

Multicontact semilattices

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ABSTRACT. We characterize those join semilattices with a multicontact relation which can be embedded into (the reduct of) a Boolean algebra with multicontact given by overlap, equivalently, into a distributive lattice with additive multicontact, still equivalently, into the multicontact semilattice associated to some topological space. A similar characterization is presented for embeddability into Boolean algebras or distributive lattices with a not necessarily overlap nor additive multicontact.

1. Introduction

Structures with a contact or a proximity relation are studied in topology [6, 18, 19], in region based theory of space [1, 7, 8, 9, 13, 14, 22] and, possibly with different terminology, in algebraic logic [4, 10] and computer science [21, 23, 24, 25]. As a basic example, if X is a topological space with closure K , the binary relation δ defined by $x \delta y$ if $Kx \cap Ky \neq \emptyset$, for $x, y \subseteq X$ is a proximity, called the *standard proximity* [6, Example 2.1.3]. Of course, it is also natural to define ternary, more generally, n -ary contact relations; for example, the above relation δ can be generalized to a ternary relation δ_3 defined by $\delta_3(x, y, z)$ if $Kx \cap Ky \cap Kz \neq \emptyset$.

In the case of T_1 topological spaces the topology can be retrieved from δ , since a point $p \in X$ belongs to Kx if and only if $\{p\} \delta x$; in particular, δ_3 can be retrieved from δ . On the other hand, in pointfree or algebraic contexts, generally “ n -ary contact” cannot be retrieved from the binary contact [14, Section 7]. Boolean algebras with multicontact (actually, with a more encompassing structure) are studied in [22]. In computer science a notion equivalent to “multicontact”, with a further finiteness condition, is the consistency relation in event structures [24, Subsec. 2.1.2]. See Remark 6.4 below for details.

Here we study join semilattices with multicontact and characterize those multicontact semilattices which can be multicontact- and semilattice embedded into Boolean algebras, respectively, Boolean algebras with overlap multicontact. This is parallel to the case of a binary contact, treated in [13, 15, 17]. We show that a multicontact semilattice \mathbf{S} can be embedded into a Boolean algebra with overlap multicontact if and only if \mathbf{S} can be embedded into a distributive lattice with additive multicontact, if and only if \mathbf{S} has a topological representation (compare Example 2.2(b) below), and even if and only if

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\mathbf{S} can be associated to some distributive lattice with additive pre-closure, as described in Example 2.2(c). See Theorem 4.2. Still another equivalent condition is that \mathbf{S} has additive contact and can be embedded into a distributive lattice (with not necessarily additive multicontact). Quite remarkably, here the assumption that the contact on \mathbf{S} is additive is enough, while, in the case of “binary” contact, additivity is not enough and a rather involved condition must be assumed. See clause (2'') in Theorem 4.2 here and Condition (D2) in [15] for the case of a binary contact.

The above results show that multicontact semilattices have a quite good theory, are axiomatizable by a simple set of axioms and are a mathematically interesting object of study. Moreover the representation theorems generally do not use the axiom of choice. Apart from intrinsic interest, we now present some further motivations for the study of semilattices with further structure.

From the point of view of region based theory of space, it is interesting to study structures whose algebraic basic component is just a semilattice, rather than, as a more common approach, a Boolean algebra. This is discussed in detail in [13], where the reader can find further references. For short, if one considers large but limited regions of space, then complementation and meet might turn out to be inappropriate. Complementation is even more problematic, since the complement of some region turns out to be dependent on the universe in which one considers that region. As another argument in favor of the use of the join operation only, in [16] we proposed the project of detecting which topological properties are preserved by image functions associated to continuous maps. We believe that the project is appreciably close to intuition, due to the fact that a function f between topological spaces is continuous if and only if f preserves the adherence relation between points and subsets. In this sense, a semilattice structure is a natural setting: indeed, image functions preserve unions but not necessarily intersections or complements. Notions of contact and proximity are almost invariably preserved, as well.

From a more general point of view, classical logical and topological applications of semilattices, possibly with further structure, e.g. [5, Chapters 6–8] and of contact Boolean algebras [1, 2, 3, 7, 22], among many others, suggest that multicontact semilattices might find similar kinds of applications. The above mentioned applications to computer science [21, 23, 24, 25], where essentially equivalent notions have been independently developed, further suggest the relevance of multicontact relations, possibly when the semilattice structure is weakened to just a partial order. Finally, as a project for further research, and extending a parallel comment in [15], we believe that it is interesting also to study those multicontact semilattices for which only weaker representation theorems hold. See Problem 6.2 below.

2. Multicontact semilattices

Definition 2.1. A *multicontact poset* is a structure $(P, \leq, 0, \Delta)$, where $(P, \leq, 0)$ is a poset with minimal element 0 and Δ is a subset of $\mathcal{P}_f(P)$, the family of all the finite subsets of P . Moreover, Δ is required to satisfy the following properties, for all $m \in \mathbb{N}^+$, $a_1, a_2, \dots, a_m, p \in P$ and $F, G \in \mathcal{P}_f(P)$.

$$\{a_1, a_2, \dots, a_m\} \in \Delta \text{ implies } a_1 > 0, a_2 > 0, \dots, a_m > 0, \quad (\text{Emp}_\Delta)$$

$$F \in \Delta \text{ and } G \subseteq F \text{ imply } G \in \Delta, \quad (\text{Sub}_\Delta)$$

$$\text{if } \{a_1, a_2, \dots, a_m\} \in \Delta \text{ and } a_1 \leq b, \text{ then } \{a_1, a_2, \dots, a_m, b\} \in \Delta. \quad (\text{Mon}_\Delta)$$

$$p \neq 0 \text{ implies } \{p\} \in \Delta. \quad (\text{Ref}_\Delta)$$

Conventionally, $\emptyset \in \Delta$ is always assumed (this follows from (Ref_Δ) and (Sub_Δ) , unless $P = \{0\}$).

