# IMPLICATIONS OF SOME MASS-CAPACITY INEQUALITIES

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ABSTRACT. Applying a family of mass-capacity related inequalities proved in [16], we obtain sufficient conditions that imply the nonnegativity as well as positive lower bounds of the mass, on a class of manifolds with nonnegative scalar curvature, with or without a singularity.

# 1. INTRODUCTION

A smooth Riemannian 3-manifold (M, g) is called asymptotically flat (AF) if M, outside a compact set, is diffeomorphic to  $\mathbb{R}^3$  minus a ball; the associated metric coefficients satisfy

$$g_{ij} = \delta_{ij} + O(|x|^{-\tau}), \ \partial g_{ij} = O(|x|^{-\tau-1}), \ \partial \partial g_{ij} = O(|x|^{-\tau-2}),$$

for some  $\tau > \frac{1}{2}$ ; and the scalar curvature of g is integrable. Under these AF conditions, the limit, near  $\infty$ ,

$$\mathfrak{m} = \lim_{r \to \infty} \frac{1}{16\pi} \int_{|x|=r} \sum_{j,k} (g_{jk,j} - g_{jj,k}) \frac{x^k}{|x|}$$

exists and is called the ADM mass [2] of (M, g). It is a result of Bartnik [3], and of Chruściel [9], that  $\mathfrak{m}$  is a geometric invariant, independent on the choice of the coordinates  $\{x_i\}$ .

A fundamental result on the mass and the scalar curvature is the Riemannian positive mass theorem (PMT):

**Theorem 1.1** ([20, 22]). Let (M, g) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature without boundary. Then

 $\mathfrak{m} \geq 0$ ,

and equality holds if and only if (M, q) is isometric to the Euclidean space  $\mathbb{R}^3$ .

On an asymptotically flat 3-manifold (M, g) with boundary  $\Sigma = \partial M$ , the capacity (or  $L^2$ -capacity) of  $\Sigma$  is defined by

$$\mathbf{c}_{\Sigma} = \inf_{f} \left\{ \frac{1}{4\pi} \int_{M} |\nabla f|^2 \right\},\,$$

where the infimum is taken over all locally Lipschitz functions f that vanishes on  $\Sigma$ and tend to 1 at infinity. Equivalently, if  $\phi$  denotes the function with

$$\Delta \phi = 0, \ \phi|_{\Sigma} = 1, \text{ and } \phi \to 0 \text{ at } \infty,$$

then, 
$$\mathbf{c}_{\Sigma} = \frac{1}{4\pi} \int_{M} |\nabla \phi|^2 = \frac{1}{4\pi} \int_{\Sigma} |\nabla \phi|$$
, and  
 $\phi = \mathbf{c}_{\Sigma} |x|^{-1} + o(|x|^{-1})$ , as  $x \to \infty$ .

Regarding the mass and the capacity, if  $\Sigma$  is a minimal surface, Bray showed

**Theorem 1.2** ([4]). Let (M, g) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, with minimal surface boundary  $\Sigma = \partial M$ . Then

 $\mathfrak{m} \geq \mathfrak{c}_{\Sigma},$ 

and equality holds iff (M, g) is isometric to a spatial Schwarzschild manifold outside the horizon.

In [16, Theorem 7.4], an inequality relating the mass-to-capacity ratio to the Willmore functional of the boundary was obtained:

**Theorem 1.3** ([16]). Let (M, g) be a complete, orientable, asymptotically flat 3manifold with one end, with boundary  $\Sigma$ . Suppose  $\Sigma$  is connected and  $H_2(M, \Sigma) = 0$ . If g has nonnegative scalar curvature, then

(1.1) 
$$\frac{\mathfrak{m}}{\mathfrak{c}_{\Sigma}} \ge 1 - \left(\frac{1}{16\pi} \int_{\Sigma} H^2\right)^{\frac{1}{2}}.$$

Here  $\mathfrak{m}$  is the mass of (M, g),  $\mathfrak{c}_{\Sigma}$  is the capacity of  $\Sigma$  in (M, g), and H is the mean curvature of  $\Sigma$ . Moreover, equality in (1.1) holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative mean curvature.

As shown in [16], (1.1) implies the 3-dimensional PMT. For instance, assuming M is topologically  $\mathbb{R}^3$ , applying (1.1) to the exterior of a geodesic sphere  $S_r$  with radius r centered at any point  $p \in M$ , one has

(1.2) 
$$\frac{\mathfrak{m}}{\mathfrak{c}_{S_r}} \ge 1 - \left(\frac{1}{16\pi} \int_{S_r} H^2\right)^{\frac{1}{2}}.$$

Letting  $r \to 0$ , one obtains  $\mathfrak{m} \ge 0$ . Earlier proofs of 3-d PMT via harmonic functions were given by Bray-Kazaras-Khuri-Stern [6] and Agostiniani-Mazzieri-Oronzio [1].

Theorem 1.3 follows from two other results (Corollary 7.1 and Theorem 7.3) in [16]:

**Theorem 1.4** ([16]). Let (M, g) be a complete, orientable, asymptotically flat 3manifold with one end, with connected boundary  $\Sigma$ , satisfying  $H_2(M, \Sigma) = 0$ . If g has nonnegative scalar curvature, then

(1.3) 
$$\left(\frac{1}{\pi}\int_{\Sigma}|\nabla u|^{2}\right)^{\frac{1}{2}} \leq \left(\frac{1}{16\pi}\int_{\Sigma}H^{2}\right)^{\frac{1}{2}} + 1,$$

and

(1.4) 
$$\frac{\mathfrak{m}}{2\mathfrak{c}_{\Sigma}} \ge 1 - \left(\frac{1}{4\pi} \int_{\Sigma} |\nabla u|^2\right)^{\frac{1}{2}}.$$

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Here u is the harmonic function with u = 0 at  $\Sigma$  and  $u \to 1$  near  $\infty$ . Moreover,

- equality in (1.3) holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative mean curvature;
- equality in (1.4) holds if and only if (M, g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere.

