Self-adjoint Laplace operator with translation invariance on infinite-dimensional space $\mathbb{R}^\infty$

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The standard Laplacian $-\triangle_{\mathbb{R}^n}$ in $L^2(\mathbb{R}^n)$ is self-adjoint and translation invariant on the finite-dimensional vector space $\mathbb{R}^n$. In this paper, using some quadratic form, we define a translation invariant operator $-\triangle_{\mathbb{R}^\infty}$ on $\mathbb{R}^\infty$ as a non-negative self-adjoint operator in some Hilbert space $L^2(\mathbb{R}^\infty)$, which is a subset of the set $CM(\mathbb{R}^\infty)$ of all complex measures on the product measurable space $\mathbb{R}^\infty$. Furthermore, we show that for any $f \in L^2(\mathbb{R}^n)$ and any $u \in L^2(\mathbb{R}^\infty)$, $e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty}t}(f \otimes u) = (e^{\sqrt{-1}\triangle_{\mathbb{R}^n}t}f) \otimes (e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty}t}u)$ $(t \in (-\infty, +\infty))$ and $e^{\triangle_{\mathbb{R}^\infty}t}(f \otimes u) = (e^{\triangle_{\mathbb{R}^n}t}f) \otimes (e^{\triangle_{\mathbb{R}^\infty}t}u)$ $(t \in [0, +\infty))$ hold. This clearly shows that $-\triangle_{\mathbb{R}^\infty}$ is an analog of $-\triangle_{\mathbb{R}^n}$. The starting point for the discussion in this paper is to naturally introduce a translation invariant Hilbert space structure into $CM(\mathbb{R}^\infty)$.

Schrödinger equation, heat equation, diffusion equation, infinite particle system, Gibbs measure, Feynman measure, canonical commutation relation (CCR), strongly continuous unitary representation.

1 Introduction

In this paper, we provide an analog of the Laplacian on the infinite-dimensional linear space $\mathbb{R}^\infty$ that is self-adjoint and translation invariant. There is no known previous research that has given an analog of the Laplacian on an infinite-dimensional linear space that is self-adjoint **and** translation invariant, and in this sense there does not seem to be any clear related literature. On the other hand, the amount of previous research on analogs of the Laplacian on infinite-dimensional linear spaces that are self-adjoint **or** translation invariant, and we are unable to provide an unbiased citation.

As $\mathbb{R}$ is the measurable space whose set of all measurable sets is the topological σ-algebra, let $\mathbb{R}^\infty$ denote the countable product measurable space $\prod_{n\in\mathbb{N}}\mathbb{R}$. There does not exist a σ-finite measure μ with translation invariance on $\mathbb{R}^\infty$ that satisfies $\mu([0,1)^\infty)=1$ (i.e., ideal Lebesgue measure on $\mathbb{R}^\infty$). Therefore, the space of square-integrable functions on $\mathbb{R}^\infty$ seems indefinable. On the other hand, according to Born and Heisenberg probabilistic interpretation of quantum mechanical wavefunction, as Ω is a measurable space, for a measure μ on Ω and a function $f\in L^2(\mu)$ that satisfies $\|f\|_{L^2(\mu)}\neq 0$, the probabilistic interpretation of the state vector f for position measurement is the probability measure

$$\frac{1}{\|f\|_{L^2(\mu)}^2}|f|^2d\mu$$

on Ω. This probability measure is the normalization of the total variation of the complex measure

$$f|f|d\mu$$

on Ω. So, in the set of all complex measures on Ω (denoted by $CM(\Omega)$), is it possible to introduce the structure of Hilbert space in which, for complex measures $f|f|d\mu$ and $g|g|d\mu$ on Ω, the inner product is

$$\langle f|f|d\mu, g|g|d\mu\rangle = \int_\Omega f\overline{g}d\mu$$

? As a matter of fact, (given limiting μ to be σ-finite,) this is easy to do and is done in Section 2. Since the sum that fits the inner product is defined in Definition 4 and it is simple, readers who have doubts here should take a look at Definition 4. In Section 3, we show that the Hilbert space structure of $CM(\mathbb{R}^\infty)$ is translation invariant and we define the generalized partial differential operator $\sqrt{-1}\frac{\partial}{\partial x_k}$ as a self-adjoint operator in the closed linear subspace of $CM(\mathbb{R}^\infty)$ in which the translation group in the x_k-direction is strongly continuous. In Section 4, from the very simple quadratic form (Hermitian form)

$$\sum_{k\in\mathbb{N}}\langle\sqrt{-1}\frac{\partial u}{\partial x_k},\sqrt{-1}\frac{\partial v}{\partial x_k}\rangle,$$

we define some closed linear subspace of $CM(\mathbb{R}^\infty)$ (denoted by $L^2(\mathbb{R}^\infty)$) and a non-negative self-adjoint operator $-\triangle_{\mathbb{R}^\infty}$ in $L^2(\mathbb{R}^\infty)$. In Section 5, when Ω_1 and Ω_2 are measurable spaces, u_1 is a complex measure on Ω_1 and u_2 is a complex measure on Ω_2, it is shown that the product $u_1 \cdot u_2$ can be naturally defined as a complex measure on $\Omega_1 \times \Omega_2$ and $\|u_1 \cdot u_2\| = \|u_1\|\|u_2\|$ holds. Since the product is defined in Definition 26 and it is simple, readers who have doubts here should take a look at Definition 26. In Section 6, when for $f \in L^2(\mathbb{R}^n)$ and $u \in CM(\mathbb{R}^\infty)$, $f \otimes u \in CM(\mathbb{R}^\infty)$ is defined as

$$(f\otimes u)(x_1,x_2,\cdots)$$

$$:= (f(x_1,x_2,\cdots,x_n)|f(x_1,x_2,\cdots,x_n)|dx_1dx_2\cdots dx_n)\cdot u(x_{n+1},x_{n+2},\cdots),$$

it is show that for any $f_0 \in L^2(\mathbb{R}^n)$ and any $u_0 \in L^2(\mathbb{R}^\infty)$,

$$e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty}t}(f_0\otimes u_0)=(e^{\sqrt{-1}\triangle_{\mathbb{R}^n}t}f_0)\otimes(e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty}t}u_0)\quad(t\in(-\infty,+\infty)),$$

$$e^{\triangle_{\mathbb{R}^\infty}t}(f_0\otimes u_0)=(e^{\triangle_{\mathbb{R}^n}t}f_0)\otimes(e^{\triangle_{\mathbb{R}^\infty}t}u_0)\quad(t\in[0,+\infty))$$

hold. This clearly shows that $\triangle_{\mathbb{R}^\infty}$ is an analog of the standard Laplacian $\triangle_{\mathbb{R}^n}$. In Section 7, we show that $L^2(\mathbb{R}^\infty)$ has an uncountable orthogonal system (i.e., $L^2(\mathbb{R}^\infty)$ is not separable). In Section 8, just to make sure, we confirm that $\triangle_{\mathbb{R}^\infty}$ is translation invariant. Section 6, Section 7 and Section 8 can each be read independently. In addition, we once again state the proof of some fundamental facts that should hold, because there does not seem to be much accessible well-known literature that explicitly states the proof.

2 Square root of density

Throughout this paper, we may use the following three simple facts (Lemma 1, Lemma 2 and Remark) without specific mention.

Lemma 1: Let η be the map from $\mathbb{C}$ to $\mathbb{C}$ defined for $z \in \mathbb{C}$, as $\eta(z) := z|z|$. Let ζ be the map from $\mathbb{C}$ to $\mathbb{C}$ defined for $w \in \mathbb{C}$, as $\zeta(w) := w|w|^{-\frac{1}{2}}$ when $w \neq 0$ holds and $\zeta(w) := 0$ when $w = 0$ holds. Then, η and ζ are continuous. $\zeta \circ \eta$ and $\eta \circ \zeta$ are the identity map. $\square$

Lemma 2: Let Ω be a measurable space. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of σ-finite measures on Ω. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of complex measures on Ω. Then, there exists a finite measure ν on Ω such that for any $n \in \mathbb{N}$, μ_n and u_n are absolutely continuous with respect to ν.

Proof: There exists $\{E_{n,m}\}_{n,m=1}^{\infty}$ such that for any n, $\Omega = \sqcup_{m=1}^{\infty} E_{n,m}$ holds and for any m, $\mu_n(E_{n,m}) < +\infty$ holds. As $|u_n|$ is the total variation of u_n, let

$$\nu(E) := \sum_n \left(\frac{1}{2^n} \left(\frac{|u_n|(E)}{1 + |u_n|(\Omega)} + \sum_m \left(\frac{1}{2^m} \frac{\mu_n(E \cap E_{n,m})}{1 + \mu_n(E_{n,m})} \right) \right) \right).$$

$\blacksquare$

Remark: Let Ω be a measurable space. Let μ be a measure on Ω. Let ρ be a $[0, +\infty)$-valued measurable function on Ω. Let f be a complex measurable function on Ω. Then, the followings hold.

(1) Suppose that f is non-negative. Then, $f(\rho d\mu) = (f\rho)d\mu$ holds.

(2) Suppose that $f \in L^1(\rho d\mu)$ or $f\rho \in L^1(\mu)$ holds. Then, $f \in L^1(\rho d\mu)$, $f\rho \in L^1(\mu)$ and $f(\rho d\mu) = (f\rho)d\mu$ hold.

Proof: Although it is a natural result, we include the proof just in case.

(1) There exists a monotonic non-decreasing sequence $\{g_n\}_n$ of non-negative simple measurable functions such that for any x, $\lim_n |g_n(x) - f(x)| = 0$ holds. Then, for any measurable set E, $(f(\rho d\mu))(E) = \int_E f(\rho d\mu) = \lim_n(\int_E g_n(\rho d\mu)) = \lim_n(\int_E (g_n\rho)d\mu) = \int_E (f\rho)d\mu = ((f\rho)d\mu)(E)$ holds.

(2) From (1), $f \in L^1(\rho d\mu)$ and $f\rho \in L^1(\mu)$ hold. So, there exist $h_1, h_2, h_3, h_4 \in L^1(\rho d\mu)$ such that h_1, h_2, h_3 and h_4 are non-negative and

$$f = (h_1 - h_2) + \sqrt{-1}(h_3 - h_4)$$

holds. Then, from (1), for any measurable set E, $(f(\rho d\mu))(E) = \int_E f(\rho d\mu) = (\int_E h_1(\rho d\mu) - \int_E h_2(\rho d\mu)) + \sqrt{-1}(\int_E h_3(\rho d\mu) - \int_E h_4(\rho d\mu)) = (\int_E (h_1\rho)d\mu -$

$\int_E (h_2\rho)d\mu) + \sqrt{-1}(\int_E (h_3\rho)d\mu - \int_E (h_4\rho)d\mu) = \int_E (f\rho)d\mu = ((f\rho)d\mu)(E)$ holds. ■

Definition 3 ($CM(\Omega)$): Let Ω be a measurable space. Then, let $CM(\Omega)$ denote the set of all complex measures on Ω. □

Definition 4 (sum): Let Ω be a measurable space. Let $u_1, u_2 \in CM(\Omega)$. Then, there uniquely exists $v \in CM(\Omega)$ such that the following holds. There exist a σ-finite measure μ on Ω and $f_1, f_2 \in L^2(\mu)$ such that $u_1 = f_1|f_1|d\mu$, $u_2 = f_2|f_2|d\mu$ and $v = (f_1 + f_2)|f_1 + f_2|d\mu$ hold. Let $u_1 + u_2 \in CM(\Omega)$ be defined as $u_1 + u_2 := v$.

