \in \mathcal{R}} p_i \log(p_i) + \lambda \left(1 - \sum_{i \in \mathcal{R}} p_i \right), \text{ on domain } p \ge 0, \lambda \in \mathbb{R}.$$

Because the objective $\mathcal{E}_{ptt}(\mathcal{R}, .)$ is convex and the constraint is linear, it is sufficient for optimality to find a probability vector $p^* \in P$ and a multiplier $\lambda^* \in \mathbb{R}$ such that the stationarity condition

$$\frac{d\mathcal{L}(\mathcal{R}, p, \lambda)}{dp_i} = l_i + \frac{1}{\beta} \left(\log p_i + 1 \right) - \lambda = 0, \quad \forall i \in \mathcal{R}$$

holds. It is straightforward to verify that this is the case for

$$p_i^* = \frac{e^{-\beta l_i}}{\sum_{j \in \mathcal{R}} e^{-\beta l_j}} = p_i^{logit} \text{ and } \lambda^* = \frac{1}{\beta} - \frac{1}{\beta} \log \left(\sum_{j \in \mathcal{R}} e^{-\beta l_j} \right),$$

which proves consistency for logit routing with perceived travel time evaluation. Finally, we prove that shortest path routing with perceived travel time evaluation is inconsistent through an example. Consider route set $\mathcal{R} = \{1, 2\}$ with route lengths $l_1 = 1$, $l_2 = 2$ and parameter $\beta = 1$. The shortest path routing is given by $p_1^{sp} = 1$, $p_2^{sp} = 0$, resulting in an evaluation of $\mathcal{E}(\mathcal{R}, p^{sp}) = 1$. However, the solution $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$ provides a value of $\mathcal{E}(\mathcal{R}, p) = \frac{3}{2} - \log(2) < 1$, which proves that this combination is inconsistent.

B Proofs Section **3.2** (Timetables)

Proposition 1 (Constant Routing between Departures). For shortest path routing and logit routing the route choice does not change between departures. That is, for route $i \in \mathcal{R}$ and successor $\sigma(i) \in \mathcal{R}$ it holds that $\mathbf{p}(\mathcal{R}_{\theta}(t))$ is constant for $t \in (\theta_i, \theta_{\sigma(i)}]$.

Proof. Until the next departure, all waiting times decrease by the same amount as time progresses. Subtracting a constant from all route lengths does not affect which length is the minimum, which proves the proposition for shortest path routing. For logit routing we have for any route $j \in \mathcal{R}$ and constant $C \in \mathbb{R}$ that

$$p_j^{logit} = \frac{e^{-\beta l_j}}{\sum_{k \in \mathcal{R}} e^{-\beta l_k}} = \frac{e^{-\beta(l_j + C)}}{\sum_{k \in \mathcal{R}} e^{-\beta(l_k + C)}},$$

which completes the proof.

Proposition 2 (Translation Invariance). Both \mathcal{M}_{sp}^{RS} and \mathcal{M}_{logit}^{RS} are translation invariant between departures. That is, for route $i \in \mathcal{R}$ and successor $\sigma(i) \in \mathcal{R}$ it holds that

$$\mathcal{M}^{RS}(\mathcal{R}_{\theta}(t)) = [\theta_{\sigma(i)} - t]_T + \mathcal{M}^{RS}(\mathcal{R}_{\theta}(\theta_{\sigma(i)})), \quad \forall t \in (\theta_i, \theta_{\sigma(i)}]$$

Proof. By Proposition 1, the routing $p \in \mathcal{P}$ is constant on the interval $t \in (\theta_i, \theta_{\sigma(i)}]$. The route set measure on the interval is therefore given by the evaluation function for this fixed p. It follows that

$$\mathcal{M}_{logit}^{RS}(\mathcal{R}_{\theta}(t)) = \mathcal{E}_{ptt}(\mathcal{R}_{\theta}(t), p)$$

$$= \sum_{j \in \mathcal{R}} (l_{j} + [\theta_{j} - t]_{T}) p_{j} + \frac{1}{\beta} \sum_{j \in \mathcal{R}} p_{j} \log(p_{j}) \qquad \text{(Definition } \mathcal{E}_{ptt} \text{ and } \mathcal{R}_{\theta}(t))$$

$$= \sum_{j \in \mathcal{R}} (l_{j} + [\theta_{j} - \theta_{\sigma(i)}]_{T} + [\theta_{\sigma(i)} - t]_{T}) p_{j} + \frac{1}{\beta} \sum_{j \in \mathcal{R}} p_{j} \log(p_{j}) \qquad \text{(split the wait time)}$$