A corollary of (1.3) (see [16, Theorem 7.2]) is an upper bound on the capacity-toarea-radius ratio, first derived by Bray and the author [7].

**Theorem 1.5** ([7]). Let (M, g) be a complete, orientable, asymptotically flat 3manifold with one end, with boundary  $\Sigma$ . Suppose  $\Sigma$  is connected and  $H_2(M, \Sigma) = 0$ . If g has nonnegative scalar curvature, then

(1.5) 
$$\frac{2\mathfrak{c}_{\Sigma}}{r_{\Sigma}} \le \left(\frac{1}{16\pi} \int_{\Sigma} H^2\right)^{\frac{1}{2}} + 1.$$

Here  $\mathbf{c}_{\Sigma}$  is the capacity of  $\Sigma$  in (M,g) and  $r_{\Sigma} = \left(\frac{|\Sigma|}{4\pi}\right)^{\frac{1}{2}}$  is the area-radius of  $\Sigma$ . Moreover, equality holds if and only if (M,g) is isometric to a spatial Schwarzschild manifold outside a rotationally symmetric sphere with nonnegative mean curvature.

In this paper, we give some other applications of (1.1), (1.3) and (1.4).

First, for later purposes, we remark on the topological assumption " $H_2(M, \Sigma) = 0$ " in Theorems 1.3 – 1.5 above: the assumption is imposed only to ensure each regular level set of the harmonic function u, vanishing at the boundary and tending to 1 near  $\infty$ , to be connected in the interior of M (see the paragraph preceding the proof of Theorem 3.1 in [16]); indeed, (1.1), (1.3) and (1.4) (and all other results from [16]) hold if " $H_2(M, \Sigma) = 0$ " is replaced by assuming

(\*) each closed, connected, orientable surface in the interior of M either is the boundary of a bounded domain, or together with  $\Sigma$  forms the boundary of a bounded domain.

Now we motivate the main tasks in this paper. Let us first return to the setting of (1.2), in which the surface  $S_r$  "closes up nicely" (to bound a geodesic ball). In this setting, by a result of Mondino and Templeton-Browne [18],  $\{S_r\}$  can be perturbed to yield another family of surfaces  $\{\Sigma_r\}$  so that, as  $r \to 0$ ,

(1.6) 
$$\int_{\Sigma_r} H^2 = 16\pi - \frac{8\pi}{3}R(p)r^2 + \frac{4\pi}{3} \left[\frac{1}{9}R(p)^2 - \frac{4}{15}|\mathring{\operatorname{Ric}}(p)|^2 - \frac{1}{5}\Delta R(p)\right]r^4 + O(r^5).$$

Here R denotes the scalar curvature and  $\operatorname{Ric} = \operatorname{Ric} - \frac{1}{3}Rg$  is the traceless part of Ric, the Ricci tensor. Applying (1.1) to the exterior of these  $\Sigma_r$  in (M, g), one obtains

(1.7) 
$$\frac{\mathfrak{m}}{\mathfrak{c}_{\Sigma_r}} \ge \frac{1}{12}R(p)r^2 + \left[\frac{1}{90}|\mathring{\operatorname{Ric}}(p)|^2 - \frac{1}{864}R(p)^2 + \frac{1}{120}\Delta R(p)\right]r^4 + O(r^5).$$

If  $R \ge 0$ , (1.7) shows the inequality  $\mathfrak{m} \ge 0$  as well as the rigidity of  $\mathfrak{m} = 0$ .

In general, (1.1) suggests that, if it is applied to obtain  $\mathfrak{m} \geq 0$  on an (M, g), the manifold boundary  $\Sigma$  does not need to admit a "nice fill-in". Rewriting (1.1) as

$$\mathfrak{m} \geq \mathfrak{c}_{_{\Sigma}} \left[ 1 - \left( \frac{1}{16\pi} \int_{S_r} H^2 \right)^{\frac{1}{2}} \right],$$

one may seek conditions on metrics g with a "singularity" so that  $\mathfrak{m} \geq 0$  while g is allowed to be incomplete.

Similarly, on an (M, g) with two ends, one of which is asymptotically flat (AF), assuming it admits a harmonic function u that tends to 1 at the AF end and tends to 0 at the other end, one may aim to apply (1.4), i.e.

$$\mathfrak{m} \geq 2\mathfrak{c}_{\Sigma} \left[ 1 - \left( \frac{1}{4\pi} \int_{\Sigma} |\nabla u|^2 \right)^{\frac{1}{2}} \right],$$

to bound  $\mathfrak{m}$  via the energy of u on the entire (M, g).

Below we formulate a class of manifolds to carry out the above mentioned tasks. Throughout the paper, let N be a noncompact, connected, orientable 3-manifold. We assume N admits an increasing exhaustion sequence of bounded domains with connected boundary. Precisely, this means there exists a sequence of closed, orientable surfaces  $\{\Sigma_k\}_{k=1}^{\infty}$  in N such that

- $\Sigma_k$  is connected;
- $\Sigma_k = \partial D_k$  for a precompact domain  $D_k \subset N$ ; and
- $\overline{D}_k \subset D_{k+1}$  and  $N = \bigcup_{k=1}^{\infty} D_k$ . Here  $\overline{D}_k = D_k \cup \Sigma_k$  is the closure of  $D_k$  in N.