Proof: From Lemma 1 and Lemma 2, existence is easy. We show uniqueness. Suppose that $(v_1, \mu_1, f_{1,1}, f_{2,1})$ and $(v_2, \mu_2, f_{1,2}, f_{2,2})$ each satisfy the condition of the definition. Then, from Lemma 2, there exist a finite measure ν and $[0, +\infty)$-valued measurable functions ρ_1, ρ_2 such that $\mu_1 = \rho_1 d\nu$ and $\mu_2 = \rho_2 d\nu$ hold. So, $u_1 = f_{1,1}|f_{1,1}|d\mu_1 = (f_{1,1}\sqrt{\rho_1})|f_{1,1}\sqrt{\rho_1}|d\nu$ and $u_1 = f_{1,2}|f_{1,2}|d\mu_2 = (f_{1,2}\sqrt{\rho_2})|f_{1,2}\sqrt{\rho_2}|d\nu$ hold. From Lemma 1, ν-a.e., $f_{1,1}\sqrt{\rho_1} = f_{1,2}\sqrt{\rho_2}$ holds. Similarly, ν-a.e., $f_{2,1}\sqrt{\rho_1} = f_{2,2}\sqrt{\rho_2}$ holds. $v_1 = (f_{1,1} + f_{2,1})|f_{1,1} + f_{2,1}|d\mu_1 = (f_{1,1}\sqrt{\rho_1} + f_{2,1}\sqrt{\rho_1})|f_{1,1}\sqrt{\rho_1} + f_{2,1}\sqrt{\rho_1}|d\nu = (f_{1,2}\sqrt{\rho_2} + f_{2,2}\sqrt{\rho_2})|f_{1,2}\sqrt{\rho_2} + f_{2,2}\sqrt{\rho_2}|d\nu = (f_{1,2} + f_{2,2})|f_{1,2} + f_{2,2}|d\mu_2 = v_2$ holds. ■

Definition 5 (scalar multiple): Let Ω be a measurable space. Let $c \in \mathbb{C}$ and $u \in CM(\Omega)$. Then, there uniquely exists $v \in CM(\Omega)$ such that the following holds. There exist a σ-finite measure μ on Ω and $f \in L^2(\mu)$ such that $u = f|f|d\mu$ and $v = (cf)|cf|d\mu$ hold. Let $cu \in CM(\Omega)$ be defined as $cu := v$.

Proof: From Lemma 1, existence is easy. Similar to Definition 4, uniqueness can be confirmed. ■

Definition 6 (inner product complex measure): Let Ω be a measurable space. Let $u_1, u_2 \in CM(\Omega)$. Then, there uniquely exists $v \in CM(\Omega)$ such that the following holds. There exist a σ-finite measure μ on Ω and $f_1, f_2 \in L^2(\mu)$ such that $u_1 = f_1|f_1|d\mu$, $u_2 = f_2|f_2|d\mu$ and $v = f_1\overline{f_2}d\mu$ hold. Let $\langle\langle u_1, u_2\rangle\rangle \in CM(\Omega)$ be defined as $\langle\langle u_1, u_2\rangle\rangle := v$.

Proof: Similar to Definition 4, existence and uniqueness can be confirmed. ■

Definition 7 (inner product): Let Ω be a measurable space. Let $u_1, u_2 \in CM(\Omega)$. Then, let $\langle u_1, u_2\rangle_{CM(\Omega)} \in \mathbb{C}$ denote $\langle\langle u_1, u_2\rangle\rangle(\Omega)$. □

Proposition 8: Let Ω be a measurable space. Then, $CM(\Omega)$ is Hilbert space.

Proof: Similar to Definition 4, it can be confirmed that $CM(\Omega)$ is an inner product space. Let $\{u_n\}_n$ be Cauchy sequence in $CM(\Omega)$. We show that there exists $v \in CM(\Omega)$ such that $\lim_n \|u_n - v\|_{CM(\Omega)} = 0$ holds. From Lemma 1 and Lemma 2, there exist a finite measure ν and a sequence $\{f_n\}_n$ in $L^2(\nu)$ such that for any n, $u_n = f_n|f_n|d\nu$ holds. So, because of $\|f_k - f_l\|_{L^2(\nu)} = \|u_k - u_l\|_{CM(\Omega)}$, $\{f_n\}_n$ is Cauchy sequence in $L^2(\nu)$. There exists $g \in L^2(\nu)$ such that $\lim_n \|f_n - g\|_{L^2(\nu)} = 0$ holds. Then, because of $\|u_n - g|g|d\nu\|_{CM(\Omega)} = \|f_n - g\|_{L^2(\nu)}$, $\lim_n \|u_n - g|g|d\nu\|_{CM(\Omega)} = 0$ holds. ■

Remark: Let M be a C^∞-manifold. Hörmander ([2]) defined densities of order $\frac{1}{p}$ on M (§2.4) and further defined an inner product for densities of order $\frac{1}{2}$ on M (§4.2). Let $(\cdot,\cdot)_M$ be the inner product defined by Hörmander. Then, $(\cdot,\cdot)_M$ can be understood as a special case of the inner product of $CM(M)$. For simplicity, assume that M is compact. Let ω be a volume element of M. Then, the positive square root $\sqrt{\omega}$ is a C^∞-density of order $\frac{1}{2}$. For any C^∞-density u of order $\frac{1}{2}$, there uniquely exists a complex-valued C^∞-function f such that $u = f\sqrt{\omega}$ holds. For any complex-valued C^∞-functions f_1 and f_2, $(f_1\sqrt{\omega}, f_2\sqrt{\omega})_M = \int_M f_1\overline{f_2}d\omega = \langle f_1|f_1|d\omega, f_2|f_2|d\omega\rangle_{CM(M)}$ holds. □

3 Self-adjoint operator $\sqrt{-1}\frac{\partial}{\partial x_k}$

As $\mathbb{R}$ is the measurable space whose set of all measurable sets is the topological σ-algebra, $\mathbb{R}^\infty$ denotes the product measurable space $\prod_{n\in\mathbb{N}}\mathbb{R}$.

Lemma 9: Let $\alpha, \beta \in \mathbb{R}^\infty$. Let E be a subset of $\mathbb{R}^\infty$. Let $E_\alpha := \{x \in \mathbb{R}^\infty | x - \alpha \in E\}$ and $E_\beta := \{x \in \mathbb{R}^\infty | x - \beta \in E\}$. Then, if E_α is a measurable set of $\mathbb{R}^\infty$, then E_β is a measurable set of $\mathbb{R}^\infty$.

Proof: Let $\mathcal{M}$ be the set of all subsets D of $\mathbb{R}^\infty$ such that $\{x \in \mathbb{R}^\infty | x - (\beta - \alpha) \in D\}$ is a measurable set of $\mathbb{R}^\infty$. Then, $\mathcal{M}$ is a σ-algebra on $\mathbb{R}^\infty$. Furthermore, for any $k \in \mathbb{N}$ and any measurable set B of $\mathbb{R}$, $(\prod_{n\in\{k\}} B) \times (\prod_{n\in\mathbb{N}\setminus\{k\}} \mathbb{R}) \in \mathcal{M}$ holds. So, if D is a measurable set of $\mathbb{R}^\infty$, then $D \in \mathcal{M}$ holds. In particular, $E_\alpha \in \mathcal{M}$ holds. Therefore, since on the other hand, $E_\beta = \{x \in \mathbb{R}^\infty | x - (\beta - \alpha) \in E_\alpha\}$ holds, E_β is a measurable set of $\mathbb{R}^\infty$. ■

Lemma 10: Let $a \in \mathbb{R}^\infty$ and $T \in CM(\mathbb{R}^\infty)$. Let T_a be the map from the set of all subsets E of $\mathbb{R}^\infty$ such that $\{x \in \mathbb{R}^\infty | x + a \in E\}$ is a measurable set of $\mathbb{R}^\infty$ to $\mathbb{C}$ defined as

$$T_a(E) := T(\{x \in \mathbb{R}^\infty | x + a \in E\}).$$

Then, $T_a \in CM(\mathbb{R}^\infty)$ holds.

Proof: From Lemma 9, it is easy. ■

Definition 11 (translation): Let $a \in \mathbb{R}^\infty$. Then, let a map τ_a from $CM(\mathbb{R}^\infty)$ to $CM(\mathbb{R}^\infty)$ be defined as

$$(\tau_a T)(E) := T(\{x \in \mathbb{R}^\infty | x + a \in E\}).$$

□

Lemma 12: Let $a \in \mathbb{R}^\infty$. Let μ be a measure on $\mathbb{R}^\infty$. Let μ_a be the map from the set of all subsets E of $\mathbb{R}^\infty$ such that $\{x \in \mathbb{R}^\infty | x + a \in E\}$ is a measurable set of $\mathbb{R}^\infty$ to $[0, +\infty]$ defined as

$$\mu_a(E) := \mu(\{x \in \mathbb{R}^\infty | x + a \in E\}).$$

Then, μ_a is a measure on $\mathbb{R}^\infty$. Let f be a complex measurable function on $\mathbb{R}^\infty$. Let f_a be the map from $\mathbb{R}^\infty$ to $\mathbb{C}$ defined as

$$f_a(x) := f(x - a).$$

Then, f_a is a complex measurable function on $\mathbb{R}^\infty$. If $f \in L^1(\mu)$ holds, then $f_a \in L^1(\mu_a)$ and $\tau_a(f d\mu) = f_a d\mu_a$ hold.

Proof: Although it is a natural result, we include the proof just in case. From Lemma 9, μ_a is a measure and f_a is a measurable function.

We show that if f is non-negative, then $\tau_a(fd\mu) = f_a d\mu_a$ holds. There exists a monotonic non-decreasing sequence $\{g_n\}_n$ of non-negative simple measurable functions such that for any x, $\lim_n |g_n(x) - f(x)| = 0$ holds. For n, let $g_{a,n}$ be the map defined as $g_{a,n}(x) := g_n(x-a)$. Then, $\{g_{a,n}\}_n$ is a monotonic non-decreasing sequence of non-negative simple measurable functions such that for any x, $\lim_n |g_{a,n}(x) - f_a(x)| = 0$ holds. So, for any measurable set E, $(\tau_a(fd\mu))(E) = (fd\mu)(\{x \in \mathbb{R}^\infty | x + a \in E\}) = \int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} fd\mu = \lim_n(\int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} g_n d\mu) = \lim_n(\int_E g_{a,n} d\mu_a) = \int_E f_a d\mu_a$ holds. If f is non-negative, then $\tau_a(fd\mu) = f_a d\mu_a$ holds.