$$= [\theta_{\sigma(i)} - t]_{T} + \sum_{j \in \mathcal{R}} (l_{j} + [\theta_{j} - \theta_{\sigma(i)}]_{T}) p_{j} + \frac{1}{\beta} \sum_{j \in \mathcal{R}} p_{j} \log(p_{j}) \qquad (\sum_{j \in \mathcal{R}} p_{j} = 1)$$

$$= [\theta_{\sigma(i)} - t]_{T} + \mathcal{M}_{logit}^{RS}(\mathcal{R}_{\theta}(\theta_{\sigma(i)})). \qquad \text{(Definition } \mathcal{E}_{ptt} \text{ and } \mathcal{R}_{\theta}(t))$$

For \mathcal{M}_{sp}^{RS} the evaluation function omits the term $\frac{1}{\beta} \sum_{j \in \mathcal{R}} p_j \log(p_j)$, but the proof is identical.

Proposition 3. Algorithm 1 correctly calculates the characteristic values.

Proof. The main steps of the algorithm are straightforward, as indicated in the main text. It remains to go over the edge cases and justify the use of the waiting time correction w(i, j). The first edge case is when $|\mathcal{R}| = 1$. With a single route, the regular definition of δ_1 would evaluate to $[\theta_i - \theta \pi(i)]_T = 0$. Hence, the algorithm sets the correct value $\delta_1 = T$ and the corresponding $\tau_1 = l_1$ as a special case. The vertical jump evaluates to $\Delta_1 = \delta_1 + \tau_1 - \tau_1 = T$ as expected.

Next, consider the waiting time correction

$$w(i,j) = \begin{cases} T & \text{if } \theta_i = \theta_j, \ j < i, \\ [\theta_j - \theta_i]_T & \text{else.} \end{cases}$$
(18)

This correction ensures that route with the same departure time $\theta_i = \theta_j$ are processed sequentially according to their ordering, essentially as if there were small gaps between the departures. Recall that routes that depart at the same time are ordered according to their index (Section 3.2). Consider processing route $i \in \mathcal{R}$. If route $j \in \mathcal{R}$ departs at the same time but is processed later $j \ge i$, then the regular waiting time $[\theta_j - \theta_i]_T = 0$ applies. If route j has already been processed, then it will take time T before j is reached again, and hence a waiting time of T is used. After all routes at this particular time have been processed, all waiting times have been increased by T. The correction also ensures that the total gap between departures $\sum_{i\in\mathcal{R}} \delta_i$ adds up to T as expected. This is straightforward for the regular case, but for the edge case where all routes $\mathcal{R} = \{1, \ldots, n\}$ depart at the same time, the correction is necessary to ensure that the horizontal gap from n back to 1 is recorded as T.

Corollary 4. The output of Algorithm 1 satisfies $\sum_{i \in \mathcal{R}} \delta_i = T$ and $\sum_{i \in \mathcal{R}} \Delta_i = T$.

Proof. The first summation follows from the definition of δ_i , and the discussion of the edge cases in the proof of Proposition 3. The second summation follows from the identity $\Delta_i = \delta_{\sigma(i)} + \tau_{\sigma(i)} - \tau_i$ in Line 19 of Algorithm 1, and the fact that the τ -terms cancel in the summation. It follows that $\sum_{i \in \mathcal{R}} \Delta_i = \sum_{i \in \mathcal{R}} \delta_i = T$.

Proposition 5. Under uniform arrivals, the timetable measure for shortest path routing or logit routing can be calculated as

$$\mathcal{M}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \delta_i^2 + \tau_i \delta_i \right) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \Delta_i^2 + \tau_i \Delta_i \right), \tag{6}$$

where the characteristic values δ , τ , Δ are obtained from Algorithm 1.

