# Local version of Vizing's theorem for multi-graphs 

Clinton T. Conley<br>Department of Mathematical Sciences<br>Carnegie Mellon University<br>Pittsburgh, PA 15213, USA<br>Email: clintonc@andrew.cmu.edu<br>Jan Grebík<br>UCLA Mathematics, Los Angeles, CA 90095, USA, and<br>Faculty of Informatics, Masaryk University, 60200 Brno, Czech Republic<br>Email: grebikj@math.ucla.edu<br>Oleg Pikhurko<br>Mathematics Institute and DIMAP<br>University of Warwick<br>Coventry CV4 7AL, UK<br>Email: o.pikhurko@warwick.ac.uk

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#### Abstract

Extending a result of Christiansen, we prove that every mutli-graph $G=(V, E)$ admits a proper edge colouring $\phi: E \rightarrow\{1,2, \ldots\}$ which is local, that is, $\phi(e) \leqslant \max \{d(x)+$ $\pi(x), d(y)+\pi(y)\}$ for every edge $e$ with end-points $x, y \in V$, where $d(z)$ (resp. $\pi(z))$ denotes the degree of a vertex $z$ (resp. the maximum edge multiplicity at $z$ ). This is derived from a local version of the Fan Equation.


## 1 Introduction

Edge-colouring is an important and active area of graph theory; for an overview see, for example, the book by Stiebitz, Scheide, Toft and Favrholdt [8]. One of the key results here is the remarkable theorem of Vizing [9, proved independently by Gupta [6], that every multi-graph $G$ admits a proper edge colouring with at most $\Delta(G)+\pi(G)$ colours, where $\Delta(G)$ and $\pi(G)$ denote respectively the maximum degree and the maximum edge multiplicity of $G$, and an edge colouring is called proper if no two different edges sharing a vertex get the same colour.

Some local versions of edge colouring problems (where a possible colour of an edge $x y$ depends on local information such as the degrees of $x$ and $y$ rather than the global maximum degree) were introduced already in the influential paper of Erdős, Rubin and Taylor [5. One of the strongest
asymptotic results here is by Bonamy, Delcourt, Lang and Postle [1] giving the following local list version of Vizing's theorem: for every $\varepsilon>0$ if the maximum degree $\Delta(G)$ of a graph $G$ is sufficiently large, the minimum degree $\delta(G)$ is at least $\ln ^{25} \Delta(G)$ and we have an assignment of colour lists to edges $e \mapsto L(e)$ so that each edge $\{x, y\}$ gets at least $(1+\varepsilon) \max \{d(x), d(y)\}$ colours then there is a proper edge colouring $\phi$ of $G$ with $\phi(e) \in L(e)$ for each edge $e$, where $d(x)$ denotes the degree of a vertex $x$.

Resolving a conjecture posed in [1], Christiansen [3, 4] proved that every simple graph $G$ admits a proper edge colouring into $\mathbb{N}:=\{1,2,3, \ldots\}$ which is local, meaning that the colour of any edge $\{x, y\}$ is at $\operatorname{most} \max \{d(x), d(y)\}+1$. In the special case when the graph $G$ is biparite, the stronger conclusion that the colour of $\{x, y\}$ is at $\operatorname{most} \max \{d(x), d(y)\}$ (which may be called the local König Theorem) follows from the more general results of Borodin, Kostochka and Woodall [2].
The purpose of this note is to observe that the proof of Christiansen extends to multi-graphs. The following definition seems to be the "right" one in this context: let us call an $\mathbb{N}$-valued edge colouring $\phi$ of a multi-graph local if for every edge $e$ with endpoints $x$ and $y$ it holds that

$$
\begin{equation*}
\phi(e) \leqslant \max \{d(x)+\pi(x), d(y)+\pi(y)\}, \tag{1}
\end{equation*}
$$

where $\pi(z)$ denotes the maximum edge multiplicity at a vertex $z$. Here is a local version of Vizing's theorem for multi-graphs.

Theorem 1 Every multi-graph admits a proper local edge colouring.
The standard example (a 3 -vertex multi-graph where every pair has the same multiplicity) shows that one cannot decrease the right-hand size of (11) in general. Note that, even without the extra locality restriction, we do not know the minimum number of colours as a function of $\Delta$ and $\pi$ that suffices for proper edge colouring of every multi-graph with maximum degree $\Delta$ and maximum edge multiplicity $\pi$; see Scheide and Stiebitz [7 for our current knowledge on this question.

The original Vizing's theorem as well as some other edge colouring results can be derived from the so-called Fan Equation of Vizing [10] (which is discussed in detail in [8, Section 2]). Here we present a local version of the Fan Equation (which, in this more general setting, becomes an inequality, see Theorem(4) and derive Theorem 1 from it. As another consequence of Theorem (4, we have the following result.

Theorem 2 Let $G=(V, E)$ be a simple graph and $k \in \mathbb{N}$. If the maximum degree of $G$ is at most $k$ and the set of vertices of degree exactly $k$ spans no cycle in $G$, then there is a proper local colouring $E \rightarrow\{1, \ldots, k\}$.

## 2 Notation and preliminaries

By $\mathbb{N}:=\{1,2, \ldots\}$ we denote the set of positive integers. For integers $k \geqslant \ell \geqslant 0$, we denote $[\ell, k]:=\{\ell, \ldots, k\}$ and $[k]:=\{1, \ldots, k\}$.