There exist $h_1, h_2, h_3, h_4 \in L^1(\mu)$ such that h_1, h_2, h_3 and h_4 are non-negative and

$$f = (h_1 - h_2) + \sqrt{-1}(h_3 - h_4)$$

holds. Let $h_{a,1}(x) := h_1(x-a), h_{a,2}(x) := h_2(x-a), h_{a,3}(x) := h_3(x-a)$ and $h_{a,4}(x) := h_4(x-a)$. Then, $\tau_a(h_1 d\mu) = h_{a,1} d\mu_a$, $\tau_a(h_2 d\mu) = h_{a,2} d\mu_a$, $\tau_a(h_3 d\mu) = h_{a,3} d\mu_a$ and $\tau_a(h_4 d\mu) = h_{a,4} d\mu_a$ hold. Furthermore,

$$f_a = (h_{a,1} - h_{a,2}) + \sqrt{-1}(h_{a,3} - h_{a,4})$$

holds. So, for any measurable set E, $(\tau_a(fd\mu))(E) = (fd\mu)(\{x \in \mathbb{R}^\infty | x + a \in E\}) = \int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} fd\mu = (\int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} h_1 d\mu - \int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} h_2 d\mu) + \sqrt{-1}(\int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} h_3 d\mu - \int_{\{x\in\mathbb{R}^\infty|x+a\in E\}} h_4 d\mu) = ((h_1 d\mu)(\{x \in \mathbb{R}^\infty | x + a \in E\}) - (h_2 d\mu)(\{x \in \mathbb{R}^\infty | x + a \in E\})) + \sqrt{-1}((h_3 d\mu)(\{x \in \mathbb{R}^\infty | x + a \in E\}) - (h_4 d\mu)(\{x \in \mathbb{R}^\infty | x + a \in E\})) = ((\tau_a(h_1 d\mu))(E) - (\tau_a(h_2 d\mu))(E)) + \sqrt{-1}((\tau_a(h_3 d\mu))(E) - (\tau_a(h_4 d\mu))(E)) = ((h_{a,1} d\mu_a)(E) - (h_{a,2} d\mu_a)(E)) + \sqrt{-1}((h_{a,3} d\mu_a)(E) - (h_{a,4} d\mu_a)(E)) = (\int_E h_{a,1} d\mu_a - \int_E h_{a,2} d\mu_a) + \sqrt{-1}(\int_E h_{a,3} d\mu_a - \int_E h_{a,4} d\mu_a) = \int_E f_a d\mu_a = (f_a d\mu_a)(E)$ holds. ■

Proposition 13: Let $a \in \mathbb{R}^\infty$. Then, τ_a is a unitary operator in $CM(\mathbb{R}^\infty)$.

Proof: From Lemma 12, it is easy. ■

Definition 14 ($L^2_k(\mathbb{R}^\infty)$, $(H^1_k(\mathbb{R}^\infty), \frac{\partial}{\partial x_k})$): Let $k \in \mathbb{N}$. Then, there uniquely exists $e_k \in \mathbb{R}^\infty$ such that $e_{k,k} = 1$ holds and for any $n \in \mathbb{N} \setminus \{k\}$, $e_{k,n} = 0$ holds.

(1) Let $L^2_k(\mathbb{R}^\infty)$ denote the set of all $u \in CM(\mathbb{R}^\infty)$ such that

$$\lim_{h\downarrow+0} \|\tau_{he_k} u - u\|_{CM(\mathbb{R}^\infty)} = 0$$

holds.

(2) For $u_1, u_2 \in L^2_k(\mathbb{R}^\infty)$, let $\langle u_1, u_2 \rangle_{L^2_k(\mathbb{R}^\infty)} \in \mathbb{C}$ be defined as

$$\langle u_1, u_2 \rangle_{L^2_k(\mathbb{R}^\infty)} := \langle u_1, u_2 \rangle_{CM(\mathbb{R}^\infty)}.$$

(3) Let $H^1_k(\mathbb{R}^\infty)$ denote the set of all $u \in CM(\mathbb{R}^\infty)$ such that the following holds. There uniquely exists $v \in CM(\mathbb{R}^\infty)$ such that

$$\lim_{h \downarrow +0} \| \frac{\tau_{he_k} u - u}{h} - v \|_{CM(\mathbb{R}^\infty)} = 0$$

holds.

(4) Let $u \in H^1_k(\mathbb{R}^\infty)$. Then, there uniquely exists $v \in CM(\mathbb{R}^\infty)$ such that

$$\lim_{h \downarrow +0} \| \frac{\tau_{he_k} u - u}{h} - v \|_{CM(\mathbb{R}^\infty)} = 0$$

holds. Let $\frac{\partial u}{\partial x_k} \in CM(\mathbb{R}^\infty)$ be defined as $\frac{\partial u}{\partial x_k} := -v$. □

Lemma 15: Let $k \in \mathbb{N}$. Then, the followings hold.

(1) $H^1_k(\mathbb{R}^\infty) \subset L^2_k(\mathbb{R}^\infty)$ holds.

(2) $L^2_k(\mathbb{R}^\infty)$ is the set of all $u \in CM(\mathbb{R}^\infty)$ such that the following holds. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_0, t_1, t_2 \in \mathbb{R}$, if $|t_1 - t_2| < \delta$ holds, then $\|\tau_{t_1 e_k}(\tau_{t_0 e_k} u) - \tau_{t_2 e_k}(\tau_{t_0 e_k} u)\|_{CM(\mathbb{R}^\infty)} < \varepsilon$ holds.

(3) $L^2_k(\mathbb{R}^\infty)$ is a closed linear subspace of $CM(\mathbb{R}^\infty)$.

(4) Let $u \in H^1_k(\mathbb{R}^\infty)$. Then, $-\frac{\partial u}{\partial x_k} \in L^2_k(\mathbb{R}^\infty)$ holds.

Proof: (1) It is easy.

(2) Because of $\tau_{t_1 e_k}(\tau_{t_0 e_k} u) - \tau_{t_2 e_k}(\tau_{t_0 e_k} u) = -\tau_{t_0 e_k} \tau_{t_1 e_k}(\tau_{(t_2 - t_1) e_k} u - u) = +\tau_{t_0 e_k} \tau_{t_2 e_k}(\tau_{(t_1 - t_2) e_k} u - u)$, from Proposition 13, it follows.

(3) From Proposition 13, it is easy.

(4) From (1), (2) and (3), it follows. ■

Theorem 16: Let $k \in \mathbb{N}$. Then, the followings hold.

(1) The domain of $\sqrt{-1}\frac{\partial}{\partial x_k}$ is $H^1_k(\mathbb{R}^\infty)$.

(2) $\sqrt{-1}\frac{\partial}{\partial x_k}$ is a self-adjoint operator in $L^2_k(\mathbb{R}^\infty)$.

Proof: (1) It is obvious.

(2) From Proposition 8, Proposition 13 and Lemma 15, it follows. ■

4 Non-negative self-adjoint operator $-\triangle_{\mathbb{R}^\infty}$

Definition 17 ($H^1(\mathbb{R}^\infty)$): (1) Let $H^1(\mathbb{R}^\infty)$ denote the set of all $u \in \cap_{k\in\mathbb{N}} H^1_k(\mathbb{R}^\infty)$ such that

$$\sum_{k\in\mathbb{N}} \langle \sqrt{-1}\frac{\partial u}{\partial x_k}, \sqrt{-1}\frac{\partial u}{\partial x_k}\rangle_{L^2_k(\mathbb{R}^\infty)} < +\infty$$

holds.

(2) For $u_1, u_2 \in H^1(\mathbb{R}^\infty)$, let $\langle u_1, u_2\rangle_{H^1(\mathbb{R}^\infty)} \in \mathbb{C}$ be defined as

$$\langle u_1, u_2\rangle_{H^1(\mathbb{R}^\infty)} := \langle u_1, u_2\rangle_{CM(\mathbb{R}^\infty)} + \sum_{k\in\mathbb{N}} \langle \sqrt{-1}\frac{\partial u_1}{\partial x_k}, \sqrt{-1}\frac{\partial u_2}{\partial x_k}\rangle_{L^2_k(\mathbb{R}^\infty)}.$$

$\square$

Proposition 18: $H^1(\mathbb{R}^\infty)$ is Hilbert space.

Proof: It is easy that $H^1(\mathbb{R}^\infty)$ is an inner product space. Let $\{u_n\}_n$ be Cauchy sequence in $H^1(\mathbb{R}^\infty)$. Then, we show that there exists $v \in H^1(\mathbb{R}^\infty)$ such that $\lim_n \|u_n - v\|_{H^1(\mathbb{R}^\infty)} = 0$ holds.

There exist v and $\{w_k\}_{k\in\mathbb{N}}$ such that $\lim_n \|u_n - v\|_{CM(\mathbb{R}^\infty)} = 0$ holds and for any $k \in \mathbb{N}$, $\lim_n \|\sqrt{-1}\frac{\partial u_n}{\partial x_k} - w_k\|_{L^2_k(\mathbb{R}^\infty)} = 0$ holds. For any $k \in \mathbb{N}$, because $\{u_n\}_n$ is Cauchy sequence in $L^2_k(\mathbb{R}^\infty)$, $\lim_n \|u_n - v\|_{L^2_k(\mathbb{R}^\infty)} = 0$ holds. So, from Theorem 16, (because self-adjoint operators are closed,) for any $k \in \mathbb{N}$, $\sqrt{-1}\frac{\partial v}{\partial x_k} = w_k$ holds. Hence, for any $k \in \mathbb{N}$, $\lim_n \|\sqrt{-1}\frac{\partial u_n}{\partial x_k} - \sqrt{-1}\frac{\partial v}{\partial x_k}\|_{L^2_k(\mathbb{R}^\infty)} = 0$ holds. Because for any $n, K \in \mathbb{N}$,

$$\sum_{k=1}^{K} \langle \sqrt{-1}\frac{\partial u_n}{\partial x_k}, \sqrt{-1}\frac{\partial u_n}{\partial x_k}\rangle_{L^2_k(\mathbb{R}^\infty)} \le \sup_{m\in\mathbb{N}} \langle u_m, u_m\rangle_{H^1(\mathbb{R}^\infty)} < +\infty$$

holds, for any $K \in \mathbb{N}$,

$$\sum_{k=1}^{K} \langle \sqrt{-1}\frac{\partial v}{\partial x_k}, \sqrt{-1}\frac{\partial v}{\partial x_k}\rangle_{L^2_k(\mathbb{R}^\infty)} \le \sup_{m\in\mathbb{N}} \langle u_m, u_m\rangle_{H^1(\mathbb{R}^\infty)} < +\infty$$

holds. Therefore, $v \in H^1(\mathbb{R}^\infty)$ holds.

Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$n \ge N, m \ge N \quad \Longrightarrow \quad \|u_n - u_m\|_{H^1(\mathbb{R}^\infty)} < \frac{\varepsilon}{2}$$

holds. For any $K \in \mathbb{N}$,

$$n \geq N, m \geq N$$
$$\Longrightarrow$$
$$\langle u_n - u_m, u_n - u_m \rangle_{CM(\mathbb{R}^\infty)} + \sum_{k=1}^{K} \langle \sqrt{-1} \frac{\partial (u_n - u_m)}{\partial x_k}, \sqrt{-1} \frac{\partial (u_n - u_m)}{\partial x_k} \rangle_{L^2_k(\mathbb{R}^\infty)}$$
$$< (\frac{\varepsilon}{2})^2$$

holds. So, because of

$$n \geq N$$
$$\Longrightarrow$$
$$\langle u_n - v, u_n - v \rangle_{CM(\mathbb{R}^\infty)} + \sum_{k=1}^{K} \langle \sqrt{-1} \frac{\partial (u_n - v)}{\partial x_k}, \sqrt{-1} \frac{\partial (u_n - v)}{\partial x_k} \rangle_{L^2_k(\mathbb{R}^\infty)}$$
$$\leq (\frac{\varepsilon}{2})^2,$$

$$n \geq N \quad \Longrightarrow \quad \|u_n - v\|_{H^1(\mathbb{R}^\infty)} \leq \frac{\varepsilon}{2} < \varepsilon$$

holds. ■

Definition 19 ($L^2(\mathbb{R}^\infty)$): Let $L^2(\mathbb{R}^\infty)$ denote the closure of $H^1(\mathbb{R}^\infty)$ in $CM(\mathbb{R}^\infty)$. For $u_1, u_2 \in L^2(\mathbb{R}^\infty)$, let $\langle u_1, u_2 \rangle_{L^2(\mathbb{R}^\infty)} \in \mathbb{C}$ be defined as $\langle u_1, u_2 \rangle_{L^2(\mathbb{R}^\infty)} := \langle u_1, u_2 \rangle_{CM(\mathbb{R}^\infty)}$. □

Proposition 20: $L^2(\mathbb{R}^\infty)$ is a closed linear subspace of $CM(\mathbb{R}^\infty)$. $L^2(\mathbb{R}^\infty)$ is Hilbert space. $H^1(\mathbb{R}^\infty)$ is a dense linear subspace of $L^2(\mathbb{R}^\infty)$. For any $u \in H^1(\mathbb{R}^\infty)$, $\|u\|_{L^2(\mathbb{R}^\infty)} \leq \|u\|_{H^1(\mathbb{R}^\infty)}$ holds.

Proof: It is easy. ■

Definition 21 (elliptic equation, $(1 - \triangle_{\mathbb{R}^\infty})^{-1}$): Let $f \in L^2(\mathbb{R}^\infty)$. Then, there uniquely exists $u \in H^1(\mathbb{R}^\infty)$ such that for any $\varphi \in H^1(\mathbb{R}^\infty)$,

$$\langle u, \varphi \rangle_{H^1(\mathbb{R}^\infty)} = \langle f, \varphi \rangle_{L^2(\mathbb{R}^\infty)}$$

holds. Let $(1 - \triangle_{\mathbb{R}^\infty})^{-1} f \in H^1(\mathbb{R}^\infty)$ be defined as $(1 - \triangle_{\mathbb{R}^\infty})^{-1} f := u$.

Proof: From Proposition 18 and Proposition 20, existence and uniqueness are easy (by Riesz theorem). ■

Proposition 22: For any $f \in L^2(\mathbb{R}^\infty)$, $\|(1-\triangle_{\mathbb{R}^\infty})^{-1} f\|_{H^1(\mathbb{R}^\infty)} \leq \|f\|_{L^2(\mathbb{R}^\infty)}$ holds. $(1 - \triangle_{\mathbb{R}^\infty})^{-1}$ is an injective self-adjoint operator in $L^2(\mathbb{R}^\infty)$.

Proof: $\langle f_1, (1-\triangle_{\mathbb{R}^\infty})^{-1} f_2\rangle_{L^2(\mathbb{R}^\infty)} = \langle (1-\triangle_{\mathbb{R}^\infty})^{-1} f_1, (1-\triangle_{\mathbb{R}^\infty})^{-1} f_2\rangle_{H^1(\mathbb{R}^\infty)} = \overline{\langle (1-\triangle_{\mathbb{R}^\infty})^{-1} f_2, (1-\triangle_{\mathbb{R}^\infty})^{-1} f_1\rangle_{H^1(\mathbb{R}^\infty)}} = \overline{\langle f_2, (1-\triangle_{\mathbb{R}^\infty})^{-1} f_1\rangle_{L^2(\mathbb{R}^\infty)}} = \langle (1-\triangle_{\mathbb{R}^\infty})^{-1} f_1, f_2\rangle_{L^2(\mathbb{R}^\infty)}$ holds.

$\|(1-\triangle_{\mathbb{R}^\infty})^{-1} f\|^2_{H^1(\mathbb{R}^\infty)} = \langle f, (1-\triangle_{\mathbb{R}^\infty})^{-1} f\rangle_{L^2(\mathbb{R}^\infty)} \leq \|f\|_{L^2(\mathbb{R}^\infty)} \|(1-\triangle_{\mathbb{R}^\infty})^{-1} f\|_{L^2(\mathbb{R}^\infty)}$ holds.

There exists a sequence $\{g_n\}_n$ in $H^1(\mathbb{R}^\infty)$ such that $\lim_n \|g_n - f\|_{L^2(\mathbb{R}^\infty)} = 0$ holds. So, if $(1-\triangle_{\mathbb{R}^\infty})^{-1} f = 0$ holds, then $\langle f, f\rangle_{L^2(\mathbb{R}^\infty)} = \lim_n \langle f, g_n\rangle_{L^2(\mathbb{R}^\infty)} = \lim_n \langle (1-\triangle_{\mathbb{R}^\infty})^{-1} f, g_n\rangle_{H^1(\mathbb{R}^\infty)} = 0$ holds. ■

Definition 23 ($\triangle_{\mathbb{R}^\infty}$): Let u be an element of the range of $(1-\triangle_{\mathbb{R}^\infty})^{-1}$. Then, let $\triangle_{\mathbb{R}^\infty} u \in L^2(\mathbb{R}^\infty)$ be defined as

$$\triangle_{\mathbb{R}^\infty} u := u - ((1-\triangle_{\mathbb{R}^\infty})^{-1})^{-1} u.$$

□

Theorem 24: (1) The domain of $-\triangle_{\mathbb{R}^\infty}$ is a linear subspace of $H^1(\mathbb{R}^\infty)$. (2) $-\triangle_{\mathbb{R}^\infty}$ is a non-negative self-adjoint operator in $L^2(\mathbb{R}^\infty)$.

Proof: (1) The range of $(1-\triangle_{\mathbb{R}^\infty})^{-1}$ is a linear subspace of $H^1(\mathbb{R}^\infty)$.

(2) From Proposition 22, $-\triangle_{\mathbb{R}^\infty}$ is self-adjoint. $\langle -\triangle_{\mathbb{R}^\infty} u, u\rangle_{L^2(\mathbb{R}^\infty)} = \langle ((1-\triangle_{\mathbb{R}^\infty})^{-1})^{-1} u, u\rangle_{L^2(\mathbb{R}^\infty)} - \langle u, u\rangle_{L^2(\mathbb{R}^\infty)} = \langle u, u\rangle_{H^1(\mathbb{R}^\infty)} - \langle u, u\rangle_{L^2(\mathbb{R}^\infty)} \geq 0$ holds. ■

5 Product complex measure

We may use the following simple lemma without specific mention.

Lemma 25: Let Ω_1 and Ω_2 be measurable spaces. Let μ_1 be a σ-finite measure on Ω_1. Let μ_2 be a σ-finite measure on Ω_2. Let ρ_1 be a $[0,+\infty)$-valued measurable function on Ω_1. Let ρ_2 be a $[0,+\infty)$-valued measurable function on Ω_2. Then, $\rho_1\rho_2 d\mu_1 d\mu_2 = (\rho_1 d\mu_1)(\rho_2 d\mu_2)$ holds.

Proof: Although it is a natural result, we include the proof just in case. Let E_1 be a measurable set of Ω_1. Let E_2 be a measurable set of Ω_2. Then, $(\rho_1\rho_2 d\mu_1 d\mu_2)(E_1 \times E_2) = \int_{E_1\times E_2} \rho_1\rho_2 d\mu_1 d\mu_2 = \int_{E_2}(\int_{E_1} \rho_1\rho_2 d\mu_1)d\mu_2 = (\int_{E_1} \rho_1 d\mu_1)(\int_{E_2} \rho_2 d\mu_2) = ((\rho_1 d\mu_1)(E_1))((\rho_2 d\mu_2)(E_2))$ holds. ■

Definition 26 (product complex measure): Let u_1 and u_2 be complex measures. Then, there uniquely exists a complex measure v such that the following holds. There exist σ-finite measures μ_1, μ_2 and $f_1 \in L^1(\mu_1)$, $f_2 \in L^1(\mu_2)$ such that $u_1 = f_1 d\mu_1$, $u_2 = f_2 d\mu_2$ and $v = f_1 f_2 d\mu_1 d\mu_2$ hold. Let the complex measure $u_1 \cdot u_2$ be defined as $u_1 \cdot u_2 := v$.

Proof: Existence is easy. We show uniqueness. Suppose that $(v_1, \mu_{1,1}, \mu_{2,1}, f_{1,1}, f_{2,1})$ and $(v_2, \mu_{1,2}, \mu_{2,2}, f_{1,2}, f_{2,2})$ each satisfy the condition of the definition. Then, there exist finite measures ν_1, ν_2 and $[0,+\infty)$-valued measurable functions $\rho_{1,1}, \rho_{2,1}, \rho_{1,2}, \rho_{2,2}$ such that $\mu_{1,1} = \rho_{1,1} d\nu_1$, $\mu_{1,2} = \rho_{1,2} d\nu_1$, $\mu_{2,1} = \rho_{2,1} d\nu_2$ and $\mu_{2,2} = \rho_{2,2} d\nu_2$ hold. So, because $f_{1,1}\rho_{1,1} d\nu_1 = u_1 = f_{1,2}\rho_{1,2} d\nu_1$ and $f_{2,1}\rho_{2,1} d\nu_2 = u_2 = f_{2,2}\rho_{2,2} d\nu_2$ hold, $d\nu_1 d\nu_2$-a.e., $f_{1,1}\rho_{1,1} f_{2,1}\rho_{2,1} = f_{1,2}\rho_{1,2} f_{2,2}\rho_{2,2}$ holds. From Lemma 25, $v_1 = f_{1,1} f_{2,1} d\mu_{1,1} d\mu_{2,1} = f_{1,1}\rho_{1,1} f_{2,1}\rho_{2,1} d\nu_1 d\nu_2 = f_{1,2}\rho_{1,2} f_{2,2}\rho_{2,2} d\nu_1 d\nu_2 = f_{1,2} f_{2,2} d\mu_{1,2} d\mu_{2,2} = v_2$ holds. ■

Proposition 27: Let u_1, u_2 and u_3 be complex measures. Then,

$$(u_1 \cdot u_2) \cdot u_3 = u_1 \cdot (u_2 \cdot u_3)$$

holds.