If $p \neq 0$ and $p \leq b_1, \dots, p \leq b_m$, then, by (Ref_Δ) and repeated applications of (Mon_Δ) we get $\{p, b_1, b_2, \dots, b_m\} \in \Delta$, thus $\{b_1, b_2, \dots, b_m\} \in \Delta$, by (Sub_Δ) . Notice that Δ is a set of unordered n -uples, hence the element a_1 in (Mon_Δ) plays no special role. We have showed that every multicontact poset satisfies the following condition.

$$p \neq 0 \text{ and } p \leq b_1, \dots, p \leq b_m \text{ imply } \{b_1, b_2, \dots, b_m\} \in \Delta. \quad (\text{Ov}_\Delta)$$

More generally, by the same argument, we get

$$(\text{Cof}) \text{ Suppose that } \{p_1, \dots, p_m\} \text{ and } \{q_1, \dots, q_n\} \text{ are two finite subsets of } P \text{ and, for every } j \leq n, \text{ there is some } i \leq m \text{ such that } p_i \leq q_j. \text{ If } \{p_1, \dots, p_m\} \in \Delta, \text{ then } \{q_1, \dots, q_n\} \in \Delta.$$

In particular,

$$\{a_1, \dots, a_m\} \in \Delta \text{ and } a_1 \leq b_1, \dots, a_m \leq b_m \text{ imply } \{b_1, \dots, b_m\} \in \Delta \quad (\text{Ext}_\Delta)$$

A *multicontact semilattice* \mathbf{S} is a join semilattice with 0 together with a family $\Delta \subseteq \mathcal{P}_f(S)$ satisfying the above properties, where \leq is the partial order induced by the semilattice operation, namely, $a \leq b$ if $a + b = b$. *Multicontact lattices, Boolean algebras*, etc. are defined in an analogous way.

A multicontact semilattice is *additive* if the following holds.

$$\begin{aligned} &\text{If } \{p + q, p_2, \dots, p_m\} \in \Delta, \text{ then} \\ &\text{either } \{p, p_2, \dots, p_m\} \in \Delta, \text{ or } \{q, p_2, \dots, p_m\} \in \Delta. \end{aligned} \quad (\text{Add}_\Delta)$$

Notice that we need at least a semilattice operation in order to define the notion of additivity; a partial order is not enough.

Examples 2.2. (a) If $(P, \leq, 0)$ is a poset and we let $\{a_1, a_2, \dots, a_m\} \in \Delta$ if there is $p \in P$, $p > 0$ such that $p \leq a_1, \dots, p \leq a_m$, then Δ is a multicontact on \mathbf{P} . Such a Δ will be called the *overlap* multicontact over \mathbf{P} . Thus in a poset with overlap multicontact (Ov_Δ) becomes an if and only if condition.

We will show in Lemma 3.4(a) that a *distributive* lattice with overlap multicontact is additive. In general, this is not true: if \mathbf{M}_3 is the 5-element

nondistributive modular lattice with 3 atoms, then the overlap multicontact on \mathbf{M}_3 is not additive.

(b) If X is a topological space with closure K , $S = \mathcal{P}(X)$ and, for $a_1, a_2, \dots, a_m \subseteq X$, we set $\{a_1, a_2, \dots, a_m\} \in \Delta$ if $Ka_1 \cap Ka_2 \cap \dots \cap Ka_m \neq \emptyset$, then $(S, \cup, \emptyset, \Delta)$ is an additive multicontact semilattice, which will be called the *multicontact semilattice associated to X* . Actually, we get a Boolean algebra, if we consider also union and complementation.

(c) More generally, assume that $(P, \leq, 0, K)$ is a *normal pre-closure poset*. This means that K is a unary normal, extensive and isotone operation on P , namely, K satisfies $K0 = 0$, $Kx \geq x$, for all $x \in P$ and, moreover, $x \leq y$ implies $Kx \leq Ky$. If K is also idempotent, that is, $KKx = Kx$, then K is called a (normal) *closure operation*.

In a normal pre-closure poset, setting $\{a_1, a_2, \dots, a_m\} \in \Delta$ if there is $p \in P$, $p > 0$ such that $p \leq Ka_1, \dots, p \leq Ka_m$, we get a multicontact Δ on \mathbf{P} , which shall be called the *multicontact associated to K* . The assumption that K is extensive can be weakened; it is enough to assume that $p > 0$ implies $Kp > 0$.

(d) Recall that a *weak contact* relation on a poset \mathbf{P} with 0 is a symmetric and reflexive binary relation δ on $P \setminus \{0\}$ such that

$$a \delta b, a \leq a_1 \text{ and } b \leq b_1 \text{ imply } a_1 \delta b_1. \quad (\text{Ext})$$

The relation δ is *additive* if

$$a \delta b + c \text{ implies } a \delta b \text{ or } a \delta c. \quad (\text{Add})$$

The *overlap weak contact* relation is defined by $a \delta b$ if there is $p > 0$ such that $p \leq a$ and $p \leq b$.

(e) If δ is a weak contact on a poset \mathbf{P} with 0, then, setting $\{a_1, a_2, \dots, a_m\} \in \Delta_\ell$ if $a_i \delta a_j$, for all $i, j \leq m$, we get a multicontact Δ_ℓ on \mathbf{P} .

Notice that, even when δ is the weak contact overlap, it might happen that Δ_ℓ , as defined above, is not the *multicontact* overlap. For example, in the 8-element Boolean algebra the three coatoms are pairwise in contact (for every weak contact relation), hence are in Δ_ℓ , as defined above. On the other hand, the set of the three coatoms is not in the overlap multicontact, since their meet is 0.

The same example shows that even when δ is additive, then Δ_ℓ , as defined above, is not necessarily additive. The weak contact overlap on a distributive lattice is additive [9, Lemma 2, item 1], the proof is similar to the proof of Lemma 3.4 below. Thus the overlap weak contact on the 8-element Boolean algebra \mathbf{B} is additive. Let the coatoms of \mathbf{B} be c_1, c_2, c_3 and the atoms a_1, a_2, a_3 , with $c_i = a_j + a_k$, for $\{i, j, k\} = \{1, 2, 3\}$. As in the previous paragraph, $\{c_1, c_2, c_3\} \in \Delta_\ell$, that is, $\{a_2 + a_3, c_2, c_3\} \in \Delta_\ell$, but neither $\{a_2, c_2, c_3\} \in \Delta_\ell$, nor $\{a_3, c_2, c_3\} \in \Delta_\ell$, since $a_2c_2 = 0$ and $a_3c_3 = 0$, that is, $a_2 \not\delta c_2$ and $a_3 \not\delta c_3$.

(f) Suppose again that δ is a weak contact on a poset \mathbf{P} . Set $\{a_1, a_2, \dots, a_m\} \in \Delta_s$ if there are $p, q \in P$ (possibly, $p = q$) such that $p \delta q$ and, for every $i \leq m$, either $p \leq a_i$, or $q \leq a_i$. Then Δ_s is a multicontact on \mathbf{P} .

In this case, if δ is the overlap weak contact, then Δ_s is the overlap multicontact.

On the other hand, it may happen that δ is additive but Δ_s is not. For example, let \mathbf{M}_4 be the 6-element modular lattice with 4 atoms a_1, a_2, a_3, a_4 and let all pairs of nonzero elements be δ -related. Then δ is additive. However, $\{a_1, a_2, a_3 + a_4\} \in \Delta_s$, but neither $\{a_1, a_2, a_3\} \in \Delta_s$, nor $\{a_1, a_2, a_4\} \in \Delta_s$, thus Δ_s is not additive.