Let $G=(V, E, 1)$ be a multi-graph, that is, $V$ and $E$ are the sets of vertices and edges respectively, and 1 is a (not necessarily injective) function from $E$ to $\binom{V}{2}$, the set of unordered pairs of vertices. The (edge) multiplicity $\pi(x, y):=\left|1^{-1}(\{x, y\})\right|$ of a pair $\{x, y\} \in\binom{V}{2}$ is the number of edges whose end-points are $x$ and $y$. For $x, y \in V$ and $e \in E$, we write $x \in e$ (resp. $e=x y$ ) to mean that $x \in 1(e)$ (resp. $1(e)=\{x, y\})$.
The degree of $x \in V$ is $d(x):=\sum_{y \in V \backslash\{x\}} \pi(x, y)$ which is the number of edges incident to $x$. Also, let $\pi(x):=\max _{y \in V \backslash\{x\}} \pi(x, y)$ denote the maximum multiplicity at $x$.
As only edges will be coloured, we will usually say just "colouring" instead of "edge colouring". A partial colouring of $G$ is a function $\phi: \operatorname{dom}(\phi) \rightarrow \mathbb{N}$ where the domain $\operatorname{dom}(\phi)$ of $\phi$ is a subset of $E$. We may also write this as $\phi: E \rightharpoonup \mathbb{N}$. If $\operatorname{dom}(\phi)=E$ (that is, every edge is coloured) then we call $\phi$ a colouring (and write $\phi: E \rightarrow \mathbb{N}$ ). A (partial) colouring $\phi$ is proper if no two distinct edges sharing at least one vertex get the same colour. It is called local if

$$
\begin{equation*}
\phi(e) \leqslant \max \{d(x)+\pi(x): x \in e\}, \quad \text { for every } e \in \operatorname{dom}(\phi), \tag{2}
\end{equation*}
$$

that is, for every coloured edge $e$ there is $x \in e$ with $\phi(e) \leqslant d(x)+\pi(x)$.
Given a partial colouring $\phi$, a $\phi$-chain is a sequence of distinct edges $C=\left(e_{1}, \ldots, e_{p}\right)$ such that $\left\{e_{1}, \ldots, e_{p}\right\} \cap \operatorname{dom}(\phi)=\left\{e_{2}, \ldots, e_{p}\right\}$ (i.e. $e_{1}$ is the only uncoloured edge) and, for every $i \in[p-1]$, the edges $e_{i}$ and $e_{i+1}$ share exactly one vertex. The $C$-shift (or shift along $C$ ) of $\phi$ is the partial colouring $\phi^{\prime}$ with $\operatorname{dom}\left(\phi^{\prime}\right)=\left(\operatorname{dom}(\phi) \backslash\left\{e_{p}\right\}\right) \cup\left\{e_{1}\right\}$ which coincides with $\phi$, except $\phi^{\prime}\left(e_{i}\right):=\phi\left(e_{i+1}\right)$ for $i \in[p-1]$ (while $e_{p}$ is uncoloured under $\phi^{\prime}$ ). Informally speaking, we shift colours one step down along $C$. A $\phi$-chain $\left(e, e^{\prime}\right)$ with $e=x y$ and $e^{\prime}=x z$ (thus, $e$ is uncoloured, $e^{\prime}$ is coloured and $y \neq z$ ) is called $\phi$-safe if $\phi\left(e^{\prime}\right) \leqslant d(y)+\pi(y)$ and no edge incident to $y$ has colour $\phi\left(e^{\prime}\right)$. A $\phi$-chain $C=\left(e_{1}, \ldots, e_{m}\right)$ with $m \geqslant 3$ is $\phi$-safe if, for every $i \in[m-1]$, the $\phi_{i}$-chain $\left(e_{i}, e_{i+1}\right)$ is $\phi_{i}$-safe, where $\phi_{i}$ denotes the $\left(e_{1}, \ldots, e_{i}\right)$-shift of $\phi$. In other words, $C$ is $\phi$-safe if each individual 1-edge step of the $C$-shift is safe with respect to the current colouring. Note that the sequence $C=\left(e_{1}\right)$ made of a single edge $e_{1}$ uncoloured under $\phi$ is a $\phi$-chain (and its shift keeps the partial colouring $\phi$ unchanged); also, let us agree that every such single-edge chain is $\phi$-safe. When the colouring $\phi$ is understood, we may say just "chain" instead of " $\phi$-chain", etc.

For a vertex $x \in V$ and distinct colours $\alpha, \beta \in \mathbb{N}$ with $\beta$ not present at $x$, let $P_{x}(\alpha, \beta, \phi)$ denote the sequence of edges on the maximal $(\alpha, \beta)$-bichromatic path that starts at the vertex $x$. Thus the values of $\phi$ on $P_{x}(\alpha, \beta, \phi)$ alternate between $\alpha$ and $\beta$, starting with $\alpha$; if the colour $\alpha$ is not present at $x$, then $P_{x}(\alpha, \beta, \phi)$ is the empty sequence. Note that we exclude the case that $P_{x}(\alpha, \beta, \phi)$ is a double edge by requiring that $\beta$ is missing at $x$ in this definition.