Proof. By definition of the expectation value, the timetable measure is obtained as the area under the observed route set measure function, divided by T. For the first expression, Figure 6a provides a graphical proof: For a given route $i \in \mathcal{R}$ the shaded area is the sum of the area of half a square $(\frac{1}{2}\delta_i^2)$ and a rectangle $(\delta_i\tau_i)$. If routes $i, \sigma(i)$ depart at the same time it follows that $\delta_{\sigma(i)} = 0$, which does not affect the argument. Summing over the areas and dividing by T gives the result. The second expression is obtained by substituting $\Delta_i = \delta_{\sigma(i)} + \tau_{\sigma(i)} - \tau_i$ (Line 19, Algorithm 1) into the first expression:

$$\frac{1}{T}\sum_{i\in\mathcal{R}} \left(\frac{1}{2}\delta_i^2 + \tau_i\delta_i\right) = \frac{1}{T}\sum_{i\in\mathcal{R}} \left(\frac{1}{2}\delta_{\sigma(i)}^2 + \tau_{\sigma(i)}\delta_{\sigma(i)}\right) \\
= \frac{1}{T}\sum_{i\in\mathcal{R}} \left(\frac{1}{2}\left(\Delta_i - \tau_{\sigma(i)} + \tau_i\right)^2 + \tau_{\sigma(i)}\left(\Delta_i - \tau_{\sigma(i)} + \tau_i\right)\right) \\
= \frac{1}{T}\sum_{i\in\mathcal{R}} \left(\frac{1}{2}\Delta_i^2 + \tau_i\Delta_i + \frac{1}{2}\left(\tau_i^2 - \tau_{\sigma(i)}^2\right)\right) \\
= \frac{1}{T}\sum_{i\in\mathcal{R}} \left(\frac{1}{2}\Delta_i^2 + \tau_i\Delta_i\right).$$

Proposition 6. For logit routing, the output of Algorithm 1 satisfies

$$\tau_i = l_i + \frac{1}{\beta} \log \left(\frac{1 - e^{-\beta \Delta_i}}{1 - e^{-\beta T}} \right) \quad \forall i \in \mathcal{R}.$$

Proof. By definition, the value Δ_i is the jump in the observed route set measure due to missing route $i \in \mathcal{R}$. Before the jump, the value of the observed route set measure is τ_i . Missing route $i \in \mathcal{R}$ increases the waiting time for this route from 0 to T. It follows from Line 17 that

$$\Delta_i = -\frac{1}{\beta} \log \left(\sum_{j \in \mathcal{R}} e^{-\beta(l_j + w(i,j))} + e^{-\beta(l_i + T)} - e^{-\beta(l_i + 0)} \right) - \tau_i,$$
$$= -\frac{1}{\beta} \log \left(e^{-\beta\tau_i} + e^{-\beta l_i} \left(e^{-\beta T} - 1 \right) \right) - \tau_i.$$

Note that $\Delta_i > 0$ by strict monotonicity of the route set measure. Take exponents on both sides and rearrange to obtain

$$e^{-\beta\Delta_{i}} = \frac{e^{-\beta\tau_{i}} + e^{-\beta l_{i}} \left(e^{-\beta T} - 1\right)}{e^{-\beta\tau_{i}}} = 1 + e^{-\beta l_{i}} \frac{e^{-\beta T} - 1}{e^{-\beta\tau_{i}}},$$
$$e^{-\beta\tau_{i}} = e^{-\beta l_{i}} \frac{1 - e^{-\beta T}}{1 - e^{-\beta\Delta_{i}}}.$$

Finally, take logarithms to recover τ_i :

$$\tau_i = l_i - \frac{1}{\beta} \log\left(\frac{1 - e^{-\beta T}}{1 - e^{-\beta \Delta_i}}\right) = l_i + \frac{1}{\beta} \log\left(\frac{1 - e^{-\beta \Delta_i}}{1 - e^{-\beta T}}\right).$$

Proposition 7. The timetable measure for logit routing under uniform arrivals,

$$\mathcal{M}_{logit}^{TT}(\mathcal{R}, T, \boldsymbol{\theta}) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} \Delta_i^2 + l_i \Delta_i + \frac{\Delta_i}{\beta} \log\left(\frac{1 - e^{-\beta \Delta_i}}{1 - e^{-\beta T}}\right) \right),\tag{8}$$

is strictly convex on the domain $\Delta \ge 0$ for parameters $\beta > 0$ and T > 0.

Proof. Due to symmetry, it is sufficient to prove convexity for a single Δ_i . To simplify the proof, we split the logarithm and remove the linear terms $l_i\Delta_i$ and $-\frac{\Delta_i}{\beta}\log(1-e^{-\beta T})$. Furthermore, we use the linear transformation $x = \beta \Delta_i$ and scale the function by $\beta^2 T > 0$. None of these transformations affect convexity. It remains to prove that

$$f(x) = \beta^2 T \frac{1}{T} \left(\frac{1}{2} \frac{x^2}{\beta^2} + \frac{x}{\beta^2} \log\left(1 - e^{-x}\right) \right) = \frac{1}{2} x^2 + x \log\left(1 - e^{-x}\right)$$

is convex. Interestingly, the term $x \log (1 - e^{-x})$ is not convex by itself.