Proof: It is easy. ■

Proposition 28: Let Ω_1 and Ω_2 be measurable spaces. Then, the followings hold.

(1) Let $u_1, v_1 \in CM(\Omega_1)$ and $u_2, v_2 \in CM(\Omega_2)$. Then,

$$\langle\langle u_1 \cdot u_2, v_1 \cdot v_2 \rangle\rangle = \langle\langle u_1, v_1 \rangle\rangle \cdot \langle\langle u_2, v_2 \rangle\rangle,$$

$$\langle u_1 \cdot u_2, v_1 \cdot v_2 \rangle_{CM(\Omega_1\times\Omega_2)} = \langle u_1, v_1 \rangle_{CM(\Omega_1)} \langle u_2, v_2 \rangle_{CM(\Omega_2)}$$

hold.

(2) The map $(u_1, u_2) \mapsto u_1 \cdot u_2$ from $CM(\Omega_1) \times CM(\Omega_2)$ to $CM(\Omega_1 \times \Omega_2)$ is bi-linear.

Proof: (1) There exist finite measures μ_1, μ_2 and $f_1, g_1 \in L^2(\mu_1), f_2, g_2 \in L^2(\mu_2)$ such that $u_1 = f_1|f_1|d\mu_1, v_1 = g_1|g_1|d\mu_1, u_2 = f_2|f_2|d\mu_2$ and $v_2 = g_2|g_2|d\mu_2$ hold. Then, from $\langle\langle u_1, v_1\rangle\rangle = f_1\overline{g_1}d\mu_1$, $\langle\langle u_2, v_2\rangle\rangle = f_2\overline{g_2}d\mu_2$ and $\langle\langle u_1 \cdot u_2, v_1 \cdot v_2\rangle\rangle = f_1\overline{g_1}f_2\overline{g_2}d\mu_1 d\mu_2$, it follows.

(2) It is easy. ■

We may use (2) of the following remark without specific mention, as it is a natural result.

Remark: (1) Let Ω be a set. Let $\mathcal{A}$ be an algebra on Ω. Let $\mathcal{B}$ be the smallest σ-algebra on Ω such that $\mathcal{A} \subset \mathcal{B}$ holds. Let u_1 and u_2 be real measures on the measurable space whose set of all measurable sets is $\mathcal{B}$. Suppose that for any $E \in \mathcal{A}$, $u_1(E) = u_2(E)$ holds. Then, $u_1 = u_2$ holds.

(2) Let Ω_1 and Ω_2 be measurable spaces. Let $u_1, u_2 \in CM(\Omega_1 \times \Omega_2)$. Suppose that for any measurable set E_1 of Ω_1 and any measurable set E_2 of Ω_2, $u_1(E_1 \times E_2) = u_2(E_1 \times E_2)$ holds. Then, $u_1 = u_2$ holds.

Proof: (1) Let $|u_1|$ be the total variation of u_1. Let $|u_2|$ be the total variation of u_2. For a measurable set E, let $\mu(E) := |u_1|(E) + |u_2|(E)$. Then, there exist $[-1, +1]$-valued measurable functions f_1 and f_2 such that $u_1 = f_1 d\mu$ and $u_2 = f_2 d\mu$ hold. Let $E_{f_1<f_2} := \{x \in \Omega | f_1(x) < f_2(x)\}$ and $E_{f_2<f_1} := \{x \in \Omega | f_2(x) < f_1(x)\}$.

Let $\varepsilon > 0$. Then, from basic facts about Carathéodory outer measure (especially, Hopf extension theorem), there exists a sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that $E_{f_1<f_2} \subset \cup_n E_n$ and $\sum_n \mu(E_n) < \mu(E_{f_1<f_2}) + \frac{\varepsilon}{2}$ hold. Let $E'_n := E_n \backslash \cup_{k=1}^{n-1} E_k$. Then, because $\sum_n \mu(E'_n \backslash E_{f_1<f_2}) = \mu(\cup_n (E'_n \backslash E_{f_1<f_2})) = \mu((\cup_n E_n) \backslash E_{f_1<f_2}) = \mu(\cup_n E_n) - \mu(E_{f_1<f_2}) < \frac{\varepsilon}{2}$ and $E'_n \in \mathcal{A}$ hold, $\int_{E_{f_1<f_2}} |f_1 - f_2| d\mu = \sum_n \int_{E'_n \cap E_{f_1<f_2}} (f_2 - f_1) d\mu = \sum_n (\int_{E'_n} (f_2 - f_1) d\mu - \int_{E'_n \backslash E_{f_1<f_2}} (f_2 - f_1) d\mu) = \sum_n \int_{E'_n \backslash E_{f_1<f_2}} (f_1 - f_2) d\mu \leq \sum_n \int_{E'_n \backslash E_{f_1<f_2}} 2 d\mu < \varepsilon$ holds. Therefore, $\int_{E_{f_1<f_2}} |f_1 - f_2| d\mu = 0$ holds. Similarly, $\int_{E_{f_2<f_1}} |f_1 - f_2| d\mu = 0$ holds.

(2) From (1), it follows ■

6 Embedding finite-dimensional evolution $\{e^{\sqrt{-1}\triangle_{\mathbb{R}^N}t}\}_{t\in(-\infty,+\infty)}$ and $\{e^{\triangle_{\mathbb{R}^N}t}\}_{t\in[0,+\infty)}$

Let $N \in \mathbb{N}$. As $\mathbb{R}$ is the measurable space whose set of all measurable sets is the topological σ-algebra, let $\mathbb{R}^N$ denote the product measurable space $\prod_{n=1}^{N} \mathbb{R}$. Let dx denote the (ordinary) measure $\prod_{n=1}^{N} dx_n$ on $\mathbb{R}^N$. Let $\triangle_{\mathbb{R}^N}$ denote the (ordinary) Laplacian in $L^2(\mathbb{R}^N)$.

In Section 3, we defined $\{\frac{\partial}{\partial x_k}\}_{k\in\mathbb{N}}$. However, in $\mathbb{R}^N \times \mathbb{R}^\infty$, the variables $x_1, x_2, \cdots, x_N$ conflict. For notational consistency, we introduce the following definition.

Definition 29 (N-shift): Let $u \in CM(\mathbb{R}^\infty)$. Then, let u_N^+ denote the map from the set of all subsets E of $\prod_{n=N+1}^{\infty} \mathbb{R}$ such that $\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\infty | \{x_{n-N}\}_{n=N+1}^{\infty} \in E\}$ is a measurable set of $\mathbb{R}^\infty$ to $\mathbb{C}$ defined as

$$u_N^+(E) := u(\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\infty | \{x_{n-N}\}_{n=N+1}^{\infty} \in E\}).$$

□

Lemma 30: For any $u \in CM(\mathbb{R}^\infty)$, $u_N^+ \in CM(\prod_{n=N+1}^{\infty} \mathbb{R})$ holds. The map $u \mapsto u_N^+$ from $CM(\mathbb{R}^\infty)$ to $CM(\prod_{n=N+1}^{\infty} \mathbb{R})$ is a unitary operator.

Proof: It is easy. ■

Definition 31 ($\otimes$): Let $f \in L^2(\mathbb{R}^N)$ and $u \in CM(\mathbb{R}^\infty)$. Then, let $f \otimes u \in CM(\mathbb{R}^\infty)$ defined as

$$f \otimes u := (f|f|dx) \cdot (u_N^+).$$

□

Lemma 32: Let $f_1, f_2 \in L^2(\mathbb{R}^N)$ and $u_1, u_2 \in CM(\mathbb{R}^\infty)$. Then,

$$\langle f_1 \otimes u_1, f_2 \otimes u_2\rangle_{CM(\mathbb{R}^\infty)} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^N)}\langle u_1, u_2\rangle_{CM(\mathbb{R}^\infty)}$$

holds.

Proof: From Proposition 28 (1) and Lemma 30, it is easy. ■

Lemma 33: The map $(f, u) \mapsto f \otimes u$ from $L^2(\mathbb{R}^N) \times CM(\mathbb{R}^\infty)$ to $CM(\mathbb{R}^\infty)$ is bi-linear.

Proof: From Proposition 28 (2) and Lemma 30, it is easy. ■

Lemma 34: Let $a \in \mathbb{R}^N$. Let f be a complex measurable function on $\mathbb{R}^N$. Let f_a be the map from $\mathbb{R}^N$ to $\mathbb{C}$ defined as

$$f_a(x) := f(x - a).$$

Then, f_a is a complex measurable function on $\mathbb{R}^N$. Let E be a measurable set of $\mathbb{R}^N$. Let

$$E_a := \{x \in \mathbb{R}^N | x - a \in E\}.$$

Then, E_a is a measurable set of $\mathbb{R}^N$. If $f \in L^1(\mathbb{R}^N)$ holds, then $f_a \in L^1(\mathbb{R}^N)$ and $\int_{E_a} f_a dx = \int_E f dx$ hold.

Proof: It is well known. ■

Lemma 35: Let $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\infty$. Let $f \in L^2(\mathbb{R}^N)$ and $u \in CM(\mathbb{R}^\infty)$. Let $f_{\{a_n\}_{n=1}^N}$ be the map from $\mathbb{R}^N$ to $\mathbb{C}$ defined as

$$f_{\{a_n\}_{n=1}^N}(x) := f(x - \{a_n\}_{n=1}^N).$$

Then,

$$\tau_{\{a_n\}_{n\in\mathbb{N}}}(f \otimes u) = f_{\{a_n\}_{n=1}^N} \otimes (\tau_{\{a_{N+n}\}_{n\in\mathbb{N}}} u)$$

holds.

Proof: Although it is a natural result, we include the proof just in case.