In some proofs, we will be given an integer $k$ and the set of possible colours will be restricted to $[k]$. Then we will use the following definitions that implicitly depend on $k$. Given $\phi: E \rightharpoonup[k]$, we let for a vertex $x \in V$

$$
\bar{\phi}(x):=[k] \backslash\{\phi(e): e \ni x\},
$$

to be the set of colours missing at $x$, and

$$
\widetilde{\phi}(x):=\bar{\phi}(x) \cap[d(x)+\pi(x)],
$$

to be the set of colours which are safe at $x$. Thus a colour is safe at $x$ if it is missing at $x$ and is at most $d(x)+\pi(x)$. Also, the potential (function) of $\phi: E \rightharpoonup[k]$ is

$$
\Phi(\phi):=\sum_{x \in V}|\widetilde{\phi}(x)| .
$$

Lemma 3 Let $G=(V, E, 1)$ be a multigraph, $k \in \mathbb{N}$ be a positive integer, $\phi: E \rightarrow[k]$ be a proper local partial colouring, and $\phi^{\prime}$ be the shift of $\phi$ along a $\phi$-safe chain $\left(e, e^{\prime}\right)$, with $e=x y$ and $e^{\prime}=x z$. Then $\phi^{\prime}$ is a proper local partial colouring and $\Phi\left(\phi^{\prime}\right) \leqslant \Phi(\phi)$. Moreover, we have equality if and only if $\phi\left(e^{\prime}\right) \leqslant d(z)+\pi(z)$.

Proof. The edge $e=x y$ gets coloured during the shift. The new colouring $\phi^{\prime}$ is proper because the set of colours at $x$ does not change while the new colour on $e=x y$ was missing at $y$. Also, the locality condition for $e$ holds from the $y$-side, that is, $\phi^{\prime}(e)=\phi\left(e^{\prime}\right) \leqslant d(y)+\pi(y)$.

Consider the difference $\Phi\left(\phi^{\prime}\right)-\Phi(\phi)$. Since the contributions of a vertex that sees the same sets of colours on incident edges in $\phi$ and $\phi^{\prime}$ cancel each other, we have to look at $y$ and $z$ only. The contribution of $y$ to $\Phi\left(\phi^{\prime}\right)-\Phi(\phi)$ is exactly -1 , because the shift reduces the number of safe colours at $y$ by 1 (namely, $\phi\left(e^{\prime}\right)$ is now gone from this list). The contribution of $z$ to $\Phi\left(\phi^{\prime}\right)-\Phi(\phi)$ is at most 1 , and it is equal to 1 if and only if the colour moved out from $z$ is in $[d(z)+\pi(z)]$, that is, if and only if $\phi\left(e^{\prime}\right) \leqslant d(z)+\pi(z)$. This finishes the proof of the lemma.

## 3 Multi-fans

Let $k \in \mathbb{N}$ and a multigraph $G=(V, E, 1)$ be given. Let $\phi: E \rightharpoonup[k]$ be any proper local partial colouring. Let $e \in E$ be an uncoloured edge and let $x \in e$.

A multi-fan at $x$ with respect to $e$ and $\phi$ (and $k$ ) is a sequence $F=\left(e_{1}, y_{1}, \ldots, e_{p}, y_{p}\right)$ with $p \geqslant 1$ consisting of edges $e_{1}, \ldots, e_{p}$ and vertices $y_{1}, \ldots, y_{p}$ satisfying the following conditions.
(F1) The edges $e_{1}, \ldots, e_{p}$ are distinct, $e_{1}=e$, and $e_{i}=x y_{i}$ for $i \in[p]$.
(F2) For every $i \in[2, p]$ there is $j \in[i-1]$ such that $\phi\left(e_{i}\right) \in \widetilde{\phi}\left(y_{j}\right)$; in particular, $e_{i}$ is coloured by $\phi$.

Also, we denote $V(F):=\left\{y_{1}, \ldots, y_{p}\right\}$. Note that $V(F)$ does not include $x$.
Usually, the Fan Equation is stated for an edge critical multi-graph (when a desired colouring of the whole edge set exists when we remove any edge). Having in mind possible algorithmic and descriptive set theory applications, we state a version where the multi-graph $G$ we want to colour need not be edge critical and the presented result can be used to "improve" the current partial colouring using chains of special kind. With this in mind, we make the following definitions.

A chain $C=\left(e_{1}, \ldots, e_{p}\right)$ is improving if it is $\phi$-safe and the $C$-shift $\phi^{\prime}$ of $\phi$ has strictly smaller potential, or there is a way to extend $\phi^{\prime}$ to $e_{p}$ keeping $\phi^{\prime}$ proper and local (in particular, strictly
decreasing the potential). Note that the new proper local colouring $\phi^{\prime}$ satisfies $\Phi\left(\phi^{\prime}\right)<\Phi(\phi)$ in either case. We say that a chain $C$ is Vizing if it starts with some edges $f_{1}, \ldots, f_{q}, q \geqslant 1$, all containing the same vertex $z$ and then continues with a (possibly empty) initial segment of a bichromatic path starting at $y$, where $y \neq z$ is the other endpoint of $f_{q}$. (Some further restrictions can be put on possible chains arising from the proof of Theorem 4 but we would like to keep this definition simple and short.) Also, for $z \in V$ let $d_{\phi}(z):=|\{e \in \operatorname{dom}(\phi): e \ni z\}|$ denote the number of edges at $z$ that are assigned a colour by $\phi$.