We prove strictly convexity by showing that the derivative

$$f'(x) = x - \frac{x}{1 - e^{-x}} e^{-x} + \log(1 - e^{-x})$$

= $\frac{x}{1 - e^{-x}} (1 - e^{-x}) - \frac{x}{1 - e^{-x}} e^{-x} + \log(1 - e^{-x})$
= $\frac{x}{1 - e^{-x}} + \log(1 - e^{-x})$

is strictly increasing. The first term is non-decreasing, as the derivative

$$\frac{d}{dx}\frac{x}{1-e^{-x}} = \frac{(1-e^{-x})-xe^{-x}}{(1-e^{-x})^2} = \frac{e^{-x}}{(1-e^{-x})^2}(e^x-1-x) \ge 0.$$
 (19)

This follows from the well-known inequality $e^x \ge 1 + x$. The second term of f'(x) is strictly increasing by inspection. It follows that f'(x) is strictly increasing and therefore f(x) is convex.

Corollary 8. The derivative of the timetable measure for logit routing under uniform arrivals tends to $-\infty$ when $\Delta_i \to 0$ for any $i \in \mathcal{R}$.

Proof. The transformations applied in the proof above do not affect whether or not the limits at $\Delta_i \to 0$ are finite. For $x \to 0$, the first term of f'(x) is finite, and the second term tends to $-\infty$. It follows that the derivative tends to $-\infty$ when $\Delta_i \to 0$ for any $i \in \mathcal{R}$.

C Proofs Section 3.3 (Line Plans)

Proposition 9. The output of the procedure TIMETABLESHORTESTPATH (Algorithm 2) is a feasible timetable for which the characteristic values match the desired values set during the execution of the algorithm.

Proof. The input satisfies $x_i \ge 0$ for every route $i \in \mathcal{R}$. The timetable is constructed according to $\hat{\delta}_i \leftarrow x_i$ and $\theta_i \leftarrow \theta_{\pi(i)} + \hat{\delta}_i$. This timetable is feasible, as the gaps are non-negative and sum to $\sum_{i \in \mathcal{R}} \hat{\delta}_i = \sum_{i \in \mathcal{R}} x_i = T$ as expected. It is clear from the definition that when δ_i is calculated by Algorithm 1, indeed $\delta_i = \hat{\delta}_i$ matches.

Lemma 10. Every optimal solution to Problem (12) satisfies y > 0.

Proof. The statement is trivial for a single route, as $y_1 = T$ by constraint (12b). Now assume that there are at least two routes $|\mathcal{R}| \geq 2$ and that $y_i = 0$. Due to (12b), there must be a

route $j \in \mathcal{R}, j \neq i$, for which $y_j > 0$. By Corollary (8), the derivative of the timetable measure tends to $-\infty$ as $y_i \to 0$. It follows that the solution can strictly be improved by adding a small amount $\varepsilon > 0$ to y_i while subtracting it from y_j . This contradicts the assumption that y is optimal, so it must be that $y_i > 0$.

Lemma 11. The desired values set by TIMETABLELOGIT (Algorithm 2) satisfy $\hat{\tau}_i + \frac{\hat{\Delta}_i}{1 - e^{-\beta \hat{\Delta}_i}} = \mu$ for every route $i \in \mathcal{R}$, for some constant $\mu \in \mathbb{R}$.

Proof. The procedure sets $\hat{\Delta}_i \leftarrow y_i$ and $\hat{\tau}_i \leftarrow l_i + \frac{1}{\beta} \log \left(\frac{1-e^{-\beta \hat{\Delta}_i}}{1-e^{-\beta T}} \right) = l_i + \frac{1}{\beta} \log \left(\frac{1-e^{-\beta y_i}}{1-e^{-\beta T}} \right)$, where y is an optimal solution to Problem (12). Lemma 10 states that y > 0, which also implies y < T. Then it must be that the partial derivatives of objective (12a) are all equal: if the derivatives with respect to routes $i, j \in \mathcal{R}, i \neq j$, are not equal, then a better solution is obtained by moving a small amount ε between the routes without affecting feasibility. Hence, the value of