Let F be a measurable set of $\mathbb{R}^N$. Let G be a measurable set of $\prod_{n=N+1}^\infty \mathbb{R}$. Then, from $\{x \in \mathbb{R}^\infty | x + \{a_n\}_{n=1}^\infty \in F \times G\} = \{x \in \mathbb{R}^N | x + \{a_n\}_{n=1}^N \in F\} \times \{x \in \prod_{n=N+1}^\infty \mathbb{R} | x + \{a_n\}_{n=N+1}^\infty \in G\}$ and Lemma 34, $(\tau_{\{a_n\}_{n=1}^\infty}(f \otimes u))(F \times G) = (f \otimes u)(\{x \in \mathbb{R}^\infty | x + \{a_n\}_{n=1}^\infty \in F \times G\}) = (\int_{x+\{a_n\}_{n=1}^N \in F} f|f|dx)(u_N^+(\{x \in \prod_{n=N+1}^\infty \mathbb{R} | x + \{a_n\}_{n=N+1}^\infty \in G\})) = (\int_F f_{\{a_n\}_{n=1}^N} |f_{\{a_n\}_{n=1}^N}| dx)((\tau_{\{a_{N+n}\}_{n=1}^\infty} u)_N^+(G))$ holds. So, from Remark (2) at the end of Section 5, $\tau_{\{a_n\}_{n=1}^\infty}(f \otimes u) = (f_{\{a_n\}_{n=1}^N} |f_{\{a_n\}_{n=1}^N}| dx) \cdot ((\tau_{\{a_{N+n}\}_{n=1}^\infty} u)_N^+)$ holds. ■

Lemma 36: Let $k \in \mathbb{N}$, $u \in H_k^1(\mathbb{R}^\infty)$ and $f \in L^2(\mathbb{R}^N)$. Then, $f \otimes u \in H_{N+k}^1(\mathbb{R}^\infty)$ and

$$\frac{\partial(f \otimes u)}{\partial x_{N+k}} = f \otimes \frac{\partial u}{\partial x_k}$$

hold.

Proof: There uniquely exists $e_k \in \mathbb{R}^\infty$ such that $e_{k,k} = 1$ holds and for any $n \in \mathbb{N} \setminus \{k\}$, $e_{k,n} = 0$ holds. There uniquely exists $e_{N+k} \in \mathbb{R}^\infty$ such that $e_{N+k,N+k} = 1$ holds and for any $n \in \mathbb{N} \setminus \{N+k\}$, $e_{N+k,n} = 0$ holds. Then, from Lemma 35, Lemma 33 and Lemma 32, $\|\frac{\tau_{he_{N+k}}(f\otimes u) - f \otimes u}{h} + f \otimes \frac{\partial u}{\partial x_k}\|_{CM(\mathbb{R}^\infty)} = \|\frac{f \otimes (\tau_{he_k} u) - f \otimes u}{h} + f \otimes \frac{\partial u}{\partial x_k}\|_{CM(\mathbb{R}^\infty)} = \|f \otimes (\frac{\tau_{he_k} u - u}{h} + \frac{\partial u}{\partial x_k})\|_{CM(\mathbb{R}^\infty)} = \|f\|_{L^2(\mathbb{R}^N)} \|\frac{\tau_{he_k} u - u}{h} + \frac{\partial u}{\partial x_k}\|_{CM(\mathbb{R}^\infty)}$ holds. ■

Lemma 37: Let $k \in \{1, 2, \cdots, N\}$. Then, there uniquely exists $e_k^N \in \mathbb{R}^N$ such that $e_{k,k}^N = 1$ holds and for any $n \in \{1, 2, \cdots, N\} \setminus \{k\}$, $e_{k,n}^N = 0$ holds.

Let $f, g \in L^2(\mathbb{R}^N)$. Suppose that

$$\lim_{h \downarrow +0} \| \frac{f(x - he_k^N) - f(x)}{h} + g(x) \|_{L^2(\mathbb{R}^N)} = 0$$

holds. Let $u \in CM(\mathbb{R}^\infty)$. Then, $f \otimes u \in H_k^1(\mathbb{R}^\infty)$ and

$$\frac{\partial (f \otimes u)}{\partial x_k} = g \otimes u$$

hold.

Proof: There uniquely exists $e_k \in \mathbb{R}^\infty$ such that $e_{k,k} = 1$ holds and for any $n \in \mathbb{N} \setminus \{k\}$, $e_{k,n} = 0$ holds. For $a \in \mathbb{R}^N$, let $f_a(x) := f(x - a)$. Then, from Lemma 35, Lemma 33 and Lemma 32, $\| \frac{\tau_{he_k}(f \otimes u) - f \otimes u}{h} + g \otimes u \|_{CM(\mathbb{R}^\infty)} = \| \frac{f_{he_k^N} \otimes u - f \otimes u}{h} + g \otimes u \|_{CM(\mathbb{R}^\infty)} = \| (\frac{f_{he_k^N} - f}{h} + g) \otimes u \|_{CM(\mathbb{R}^\infty)} = \| \frac{f_{he_k^N} - f}{h} + g \|_{L^2(\mathbb{R}^N)} \|u\|_{CM(\mathbb{R}^\infty)}$ holds. ■

Lemma 38: (1) Let $f \in H^1(\mathbb{R}^N)$ and $u \in H^1(\mathbb{R}^\infty)$. Then, $f \otimes u \in H^1(\mathbb{R}^\infty)$ holds.

(2) Let $f \in L^2(\mathbb{R}^N)$ and $u \in L^2(\mathbb{R}^\infty)$. Then, $f \otimes u \in L^2(\mathbb{R}^\infty)$ holds.

Proof: (1) From Lemma 37 and Lemma 36, $f \otimes u \in \cap_{k \in \mathbb{N}} H_k^1(\mathbb{R}^\infty)$ holds and for any $k \in \{N+1, N+2, \cdots\}$, $\frac{\partial (f \otimes u)}{\partial x_k} = f \otimes \frac{\partial u}{\partial x_{k-N}}$ holds. So, from Lemma 32, $\sum_{k=N+1}^\infty \| \frac{\partial (f \otimes u)}{\partial x_k} \|_{CM(\mathbb{R}^\infty)}^2 = \|f\|_{L^2(\mathbb{R}^N)}^2 (\sum_{k \in \mathbb{N}} \| \frac{\partial u}{\partial x_k} \|_{CM(\mathbb{R}^\infty)}^2) \leq \|f\|_{L^2(\mathbb{R}^N)}^2$ $\|u\|_{H^1(\mathbb{R}^\infty)}^2$ holds.

(2) There exists a sequence $\{g_n\}_n$ in $H^1(\mathbb{R}^N)$ such that $\lim_n \|g_n - f\|_{L^2(\mathbb{R}^N)} = 0$ holds. There exists a sequence $\{v_n\}_n$ in $H^1(\mathbb{R}^\infty)$ such that $\lim_n \|v_n - u\|_{L^2(\mathbb{R}^\infty)} = 0$ holds. Then, from (1), for any n, $g_n \otimes v_n \in H^1(\mathbb{R}^\infty)$ holds. On the other hand, from Lemma 33 and Lemma 32, $\lim_n \|g_n \otimes v_n - f \otimes u\|_{CM(\mathbb{R}^\infty)} = 0$ holds. ■

In order to examine $(\triangle_{\mathbb{R}^N} f) \otimes u$ and $f \otimes (\triangle_{\mathbb{R}^\infty} u)$, we introduce $\otimes$-contraction ($\diamond_N$ and $\diamond_\infty$).

Definition 39 ($\diamond_N$, $\diamond_\infty$): Let $v \in CM(\mathbb{R}^\infty)$.

(1) Let $f \in L^2(\mathbb{R}^N)$. Then, there uniquely exists $w \in CM(\mathbb{R}^\infty)$ such that for any $u \in CM(\mathbb{R}^\infty)$,

$$\langle f \otimes u, v \rangle_{CM(\mathbb{R}^\infty)} = \langle u, w \rangle_{CM(\mathbb{R}^\infty)}$$

holds. Let $f \diamond_N v \in CM(\mathbb{R}^\infty)$ be defined as $f \diamond_N v := w$.

(2) Let $u \in CM(\mathbb{R}^\infty)$. Then, there uniquely exists $g \in L^2(\mathbb{R}^N)$ such that for any $f \in L^2(\mathbb{R}^N)$,

$$\langle f \otimes u, v\rangle_{CM(\mathbb{R}^\infty)} = \langle f, g\rangle_{L^2(\mathbb{R}^N)}$$

holds. Let $u \diamond_\infty v \in L^2(\mathbb{R}^N)$ be defined as $u \diamond_\infty v := g$.

Proof: Existence and uniqueness are easy (by Riesz Theorem). ■

Lemma 40: (1) Let $f \in L^2(\mathbb{R}^N)$ and $v \in CM(\mathbb{R}^\infty)$. Then, $\|f \diamond_N v\|_{CM(\mathbb{R}^\infty)} \leq \|f\|_{L^2(\mathbb{R}^N)}\|v\|_{CM(\mathbb{R}^\infty)}$ holds.

(2) Let $u, v \in CM(\mathbb{R}^\infty)$. Then, $\|u \diamond_\infty v\|_{L^2(\mathbb{R}^N)} \leq \|u\|_{CM(\mathbb{R}^\infty)}\|v\|_{CM(\mathbb{R}^\infty)}$ holds.

Proof: It is easy. ■

Proposition 41: (1) Let $k \in \mathbb{N}$, $v \in H^1_{N+k}(\mathbb{R}^\infty)$ and $f \in L^2(\mathbb{R}^N)$. Then, $f \diamond_N v \in H^1_k(\mathbb{R}^\infty)$ and

$$\frac{\partial(f \diamond_N v)}{\partial x_k} = f \diamond_N \frac{\partial v}{\partial x_{N+k}}$$

hold.

(2) Let $k \in \{1, 2, \cdots, N\}$, $v \in H^1_k(\mathbb{R}^\infty)$ and $u \in CM(\mathbb{R}^\infty)$. Let $\frac{\partial}{\partial x_{N,k}}$ be the (ordinary) generalized partial differential operator in $L^2(\mathbb{R}^N)$. Then, $u \diamond_\infty v$ is an element of the domain of $\frac{\partial}{\partial x_{N,k}}$ and

$$\frac{\partial(u \diamond_\infty v)}{\partial x_{N,k}} = u \diamond_\infty \frac{\partial v}{\partial x_k}$$

holds.

Proof: (1) From Lemma 36 and Theorem 16, because for any $u \in H^1_k(\mathbb{R}^\infty)$, $\langle u, f\diamond_N \frac{\partial v}{\partial x_{N+k}}\rangle_{CM(\mathbb{R}^\infty)} = \langle f\otimes u, \frac{\partial v}{\partial x_{N+k}}\rangle_{CM(\mathbb{R}^\infty)} = -\langle f\otimes \frac{\partial u}{\partial x_k}, v\rangle_{CM(\mathbb{R}^\infty)} = -\langle \frac{\partial u}{\partial x_k}, f \diamond_N v\rangle_{CM(\mathbb{R}^\infty)}$ holds, $f \diamond_N \frac{\partial v}{\partial x_{N+k}} = -(\frac{\partial}{\partial x_k})^*(f \diamond_N v) = \frac{\partial(f\diamond_N v)}{\partial x_k}$ holds.