Theorem 4 (Local Fan Inequality) Let $k \geqslant 2$, let $G=(V, E, 1)$ be a multi-graph with $\Delta(G) \leqslant k$, and let $\phi: E \rightharpoonup[k]$ be a partial proper local colouring that admits no improving Vizing chain. Suppose that $e \in E$ is uncoloured. Let $x \in e$ and let $F=\left(e_{1}, y_{1}, \ldots, e_{p}, y_{p}\right)$ be a multi-fan at $x$ with respect to $e$ and $\phi$. Then all of the following properties hold.
(a) For every $i \in[p]$ there are $m \geqslant 0$ and a sequence $i_{0}>i_{1}>\cdots>i_{m}$ such that $i_{0}=i$, $i_{m}=1$ and $\left(e_{i_{m}}, e_{i_{m-1}}, \ldots, e_{i_{0}}\right)$ is a $\phi$-safe chain.
(b) For every $i \in[p]$, we have

$$
\begin{equation*}
\bar{\phi}(x) \cap \bar{\phi}\left(y_{i}\right) \cap\left[\max \left\{d(x)+\pi(x), d\left(y_{i}\right)+\pi\left(y_{i}\right)\right\}\right]=\emptyset . \tag{3}
\end{equation*}
$$

(c) For all choices of $i \in[p], \alpha \in \widetilde{\phi}(x)$, and $\beta \in \widetilde{\phi}\left(y_{i}\right)$, the $(\alpha, \beta)$-bichromatic path $P_{y_{i}}(\alpha, \beta, \phi)$ from $y_{i}$ ends in $x$.
(d) For every $i, j \in[p]$, if $y_{i} \neq y_{j}$ then

$$
\begin{equation*}
\widetilde{\phi}\left(y_{i}\right) \cap \widetilde{\phi}\left(y_{j}\right)=\emptyset . \tag{4}
\end{equation*}
$$

(e) If the multi-fan $F$ is maximal then $\left|\left\{y_{1}, \ldots, y_{p}\right\}\right| \geqslant 2$ and

$$
\begin{equation*}
\sum_{z \in V(F)}\left(\pi_{F}(x, z)-\min \{k, d(z)+\pi(z)\}+d_{\phi}(z)\right) \geqslant 1 . \tag{5}
\end{equation*}
$$

(Here $\pi_{F}$ denotes the edge multiplicity in the multi-graph $F$, while $\pi=\pi_{G}$ is taken with respect to the whole multi-graph $G$.)

Proof. To prove Part (a) for any given $i \in[p]$, we construct a required sequence as follows. Initially, let $i_{0}:=i$ and $m:=0$. If the current $i_{m}$ is equal to 1 then stop. Otherwise let $i_{m+1} \in\left[i_{m}-1\right]$ be any index satisfying (F2) for $i_{m}$ (that is, $\left.\phi\left(e_{i_{m}}\right) \in \widetilde{\phi}\left(e_{i_{m+1}}\right)\right)$, increase $m$ by 1 , and repeat. Let $i_{0}>i_{1}>\cdots>i_{m}$ be the final sequence (with $i_{0}=i, i_{m}=1$, and $m \geqslant 0$ ). Clearly, $\left(e_{i_{m}}, \ldots, e_{i_{0}}\right)$ is a chain.
Although this is intuitively obvious, let us formally check that the chain $\left(e_{i_{m}}, \ldots, e_{i_{0}}\right)$ is $\phi$-safe. This vacuously holds if $m=0$, so assume that $m \geqslant 1$. Let $\phi_{1}:=\phi$ and, inductively for $j \in[m]$, let $\phi_{j+1}$ be the ( $e_{i_{m-j+1}}, e_{i_{m-j}}$ )-shift of $\phi_{j}$; equivalently, $\phi_{j+1}$ is obtained from $\phi$ by shifting along $\left(e_{i_{m}}, \ldots, e_{i_{m-j}}\right)$. We show by induction on $j \in[m]$ that the $\phi_{j}$-chain $\left(e_{i_{m-j+1}}, e_{i_{m-j}}\right)$ is $\phi_{j}$-safe and the partial colouring $\phi_{j+1}$ is proper and local. So take any $j \in[m]$. We know that $\phi_{j}$ is
proper and local (by induction if $j \geqslant 2$ and by $\phi_{1}=\phi$ if $j=1$ ). Note that $\phi_{j}\left(e_{i_{m-j}}\right)=\phi\left(e_{i_{m-j}}\right)$ (since, by (F1), the edge $e_{i_{m-j}}$ is distinct from any edge whose colour changes when we construct $\phi_{j}$ from $\phi_{1}=\phi$ ). By the choice of $i_{m-j+1}$ (that is, by (F2)], $\phi\left(e_{i_{m-j}}\right)$ is in $\widetilde{\phi}\left(y_{i_{m-j+1}}\right)$. Since all colours at $x$ are pairwise distinct, the colour $\phi\left(e_{i_{m-j}}\right)$ cannot appear at $y_{i_{m-j+1}}$ when we pass from $\phi_{1}=\phi$ to $\phi_{j}$. Thus $\phi_{j}\left(e_{i_{m-j}}\right)=\phi\left(e_{i_{m-j}}\right)$ is also in $\widetilde{\phi}_{j}\left(y_{i_{m-j+1}}\right)$ and the $\phi_{j}$-chain $\left(e_{i_{m-j+1}}, e_{i_{m-j}}\right)$ is $\phi_{j}$-safe. It follows that the partial colouring $\phi_{j+1}$ is proper and local by Lemma 3, as required.
Suppose that Part (b) is false. Let $i \in[p]$ be the smallest index violating it. Let $\alpha$ be a colour present in the left-hand side of (3)) Let $\left(i_{0}, \ldots, i_{m}\right)$ be the sequence returned by Part (a) for this index $i$. Let $\phi^{\prime}$ be the shift of $\phi$ along $\left(e_{i_{m}}, e_{i_{m-1}}, \ldots, e_{i_{0}}\right)$. By Lemma 3 the new colouring $\phi^{\prime}$ is proper and local (and satisfies $\Phi\left(\phi^{\prime}\right) \leqslant \Phi(\phi)$ ). Under $\phi^{\prime}$, the edge $e_{i}=x y_{i}$ is uncoloured; also, the colour $\alpha$ is missing at both $x$ and $y_{i}$. Indeed, $\alpha$ was missing at both $x$ and $y_{i}$ before the shift (i.e. under $\phi$ ) while the shift, that affects only edges at $x$, does not move colour $\alpha$ at all.