$$T\frac{d}{dy_i}(12\mathbf{a}) = y_i + y_i \frac{1 - e^{-\beta T}}{1 - e^{-\beta y_i}} \frac{e^{-\beta y_i}}{1 - e^{-\beta T}} + \left(l_i + \frac{1}{\beta}\log\left(\frac{1 - e^{-\beta\hat{\Delta}_i}}{1 - e^{-\beta\hat{\Delta}_i}}\right)\right)$$
$$= \hat{\Delta}_i + \hat{\Delta}_i \frac{e^{-\beta\hat{\Delta}_i}}{1 - e^{-\beta\hat{\Delta}_i}} + \hat{\tau}_i = \hat{\tau}_i + \frac{\hat{\Delta}_i}{1 - e^{-\beta\hat{\Delta}_i}}$$

is the same for each route $i \in \mathcal{R}$.

Lemma 12. The desired values set by TIMETABLELOGIT (Algorithm 2) satisfy $\hat{\tau}_i + \hat{\Delta}_i > \hat{\tau}_j$ for every pair of routes $i, j \in \mathcal{R}$.

Proof. By Lemma 11, it is equivalent to prove

$$\mu - \frac{\hat{\Delta}_i}{1 - e^{-\beta \hat{\Delta}_i}} + \hat{\Delta}_i > \mu - \frac{\hat{\Delta}_j}{1 - e^{-\beta \hat{\Delta}_j}} \Longleftrightarrow \frac{\hat{\Delta}_i}{e^{\beta \hat{\Delta}_i} - 1} < \frac{\hat{\Delta}_j}{1 - e^{-\beta \hat{\Delta}_j}}$$

Let $z_i = \beta \hat{\Delta}_i$ and $z_j = \beta \hat{\Delta}_j$. By Lemma 10, $\hat{\Delta} = y > 0$. Hence, it is sufficient to prove the following for $z_i > 0, z_j > 0$:

$$\frac{z_i}{e^{z_i} - 1} < \frac{z_j}{1 - e^{-z_j}}.$$

Now use the well-known inequality $e^z > 1 + z$ for $z \neq 0$ as follows. On the left-hand side, the numerator and denominator are both positive. The inequality shows that the denominator is larger, and thus that the fraction is less than one. Similarly, the right-hand side has a larger numerator, such that the fraction is larger than one. This completes the proof.

Proposition 13. The output of the procedure TIMETABLELOGIT (Algorithm 2) is a feasible timetable for which the characteristic values match the desired values set during the execution of the algorithm.

Proof. The procedure sets $\hat{\Delta}_i \leftarrow y_i$, $\hat{\tau}_i \leftarrow l_i + \frac{1}{\beta} \log \left(\frac{1-e^{-\beta \hat{\Delta}_i}}{1-e^{-\beta T}} \right) = l_i + \frac{1}{\beta} \log \left(\frac{1-e^{-\beta y_i}}{1-e^{-\beta T}} \right)$, and $\hat{\delta}_i \leftarrow \hat{\tau}_{\pi(i)} + \hat{\Delta}_{\pi(i)} - \hat{\tau}_i$, where y is an optimal solution to Problem (12). Then the timetable is constructed according to $\theta_i \leftarrow \theta_{\pi(i)} + \hat{\delta}_i$. It follows directly from Lemma 12 that $\hat{\delta} > 0$. Furthermore, we have $\sum_{i \in \mathcal{R}} \hat{\delta}_i = \sum_{i \in \mathcal{R}} \hat{\Delta}_i$ due to the telescoping sum. This sum equals T as expected, due to constraint (12b). It follows that the timetable is feasible, and it is easy to see that indeed $\delta = \hat{\delta}$ matches when δ is calculated by Algorithm 1.

Next we prove that $\tau = \hat{\tau}$ matches when τ is calculated by Algorithm 1. Without loss of generality, we prove the statement for i = 1, and we relabel the other routes such that i = 1, $\sigma(i) = 2$, etc. Note that all routes depart at different times $(\hat{\delta} > 0)$, so no wait time correction is necessary. We obtain the wait time function $w(1, j) = \sum_{k=2}^{j} \delta_k = \hat{\tau}_1 + \sum_{k=1}^{j-1} y_k - \hat{\tau}_j$, which is due to the telescoping sum, $\hat{\delta} = \delta$ (as proven previously), and $\hat{\Delta} = y$ (by definition).