(g) In the other direction, if Δ is a multicontact on \mathbf{P} , then δ defined by $a \delta b$ if $\{a, b\} \in \Delta$ is a weak contact relation, the *binary reduct* of Δ . If this is the case, we shall also say that Δ is an *expansion* of δ .

(h) If δ is a weak contact on \mathbf{P} , then Δ_ℓ as defined in (e), resp., Δ_s as defined in (f), are the largest, resp., the smallest multicontact on \mathbf{P} such that the binary reduct of Δ_ℓ , resp., Δ_s is again δ .

Some of the constructions in (e) - (g) are known in the framework of event structures [11, 23, 24, 25].

(i) Suppose that \mathbf{S} is a poset with 0 and such that every nonzero element of \mathbf{S} is larger than some atom of \mathbf{S} . Let A be the set of atoms of \mathbf{S} and let Δ_A be a family of finite subsets of A such that Δ_A is closed under subsets and contains all singletons from A . Let Δ be the family of those finite subsets x of S such that there is $y \in \Delta_A$ such that for every $b \in x$ there is $a \in \Delta_A$ such that $a \leq b$. Then Δ is a multicontact on \mathbf{S} .

(j) Recall that a join semilattice is *distributive* if, whenever $a \leq b + c$, there are $b^* \leq b$ and $c^* \leq c$ such that $a = b^* + c^*$. In addition to the assumptions from (i), suppose further that \mathbf{S} is a distributive join semilattice (in particular, this applies if \mathbf{S} satisfies the assumptions from (i) and has the structure of a distributive lattice). Then Δ , as defined in (i), is additive.

Indeed, if $\{p + q, p_2, \dots, p_m\} \in \Delta$, then, by definition, there is $y \in \Delta_A$ such that $a_1 \leq p + q$, $a_2 \leq p_2$, for certain elements $a_1, a_2, \dots \in y$. From $a_1 \leq p + q$, by distributivity, we get $a_1 = p^* + q^*$, for some $p^* \leq p$ and $q^* \leq q$. Since $a_1 \in A$ is an atom, then either $a_1 = p^*$ or $a_1 = q^*$, hence, say in the former case $a_1 = p^* \leq p$. Thus $\{p, p_2, \dots, p_m\} \in \Delta$, as witnessed by the same $y \in \Delta_A$.

(k) By (Sub_Δ) and (Mon_Δ) , a multicontact Δ is determined by the set of the antichains in Δ .

(ℓ) Let \mathbf{P} be a poset with 0. For every $n \geq 1$, there is the smallest multicontact Δ_n containing all subsets of $P \setminus \{0\}$ of cardinality $\leq n$. Explicitly, some set $\{b_1, b_2, \dots, b_m\}$ lies in Δ_n if and only if, for some $h \leq n$, there is a set p_1, \dots, p_h of nonzero elements such that, for every $i \leq m$, there is $j \leq h$ such that $p_j \leq b_i$. In particular, Δ_1 is the overlap multicontact.

By (k) above, Δ_n is “generated” by the antichains of cardinality $\leq n$, excluding the “zero” antichain $\{0\}$.

3. Some auxiliary notions and lemmas

We now consider some conditions a multicontact semilattice \mathbf{S} might or might not satisfy.

$$\begin{aligned} &\text{For every } n \in \mathbb{N}^+, a, b, p_1, \dots, p_n \in S, \\ &\text{if } b \leq a + p_1, \dots, b \leq a + p_n \text{ and } \{p_1, \dots, p_n\} \notin \Delta, \text{ then } b \leq a. \end{aligned} \quad (\text{M1})$$

Note that, by taking $a = 0$ and $p = p_1 = \dots = p_n > 0$, (M1) implies (Ov_Δ) .

In many of the following conditions we shall consider finite sequences $(c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}), \dots, (c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n})$ of varying lengths, and functions $f : \{1, \dots, n\} \rightarrow \mathbb{N}^+$ such that $f(i)$ is a possible second index of the i th sequence, namely, $f(i) \leq \ell_i$, for each $i \leq n$. For the sake of brevity, a function satisfying the above condition will be called *compatible* (the sequences under consideration will be always clear from the context).

Lemma 3.1. *Suppose that \mathbf{S} is a multicontact semilattice and \mathbf{S} satisfies (M1). Then \mathbf{S} satisfies the following condition.*

$$\begin{aligned} &\text{For all } n, \ell_1, \ell_2, \dots, \ell_n \in \mathbb{N}^+ \text{ and} \\ &a, b, c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}, c_{2,1}, c_{2,2}, \dots, c_{2,\ell_2}, \dots, c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n} \in S, \\ &\text{IF } \{c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}\} \notin \Delta, \dots, \{c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n}\} \notin \Delta, \text{ and} \quad (\text{M1+}) \\ &b \leq a + c_{1,f(1)} + \dots + c_{n,f(n)}, \text{ for every compatible } f, \\ &\text{THEN } b \leq a. \end{aligned}$$

Proof. The proof is by induction on $n \geq 1$. As for the base step, (M1) is the special case $n = 1$ of (M1+). For the induction step, suppose that all instances of (M1+) hold for some specific $n > 0$, and suppose that the assumptions of (M1+) are satisfied for $n + 1$, say, for certain elements $a, b, \dots, c_{1,1}, \dots, c_{n,\ell_n}, c_{n+1,1}, \dots, c_{n+1,\ell_{n+1}}$. From $b \leq a + c_{1,f(1)} + \dots + c_{n,f(n)} + c_{n+1,f(n+1)}$, for all compatible functions $f : \{1, \dots, n, n + 1\} \rightarrow \mathbb{N}^+$, we get $b \leq a + c_{n+1,1} + c_{1,g(1)} + \dots + c_{n,g(n)}$, for all compatible $g : \{1, \dots, n\} \rightarrow \mathbb{N}^+$. By applying (M1+) in case n with $a + c_{n+1,1}$ in place of a , we get $b \leq a + c_{n+1,1}$. Similarly, $b \leq a + c_{n+1,2}, \dots, b \leq a + c_{n+1,\ell_{n+1}}$. Then apply (M1) with ℓ_{n+1} in place of n and $p_i = c_{n+1,i}$, for $i \leq \ell_{n+1}$, getting $b \leq a$. \square

We now introduce another relevant condition.