Obtain $\phi^{\prime \prime}$ from $\phi^{\prime}$ by colouring $e_{i}$ with the colour $\alpha$. By above, the new colouring $\phi^{\prime \prime}$ is still proper. By definition, $\alpha$ is at $\operatorname{most} \max \left\{d(x)+\pi(x), d\left(y_{i}\right)+\pi\left(y_{i}\right)\right\}$, so $\phi^{\prime \prime}$ is also local. Thus $\left(e_{i_{m}}, \ldots, e_{i_{0}}\right)$ is an improving Vizing chain for $\phi$. This contradiction proves Part (b),
Let us turn to Part (c). Suppose that the claim is false for some vertex $y_{i}$ and colours $\alpha$ and $\beta$. Choose the smallest possible such index $i \in[p]$. Recall that $\beta$ is missing at $y_{i}, P:=P_{y_{i}}(\alpha, \beta, \phi)$ is the $(\alpha, \beta)$-bichromatic path starting at $y_{i}$, and we assumed on the contrary that the final endpoint of $P$ is a vertex $x^{\prime} \neq x$. The vertex set $V(P)$ of $P$ does not contain $x$, as otherwise $x$ would be an endpoint (since $\alpha$ is missing at $x$ ). It follows that none of the edges incident to $x$ (in particular, none of $e_{1}, \ldots, e_{p}$ ) can belong to $P$. Let $i_{0}=i>\cdots>i_{m}=1$ be the sequence returned by Part (a) for this $i$ and let $\phi^{\prime}$ be the shift of $\phi$ along the chain $\left(e_{i_{m}}, \ldots, e_{i_{0}}\right)$. By Lemma 3, the partial colouring $\phi^{\prime}$ is proper and local, and satisfies $\Phi\left(\phi^{\prime}\right) \leqslant \Phi(\phi)$. Note that the edge $e_{i}$ is not coloured by $\phi^{\prime}$.

Let the path $P$ traverse edges $\left(f_{1}, \ldots, f_{\ell}\right)$ and vertices $\left(u_{1}, u_{2}, \ldots, u_{\ell+1}\right)$ in this order. Thus $u_{1}=y_{i}$ and $f_{s}=u_{s} u_{s+1}$ for every $s \in[\ell]$. By Part (b), the colour $\alpha$ is present at $y_{i}$ under $\phi$ and also under $\phi^{\prime}$ (since the other endpoint of the colour- $\alpha$ edge at $y$ cannot be $x$ ). Thus $\ell \geqslant 1$. Let $f_{0}:=e_{i}$ denote the edge coming before the path $P$ in our shifting procedure. Let $P^{\prime}:=\left(f_{0}, \ldots, f_{\ell}\right)$. It is a path (since $\left.V(P) \not \supset x\right)$ and a $\phi^{\prime}$-chain (since $f_{0} \notin \operatorname{dom}\left(\phi^{\prime}\right)$ ). Also, let $u_{0}:=x$ (so that $\left.f_{0}=u_{0} u_{1}\right)$ and let $C$ be the concatenation of $\left(e_{i_{m}}, \ldots, e_{i_{0}}\right)$ and $P$. Clearly, $C$ is a Vizing chain.

Observe that if we were interested in only proper edge-colourings (that is, not requiring that the extra locality property in (21) holds), then we could have taken the $P^{\prime}$-shift of $\phi^{\prime}$ and assign one of $\alpha$ or $\beta$ to the (now uncoloured) edge $f_{\ell}$, thus finding a colouring with strictly larger domain than $\phi$. Instead, we proceed as follows.
Starting with $\phi^{\prime}$, we iteratively apply the $\left(f_{s}, f_{s+1}\right)$-shifts for $s=0,1, \ldots, \ell-1$ as long as each of them is safe. Lemma 3 implies that the potential $\Phi$ cannot increase at any step. Thus $\Phi$ stays constant by our assumptions on $\phi$.
Suppose first that we cannot perform the above shift for some $s \leqslant \ell-1$, that is, before doing the
whole chain $P^{\prime}$. Denote the obtained colouring $\phi^{\prime \prime}$. Thus $\phi^{\prime \prime}$ is the $\left(f_{0}, \ldots, f_{s}\right)$-shift of $\phi^{\prime}$, where $s$ is the maximum index (assumed to be at most $\ell-1)$ such that $\left(f_{0}, \ldots, f_{s}\right)$ is a $\phi^{\prime}$-safe chain. Note that $s \geqslant 1$ since the first shift (along $\left.\left(e_{i}, f_{1}\right)\right)$ moves the colour $\alpha$ to $e_{i}=x y_{i}$ and cannot violate the locality condition at $x$ by $\alpha \leqslant d(x)+\pi(x)$. Since the ( $\left.f_{s}, f_{s+1}\right)$-shift is not safe in the current colouring $\phi^{\prime \prime}$ but results in a proper colouring (since we shift along a bichromatic path), we have that