Now calculate τ_1 according to Algorithm 1, with the goal to prove that it matches $\hat{\tau}_1$:

$$\begin{aligned} \tau_1 &= -\frac{1}{\beta} \log \left(\sum_{j \in \mathcal{R}} e^{-\beta(l_j + w(i,j))} \right) = -\frac{1}{\beta} \log \left(\sum_{j \in \mathcal{R}} e^{-\beta(l_j + \hat{\tau}_1 + \sum_{k=1}^{j-1} y_k - \hat{\tau}_j)} \right) \\ &= \hat{\tau}_1 - \frac{1}{\beta} \log \left(\sum_{j \in \mathcal{R}} e^{-\beta(l_j + \sum_{k=1}^{j-1} y_k - \hat{\tau}_j)} \right) \stackrel{?}{=} \hat{\tau}_1. \end{aligned}$$

As the $\hat{\tau}_i$ s cancel, it remains to prove that the argument of the logarithm is equal to one:

$$\sum_{j \in \mathcal{R}} e^{-\beta(l_j + \sum_{k=1}^{j-1} y_k - \hat{\tau}_j)} = 1$$

$$\sum_{j \in \mathcal{R}} e^{-\beta \sum_{k=1}^{j-1} y_k} \left(\frac{1 - e^{-\beta y_j}}{1 - e^{-\beta T}} \right) = 1 \qquad \text{(by definition } \hat{\tau}_j)$$

$$\sum_{j \in \mathcal{R}} \left(e^{-\beta \sum_{k=1}^{j-1} y_k} - e^{-\beta \sum_{k=1}^{j} y_k} \right) = 1 - e^{-\beta T}$$

$$e^{-\beta 0} - e^{-\beta \sum_{k=1}^{n} y_k} = 1 - e^{-\beta T}. \qquad \text{(telescoping sum)}$$

This holds because $\sum_{k=1}^{n} y_k = T$ by constraint (12b). We conclude that indeed $\tau = \hat{\tau}$.

It only remains to prove that $\Delta = \hat{\Delta}$ matches, given that $\delta = \hat{\delta}$ and $\tau = \hat{\tau}$ already match. Algorithm 1 constructs Δ_i according to $\Delta_i \leftarrow \delta_{\sigma(i)} + \tau_{\sigma(i)} - \tau_i = \hat{\delta}_{\sigma(i)} + \hat{\tau}_{\sigma(i)} - \hat{\tau}_i$. By definition, Algorithm 2 constructs $\hat{\delta}_i \leftarrow \hat{\tau}_{\pi(i)} + \hat{\Delta}_{\pi(i)} - \hat{\tau}_i$. This substitution cancels out $\hat{\tau}_{\sigma(i)}$ and $\hat{\tau}_i$ such that indeed $\Delta_i = \hat{\Delta}_i$. This completes the proof.

Proposition 14. The output of the procedure TIMETABLESHORTESTPATH (Algorithm 2) is an optimal timetable, and its measure matches the optimal objective value of Problem (11).

Proof. Consider an optimal timetable with measure \mathcal{M}^* and preprocess it by removing all suboptimal routes. As explained in the main text, the resulting timetable satisfies $\tau_i = l_i$ for all routes $i \in \mathcal{R}$ while the measure stays the same. A corresponding solution to Problem (11) is obtained by setting $x_i = 0$ for all routes that have been removed in preprocessing, and $x_i = \delta_i^*$ otherwise, where δ^* are the δ -values of the optimal timetable. Feasibility follows directly from Corollary 4, and it is easy that this solution has an objective value (11a) that matches \mathcal{M}^* . We conclude that for the optimal objective value v^* , we have $v^* \leq \mathcal{M}^*$.

Now solve Problem (11) and use Algorithm 2 to construct a timetable based on this solution. Proposition 9 states that this is always possible, and that $\delta = x$. The resulting timetable is evaluated with (6), which assigns value $\frac{1}{T} \left(\frac{1}{2} \delta_i^2 + \tau_i \delta_i \right) = \frac{1}{T} \left(\frac{1}{2} x_i^2 + \tau_i x_i \right)$ to each route $i \in \mathcal{R}$. This value cannot be worse than the value assigned by Problem (11), as the objective function (11a) instead uses $\frac{1}{T} \left(\frac{1}{2} x_i^2 + l_i x_i \right)$, with $\tau_i \leq l_i$ by definition. We conclude that \mathcal{M} , the measure of the constructed timetable, satisfies $\mathcal{M} \leq v^*$. Combining this with $\mathcal{M}^* \leq \mathcal{M}$ (by definition) and $v^* \leq \mathcal{M}^*$ (proven above) implies that $\mathcal{M} = \mathcal{M}^* = v^*$. That is, the constructed timetable is optimal, and its measure matches the optimal objective value.