Fix a point  $p \in N$ , let  $M = N \setminus \{p\}$ . On M, let g be a smooth metric that is asymptotically flat near p. We refer p as the asymptotically flat (AF)  $\infty$  of (M, g). Unless otherwise specified, we do not impose assumptions on the behavior of g near  $\Sigma_k$  as  $k \to \infty$ . In particular, (M, g) does not need to be complete,

Given any closed, connected surface  $S \subset M$ , we say S encloses p if  $S = \partial D_s$  for some precompact domain  $D_s \subset N$  such that  $p \in D_s$ . Let S denote the set of all such surfaces  $S \subset M$  enclosing p. Clearly,  $\Sigma_k \in S$  for large k. Define

(1.8) 
$$\mathbf{c}(M,g) = \inf_{S \in S} \mathbf{c}_S.$$

Here  $\mathfrak{c}_{s}$  is the capacity of S in the asymptotically flat  $(E_{s}, g)$ , where

$$E_s = (D_s \setminus \{p\}) \cup S.$$

As a functional on  $\mathcal{S}$ , the capacity  $\mathfrak{c}_s$  has a monotone property, that is if  $S_1, S_2 \in \mathcal{S}$ and  $D_{s_1} \subset D_{s_2}$ , then  $\mathfrak{c}_{s_1} \geq \mathfrak{c}_{s_2}$ . Such a property readily implies  $\{\mathfrak{c}_{\Sigma_k}\}$  is monotone non-increasing and

(1.9) 
$$\mathbf{c}(M,g) = \lim_{k \to \infty} \mathbf{c}_{\Sigma_k}.$$

Standard arguments show  $\mathfrak{c}(M,g) > 0$  if and only if there exists a harmonic function w on (M,g) such that 0 < w < 1 on M and  $w(x) \to 1$  at  $\infty$  (i.e. as  $x \to p$ ). (See Proposition 3.1 in Section 3.)

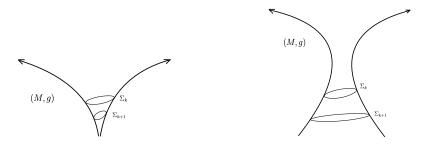


FIGURE 1. On the left is an examples of (M, g) with  $\mathfrak{c}(M, g) = 0$ ; the arrow denotes the AF end;  $\{\Sigma_k\}$  may approach a "singularity" as  $k \to \infty$ . On the right is an example of (M, g) with  $\mathfrak{c}(M, g) > 0$ ; besides the AF end, (M, g) has another end with suitable growth.

For manifolds (M, g) with  $\mathfrak{c}(M, g) = 0$ , we seek conditions that imply the AF end of (M, g) has mass  $\mathfrak{m} \geq 0$ , see Theorem 2.1 and Remark 2.2. For (M, g) with  $\mathfrak{c}(M, g) > 0$ , we explore for sufficient conditions that bound  $\mathfrak{m}$  from below via  $\mathfrak{c}(M, g)$ , see Theorem 3.1 and Corollary 3.1.

# 2. Singular metrics with $\mathfrak{m} \geq 0$

Let N, M and g be given in the definition of  $\mathfrak{c}(M,g)$  in (1.8). Given  $S \in \mathfrak{S}$ , let

$$W(S) = \int_S H^2.$$

We want to apply (1.1) to  $(E_s, g)$ . For this purpose, we assume the background manifold N satisfies  $H_2(N) = 0$ . Under this assumption, any closed, connected surface S' in  $M = N \setminus \{p\}$  is the boundary of a bounded domain  $D \subset N$ . If  $p \notin D$ , then  $D \subset M$ ; if  $p \in D$ , then S' is homologous to  $S \in S$ . Therefore, condition (\*) holds on  $E_s$ .

The following is a direct corollary of (1.1).

**Proposition 2.1.** Suppose  $H_2(N) = 0$  and (M, g) has nonnegative scalar curvature. Then

(2.1) 
$$\mathfrak{c}_{S_k} W(S_k)^{\frac{1}{2}} \to 0 \text{ along a sequence } \{S_k\} \subset \mathfrak{S} \Rightarrow \mathfrak{m} \geq 0.$$

*Proof.* If  $W(S_k) \leq 16\pi$  for some k, then (1.1) implies  $\mathfrak{m} \geq 0$ .

Suppose  $W(S_k) > 16\pi$  for every k, then " $\mathfrak{c}_{S_k} W(S_k)^{\frac{1}{2}} \to 0$ " implies " $\mathfrak{c}_{S_k} \to 0$ ". Rewriting (1.1) as

(2.2) 
$$\mathfrak{m} \ge \mathfrak{c}_{S_k} \left[ 1 - \left( \frac{1}{16\pi} W(S_k) \right)^{\frac{1}{2}} \right]$$

and letting  $k \to \infty$ , we have  $\mathfrak{m} \ge 0$ .

Given  $S \in S$ , let  $\mathfrak{m}_{H}(S) = \frac{r_{s}}{2} \left(1 - \frac{1}{16\pi}W(S)\right)$  denote the Hawking mass of S ([11]). Inequality (2.3) in the next Proposition is comparable to the result of Huisken and Ilmanen [13] on the relation between  $\mathfrak{m}$  and  $\mathfrak{m}_{H}(S)$ .

**Proposition 2.2.** Suppose  $H_2(N) = 0$  and (M, g) has nonnegative scalar curvature. If a surface  $S \in S$  satisfies  $W(S) \ge 16\pi$ , then

(2.3) 
$$\mathfrak{m} \ge \mathfrak{c}_{S} \left[ 1 - \left( \frac{1}{16\pi} W(S) \right)^{\frac{1}{2}} \right] \ge \mathfrak{m}_{H}(S).$$

*Proof.* If  $W(S) \ge 16\pi$ , then (1.5) implies

(2.4) 
$$\mathfrak{c}_{S}\left[1 - \left(\frac{1}{16\pi}\int_{S}H^{2}\right)^{\frac{1}{2}}\right] \geq \frac{r_{S}}{2}\left[1 - \frac{1}{16\pi}\int_{S}H^{2}\right] = \mathfrak{m}_{H}(S).$$

This combined with (1.1) proves (2.3).