$$
\begin{equation*}
\gamma:=\phi^{\prime \prime}\left(f_{s+1}\right)>d\left(u_{s}\right)+\pi\left(u_{s}\right) . \tag{6}
\end{equation*}
$$

Suppose first that $s \geqslant 2$. Consider the $\left(f_{s-2}, f_{s-1}\right)$-shift (that is, the penultimate one before we obtained $\left.\phi^{\prime \prime}\right)$. This shift moves the colour from $f_{s-1}=u_{s-1} u_{s}$ to $f_{s-2}=u_{s-2} u_{s-1}$, with the moved colour being $\gamma$. Thus by (6) and the last claim of Lemma 3, the $\left(f_{s-2}, f_{s-1}\right)$-shift strictly decreased the value of the potential, while the chain $C$ truncated at $f_{s-1}$ is safe and thus improving, a contradiction. So, suppose that $s=1$. Here when we shift the colouring $\phi^{\prime}$ along the path $P^{\prime}$, we perform the first safe shift $\left(f_{0}, f_{1}\right)$ but we cannot proceed with the second shift, namely along $\left(f_{1}, f_{2}\right)$. However, then we have that $u_{s}=y_{i}$ and $\gamma=\beta$ in (6), contradicting the assumption of Part (c) that $\beta$ lies in $\widetilde{\phi}\left(y_{i}\right) \subseteq\left[d\left(y_{i}\right)+\pi\left(y_{i}\right)\right]$.
Thus, we can assume that the whole chain $P^{\prime}$ is $\phi^{\prime}$-safe (and thus $C$ is $\phi$-safe). We shift $\phi^{\prime}$ all way along it to obtain $\phi^{\prime \prime}$; equivalently, $\phi^{\prime \prime}$ is the $C$-shift of $\phi$. The edge $f_{\ell}$ is uncoloured by $\phi^{\prime \prime}$. Let $\gamma$ be the element of $\{\alpha, \beta\}$ different from $\phi^{\prime \prime}\left(f_{\ell-1}\right)=\phi^{\prime}\left(f_{\ell}\right)$. Recall that $f_{\ell}$ is the last edge of the maximal $\alpha / \beta$-bichromatic path $P$ with respect to $\phi$ that starts with $y_{i}$. Clearly, the colour $\gamma$ is missing at the penultimate vertex $u_{\ell}$ under $\phi^{\prime \prime}$.
Let us show that $\gamma$ cannot occur at $u_{\ell+1}$ under $\phi^{\prime \prime}$. Suppose otherwise. As $\gamma$ is not present at $u_{\ell+1}$ under $\phi$ (by the maximality of $P$ ) but is present at $u_{\ell+1}$ under $\phi^{\prime}$, it must be the case that $\gamma=\beta$ and $u_{\ell+1}=y_{i_{j}}$ for some $j \in[m]$ (and the colour $\beta$ appeared at $y_{i_{j}}$ because it was shifted to it from $y_{i_{j-1}}$ during the ( $e_{i_{m}}, \ldots, e_{i_{0}}$ )-shift of $\phi$ ). We conclude that $i_{j}$ is strictly less than $i=i_{0}$ and the reversal of the path $P$ is the maximal $\alpha / \beta$-bichromatic path starting at $y_{i_{j}}$ under $\phi$. This means that $P_{y_{i_{j}}}(\alpha, \beta, \phi)$ ends at $y_{i} \neq x$ and the index $i_{j}$ contradicts the minimality of $i$. This contradiction proves that $\gamma$ is not present at $u_{\ell+1}$ under $\phi^{\prime \prime}$, as claimed.

We conclude that, under $\phi^{\prime \prime}$, the colour $\gamma$ is missing at both $u_{\ell}$ and $u_{\ell+1}$. Next, one can show similarly to above that $\gamma \leqslant d\left(u_{\ell}\right)+\pi\left(u_{\ell}\right)$ : otherwise, for $\ell \geqslant 2$, the shift for $s=\ell-2$ would strictly decrease $\Phi$ while, for $\ell=1$, this would contradict the choice of $\beta$ by $\gamma=\beta$ and $u_{\ell}=y_{i}$. Thus if we colour $f_{\ell}$ by $\gamma$ then we obtain a proper local colouring. Hence, $C$ is an improving Vizing chain. This contradiction proves Part (c),
Let us turn to Part (d), Suppose on the contrary there are $i, j \in[p]$ such that $y_{i} \neq y_{j}$ and there is a colour $\beta \in \widetilde{\phi}\left(y_{i}\right) \cap \widetilde{\phi}\left(y_{j}\right)$. Let $\alpha$ be any element in $\widetilde{\phi}(x)$. The last set is non-empty since at most $d(x)-1<k$ edges at $x$ are coloured. Part (c) applies to $i$ (resp. $j$ ) and gives that the $\alpha / \beta$-alternating chain starting at $y_{i}$ (resp. $y_{j}$ ) ends at $x$. This means that, in the multi-graph formed by the edges coloured $\alpha$ or $\beta$, the three distinct vertices $y_{i}, y_{j}$ and $x$ lie in the same connectivity component and each has degree 1 . This is contradiction, since the $\{\alpha, \beta\}$-bichromatic multi-graph has maximum degree at most 2 .