$$\begin{aligned} &\text{For all } m, n, \ell_1, \ell_2, \dots, \ell_n \in \mathbb{N}^+ \text{ and } a_1, a_2, \dots, a_m, \\ &c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}, c_{2,1}, c_{2,2}, \dots, c_{2,\ell_2}, \dots, c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n} \in S, \\ &\text{IF } \{c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}\} \notin \Delta, \dots, \{c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n}\} \notin \Delta \text{ and,} \quad (\text{M2}) \\ &\text{for every compatible } f : \{1, \dots, n\} \rightarrow \mathbb{N}^+, \\ &\text{there is } j \leq m \text{ such that } a_j \leq c_{1,f(1)} + \dots + c_{n,f(n)}, \\ &\text{THEN } \{a_1, a_2, \dots, a_m\} \notin \Delta. \end{aligned}$$

Remark 3.2. The case $n = 1$ of (M2) implies (Cof). Under suitable conventions about the empty sum, (Emp_Δ) can be considered the “improper” case $n = 0$ of (M2). Moreover, (M2) implies additivity. If $m \geq 1$, take $n = 2$, $\ell_1 = \ell_2 = m$, $a_1 = c_{1,1} + c_{2,1}$, $a_2 = c_{1,2} = c_{2,2}$, $a_3 = c_{1,3} = c_{2,3}$, \dots , $a_m = c_{1,m} = c_{2,m}$. From (M2) we get that if $\{c_{1,1}, a_2, \dots, a_m\} \notin \Delta$ and $\{c_{2,1}, a_2, \dots, a_m\} \notin \Delta$, then $\{c_{1,1} + c_{2,1}, a_2, \dots, a_m\} \notin \Delta$. This is (Add_Δ) in contrapositive form. The case $m = 1$ is not covered by the above argument; however, the case $m = 1$ is immediate from (Emp_Δ) and (Ref_Δ) .

We shall now see that additivity implies (M2), thus they are in fact equivalent. This fact strongly contrasts the situation in the binary case of contact relations, where (D2), a condition analogue to (M2), does not follow from additivity [15, Example 5.2(d)]. In this respect, see also [17, Theorem 4.1].

Proposition 3.3. *A multicontact semilattice \mathbf{S} is additive if and only if \mathbf{S} satisfies (M2).*

Proof. Sufficiency has been proved in the above remark. In order to prove the other direction, we need a claim.

Claim. *If \mathbf{S} is an additive multicontact semilattice, then, for all sequences $c_{1,1}, \dots, c_{n,\ell_n}$ of elements as in the first two lines of (M2), the following conditions are equivalent.*

- (1) *Either $\{c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}\} \in \Delta$, or $\{c_{2,1}, c_{2,2}, \dots, c_{2,\ell_2}\} \in \Delta$, \dots , or $\{c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n}\} \in \Delta$,*
- (2) *$\{c_{1,f(1)} + \dots + c_{n,f(n)} \mid f \text{ a compatible function}\} \in \Delta$.*

Indeed, (1) \Rightarrow (2) is immediate from (Cof) (and here we have not used additivity).

In the other direction, if (2) holds, then, by iterating the definition of additivity, we get that there is a way of choosing one summand from each sum of the form $c_{1,f(1)} + \dots + c_{n,f(n)}$, f varying among compatible functions, in such a way that the set C of the chosen summands belongs to Δ . We claim that C contains at least one among the sets $\{c_{1,1}, \dots, c_{1,\ell_1}\}, \dots, \{c_{n,1}, \dots, c_{n,\ell_n}\}$. Indeed, if this is not the case, then, for every $i = 0, \dots, n$, there is some $c_{i,g(i)}$ which is not in C . This means that we have chosen no element from the sum $c_{1,g(1)} + \dots + c_{n,g(n)}$, a contradiction. Thus $\{c_{i,1}, \dots, c_{i,\ell_i}\} \subseteq C$, for some $i \leq n$, but $C \in \Delta$, hence $\{c_{i,1}, \dots, c_{i,\ell_i}\} \in \Delta$, by (Sub_Δ) .

Having proved the Claim, suppose that \mathbf{S} is additive and that the assumptions in (M2) are satisfied. If, by contradiction, $\{a_1, a_2, \dots, a_m\} \in \Delta$, then $\{c_{1,f(1)} + \dots + c_{n,f(n)} \mid f \text{ a compatible function}\} \in \Delta$, by (Cof). Thus $\{c_{i,1}, \dots, c_{i,\ell_i}\} \in \Delta$, for some $i \leq n$, by the above Claim, contradicting the assumptions in (2). We have proved that if \mathbf{S} is additive, then (M2) holds, thus completing the proof of the proposition. \square

A lattice is *meet semidistributive at 0* if, for all elements p, q, r , $pr = 0$ and $qr = 0$ imply $(p + q)r = 0$. More generally, a join semilattice with 0 is

semidistributive at 0 if, whenever the meets of p, p_2, \dots, p_n and of q, p_2, \dots, p_n both exist and are equal to 0, then the meet of $p + q, p_2, \dots, p_n$ exists and is equal to 0. A pre-closure K on some semilattice P is *additive*, if $K(x + y) = Kx + Ky$ holds for all $x, y \in P$.

Lemma 3.4. (a) *A distributive lattice with overlap multicontact satisfies the condition (Add_Δ) . More generally, a join semilattice semidistributive at 0 with overlap contact satisfies (Add_Δ) .*

(b) *If \mathbf{P} is a distributive lattice with a normal additive pre-closure K , then the multicontact associated to K , as defined in Example 2.2(c), is additive.*

The assumption that \mathbf{P} is a distributive lattice can be weakened; it is enough to assume that \mathbf{P} is a join semilattice with 0 and \mathbf{P} is semidistributive at 0.

Proof. (a) If $\{p, p_2, \dots, p_m\} \notin \Delta$ and $\{q, p_2, \dots, p_m\} \notin \Delta$, then the meets of p, p_2, \dots, p_m and of q, p_2, \dots, p_m exist and are equal to 0, by (Ov_Δ) . By semidistributivity at 0, the meet of $p + q, p_2, \dots, p_m$ exists and is equal to 0, thus $\{p + q, p_2, \dots, p_m\} \notin \Delta$, since Δ is overlap.