(2) In the same way as (1), from Lemma 37 and Theorem 16, it follows. ■

Lemma 42: (1) Let $\varphi \in H^1(\mathbb{R}^\infty)$ and $f \in L^2(\mathbb{R}^N)$. Then, $f \diamond_N \varphi \in H^1(\mathbb{R}^\infty)$ holds.

(2) Let $\varphi \in H^1(\mathbb{R}^\infty)$ and $u \in CM(\mathbb{R}^\infty)$. Then, $u \diamond_\infty \varphi \in H^1(\mathbb{R}^N)$ holds.

(3) Let $f \in L^2(\mathbb{R}^N)$. Let u be an element of the domain of $\triangle_{\mathbb{R}^\infty}$. Then, $f\otimes(\triangle_{\mathbb{R}^\infty} u) \in L^2(\mathbb{R}^\infty)$, $f\otimes u \in \cap_{k\in\mathbb{N}} H^1_{N+k}(\mathbb{R}^\infty)$ and $\sum_{k\in\mathbb{N}} \|\frac{\partial(f\otimes u)}{\partial x_{N+k}}\|^2_{L^2_{N+k}(\mathbb{R}^\infty)} < +\infty$ hold. For any $\varphi \in H^1(\mathbb{R}^\infty)$,

$$-\langle f \otimes (\triangle_{\mathbb{R}^\infty} u), \varphi\rangle_{L^2(\mathbb{R}^\infty)} = \sum_{k\in\mathbb{N}} \langle \frac{\partial(f \otimes u)}{\partial x_{N+k}}, \frac{\partial \varphi}{\partial x_{N+k}}\rangle_{L^2_{N+k}(\mathbb{R}^\infty)}$$

holds.

(4) Let $u \in L^2(\mathbb{R}^\infty)$ and $f \in H^2(\mathbb{R}^N)$. Then, $(\triangle_{\mathbb{R}^N} f) \otimes u \in L^2(\mathbb{R}^\infty)$ and $f \otimes u \in \cap_{k=1}^N H_k^1(\mathbb{R}^\infty)$ hold. For any $\varphi \in H^1(\mathbb{R}^\infty)$,

$$-\langle (\triangle_{\mathbb{R}^N} f) \otimes u, \varphi \rangle_{L^2(\mathbb{R}^\infty)} = \sum_{k=1}^N \langle \frac{\partial (f \otimes u)}{\partial x_k}, \frac{\partial \varphi}{\partial x_k} \rangle_{L_k^2(\mathbb{R}^\infty)}$$

holds.

Proof: (1) From Proposition 41 (1) and Lemma 40 (1), it follows.

(2) From Proposition 41 (2), it follows.

(3) From Lemma 38 (2), $f \otimes (\triangle_{\mathbb{R}^\infty} u) \in L^2(\mathbb{R}^\infty)$ holds. From Theorem 24 (1), $u \in H^1(\mathbb{R}^\infty)$ holds. So, from Lemma 36, $f \otimes u \in \cap_{k \in \mathbb{N}} H_{N+k}^1(\mathbb{R}^\infty)$ and $\sum_{k \in \mathbb{N}} \| \frac{\partial (f \otimes u)}{\partial x_{N+k}} \|_{L_{N+k}^2(\mathbb{R}^\infty)}^2 < +\infty$ hold. From (1), Proposition 41 (1) and Lemma 36, $\langle f \otimes u, \varphi \rangle_{CM(\mathbb{R}^\infty)} - \langle f \otimes (\triangle_{\mathbb{R}^\infty} u), \varphi \rangle_{L^2(\mathbb{R}^\infty)} = \langle (1 - \triangle_{\mathbb{R}^\infty}) u, f \diamond_N \varphi \rangle_{L^2(\mathbb{R}^\infty)} = \langle u, f \diamond_N \varphi \rangle_{H^1(\mathbb{R}^\infty)} = \langle u, f \diamond_N \varphi \rangle_{CM(\mathbb{R}^\infty)} + \sum_{k \in \mathbb{N}} \langle \frac{\partial u}{\partial x_k}, f \diamond_N \frac{\partial \varphi}{\partial x_{N+k}} \rangle_{CM(\mathbb{R}^\infty)} = \langle f \otimes u, \varphi \rangle_{CM(\mathbb{R}^\infty)} + \sum_{k \in \mathbb{N}} \langle \frac{\partial (f \otimes u)}{\partial x_{N+k}}, \frac{\partial \varphi}{\partial x_{N+k}} \rangle_{L_{N+k}^2(\mathbb{R}^\infty)}$ holds.

(4) Similar to (3), from (2), Lemma 37, Lemma 38 (2) and Proposition 41 (2), it is shown. ■

Lemma 43: Suppose that $-\infty < T_0 < T_1 < +\infty$ holds. Let $f \in C^1([T_0, T_1]; L^2(\mathbb{R}^N))$ and $u \in C^1([T_0, T_1]; CM(\mathbb{R}^\infty))$. Then, $f \otimes u \in C^1([T_0, T_1]$ $; CM(\mathbb{R}^\infty))$ and

$$\frac{d}{dt}(f \otimes u) = (\frac{d}{dt} f) \otimes u + f \otimes (\frac{d}{dt} u)$$

hold.

Proof: From Lemma 32 and Lemma 33, it follows. ■

Theorem 44: Let $f_0 \in L^2(\mathbb{R}^N)$ and $u_0 \in L^2(\mathbb{R}^\infty)$. Then, the followings hold.

(1) Let $t \in (-\infty, +\infty)$. Then,

$$e^{\sqrt{-1} \triangle_{\mathbb{R}^\infty} t} (f_0 \otimes u_0) = (e^{\sqrt{-1} \triangle_{\mathbb{R}^N} t} f_0) \otimes (e^{\sqrt{-1} \triangle_{\mathbb{R}^\infty} t} u_0)$$

holds.

(2) Let $t \in [0, +\infty)$. Then,

$$e^{\triangle_{\mathbb{R}^\infty} t} (f_0 \otimes u_0) = (e^{\triangle_{\mathbb{R}^N} t} f_0) \otimes (e^{\triangle_{\mathbb{R}^\infty} t} u_0)$$

holds.

Proof: About (1), we show it. About (2), it can be similarly shown. Let $f(t) := e^{\sqrt{-1}\triangle_{\mathbb{R}^N} t} f_0$ and $u(t) := e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t} u_0$.

First, we show that if $f_0 \in H^2(\mathbb{R}^N)$ holds and u_0 is an element of the domain of $\triangle_{\mathbb{R}^\infty}$, then $e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t}(f_0 \otimes u_0) = f(t) \otimes u(t)$ holds. From Lemma 43 and Lemma 38 (2),

$$f \otimes u \in C^1((-\infty, +\infty); L^2(\mathbb{R}^\infty)),$$

$$\frac{d}{dt}(f \otimes u) = \sqrt{-1}((\triangle_{\mathbb{R}^N} f) \otimes u + f \otimes (\triangle_{\mathbb{R}^\infty} u))$$

hold. On the other hand, because from Lemma 38 (1), $f \otimes u \in H^1(\mathbb{R}^\infty)$ holds, from Lemma 42 (3) and Lemma 42 (4), for any $\varphi \in H^1(\mathbb{R}^\infty)$,

$$\langle f \otimes u, \varphi \rangle_{L^2(\mathbb{R}^\infty)} - \langle (\triangle_{\mathbb{R}^N} f) \otimes u + f \otimes (\triangle_{\mathbb{R}^\infty} u), \varphi \rangle_{L^2(\mathbb{R}^\infty)} = \langle f \otimes u, \varphi \rangle_{H^1(\mathbb{R}^\infty)}$$

holds. So, $f \otimes u - ((\triangle_{\mathbb{R}^N} f) \otimes u + f \otimes (\triangle_{\mathbb{R}^\infty} u)) = (1 - \triangle_{\mathbb{R}^\infty})(f \otimes u)$ holds.

$$(\triangle_{\mathbb{R}^N} f) \otimes u + f \otimes (\triangle_{\mathbb{R}^\infty} u) = \triangle_{\mathbb{R}^\infty}(f \otimes u)$$

holds. Therefore, if $f_0 \in H^2(\mathbb{R}^N)$ holds and u_0 is an element of the domain of $\triangle_{\mathbb{R}^\infty}$, then $e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t}(f_0 \otimes u_0) = f(t) \otimes u(t)$ holds.

There exist a sequence $\{g_{0,n}\}_n$ in $H^2(\mathbb{R}^N)$ such that $\lim_n \|g_{0,n} - f_0\|_{L^2(\mathbb{R}^N)} = 0$ holds. There exist a sequence $\{v_{0,n}\}_n$ in the domain of $\triangle_{\mathbb{R}^\infty}$ such that $\lim_n \|v_{0,n} - u_0\|_{L^2(\mathbb{R}^\infty)} = 0$ holds. Let $g_n(t) := e^{\sqrt{-1}\triangle_{\mathbb{R}^N} t} g_{0,n}$ and $v_n(t) := e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t} v_{0,n}$. Then, $\lim_n \|g_n(t) - f(t)\|_{L^2(\mathbb{R}^N)} = 0$ and $\lim_n \|v_n(t) - u(t)\|_{L^2(\mathbb{R}^\infty)} = 0$ hold. From Lemma 32, Lemma 33 and Lemma 38 (2),

$$\lim_n \|g_n(t) \otimes v_n(t) - f(t) \otimes u(t)\|_{L^2(\mathbb{R}^\infty)} = 0$$

holds. On the other hand, from $\lim_n \|g_{0,n} \otimes v_{0,n} - f_0 \otimes u_0\|_{L^2(\mathbb{R}^\infty)} = 0$,

$$\lim_n \|e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t}(g_{0,n} \otimes v_{0,n}) - e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t}(f_0 \otimes u_0)\|_{L^2(\mathbb{R}^\infty)} = 0$$

holds. So, because of $e^{\sqrt{-1}\triangle_{\mathbb{R}^\infty} t}(g_{0,n} \otimes v_{0,n}) = g_n(t) \otimes v_n(t)$, it follows. ■

7 Inseparability of $L^2(\mathbb{R}^\infty)$

Lemma 45: Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $[0,+\infty)$-valued measurable functions on $\mathbb{R}$. Suppose that for any $n\in\mathbb{N}$, $\|f_n\|_{L^2(\mathbb{R})}=1$ holds. Then, the followings hold.

(1) Let $m\in\mathbb{N}$. Then,

$$(\prod_{n\in\{m\}} f_n(x_n)^2 dx_n)(\prod_{n\in\mathbb{N}\setminus\{m\}} f_n(x_n)^2 dx_n)=\prod_{n\in\mathbb{N}} f_n(x_n)^2 dx_n$$

holds.