Remark 2.1. Similar to (2.1), a condition of " $r_{s_k}W(S_k) \to 0$ " along  $\{S_k\} \subset S$  also implies " $\mathfrak{m} \geq 0$ ". However, if  $\inf_k W(S_k) \geq 16\pi$ , then

$$``r_{_{S_k}}W(S_k) \to 0" \ \Rightarrow \ ``r_{_{S_k}} \to 0 \ \text{and} \ r_{_{S_k}}W(S_k)^{\frac{1}{2}} \to 0" \ \Rightarrow \ ``\mathfrak{c}_{_{S_k}} \to 0",$$

where the last step is by (1.5). Combined with (2.4), this implies the assumption of  ${}^{"}\mathfrak{c}_{S_k}W(S_k)^{\frac{1}{2}} \to 0$ " in Proposition 2.1.

In what follows, let  $\{\Sigma_k\} \subset S$  be the sequence of surfaces given in the introduction. The numerical value of  $\mathfrak{c}_{\Sigma_k}$  depends on g near the AF end. However, a property of " $\mathfrak{c}_{\Sigma_k} \to 0$ " does not. This was shown by Bray and Jauregui [5] in the context of (M,g) having a zero area singularity. Their argument applies to " $\mathfrak{c}_{\Sigma_k} W(\Sigma_k)^{\frac{1}{2}} \to 0$ ". To illustrate this, it is convenient to adopt a notion of relative capacity (see [14] for instance). Given two surfaces  $S, \tilde{S} \in S$ , suppose  $S \cap \tilde{S} = \emptyset$  and  $D_{\tilde{S}} \subset D_S$ . The capacity of S relative to  $\tilde{S}$  is

(2.5) 
$$\mathbf{c}_{(S,\tilde{S})} = \frac{1}{4\pi} \int_{D_S \setminus D_{\tilde{S}}} |\nabla v|^2,$$

where v is the harmonic function on  $D_s \setminus D_{\tilde{s}}$  with v = 0 at S and v = 1 at  $\tilde{S}$ .

**Proposition 2.3.** Let  $\tilde{S} \in S$  be a fixed surface. Then, as  $k \to \infty$ ,

$$\mathfrak{c}_{_{\Sigma_k}}W(\Sigma_k)^{\frac{1}{2}}\to 0 \iff \mathfrak{c}_{_{(\Sigma_k,\tilde{S})}}W(\Sigma_k)^{\frac{1}{2}}\to 0.$$

*Proof.* For large k, let  $u_k$ ,  $v_k$  be the harmonic function on  $D_k \setminus \{p\}$ ,  $D_k \setminus D_{\tilde{S}}$ , with boundary values  $u_k = 0$  at  $\Sigma_k$ ,  $u_k \to 1$  at the AF  $\infty$ ,  $v_k = 0$  at  $\Sigma_k$ ,  $v_k = 1$  at  $\tilde{S}$ , respectively. Let  $\beta_k = \min_{\tilde{S}} u_k$ . By the maximum principle,  $v_k \ge u_k \ge \beta_k v_k$  on

$$\square$$

 $D_k \setminus D_{\tilde{s}}$ , which implies  $\partial_{\nu} v_k \geq \partial_{\nu} u_k \geq \beta_k \partial_{\nu} v_k$  at  $\Sigma_k$ . Here  $\nu$  denotes the unit normal to  $\Sigma_k$  pointing to  $\infty$ . Since  $4\pi \mathfrak{e}_{\Sigma_k} = \int_{\Sigma_k} \partial_{\nu} u_k$  and  $4\pi \mathfrak{e}_{(\Sigma_k, \tilde{s})} = \int_{\Sigma_k} \partial_{\nu} v_k$ , one has

$$\beta_k^{-1}\mathfrak{c}_{_{\Sigma_k}} \geq \mathfrak{c}_{_{(\Sigma_k,\tilde{S})}} \geq \mathfrak{c}_{_{\Sigma_k}}.$$

The claim follows by noting that  $\beta_k$  has a uniform positive lower bound as  $k \to \infty$ .  $\Box$ 

As an application of Propositions 2.1 and 2.3, we have

**Theorem 2.1.** Let N be a noncompact, connected, orientable 3-manifold. Suppose  $H_2(N) = 0$ . Let  $M = N \setminus \{p\}$  where p is a point in N. Let g be a smooth metric with nonnegative scalar curvature on M such that g is asymptotically flat near p. Assume there is a precompact domain  $D \subset N$  such that  $p \in D$  and  $(N \setminus D, g)$  is isometric to  $((0, \delta] \times \Sigma, \overline{q} + h)$ .

where

- $\delta > 0$  is a constant,  $\Sigma$  is a closed, connected, orientable surface;
- $\bar{g} = dr^2 + a(r)^2 \sigma$ , in which  $\sigma$  is a given metric on  $\Sigma$  and a(r) is a positive function on  $(0, \delta]$ ; and
- $\lambda^{-1} \leq |\bar{g} + h|_{\bar{q}} \leq \lambda$  for some constant  $\lambda > 0$ .

Then

(2.6) 
$$\lim_{r \to 0} \left( \int_r^{\delta} \frac{1}{a(x)^2} \, dx \right)^{-1} \left[ |a'(r)| + a(r) |\bar{\nabla}h|_{\bar{g}} \right] = 0 \implies \mathfrak{m} \ge 0.$$

Remark 2.2. If  $a(r) = r^b$  for a constant b > 0, then (2.6) translates to

$$\lim_{r \to 0} r^{3b-2} \left( 1 + r |\bar{\nabla}h|_{\bar{g}} \right) = 0 \implies \mathfrak{m} \ge 0.$$

This in particular implies, if g has a conical or  $r^b$ -horn type singularity modeled on  $\bar{g} = dr^2 + r^{2b}\sigma$  near r = 0, then, under a mild asymptotic assumption of

 $\lambda^{-1} \leq |\bar{g}+h|_{\bar{g}} \leq \lambda \ \, \text{and} \ \, r|\bar{\nabla}h|_{\bar{g}} = O(1),$ 

one has " $b > \frac{2}{3} \Rightarrow \mathfrak{m} \ge 0$ ". (Related results on PMT with isolated singularities can be found in [21, 15, 10]).