Proposition 15. The output of the procedure TIMETABLELOGIT (Algorithm 2) is an optimal timetable, and its measure matches the optimal objective value of Problem (12).

Proof. Given an optimal solution to Problem (12), Proposition 13 states that a timetable can be constructed for which all the characteristic values match. As $\Delta = \hat{\Delta} = y$, the objective function (12a) is identical to the timetable measure (8). Hence, the timetable is optimal, and its measure matches the optimal objective value.

Proposition 16. Every optimal solution to Problem (11) satisfies $x_i = \max\{0, \mu - l_i\}$ for every route $i \in \mathbb{R}$, for some constant $\mu \in \mathbb{R}$.

Proof. Associate the multiplier μ/T with constraint (11b) to create the Lagrangian:

$$\mathcal{L}(x,\mu/T) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} x_i^2 + x_i l_i \right) + \frac{\mu}{T} \left(T - \sum_{i \in \mathcal{R}} x_i \right) = \frac{1}{T} \sum_{i \in \mathcal{R}} \left(\frac{1}{2} x_i^2 + (l_i - \mu) x_i \right) + \mu.$$

It follows from the KKT conditions that every optimal solution satisfies $x^* = \operatorname{argmin}_{x\geq 0} \mathcal{L}(x, \mu/T)$ for some $\mu \in \mathbb{R}$ (Patriksson, 2008). The Lagrangian is separable, and the optimum is found by independently minimizing *n* convex parabolas over \mathbb{R}_+ . It is elementary to show that the unique optimal solution is $x_i = \max\{0, \mu - l_i\}$.

Lemma 17. The output of the procedure TIMETABLELOGIT (Algorithm 2) satisfies

$$\delta_i = \frac{\Delta_i}{1 - e^{-\beta \Delta_i}} - \frac{\Delta_{\pi(i)}}{1 - e^{-\beta \Delta_{\pi(i)}}} e^{-\beta \Delta_{\pi(i)}}.$$

Proof. By Proposition 13, the output of the procedure is a feasible timetable, and all the characteristic values match the desired values. We obtain:

$$\begin{split} \delta_{i} &= \tau_{\pi(i)} + \Delta_{\pi(i)} - \tau_{i} & \text{(Line 19, Algorithm 1)} \\ &= \mu - \frac{\Delta_{\pi(i)}}{1 - e^{-\beta\Delta_{\pi(i)}}} + \Delta_{\pi(i)} - \mu + \frac{\Delta_{i}}{1 - e^{-\beta\Delta_{i}}} & \text{(Lemma 11)} \\ &= -\frac{\Delta_{\pi(i)}}{1 - e^{-\beta\Delta_{\pi(i)}}} e^{-\beta\Delta_{\pi(i)}} + \frac{\Delta_{i}}{1 - e^{-\beta\Delta_{i}}}. \end{split}$$

Lemma 18. The output of the procedure TIMETABLELOGIT (Algorithm 2) satisfies

$$p_n^{logit}(\mathcal{R}_{\theta}(\theta_i)) = \frac{1 - e^{-\beta\Delta_n}}{1 - e^{-\beta T}} e^{-\beta \sum_{k=i}^{n-1} \Delta_k}.$$

Proof. By Proposition 13, the output of the procedure is a feasible timetable, and all the characteristic values match the desired values. Algorithm 2 sets the routes to depart in order of index, and route $n \in \mathcal{R}$ is the last to depart before the next period starts. It follows that

the waiting time between route $i \in \mathcal{R}$ and route n is given by $w(i,n) = \sum_{k=i+1}^{n} \delta_k$. Using the identity $\delta_i = \tau_{\pi(i)} + \Delta_{\pi(i)} - \tau_i$ (Line 19, Algorithm 1), the telescoping sum results in $w(i,n) = \tau_i + \sum_{k=i}^{n-1} \Delta_k - \tau_n$. We obtain:

$$p_n^{logit}(\mathcal{R}_{\theta}(\theta_i)) = \frac{e^{-\beta(l_n + w(i,n))}}{\sum_{k \in \mathcal{R}} e^{-\beta(l_k + w(i,k))}} \quad (\text{definition } p^{logit})$$