(b) If $\{p, p_2, \dots, p_m\} \notin \Delta$ and $\{q, p_2, \dots, p_m\} \notin \Delta$, then the meets of Kp, Kp_2, \dots, Kp_m and of Kq, Kp_2, \dots, Kp_m exist and are equal to 0, by the definition of Δ . By semidistributivity at 0, the meet of $Kp + Kq, Kp_2, \dots, Kp_m$ exists and is equal to 0. Since K is additive, $Kp + Kq = K(p + q)$ and again the definition of Δ gives $\{p + q, p_2, \dots, p_m\} \notin \Delta$. \square

Lemma 3.5. *If \mathbf{S} is a multicontact distributive lattice, then \mathbf{S} satisfies (M1).*

Proof. If $a, b, p_1, p_2, \dots, p_n \in S$ and $b \leq a + p_1, b \leq a + p_2, \dots, b \leq a + p_n$, then, by distributivity, $b \leq (a + p_1)(a + p_2) \dots (a + p_n) = a + p_1 p_2 \dots p_n$. If $\{p_1, p_2, \dots, p_n\} \notin \Delta$, then $p_1 p_2 \dots p_n = 0$, by (Ov_Δ) , hence $b \leq a$. This proves (M1). \square

Lemma 3.6. *If \mathbf{S} is a semilattice with overlap multicontact and \mathbf{S} satisfies (M1), then \mathbf{S} satisfies (M2), hence \mathbf{S} satisfies (Add_Δ) , by Proposition 3.3.*

Proof. Assume that the hypotheses of (M2) are satisfied. We want to show that the meet of a_1, a_2, \dots, a_m exists and is 0. Indeed, if $b \in S$ and $b \leq a_1, \dots, b \leq a_m$, then, by the hypotheses in (M2), $b \leq c_{1,f(1)} + \dots + c_{n,f(n)}$, for every compatible f . By Lemma 3.1, \mathbf{S} satisfies (M1+). By taking $a = 0$ in (M1+) we get $b = 0$. Thus 0 is the meet of a_1, a_2, \dots, a_m . Since \mathbf{S} has overlap multicontact, then $\{a_1, a_2, \dots, a_m\} \notin \Delta$, which is the conclusion of (M2), what we had to show. \square

4. Embeddings into overlap Boolean algebras

Definition 4.1. If \mathbf{P} and \mathbf{Q} are multicontact posets, an *embedding* φ from \mathbf{P} to \mathbf{Q} is an order embedding from P to Q which preserves 0 and such that

$$\{p_1, p_2, \dots, p_n\} \in \Delta \text{ if and only if } \{\varphi(p_1), \varphi(p_2), \dots, \varphi(p_n)\} \in E, \quad (4.1)$$

for every $n \in \mathbb{N}$ and $p_1, p_2, \dots, p_n \in P$, where E is the multicontact on \mathbf{Q} . A Δ -homomorphism is only required to satisfy the “only if” condition in (4.1).

When dealing with semilattices φ is also assumed to preserve $+$.

In what follows we shall frequently deal with the situation in which multicontact semilattices are embedded into models with further structure, e. g., distributive lattices or Boolean algebras. Rather than explicitly saying that a multicontact semilattice \mathbf{S} can be embedded into *the multicontact semilattice reduct* of some multicontact Boolean algebra \mathbf{B} , we shall simply say, with a slight abuse of terminology, that \mathbf{S} can be $\{\Delta, +\}$ -embedded into \mathbf{B} . Notice that, on the other hand, we are never assuming that embeddings preserve existing meets, or complements, etc.

Theorem 4.2. *For every multicontact semilattice \mathbf{S} , the following conditions are equivalent, where embeddings are always intended as $\{\Delta, +\}$ -embeddings.*

- (1) \mathbf{S} can be embedded into a Boolean algebra with overlap multicontact.
- (1') \mathbf{S} can be embedded into a Boolean algebra with additive multicontact.
- (2) \mathbf{S} can be embedded into a distributive lattice with overlap multicontact.
- (2') \mathbf{S} can be embedded into a distributive lattice with additive multicontact.
- (2'') \mathbf{S} is additive and can be embedded into a multicontact distributive lattice.
- (3) \mathbf{S} is additive and satisfies (M1).
- (4) \mathbf{S} can be embedded into a complete atomic Boolean algebra with overlap multicontact.
- (5) \mathbf{S} can be embedded into the multicontact semilattice associated to some topological space, as in Example 2.2(b).
- (6) \mathbf{S} can be embedded into the multicontact semilattice associated to some distributive lattice with additive pre-closure, as in Example 2.2(c).
- (7) \mathbf{S} can be embedded into a multicontact semilattice satisfying (M1) and with overlap multicontact.

Proof. A few arguments are similar to [15, Theorem 3.2]; we give all the details for the reader's convenience. The implications (1) \Rightarrow (1') \Rightarrow (2') and (1) \Rightarrow (2) \Rightarrow (2') are either straightforward or immediate from Lemma 3.4(a). Also (2') \Rightarrow (2'') is elementary, since if \mathbf{S} can be embedded into an additive multicontact semilattice, then \mathbf{S} is additive, as well.

(2'') \Rightarrow (3) By assumption, there is an embedding $\iota : \mathbf{S} \rightarrow \mathbf{T}$, where \mathbf{T} has the structure of a multicontact distributive lattice. By Lemma 3.5, \mathbf{T} satisfies (M1). Property (M1) is clearly preserved under substructures, hence \mathbf{S} satisfies (M1), being isomorphic to a substructure of \mathbf{T} .

(3) \Rightarrow (1) Assume that $\mathbf{S} = (S, \leq, 0, \Delta)$ is an additive multicontact semilattice satisfying (M1). By Proposition 3.3 \mathbf{S} satisfies (M2). Let \mathbf{B} be the Boolean algebra $(\mathcal{P}(S), \cup, \cap, \emptyset, S, \mathbb{C})$ and let $\varphi : S \rightarrow \mathcal{P}(S)$ be the semilattice embedding defined by $\varphi(a) = \nexists a = \{x \in S \mid a \not\leq x\}$. Let \mathbf{A} be the quotient \mathbf{B}/\mathcal{I} , where \mathcal{I} is the ideal of \mathbf{B} generated by the set of all the elements of the form $\varphi(c_1) \cap \varphi(c_2) \cap \dots \cap \varphi(c_\ell)$, where $\ell \in \mathbb{N}^+$ and $c_1, c_2, \dots, c_\ell \in S$ are such

that $\{c_1, c_2, \dots, c_\ell\} \notin \Delta$. Let $\pi : \mathbf{B} \rightarrow \mathbf{A}$ be the quotient homomorphism and $\kappa = \varphi \circ \pi$. Then κ is a semilattice homomorphism from \mathbf{S} to (the semilattice reduct of) \mathbf{A} .