Proof of Theorem 2.1. Let  $\Sigma_r = \{r\} \times \Sigma$ ,  $r \in (0, \delta]$ . For  $s \in (0, \delta)$ , let  $\bar{\mathfrak{c}}_{(\Sigma_s, \Sigma_\delta)}$ ,  $\bar{W}(\Sigma_s)$  denote the capacity of  $\Sigma_s$  relative to  $\Sigma_\delta$ , the Willmore functional of  $\Sigma_s$ , respectively, with respect to  $\bar{g}$ .

The function  $u(r) = \left(\int_s^{\delta} a(x)^{-2} dx\right)^{-1} \int_s^r a(x)^{-2} dx$  is  $\bar{g}$ -harmonic on  $[s, \delta] \times \Sigma$  with u = 0 at  $\Sigma_s$  and u = 1 at  $\Sigma_{\delta}$ . This implies

(2.7) 
$$\bar{\mathfrak{c}}_{(\Sigma_s,\Sigma_\delta)} = \frac{|\Sigma|_{\sigma}}{4\pi} \left( \int_s^{\delta} a(x)^{-2} \, dx \right)^{-1},$$

where  $|\Sigma|_{\sigma}$  is the area of  $(\Sigma, \sigma)$ . The mean curvature  $\bar{H}$  of  $\Sigma_s$  with respect to  $\bar{g}$  is  $\bar{H} = 2a^{-1}a'$ . Hence,

$$\bar{W}(\Sigma_s) = 4a'(s)^2 |\Sigma|_{\sigma}.$$

We compare  $\bar{\mathfrak{c}}_{(\Sigma_s,\Sigma_{\delta})}$  and  $\mathfrak{c}_{(\Sigma_s,\Sigma_{\delta})}$ . Let  $\bar{\nabla}$ ,  $\nabla$  and  $dV_{\bar{g}}$ ,  $dV_g$  denote the gradient, the volume form with respect to  $\bar{g}$ , g, respectively. Since  $\mathfrak{c}_{(\Sigma_s,\Sigma_{\delta})}$  equals the infimum of the g-Dirichlet energy of functions that vanish at  $\Sigma_s$  and equal 1 at  $\Sigma_{\delta}$ , we have

(2.8)  
$$\begin{aligned} \mathfrak{c}_{(\Sigma_{s},\Sigma_{\delta})} &\leq \int_{[s,\delta]\times\Sigma} |\nabla u|_{g}^{2} dV_{g} \\ &\leq C \int_{[s,\delta]\times\Sigma} |\bar{\nabla} u|_{\bar{g}}^{2} dV_{\bar{g}} = C \,\bar{\mathfrak{c}}_{(\Sigma_{s},\Sigma_{\delta})}. \end{aligned}$$

Here C > 0 denotes a constant independent on s and we have used the assumption  $\lambda^{-1} \leq |g|_{\bar{g}} \leq \lambda$ .

We also compare  $\overline{W}(\Sigma_s)$  and  $W(\Sigma_s)$ . Let  $\overline{\mathbb{II}}$  denote the second fundamental form of  $\Sigma_s$  with respect to  $\overline{g}$ . Direct calculation shows

(2.9) 
$$H - \bar{H} = |\bar{\mathbb{II}}|_{\bar{g}} O(|h|_{\bar{g}}) + O(|\bar{\nabla}h|_{\bar{g}}).$$

(For instance, see formula (2.33) in [17] and the proof therein.) Therefore,

(2.10) 
$$H^{2} = \bar{H}^{2} + \left[ |\bar{\mathbb{II}}|_{\bar{g}} O(|h|_{\bar{g}}) + O(|\bar{\nabla}h|_{\bar{g}}) \right]^{2} + \bar{H} \left[ |\bar{\mathbb{II}}|_{\bar{g}} O(|h|_{\bar{g}}) + O(|\bar{\nabla}h|_{\bar{g}}) \right]$$
$$= \bar{H}^{2} + |\bar{\mathbb{II}}|_{\bar{g}}^{2} O(|h|_{\bar{g}}) + \bar{H} O(|\bar{\nabla}h|_{\bar{g}}) + O(|\bar{\nabla}h|_{\bar{g}}^{2}).$$

Let  $d\sigma_g$ ,  $d\sigma_{\bar{g}}$  denote the area form on  $\Sigma_s$  with respect to g,  $\bar{g}$ , respectively. Then

(2.11) 
$$\int_{\Sigma_s} H^2 \, d\sigma_g \leq C \int_{\Sigma_s} H^2 \, d\sigma_{\bar{g}} \\ = C \, \bar{W}(\Sigma_s) + \left[ |\bar{\mathbb{II}}|_{\bar{g}}^2 \, O(|h|_{\bar{g}}) + \bar{H} \, O(|\bar{\nabla}h|_{\bar{g}}) + O(|\bar{\nabla}h|_{\bar{g}}^2) \right] \, |\Sigma_s|_{\bar{g}}.$$

Plugging in  $\bar{W}(\Sigma_s) = 4a'^2 |\Sigma|_{\sigma}$ ,  $|\bar{\mathbb{II}}|_{\bar{g}}^2 = 2a^{-2}a'^2$ , and  $\bar{H}^2 = 4a^{-2}a'^2$ , we have

(2.12) 
$$W(\Sigma_s) \le C |\Sigma|_{\sigma} \left[ a'^2 + a'^2 |h|_{\bar{g}} + aa' |\bar{\nabla}h|_{\bar{g}} + a^2 |\bar{\nabla}h|_{\bar{g}}^2 \right] \le C |\Sigma|_{\sigma} \left[ a'^2 (1 + |h|_{\bar{g}}) + a^2 |\bar{\nabla}h|_{\bar{g}}^2 \right].$$

As  $|h|_{\bar{g}}$  is bounded by assumption, it follows from (2.7), (2.8) and (2.12) that

(2.13) 
$$\mathfrak{c}_{(\Sigma_s,\Sigma_\delta)} W(\Sigma_s)^{\frac{1}{2}} \le C \left( \int_s^\delta a(x)^{-2} \, dx \right)^{-1} \left[ |a'| + a |\bar{\nabla}h|_{\bar{g}} \right].$$

(2.6) now follows from Propositions 2.1 and 2.3.