$$= \frac{e^{-\beta(l_n + \tau_i + \sum_{k=i}^{n-1} \Delta_k - \tau_n)}}{e^{-\beta\tau_i}} \quad (\text{definition } w(i,n); \text{ definition } \tau_i \text{ (Line 17, Alg. 1)})$$

$$= e^{-\beta(l_n - \tau_n)} e^{-\beta\sum_{k=i}^{n-1} \Delta_k}$$

$$= \frac{1 - e^{-\beta\Delta_n}}{1 - e^{-\betaT}} e^{-\beta\sum_{k=i}^{n-1} \Delta_k}. \quad (\text{definition } \tau_n \text{ by Proposition 6})$$

Proposition 19. Uniform random travelers who face the optimal timetable created by TIMETABLELOGIT (Algorithm 2) choose route $i \in \mathcal{R}$ with probability Δ_i/T .

Proof. Without loss of generality, we relabel the routes such that i = n, $\pi(i) = n - 1$, etc., and we prove the statement for route n. The probability that a uniform random traveler arrives in the interval $t \in (\theta_{\pi(i)}, \theta_i]$ is equal to δ_i/T by definition. Within this interval, the routing is constant by Proposition 1. By the law of total probability, the probability that a random traveler selects route n is then given by $\sum_{i \in \mathcal{R}} \frac{\delta_i}{T} p_n^{logit}(\mathcal{R}_{\theta}(\theta_i))$. For convenience, we premultiply this quantity by a constant to obtain:

$$\begin{split} &\left(\frac{1-e^{-\beta T}}{1-e^{-\beta\Delta_n}}T\right)\sum_{i\in\mathcal{R}}\frac{\delta_i}{T}p_n^{logit}(\mathcal{R}_{\theta}(\theta_i))\\ &=\sum_{i\in\mathcal{R}}\left(\frac{\Delta_i}{1-e^{-\beta\Delta_i}}-\frac{\Delta_{\pi(i)}}{1-e^{-\beta\Delta_{\pi(i)}}}e^{-\beta\Delta_{\pi(i)}}\right)e^{-\beta\sum_{k=i}^{n-1}\Delta_k} \quad \text{(Lemmas 17 and 18)}\\ &=\sum_{i\in\mathcal{R}}\frac{\Delta_i}{1-e^{-\beta\Delta_i}}e^{-\beta\sum_{k=i}^{n-1}\Delta_k}-\sum_{i\in\mathcal{R}}\frac{\Delta_{\pi(i)}}{1-e^{-\beta\Delta_{\pi(i)}}}e^{-\beta\left(\sum_{k=i}^{n-1}\Delta_k+\Delta_{\pi(i)}\right)}\\ &=\sum_{i\in\mathcal{R}}\frac{\Delta_i}{1-e^{-\beta\Delta_i}}e^{-\beta\sum_{k=i}^{n-1}\Delta_k}-\sum_{i\in\mathcal{R}}\frac{\Delta_i}{1-e^{-\beta\Delta_i}}e^{-\beta\left(\sum_{k=i}^{n-1}\Delta_k+\Delta_i\right)} \quad \text{(shift summation index)}\\ &=\sum_{i\in\mathcal{R}}\frac{\Delta_i}{1-e^{-\beta\Delta_i}}\left(e^{-\beta\sum_{k=i}^{n-1}\Delta_k}-e^{-\beta\left(\sum_{k=\sigma(i)}^{n-1}\Delta_k+\Delta_i\right)\right)}\right)\\ &=\frac{\Delta_n}{1-e^{-\beta\Delta_n}}\left(e^{-\beta 0}-e^{-\beta\left(\sum_{k=1}^{n-1}\Delta_k+\Delta_n\right)\right) \quad \text{(difference 0 unless } i=n)\\ &=\frac{\Delta_n}{1-e^{-\beta\Delta_n}}\left(1-e^{-\beta T}\right)=\left(\frac{1-e^{-\beta T}}{1-e^{-\beta\Delta_n}}T\right)\frac{\Delta_n}{T}. \quad \text{(Corollary 4)} \end{split}$$

This proves that $\sum_{i \in \mathcal{R}} \frac{\delta_i}{T} p_n^{logit}(\mathcal{R}_{\theta}(\theta_i)) = \Delta_n/T$. As route *n* was chosen without loss of generality, this completes the proof for all routes.