For Part (e), suppose that the multi-fan $F$ is maximal. Since at most $d\left(y_{1}\right)-1$ edges are coloured at $y_{1}$, there is $\beta \in \widetilde{\phi}\left(y_{1}\right)$. The colour $\beta$ must be present at $x$ for otherwise, by colouring $e_{1}$
with colour $\beta$, we obtain a larger proper local colouring and thus $\left(e_{1}\right)$ is an improving chain, a contradiction. Let $e^{\prime}=x y^{\prime}$ be the edge of colour $\beta$. By maximality, the fan $F$ contains the edge $e^{\prime}$. Thus $V(F)$ contains $y^{\prime} \neq y$ and has size at least 2 , as desired. It remains to prove (55). For this, define $\Gamma:=\left\{\phi\left(e_{2}\right), \ldots, \phi\left(e_{p}\right)\right\}$ and $\Gamma^{\prime}:=\bigcup_{z \in V(F)} \widetilde{\phi}(z)$. (Recall that $V(F)=\left\{y_{1}, \ldots, y_{p}\right\}$.)

Claim 1 The sets $\Gamma$ and $\Gamma^{\prime}$ are the same.

Proof of Claim. Let us show that $\Gamma \subseteq \Gamma^{\prime}$. Take any index $i \in[2, p]$. By (F2), there is $j \in[i-1]$ such that $\phi\left(e_{i}\right) \in \widetilde{\phi}\left(y_{j}\right)$. Thus $\phi\left(e_{i}\right) \in \Gamma^{\prime}$, as desired.
Conversely, pick any $i \in[p]$ and $\beta \in \widetilde{\phi}\left(y_{i}\right)$. By Part (b), we have that $\beta \notin \bar{\phi}(x)$. Thus there is $e^{\prime}=x z$ with $\phi\left(e^{\prime}\right)=\beta$. By the maximality of $F$, there is $j \in[2, p]$ with $e^{\prime}=e_{j}$. Thus $\beta=\phi\left(e_{j}\right)$ belongs to $\Gamma$. We conclude that $\Gamma \supseteq \Gamma^{\prime}$, finishing the proof of the claim. I

By Claim $\mathbb{1}$ and Part (d), we have that

$$
p-1=|\Gamma|=\left|\Gamma^{\prime}\right|=\sum_{z \in V(F)}|\widetilde{\phi}(z)| \geqslant \sum_{z \in V(F)}\left(\min \{k, d(z)+\pi(z)\}-d_{\phi}(z)\right) .
$$

Also, we have that $p=\sum_{z \in V(F)} \pi_{F}(x, z)$. Putting these two identities together, we obtain (5). This finishes the proof of the lemma.

Note that, in Theorem 4, if $e$ is the only edge of $G$ not coloured by $\phi$ then $d_{\phi}\left(y_{i}\right)=d\left(y_{i}\right)$ except $d_{\phi}\left(y_{1}\right)=d\left(y_{1}\right)-1$ and the Local Fan Inequality simplifies to

$$
\sum_{z \in V(F)}\left(\pi_{F}(x, z)-\min \{k-d(z), \pi(z)\}\right) \geqslant 2 .
$$

Now we are ready to derive the promised local version of Vizing's theorem.
Proof of Theorem 1 . Let $k:=\max \{d(x)+\pi(x): x \in V(G)\}$. Starting with the empty colouring, iteratively apply improving Vizing chains until none exists. (We stop since the potential strictly decreases each time.) Suppose that some edge $e$ is not coloured by the final colouring $\phi$. Let $x \in e$ and let $F=\left(e_{1}, y_{1}, \ldots, e_{p}, y_{p}\right)$ be a maximal multi-fan at $x$ with respect to $e$ and $\phi$. Then the Local Fan Inequality (5) holds by Theorem 4. Since $k \geqslant d(z)+\pi(z)$ for every $z \in V(G)$, the inequality states that

$$
\sum_{z \in V(F)}\left(\pi_{F}(x, z)-\pi(z)+d_{\phi}(z)-d(z)\right) \geqslant 1 .
$$

This is impossible as each summand in the left-hand side is clearly non-positive.
It easy to convert the proof of Theorem $\mathbb{1}$ into an algorithm that finds a local colouring of input multi-graph $G=(V, E, 1)$ without isolated vertices with running time polynomial in $|E|$. The algorithm starts with the empty partial colouring $\phi_{\emptyset}$ and iteratively finds an improving Vizing chain and changes the current partial colouring accordingly. Each step decreases the potential.