Remark 2.3. The negative mass Schwarzschild manifolds are known to have an  $r^{b}$ horn type singularity with  $b = \frac{2}{3}$  (see [5, 21] for instance). In [5], Bray and Jauregui
developed a theory of "zero area singularities" (ZAS) modeled on the singularity of
these manifolds. Among other things, they introduced a notion of the mass of ZAS.
In [19, Theorem 4.8], Robbins showed the ADM mass of an asymptotically flat 3manifold with a single ZAS is at least the ZAS mass. The conclusion on the  $r^{b}$ -horn
type singularity in Remark 2.2 can also be reached via the results on ZAS in [5, 19].

We end this section by applying (1.1) to obtain information of  $W(\cdot)$  in the negative mass Schwarzschild manifolds.

**Proposition 2.4.** Consider a spatial Schwarzschild manifold with negative mass, i.e.

$$(M_{\mathfrak{m}}, g_{\mathfrak{m}}) = \left( (0, \infty) \times S^2, \frac{1}{1 + \frac{2m}{r}} dr^2 + r^2 \sigma_o \right)$$

where  $(S^2, \sigma_o)$  denotes the standard unit sphere and the mass  $\mathfrak{m} = -m$  is negative. Let  $\Sigma \subset M_{\mathfrak{m}}$  be any connected, closed surface that is homologous to a slice  $\{r\} \times S^2$ . Let  $r_{max}(\Sigma) = \max_{x \in \Sigma} r(x)$ . Then

(2.14) 
$$W(\Sigma) \ge 16\pi \left(1 + \frac{2m}{r_{max}(\Sigma)}\right).$$

In particular,  $W(\Sigma) \to \infty$  as  $r_{max}(\Sigma) \to 0$ .

*Proof.* (1.1) implies

(2.15) 
$$\left(\frac{1}{16\pi}W(\Sigma)\right)^{\frac{1}{2}} \ge 1 + \frac{m}{\mathfrak{c}_{\Sigma}}.$$

Let  $\Sigma_* = \{r_{max}(\Sigma)\} \times \mathbb{S}^2$ . As  $\Sigma_*$  encloses  $\Sigma$ , (2.16)  $\mathfrak{c}_{\Sigma_*} \ge \mathfrak{c}_{\Sigma}$ .

Since  $\mathfrak{m} = -m < 0$ , the above implies

(2.17) 
$$\left(\frac{1}{16\pi}W(\Sigma)\right)^{\frac{1}{2}} \ge 1 + \frac{m}{\mathfrak{c}_{\Sigma_*}}$$

Direct calculation gives

(2.18) 
$$\mathbf{c}_{\Sigma_*} = \frac{m}{\left(1 + \frac{2m}{r_{max}(\Sigma)}\right)^{\frac{1}{2}} - 1}$$

(2.14) follows from (2.17) and (2.18).

Proposition 2.4 gives another perspective of the singularity of  $(M_{\mathfrak{m}}, g_{\mathfrak{m}})$  via the Willmore functional  $W(\cdot)$ .

# 3. Bounding $\mathfrak{m}$ via $\mathfrak{c}(M,g)$

Let (M, g) be given in the definition of  $\mathfrak{c}(M, g)$  in (1.8). In this section, we relate  $\mathfrak{m}$  and  $2\mathfrak{c}(M, g)$  assuming  $\mathfrak{c}(M, g) > 0$ . We begin with a characterization of  $\mathfrak{c}(M, g) > 0$  which follows from standard arguments on harmonic functions.

**Proposition 3.1.** Let  $\mathfrak{c}(M,g)$  be defined in (1.8). Then  $\mathfrak{c}(M,g) > 0$  if and only if there exists a harmonic function w on (M,g) such that 0 < w < 1 on M and  $w(x) \to 1$  at  $\infty$  (i.e. as  $x \to p$ ).

*Proof.* For each k, let  $u_k$  be the harmonic function on  $(E_{\Sigma_k}, g)$  with  $u_k \to 1$  as  $x \to \infty$ and  $u_k = 0$  at  $\Sigma_k$ . Given any surface  $S \in S$ , by the maximum principle,  $\{u_k\}$  forms an increasing sequence in the exterior of S relative to  $\infty$  (i.e. in  $D_S \setminus \{p\}$ ). Interior elliptic estimates imply  $\{u_k\}$  converges to a harmonic function  $u_\infty$  on M uniformly

on compact sets in any  $C^i$ -norm. The limit  $u_{\infty}$  satisfies  $0 < u_{\infty} \leq 1$  and  $u_{\infty} \to 1$  as  $x \to \infty$ . By the strong maximum principle, either  $u_{\infty} \equiv 1$  or  $0 < u_{\infty} < 1$ .

Suppose (M, g) admits a harmonic w with 0 < w < 1 and  $w \to 1$  at  $\infty$ . Then w is an upper barrier for  $\{u_k\}$ , which implies  $u_{\infty} \leq w$ , and hence  $0 < u_{\infty} < 1$ . In this case,  $\mathfrak{c}(M, g)$  must be positive. Otherwise, if  $\mathfrak{c}(M, g) = 0$ , then  $\lim_{k\to\infty} \int_{E_{\Sigma_k}} |\nabla u_k|^2 = 0$ , which would imply  $\int_K |\nabla u_{\infty}|^2 = 0$  on any compact set K in M, and hence  $u_{\infty} \equiv 1$ , a contradiction.