Let \mathbf{A} be endowed with the overlap multicontact. It is sufficient to show that κ is a multicontact embedding from \mathbf{S} to \mathbf{A} . We first prove that κ is injective. For this, it is enough to show that if $\kappa(b) \leq \kappa(a)$ in \mathbf{A} , then $b \leq a$ in \mathbf{S} . If $\kappa(b) \leq \kappa(a)$, then $\varphi(b) \subseteq \varphi(a) \cup i$, for some $i \in \mathcal{I}$, that is,

$$\begin{aligned} \varphi(b) \subseteq \varphi(a) \cup (\varphi(c_{1,1}) \cap \varphi(c_{1,2}) \cap \dots \cap \varphi(c_{1,\ell_1})) \cup \dots \\ \cup (\varphi(c_{n,1}) \cap \varphi(c_{n,2}) \cap \dots \cap \varphi(c_{n,\ell_n})) \end{aligned} \quad (4.2)$$

for some $n \in \mathbb{N}$ and $c_{1,1}, \dots, c_{n,\ell_n} \in S$ such that $\{c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}\} \notin \Delta, \dots, \{c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n}\} \notin \Delta$. By distributivity, (4.2) reads

$$\varphi(b) \subseteq \bigcap_{\substack{f: \{1, \dots, n\} \rightarrow \mathbb{N}^+ \\ f \text{ compatible}}} (\varphi(a) \cup \varphi(c_{1,f(1)}) \cup \dots \cup \varphi(c_{n,f(n)}),$$

which is equivalent to

$$\begin{aligned} \varphi(b) \subseteq \varphi(a) \cup \varphi(c_{1,f(1)}) \cup \dots \cup \varphi(c_{n,f(n)}), \\ \text{for all compatible } f : \{1, \dots, n\} \rightarrow \mathbb{N}^+. \end{aligned}$$

This holds if and only if in \mathbf{S}

$$b \leq a + c_{1,f(1)} + \dots + c_{n,f(n)}, \text{ for all compatible } f : \{1, \dots, n\} \rightarrow \mathbb{N}^+,$$

since φ is a semilattice embedding. By Lemma 3.1, \mathbf{S} satisfies (M1+), hence $b \leq a$. We have showed that κ is injective.

We now show that κ is a multicontact embedding. If $\{a_1, a_2, \dots, a_m\} \notin \Delta$, then $\kappa(a_1)\kappa(a_2) \dots \kappa(a_m) = 0$, since, by definition, $\varphi(a_1) \cap \varphi(a_2) \cap \dots \cap \varphi(a_m)$ is in \mathcal{I} . Hence $\{\kappa(a_1), \kappa(a_2), \dots, \kappa(a_m)\} \notin \Delta$, since Δ is the overlap multicontact on \mathbf{A} .

For the converse, suppose that $\{a_1, a_2, \dots, a_m\} \in \Delta$. We have to show that in \mathbf{A} $\{\kappa(a_1), \kappa(a_2), \dots, \kappa(a_m)\} \in \Delta$, that is, $\kappa(a_1)\kappa(a_2) \dots \kappa(a_m) > 0$, since Δ is overlap on \mathbf{A} . This means $\varphi(a_1) \cap \varphi(a_2) \cap \dots \cap \varphi(a_m) \notin \mathcal{I}$. Assume the contrary, that is,

$$\begin{aligned} \varphi(a_1) \cap \varphi(a_2) \cap \dots \cap \varphi(a_m) \subseteq (\varphi(c_{1,1}) \cap \varphi(c_{1,2}) \cap \dots \cap \varphi(c_{1,\ell_1})) \cup \dots \\ \cup (\varphi(c_{n,1}) \cap \varphi(c_{n,2}) \cap \dots \cap \varphi(c_{n,\ell_n})) \end{aligned} \quad (4.3)$$

for some $n \in \mathbb{N}$ and $c_{1,1}, \dots, c_{n,\ell_n} \in S$ such that $\{c_{1,1}, c_{1,2}, \dots, c_{1,\ell_1}\} \notin \Delta, \dots, \{c_{n,1}, c_{n,2}, \dots, c_{n,\ell_n}\} \notin \Delta$. Arguing as in the above proof for the injectivity of κ , the inclusion (4.3) means

$$\begin{aligned} \varphi(a_1) \cap \varphi(a_2) \cap \dots \cap \varphi(a_m) \subseteq \varphi(c_{1,f(1)} + \dots + c_{n,f(n)}), \\ \text{for all compatible } f : \{1, \dots, n\} \rightarrow \mathbb{N}^+. \end{aligned}$$

Recalling that $\varphi(a) = \nabla a$, setting $\uparrow a = \{x \in S \mid x \geq a\}$ and taking complements, we get

$$\uparrow a_1 \cup \uparrow a_2 \cup \dots \cup \uparrow a_m \supseteq \uparrow (c_{1,f(1)} + \dots + c_{n,f(n)}),$$

for all compatible $f : \{1, \dots, n\} \rightarrow \mathbb{N}^+$, that is,

$$\text{for every compatible } f : \{1, \dots, n\} \rightarrow \mathbb{N}^+, \text{ there is } j \leq m \text{ such that} \\ c_{1,f(1)} + \dots + c_{n,f(n)} \geq a_j.$$

By (M2), this implies $\{a_1, a_2, \dots, a_m\} \notin \Delta$, a contradiction.

So far, we have proved that (1) - (3) are equivalent.

The implication (4) \Rightarrow (1) is straightforward. The implication (1) \Rightarrow (4) is like the corresponding implication in [15, Theorem 3.2]. Since every Boolean algebra can be extended to a complete atomic Boolean algebra, we are allowed to embed the algebra given by (1) into a complete atomic Boolean algebra. Give this larger algebra, too, the overlap multicontact. Since Boolean embeddings preserve meets and since, with overlap multicontact, $\{a_1, a_2, \dots, a_m\} \in \Delta$ is equivalent to $a_1 a_2 \dots a_m > 0$ (in lattices, hence in Boolean algebras), then the embedding preserves the multicontact, as well.

(4) \Rightarrow (5) A complete atomic Boolean algebra \mathbf{B} is isomorphic to a field of sets, say, $\mathcal{P}(X)$. If we give X the discrete topology, then Δ as defined in Example 2.2(b) corresponds exactly to the overlap multicontact in \mathbf{B} .

(5) \Rightarrow (6) is straightforward. (6) \Rightarrow (2') follows from Lemma 3.4(b). Hence (1) - (6) are all equivalent.

(2) \Rightarrow (7). Assume (2), thus \mathbf{S} can be $\{\Delta, +\}$ -embedded into some multicontact distributive lattice \mathbf{T} with overlap multicontact. By Lemma 3.5, \mathbf{T} satisfies (M1).

(7) \Rightarrow (3). By (7), \mathbf{S} can be $\{\Delta, +\}$ -embedded into some semilattice \mathbf{T} with overlap multicontact and satisfying (M1). By Lemma 3.6, \mathbf{T} is additive. Thus \mathbf{S} is isomorphic to a substructure of \mathbf{T} , hence \mathbf{S} is additive and satisfies (M1), since both properties are preserved under taking substructures and isomorphism. \square

Remark 4.3. If in Theorem 4.2 we consider *bounded semilattices*, that is, semilattices with a maximum 1, which is supposed to be preserved by homomorphisms, the same proof carries over, by considering as \mathbf{B} the Boolean algebra on $\mathcal{P}(S \setminus \{1\})$ in the proof of (3) \Rightarrow (1).