Thus we make at most $\Phi\left(\phi_{\emptyset}\right)=O(|E|)$ improvements. Given an uncoloured edge, an improving Vizing chain as in the proof of Theorem 4 can be easily found in polynomial in $|E|$ time.
Note that if $G$ is a simple graph, then the Local Fan Inequality (5) states that

$$
\begin{equation*}
\sum_{z \in V(F)}\left(d_{\phi}(z)-\min \{k-1, d(z)\}\right) \geqslant 1 . \tag{7}
\end{equation*}
$$

Proof of Theorem [. Let $\mathcal{C}$ be the set of pairs $(\phi, e)$ such that $\phi$ is a proper local partial colouring $E(G) \rightharpoonup[k]$ with the smallest possible potential, while $e \in E$ an edge is not coloured by $\phi$. Suppose that the theorem is false. Thus $\mathcal{C}$ is non-empty.

Claim 2 For every $(\phi, e) \in \mathcal{C}$ and $x \in e$ there are at least two different choice of $f \in E$ such that $f=x y$ for some $y \in V$ with $d(y)=k$ and $\left(\phi^{\prime}, f\right) \in \mathcal{C}$ for some $\phi^{\prime}$.

Proof of Claim. Let $F=\left(e_{1}, y_{1}, \ldots, e_{p}, y_{p}\right)$ be a maximal multi-fan at $x$ with respect to $e_{1}=e$ and $\phi$. By Theorem (4, the Local Fan Inequality (77) holds. Since $d_{\phi}\left(y_{1}\right) \leqslant d\left(y_{1}\right)-1$, the contribution of $y_{1}$ to the left-hand size of (7) is at most 0 if $d\left(y_{1}\right)=k$ and at most -1 if $d\left(y_{1}\right) \leqslant k-1$. We conclude by $k \geqslant d(z)$ for every $z \in V(G)$ that there are at least two indices $i \in[p]$ such that $d\left(y_{i}\right)=k$.

Let us show that, for each such $i$, the edge $f:=e_{i}$ satisfies the claim. Indeed, let $i_{0}=i>i_{1}>$ $\cdots>i_{m}=1$ be the sequence of indices returned by Part (a) of Theorem 4. Let $\phi^{\prime}$ be the shift of $\phi$ along the $\phi$-safe sequence $\left(e_{i_{m}}, e_{i_{m-1}}, \ldots, e_{i_{0}}\right)$. By Lemma 3, $\left(\phi^{\prime}, f\right) \in \mathcal{C}$, as required. I

Start with any $\left(\phi_{1}, e_{1}\right) \in \mathcal{C}$ with $e_{1}=x_{1} x_{2}$. By applying Claim 2(twice) and changing the pair ( $\phi_{1}, e_{1}$ ) we can assume first that $d\left(x_{1}\right)=k$ and then that also $d\left(x_{2}\right)=k$ (and thus $k=\Delta(G)$ ). Starting with $\phi_{1}$ and $e_{1}=x_{1} x_{2}$, we inductively construct an infinite sequence $\phi_{1}, \phi_{2}, \ldots$ and an infinite path visiting edges $e_{1}, e_{2}, \ldots$ and vertices $x_{1}, x_{2}, x_{3}, \ldots$ in the stated order such that $\left(\phi_{i}, e_{i}\right) \in \mathcal{C}$ and $d\left(x_{i}\right)=k$ for every $i \geqslant 1$ as follows. Let $i \geqslant 1$ and suppose that we already have $\phi_{1}, \ldots, \phi_{i}$ and $e_{1}, \ldots, e_{i}$ as above. Claim 2 when applied to ( $\phi_{i}, e_{i}$ ) with $x=x_{i+1}$ gives at least two different potential choices of the next edge $e_{i+1}=x_{i+1} x_{i+2}$ and the next partial colouring $\phi_{i+1}$ (that is, such that $\left(\phi_{i+1}, e_{i+1}\right) \in \mathcal{C}$ and $d\left(x_{i+2}\right)=k$ ). For at least one of these choices we have $x_{i+2} \notin\left\{x_{1}, \ldots, x_{i}\right\}$ as otherwise this would create a cycle on vertices of degree $k$. Thus we can always extend the path by a new edge. However, this contradicts the finiteness of our graph $G$, thus proving Theorem 2.

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## References

[1] M. Bonamy, M. Delcourt, R. Lang, and L. Postle, Edge-colouring graphs with local list sizes, J. Combin. Theory (B) 165 (2024), 68-96.
[2] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, List edge and list total colourings of multigraphs, J. Combin. Theory (B) 71 (1997), 184-204.
[3] A. B. G. Christiansen, The power of multi-step Vizing chains, STOC'23-Proceedings of the 55th Annual ACM Symposium on Theory of Computing, 2023, pp. 1013-1026.
[4] A. B. G. Christiansen, The power of multi-step Vizing chains, 2022. E-print arXiv:2210.07363,
[5] P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), 1980, pp. 125157.
[6] R. P. Gupta, The chromatic index and the degree of a graph, Notices Amer. Math. Soc. 13 (1966), 719.
[7] D. Scheide and M. Stiebitz, The maximum chromatic index of multigraphs with given $\Delta$ and $\mu$, Graphs Combin. 28 (2012), 717-722.
[8] M. Stiebitz, D. Scheide, B. Toft, and L. M. Favrholdt, Graph edge coloring, Wiley Series in Discrete Mathematics and Optimization, John Wiley \& Sons, Inc., Hoboken, NJ, 2012.
[9] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz No. 3 (1964), 25-30.
[10] V. G. Vizing, The chromatic class of a multigraph, Kibernetyka (Kyiv) 3 (1965), 29-39.