Next suppose  $\mathfrak{c}(M,g) > 0$ . We want to show  $0 < u_{\infty} < 1$ . If not,  $u_{\infty} \equiv 1$  on M. Pick any surface  $S \in S$ , then  $\lim_{k\to\infty} u_k = 1$  at S. Let  $\beta_k = \min_S u_k$  for large k. Let  $\tilde{u}_k$  be the harmonic function on  $E_S$  with  $\tilde{u}_k \to 1$  at  $\infty$  and  $\tilde{u}_k = \beta_k$  at S. By the maximum principle,  $u_k \geq \tilde{u}_k$ . Therefore,  $\tilde{c}_k \geq c_k$ , where  $\tilde{c}_k$ ,  $c_k$  are the coefficients in the expansions

$$\tilde{u}_k = 1 - \tilde{c}_k |x|^{-1} + o(|x|^{-1}),$$
  
 $u_k = 1 - c_k |x|^{-1} + o(|x|^{-1}),$ 

as  $x \to \infty$ . Here we have  $c_k = \mathfrak{c}_{s_k}$  and

$$4\pi \tilde{c}_k = \lim_{r \to \infty} \int_{|x|=r} \frac{\partial \tilde{u}_k}{\partial \nu} = \int_S \frac{\partial \tilde{u}_k}{\partial \nu},$$

where  $\nu$  denotes the corresponding unit normal pointing to  $\infty$ . Elliptic boundary estimates applied to  $w_k = \tilde{u}_k - \beta_k$  shows

$$\lim_{k \to \infty} \max_{S} |\nabla w_k| = 0.$$

Consequently,  $\tilde{c}_k \to 0$  as  $k \to \infty$ . Combined with  $\tilde{c}_k \ge \mathfrak{c}_k$ , this shows

$$\mathfrak{c}(M,g) = \lim_{k \to \infty} \mathfrak{c}_{_{S_k}} = 0,$$

which is a contradiction. Therefore,  $0 < u_{\infty} < 1$ . This completes the proof.

Remark 3.1. One may further require w satisfies  $\inf_M w = 0$  in Proposition 3.1. To see this, it suffices to examine the proof beginning with assuming  $\mathfrak{c}(M,g) > 0$ . In this case, we have shown  $0 < u_{\infty} < 1$  on M. Suppose  $\inf_M u_{\infty} > 0$ , consider  $v = (1 - \inf_M u_{\infty})^{-1}(u_{\infty} - \inf_M u_{\infty})$ . Then  $v < u_{\infty}$  and v also acts as a barrier for  $\{u_k\}$ , which implies  $u_{\infty} \leq v$ , a contradiction. Hence,  $\inf_M u_{\infty} = 0$ .

Next, we focus on the case in which the function u tends to zero at "the other end".

**Proposition 3.2.** Suppose there is a harmonic function u on (M, g) with 0 < u < 1,  $u(x) \to 1$  at  $\infty$  (i.e. as  $x \to p$ ), and  $\lim_{k\to\infty} \max_{\Sigma_k} u = 0$ . Then

$$(3.1) c(M,g) = C,$$

where  $\mathfrak{C} > 0$  is the coefficient in the expansion of

 $u = 1 - \mathcal{C}|x|^{-1} + o(|x|^{-1})$ 

in the AF end.

*Proof.* Let  $u_k$  be the harmonic function on  $(E_{\Sigma_k}, g)$  with  $u_k \to 1$  at  $\infty$  and  $u_k = 0$  at  $\Sigma_k$ . Then  $u_k \leq u$ , which implies  $\mathcal{C} \leq c_k$ , where  $c_k = \mathfrak{c}_{\Sigma_k}$  is the coefficient in

$$u_k = 1 - c_k |x|^{-1} + o(|x|^{-1})$$

as  $x \to \infty$ . This shows

$$\mathfrak{C} \leq \lim_{k \to \infty} \mathfrak{c}_{_{\Sigma_k}} = \mathfrak{c}(M, g).$$

To show the other direction, consider  $\alpha_k = \max_{\Sigma_k} u$ . On  $E_{\Sigma_k}$ , by the maximum principle,  $\frac{1}{1-\alpha_k}(u-\alpha_k) \leq u_k$ , which implies  $\frac{1}{1-\alpha_k}C \geq c_k$ . As  $\alpha_k \to 0$ , this gives

$$\mathfrak{C} \geq \lim_{k \to \infty} (1 - \alpha_k) \mathfrak{c}_{\Sigma_k} = \mathfrak{c}(M, g)$$

Therefore,  $\mathfrak{C} = \mathfrak{c}(M, g)$ .

We are now in a position to derive applications of (1.4).

**Theorem 3.1.** Let N be a noncompact, connected, orientable 3-manifold admitting an exhaustion sequence of precompact domains  $D_k$  with connected boundary  $\partial D_k$ ,  $k = 1, 2, \ldots$  Suppose  $H_2(N) = 0$ . Let  $M = N \setminus \{p\}$  where p is a point in N. Let g be a smooth metric with nonnegative scalar curvature on M such that g is asymptotically flat near p. Assume there is a harmonic function u on (M, g) with  $0 < u < 1, u(x) \to 1$  as  $x \to p$ , and  $\lim_{k \to \infty} \max_{\partial D_k} u = 0$ . Then

(i) The limit 
$$\lim_{t \to 0} \int_{u^{-1}(t)} |\nabla u|^2$$
 exists (finite or  $\infty$ ), where  $t \in (0,1)$  is a regular value of  $u$ ; and  
(ii)  $\mathfrak{m} \ge 2\mathfrak{c}(M,g) \left[ 1 - \lim_{t \to 0} \left( \frac{1}{4\pi} \int_{u^{-1}(t)} |\nabla u|^2 \right)^{\frac{1}{2}} \right].$ 

Proof. Given a regular value  $t \in (0,1)$  of u, let  $\Sigma_t = u^{-1}(t)$ .  $\Sigma_t$  is a closed, orientable surface in  $M = N \setminus \{p\}$ . Let  $\Sigma_t^{(1)}$  denote any connected component of  $\Sigma_t$ . Since  $H_2(N) = 0$ ,  $\Sigma_t^{(1)}$  is the boundary of a bounded domain  $\Omega_1$  in N. If  $p \notin \Omega_1$ , then u is identically a constant by the maximum principle. Hence,  $\Sigma_t^{(1)}$  encloses p. As a result, if there are two connected components of  $\Sigma_t$ , then both of them enclose p, and thus form the boundary of a bounded domain in M. By the maximum principle, u is a constant, which is a contradiction. Therefore,  $\Sigma_t$  is connected. Since t is arbitrary, this in particular shows (1.4) is applicable to  $(E_t, g)$ , where  $E_t = \{u \ge t\} \subset M$  is the exterior of  $\Sigma_t$  with respect to  $\infty$ .