5. Embeddings into nonoverlap Boolean algebras

Condition (M1) is sufficient in order to get that a multicontact semilattice can be $\{\Delta, +\}$ -embedded into a multicontact distributive lattice. Here we are not assuming that the multicontact is overlap. We first state a handy lemma.

Lemma 5.1. *Suppose that $\mathbf{S} = (S, \leq, 0, \Delta_S)$ is a poset with multicontact, $\mathbf{Q} = (Q, \leq, 0)$ is a poset with 0 and κ is an order preserving function from \mathbf{S} to \mathbf{Q} such that $a = 0$ if and only if $\kappa(a) = 0$, for every $a \in S$.*

Let Δ_Q be defined on \mathbf{Q} by letting $\{b_1, b_2, \dots, b_m\} \in \Delta_Q$, for $b_1, b_2, \dots, b_m \in Q$, if either

(a) there is $q \in Q$ such that $0 < q, q \leq b_1, q \leq b_2, \dots, q \leq b_m$, or

(b) there are $a_1, a_2, \dots, a_r \in S$ such that $\{a_1, a_2, \dots, a_r\} \in \Delta_S$ and, for every $i \leq m$, there is $j \leq r$ such that $\kappa(a_j) \leq b_i$. Then

- (i) Δ_Q is a multicontact on \mathbf{Q} and κ is a multicontact homomorphism from \mathbf{S} to \mathbf{Q} . In fact, Δ_Q is the smallest multicontact on \mathbf{Q} which makes κ a multicontact homomorphism.
- (ii) Suppose in addition that κ is an order embedding such that, whenever $\{a_1, a_2, \dots, a_r\} \notin \Delta_S$, the meet of $\kappa(a_1), \kappa(a_2), \dots, \kappa(a_r)$ in \mathbf{Q} exists and is equal to 0. Then κ is a multicontact embedding from \mathbf{S} to \mathbf{Q} .
- (iii) In particular, if both \mathbf{S} and \mathbf{Q} have a meet semilattice structure and κ is a 0-preserving meet semilattice embedding, then κ is a multicontact embedding from \mathbf{S} to \mathbf{Q} .

Proof. (i) The properties (Sub_Δ) , (Mon_Δ) and (Ref_Δ) for Δ_Q are immediate. By assumption, if $a \neq 0$, then $\kappa(a) \neq 0$, thus (Emp_Δ) holds in \mathbf{Q} , since it holds in \mathbf{S} . Thus Δ_Q is a multicontact on \mathbf{Q} and κ is a multicontact homomorphism by construction. Every multicontact on \mathbf{Q} must contain all the m -uples $\{b_1, b_2, \dots, b_m\}$ for which (a) holds, because of (Ov_Δ) . If κ is a Δ -homomorphism and $\{a_1, a_2, \dots, a_r\} \in \Delta_S$, then $\{\kappa(a_1), \kappa(a_2), \dots, \kappa(a_r)\} \in \Delta_Q$. If $\{b_1, b_2, \dots, b_m\}$ is an m -uple for which (b) holds with respect to such a_i s, then $\{b_1, b_2, \dots, b_m\} \in \Delta_Q$ because of (Cof) . Thus Δ_Q is the smallest multicontact on \mathbf{Q} with the required property.

(ii) In view of (i), we just need to check that if $\{c_1, c_2, \dots, c_m\} \notin \Delta_S$, then $\{\kappa(c_1), \kappa(c_2), \dots, \kappa(c_m)\} \notin \Delta_Q$. By assumption, $\kappa(c_1)\kappa(c_2)\dots\kappa(c_m) = 0$, hence (a) cannot be applied in order to get $\{\kappa(c_1), \kappa(c_2), \dots, \kappa(c_m)\} \in \Delta_Q$. Were (b) applicable, there should be $a_1, a_2, \dots, a_r \in S$ such that $\{a_1, a_2, \dots, a_r\} \in \Delta_S$ and, for every $i \leq m$, there is $j \leq r$ such that $\kappa(a_j) \leq \kappa(c_i)$. Since κ is an order embedding, then $a_j \leq c_i$ for the corresponding indices. Then $\{a_1, a_2, \dots, a_r\} \in \Delta_S$ and (Cof) imply $\{c_1, c_2, \dots, c_m\} \in \Delta_S$, a contradiction.

(iii) If $\{a_1, a_2, \dots, a_r\} \notin \Delta_S$, then $a_1 a_2 \dots a_r = 0$ by (Ov_Δ) , hence $\kappa(a_1)\kappa(a_2)\dots\kappa(a_r) = 0$, since κ is a meet semilattice embedding, thus we can apply (ii). \square

Recall from Definition 4.1 that we consider embeddings which preserve the semilattice and the multicontact structure, but not necessarily further structure, even when the target structure is richer.

Theorem 5.2. *For every multicontact semilattice \mathbf{S} , the following conditions are equivalent, where embeddings are always intended as $\{\Delta, +\}$ -embeddings.*

- (1) \mathbf{S} can be embedded into a multicontact Boolean algebra.
- (2) \mathbf{S} can be embedded into a multicontact distributive lattice.
- (3) \mathbf{S} satisfies (M1).
- (4) \mathbf{S} can be embedded into a multicontact complete atomic Boolean algebra.

Proof. The implications (1) \Rightarrow (2) and (4) \Rightarrow (1) are straightforward.

(2) \Rightarrow (3) follows from Lemma 3.5, arguing as in the proof of (2'') \Rightarrow (3) in Theorem 4.2.

(3) \Rightarrow (1) Let the Boolean algebra \mathbf{A} and the embedding κ be defined as in the proof of (3) \Rightarrow (1) in Theorem 4.2. Since the proof there that κ is injective uses only (M1), we get that κ is a semilattice embedding in the present case, as well. The assumption in Lemma 5.1(ii) (with \mathbf{A} in place of \mathbf{Q}) is satisfied because of the definition of \mathcal{I} in the proof of Theorem 4.2, hence we can endow \mathbf{A} with a multicontact in such a way that κ is a $\{\Delta, +\}$ -embedding.