(2) Let $\{a_n\}_{n\in\mathbb{N}}\in\mathbb{R}^\infty$. Then,

$$\tau_{\{a_n\}_{n\in\mathbb{N}}}(\prod_{n\in\mathbb{N}} f_n(x_n)^2 dx_n)=\prod_{n\in\mathbb{N}} f_n(x_n-a_n)^2 dx_n$$

holds.

Proof: Although it is a natural result, we include the proof just in case.

Let $N\in\mathbb{N}$. Let $\{E_n\}_{n=1}^N$ be a family of measurable sets of $\mathbb{R}$.

(1) $((\prod_{n\in\{m\}} f_n(x_n)^2dx_n)(\prod_{n\in\mathbb{N}\setminus\{m\}} f_n(x_n)^2dx_n))((\prod_{n=1}^N E_n)\times(\prod_{n=N+1}^\infty \mathbb{R}))=\prod_{n=1}^N(\int_{E_n} f_n(x_n)^2dx_n)$ holds.

(2) For $n\in\{1,2,\cdots,N\}$, let $F_n:=\{x_n\in\mathbb{R}|x_n+a_n\in E_n\}$. Then, $(\tau_{\{a_n\}_{n\in\mathbb{N}}}(\prod_{n\in\mathbb{N}} f_n(x_n)^2dx_n))((\prod_{n=1}^N E_n)\times(\prod_{n=N+1}^\infty \mathbb{R}))=(\prod_{n\in\mathbb{N}} f_n(x_n)^2dx_n)((\prod_{n=1}^N F_n)\times(\prod_{n=N+1}^\infty \mathbb{R}))=\prod_{n=1}^N(\int_{F_n} f_n(x_n)^2dx_n)=\prod_{n=1}^N(\int_{E_n} f_n(x_n-a_n)^2dx_n)$ holds. ■

Lemma 46: Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be sequences of $[0,+\infty)$-valued measurable functions on $\mathbb{R}$. Suppose that for any $n\in\mathbb{N}$, $\|f_n\|_{L^2(\mathbb{R})}=\|g_n\|_{L^2(\mathbb{R})}=1$ holds. Suppose that there exists $m\in\mathbb{N}$ such that $\langle f_m,g_m\rangle_{L^2(\mathbb{R})}=0$ holds. Then, $\langle\prod_{n\in\mathbb{N}} f_n(x_n)^2dx_n,\prod_{n\in\mathbb{N}} g_n(x_n)^2dx_n\rangle_{CM(\mathbb{R}^\infty)}=0$ holds.

Proof: From Lemma 45 (1) and Proposition 28 (1), it follows. ■

Lemma 47: Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $[0,+\infty)$-valued measurable functions on $\mathbb{R}$. Suppose that for any $n\in\mathbb{N}$, $\|f_n\|_{L^2(\mathbb{R})}=1$ holds. Let $m\in\mathbb{N}$. Suppose that $f_m\in H^1(\mathbb{R})$ holds. Then, $\prod_{n\in\mathbb{N}} f_n(x_n)^2dx_n\in H^1_m(\mathbb{R}^\infty)$ and $\frac{\partial}{\partial x_m}(\prod_{n\in\mathbb{N}} f_n(x_n)^2dx_n)=(\prod_{n\in\{m\}} f_n'(x_n)|f_n'(x_n)|dx_n)\cdot(\prod_{n\in\mathbb{N}\setminus\{m\}} f_n(x_n)^2dx_n)$ hold.

Proof: There uniquely exists $e_m\in\mathbb{R}^\infty$ such that $e_{m,m}=1$ holds and for any $n\in\mathbb{N}\setminus\{m\}$, $e_{m,n}=0$ holds. From Lemma 45, for $h>0$, $\tau_{he_m}(\prod_{n\in\mathbb{N}} f_n(x_n)^2dx_n)=(\prod_{n\in\{m\}} f_n(x_n-h)^2dx_n)(\prod_{n\in\mathbb{N}\setminus\{m\}} f_n(x_n)^2dx_n)$ holds.

So, because from Proposition 28 (2), $\frac{\tau_{he_m}(\prod_{n\in\mathbb{N}} f_n(x_n)^2 dx_n) - \prod_{n\in\mathbb{N}} f_n(x_n)^2 dx_n}{h}$ + $(\prod_{n\in\{m\}} f_n'(x_n)|f_n'(x_n)|dx_n)\cdot(\prod_{n\in\mathbb{N}\setminus\{m\}} f_n(x_n)^2 dx_n) = (\prod_{n\in\{m\}}(\frac{f_n(x_n-h)-f_n(x_n)}{h}+ f_n'(x_n))|\frac{f_n(x_n-h)-f_n(x_n)}{h}+f_n'(x_n)|dx_n)\cdot(\prod_{n\in\mathbb{N}\setminus\{m\}} f_n(x_n)^2 dx_n)$ holds, from Proposition 28 (1), it follows. ■

Proposition 48: There exists an orthonormal system $\{u_\tau\}_{\tau\in\prod_{n\in\mathbb{N}}\{0,1\}}$ of $L^2(\mathbb{R}^\infty)$.

Proof: There exists $f \in C^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} |f|^2 dx = 1$, for any $x \in \mathbb{R}$, $f(x) \geq 0$ holds and for any $x \in \mathbb{R} \setminus (0,1)$, $f(x) = 0$ holds. There exists a sequence $\{L_n\}_{n\in\mathbb{N}}$ in $(0,+\infty)$ such that

$$\sum_{n\in\mathbb{N}} \frac{1}{L_n^4} < +\infty$$

holds. For $\tau \in \prod_{n\in\mathbb{N}}\{0,1\}$, let

$$u_\tau := \prod_{n\in\mathbb{N}} (\frac{1}{L_n} f(\frac{x_n}{L_n^2} - \tau(n)))^2 dx_n.$$

Then, from Lemma 46, $\{u_\tau\}_{\tau\in\prod_{n\in\mathbb{N}}\{0,1\}}$ is an orthonormal system of $CM(\mathbb{R}^\infty)$. On the other hand, because from Lemma 47 and Proposition 28 (1), for any $\tau \in \prod_{n\in\mathbb{N}}\{0,1\}$ and any $n \in \mathbb{N}$,

$$\|\frac{\partial u_\tau}{\partial x_n}\|^2_{CM(\mathbb{R}^\infty)} = \frac{1}{L_n^4}\|f'\|^2_{L^2(\mathbb{R})}$$

holds, for any $\tau \in \prod_{n\in\mathbb{N}}\{0,1\}$, $u_\tau \in H^1(\mathbb{R}^\infty)$ holds. So, $\{u_\tau\}_{\tau\in\prod_{n\in\mathbb{N}}\{0,1\}}$ is an orthonormal system of $L^2(\mathbb{R}^\infty)$. ■

8 Translation invariance of $\triangle_{\mathbb{R}^\infty}$

Lemma 49: Let $a \in \mathbb{R}^\infty$ and $k \in \mathbb{N}$. Then, the followings hold.

(1) Let $u \in H^1_k(\mathbb{R}^\infty)$. Then, $\tau_a u \in H^1_k(\mathbb{R}^\infty)$ and $\frac{\partial(\tau_a u)}{\partial x_k} = \tau_a(\frac{\partial u}{\partial x_k})$ hold.

(2) Let $u_1, u_2 \in H^1_k(\mathbb{R}^\infty)$. Then, $\langle \frac{\partial(\tau_a u_1)}{\partial x_k}, \frac{\partial(\tau_a u_2)}{\partial x_k} \rangle_{CM(\mathbb{R}^\infty)} = \langle \frac{\partial u_1}{\partial x_k}, \frac{\partial u_2}{\partial x_k} \rangle_{CM(\mathbb{R}^\infty)}$ holds.

Proof: (1) From $\tau_b \tau_c = \tau_{b+c}$ and Proposition 13, it follows.

(2) From (1) and Proposition 13, it follows. ■

Lemma 50: Let $a \in \mathbb{R}^\infty$. Then, the followings hold.

(1) Let $u \in H^1(\mathbb{R}^\infty)$. Then, $\tau_a u \in H^1(\mathbb{R}^\infty)$ holds.

(2) Let $u_1, u_2 \in H^1(\mathbb{R}^\infty)$. Then, $\langle \tau_a u_1, \tau_a u_2 \rangle_{H^1(\mathbb{R}^\infty)} = \langle u_1, u_2 \rangle_{H^1(\mathbb{R}^\infty)}$ holds.

Proof: (1) From Lemma 49 (1) and Proposition 13, it follows.

(2) From Lemma 49 (2) and Proposition 13, it follows. ■

Lemma 51: Let $a \in \mathbb{R}^\infty$ and $f \in L^2(\mathbb{R}^\infty)$. Then, $\tau_a f \in L^2(\mathbb{R}^\infty)$ holds.

Proof: From Lemma 50 (1) and Proposition 13, it follows. ■

Proposition 52: Let $a \in \mathbb{R}^\infty$. Let u be an element of the domain of $\triangle_{\mathbb{R}^\infty}$. Then, $\tau_a u$ is an element of the domain of $\triangle_{\mathbb{R}^\infty}$ and $\triangle_{\mathbb{R}^\infty}(\tau_a u) = \tau_a(\triangle_{\mathbb{R}^\infty} u)$ holds.

Proof: From Lemma 50, Lemma 51 and Proposition 13, for any $\varphi \in H^1(\mathbb{R}^\infty)$, $\langle \tau_a u - \tau_a(\triangle_{\mathbb{R}^\infty} u), \varphi \rangle_{L^2(\mathbb{R}^\infty)} = \langle (1 - \triangle_{\mathbb{R}^\infty}) u, \tau_{-a}\varphi \rangle_{L^2(\mathbb{R}^\infty)} = \langle u, \tau_{-a}\varphi \rangle_{H^1(\mathbb{R}^\infty)} = \langle \tau_a u, \varphi \rangle_{H^1(\mathbb{R}^\infty)}$ holds. So, $\tau_a u = (1 - \triangle_{\mathbb{R}^\infty})^{-1}(\tau_a u - \tau_a(\triangle_{\mathbb{R}^\infty} u))$ holds. ■

Remark: (1) Gross ([1]) considered Laplace operator on infinite-dimensional real Hilbert space. Gross Laplacian is translation invariant. However, Gross Laplacian is not a self-adjoint operator.

(2) For $p \in [1, +\infty)$, let L_p be Ornstein-Uhlenbeck operator in $L^p(\mu)$ with respect to Wiener measure μ. L_p is a generator of a contraction semigroup and one of the main characters in Malliavin calculus (e.g., [4]). L_2 is a self-adjoint operator corresponding to infinite-dimensional Dirichlet form (e.g., [3]). However, L_p is not translation invariant. □

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