Applying (1.4) to  $(E_t, g)$ , we have

(3.2) 
$$\frac{\mathfrak{m}}{2\mathfrak{c}_{\Sigma_t}} \ge 1 - \left(\frac{1}{4\pi} \int_{\Sigma_t} |\nabla u_t|^2\right)^{\frac{1}{2}}$$

Here  $u_t = \frac{1}{1-t}(u-t)$  is the harmonic function on  $(E_t, g)$  that tends to 1 at  $\infty$  and equals 0 at  $\Sigma_t$ ,  $\mathbf{c}_{\Sigma_t} = \frac{1}{1-t} \mathcal{C}$ , and  $\mathcal{C}$  is the coefficient in the expansion of

$$u = 1 - \mathcal{C}|x|^{-1} + o(|x|^{-1}).$$

It follows from (3.2) that

(3.3) 
$$\frac{\mathfrak{m}}{2\mathfrak{C}} \ge \frac{1}{1-t} - \frac{1}{(1-t)^2} \left(\frac{1}{4\pi} \int_{\Sigma_t} |\nabla u|^2\right)^{\frac{1}{2}}$$

Consider the function

$$\mathcal{B}(t) = \frac{1}{(1-t)} \left[ 4\pi - \frac{1}{(1-t)^2} \int_{\Sigma_t} |\nabla u|^2 \right].$$

In [16, Theorem 3.2 (ii)], we showed  $\mathcal{B}(t)$  is monotone nondecreasing in t if g has nonnegative scalar curvature. As a result,

$$\lim_{t \to 0} \mathcal{B}(t) \text{ exists}$$

Consequently,

$$\lim_{t \to 0} \int_{\Sigma_t} |\nabla u|^2 \text{ exists.}$$

This proves (i). (ii) follows from (3.3), (i) and Proposition 3.2.

We have not assumed g to be complete on M so far. In particular, (M, g) in Theorem 3.1 could just be the interior of an AF manifold with boundary  $\Sigma$  and the function u may simply be the restriction, to the interior, of the harmonic function that tends to 1 at  $\infty$  and vanishes at  $\Sigma$ . In that extreme case,  $\lim_{t\to 0} \int_{\Sigma_t} |\nabla u|^2 = \int_{\Sigma} |\nabla u|^2$ and (ii) reduces to (1.4).

If g is complete on M, we have the following corollary.

**Corollary 3.1.** Let N, p, M, g and u be given as in Theorem 3.1. Suppose (M, g) is complete and has Ricci curvature bounded from below. Then

 $(3.4) mtextbf{m} \ge 2\mathfrak{C},$ 

where  $\mathfrak{C} = \mathfrak{c}(M, g)$  is the coefficient in the expansion of

$$u = 1 - \mathcal{C}|x|^{-1} + o(|x|^{-1})$$

as  $x \to p$ .

Corollary 3.1 relates to a result of Bray [4]. In [4, Theorem 8], Bray proved that, if (M, g) is a complete asymptotically flat 3-manifold with nonnegative scalar curvature which has multiple AF ends and mass  $\mathfrak{m}$  in a chosen end, then

 $\mathfrak{m} \geq 2\mathfrak{C},$ 

where C is the coefficient in the expansion  $u = 1 - C|x|^{-1} + o(|x|^{-1})$  at the chose end, and u is the harmonic function that tends to 1 at the chosen end and approaches 0 at all other AF ends.

Bray's theorem allows M to have more general topology and more than two ends. Its proof made use of the 3-d PMT. Complete manifolds whose ends are all asymptotically flat necessarily have bounded Ricci curvature. In this sense, Corollary 3.1 provides a partial generalization of Bray's result.

Proof of Corollary 3.1. Let  $\Sigma_t$  be given in the proof of Theorem 3.1. Since (M, g) is complete and has Ricci curvature bounded from below, by the gradient estimate of Cheng and Yau [8],  $\max_{\Sigma_t} u^{-1} |\nabla u| \leq \Lambda$  where  $\Lambda$  is a constant independent on t. Combined with  $\int_{\Sigma_t} |\nabla u| = 4\pi \mathcal{C}$ , this shows

$$\frac{1}{4\pi} \int_{\Sigma_t} |\nabla u|^2 \le \mathcal{C} \Lambda t,$$

which implies

(3.5) 
$$\lim_{t \to 0} \int_{\Sigma_t} |\nabla u|^2 = 0.$$

It follows from (3.5) and Theorem 3.1 (ii) that  $\mathfrak{m} \geq 2\mathfrak{C}$ .

Remark 3.2. Let R, Ric denote the scalar curvature, Ricci curvature of g. Since

 $R \ge 0$  and Ric bounded from above  $\Rightarrow$  Ric bounded from below,

Corollary 3.1 also holds if the assumption of "Ric bounded from below" is replaced by "Ric bounded from above".

Remark 3.3. As used in Bray's work [4], the inequality  $\mathfrak{m} \geq 2\mathfrak{C}$  has a geometric interpretation that asserts the mass of the conformally deformed metric  $u^4g$ , which might not be complete, is nonnegative. Instead of  $\mathfrak{m} \geq 2\mathfrak{C}$ , a weaker inequality  $\mathfrak{m} \geq \mathfrak{C}$  was obtained by Hirsch, Tam and the author in [12].

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