(1) \Rightarrow (4) If \mathbf{A} is given by (1), embed the Boolean reduct of \mathbf{A} into some atomic complete Boolean algebra \mathbf{C} by some Boolean embedding χ . The assumption in Lemma 5.1(ii) is satisfied for χ , since χ is, in particular, a meet embedding. If \mathbf{C} is endowed with the multicontact defined in Lemma 5.1 (with \mathbf{A} in place of \mathbf{S} and \mathbf{C} in place of \mathbf{Q}), then χ is a multicontact embedding. Now consider the composition of χ with the embedding given by (1). \square

6. Further remarks

In this note we have not used the Axiom of choice, except for the implications (1) \Rightarrow (4) in both Theorem 4.2 and Theorem 5.2. We needed a consequence of the Axiom of choice in the proofs of the above implications. The next proposition, proved in ZF, the Zermelo-Fraenkel theory without the Axiom of choice, shows that some assumption is indeed necessary. The argument is essentially the same as in [17, Proposition 3.3].

Proposition 6.1. (ZF) *The following statements are equivalent.*

- (A) *The Prime Ideal Theorem [12, Form 14].*
- (B) *The implication (1) \Rightarrow (4) in Theorem 4.2 holds*
- (C) *The implication (1) \Rightarrow (4) in Theorem 5.2 holds.*

Proof. We needed just the Stone Representation Theorem (which in ZF is equivalent to the Prime Ideal Theorem [12, Form 14]) in the proofs of (1) \Rightarrow (4) in Theorems 4.2 and 5.2. Hence (A) implies both (B) and (C).

Suppose that \mathbf{C} is a Boolean algebra and that (B) holds. Endow \mathbf{C} with the overlap multicontact. If the implication (1) \Rightarrow (4) in Theorem 4.2 holds, then \mathbf{C} can be $\{\Delta, +\}$ -embedded into some multicontact complete atomic Boolean algebra \mathbf{D} . We are going to show that this embedding, call it χ , is also a Boolean embedding. Indeed, if $c \in C$ and c' is the complement of c , then $\{c, c'\} \notin \Delta_C$, since δ_C is overlap. Hence $\{\chi(c), \chi(c')\} \notin \Delta_D$, since χ is a Δ embedding. By (Ov_Δ) , $\chi(c)\chi(c') = 0$; moreover, $\chi(c) + \chi(c') = 1$, since χ is a semilattice homomorphism. Hence $\chi(c')$ is the complement of $\chi(c)$ in \mathbf{D} , that is, χ is a homomorphism with respect to complementation. By De Morgan law, meet is expressible in terms of join and complementation, hence χ is a Boolean homomorphism. We have proved the Stone Representation Theorem, which is equivalent to the Prime Ideal Theorem [12, Form 14], hence (B) \Rightarrow (A) follows.

(C) \Rightarrow (A) is proved in the same way. □

Problem 6.2. Characterize those multicontact (weak contact) semilattices which are embeddable into a multicontact (weak contact) modular lattice, both in the general, in the additive and in the overlap case.

Example 6.3. (a) Not every multicontact (weak contact) semilattice is embeddable into a modular lattice.

A weak contact semilattice embeddable into a modular lattice satisfies

$$d \not\delta a + c \text{ and } b \leq a + c \text{ and } b \leq a + d \quad \text{imply} \quad b \leq a. \quad (6.1)$$

Indeed, under the assumptions, $d(a + c) = 0$ in any lattice, hence $b \leq (a + c)(a + d) = a + d(a + c) = a$ in any modular lattice.

Equation (6.1) is not always true, for example, consider the five element non-modular lattice with critical interval $c = b > a$ and $d + a = 1$, $dc = 0$ and with overlap weak contact.

The above counterexample works for multicontact semilattices, as well, by replacing $d \not\delta a + c$ with $\{d, a + c\} \notin \Delta$.

(b) As in [15, Example 5.2(b)], let \mathbf{M}_3 be the 5-element modular lattice with 3 atoms a , b and c and set $a \delta b$, $a \delta c$, $b \not\delta c$, symmetrically, and all the other conditions determined by the axioms of a weak contact. With this contact, \mathbf{M}_3 is an additive contact lattice which cannot be semilattice embedded into a weak contact distributive lattice, as checked in [15, Example 5.2(b)], since it fails to satisfy the condition (D1) defined in [15]. Let Δ be any multicontact expansion of δ on \mathbf{M}_3 (such an expansion exists by Example 2.2(h)). Since condition (M1) here is stronger than (D1) from [15] (of course, interpreting $x \delta y$ as $\{x, y\} \in \Delta$), then, for every Δ expanding δ , \mathbf{M}_3 with the multicontact Δ cannot be semilattice embedded into a multicontact distributive lattice.

(c) Similarly, in [15, Example 5.2(c)] we have considered the 8-element Boolean algebra \mathbf{B}_8 with three atoms a , b and c , with $c \not\delta a$, $c \not\delta b$, the symmetric relations and all the other pairs of nonzero elements δ -related. In [15] we have noticed that the weak contact on \mathbf{B}_8 is not additive, since $c \delta a + b$ but neither $c \delta a$ nor $c \delta b$. Hence any multicontact expansion Δ of δ fails to be additive. On the other hand, by Theorem 5.2, with any such multicontact, \mathbf{B}_8 satisfies (M1). Hence (M1) does not imply additivity. This shows that Theorems 4.2 and 5.2 have distinct ranges of applications.

(d) Let $h \in \mathbb{N}^+$, $r \geq h + 2$ and let \mathbf{M}_r be the $r + 2$ -element modular lattice with r atoms. Let Δ_h be the multicontact generated by the subsets of nonzero elements of cardinality $\leq h$. See Example 2.2(ℓ). Then in \mathbf{M}_r endowed with the multicontact Δ_h all the instances of (M1) with $n \leq h$ are satisfied, since the premises never hold, unless some $c_{i,k} = 0$, but in this case the conclusion of (M1) is straightforward. Similarly, all the instances of additivity (Add_Δ) with $m \leq h$ are satisfied.

On the other hand, if p_1, \dots, p_{h+1} are distinct atoms of \mathbf{M}_r , a is still another atom (this is possible since \mathbf{M}_r has $\leq h + 2$ atoms) and b is one among the

p_i 's, then the assumptions of (M1) are satisfied, but the conclusion does not hold. Thus (M1) is not satisfied in \mathbf{M}_r . Additivity fails, as well, by taking $p, q, p_2, \dots, p_{h+1}$ distinct atoms in (Add_Δ) .

(e) A standard application of the Compactness theorem then shows that neither the class axiomatized in Theorem 4.2, nor the class axiomatized in Theorem 5.2 are finitely first-order axiomatizable.

We will not repeat the argument here; in both cases, just consider clause (3) and argue as in [17, Corollary 4.2].

(f) As in [15, Corollary 5.1], it follows from Theorem 5.2 (Theorem 4.2) that if φ is a first-order sentence in the language of multicontact semilattices, then φ is a logical consequence of the theory of Boolean algebras with an (overlap) contact relation if and only if φ is a logical consequence of (M1) (and (Add_Δ)).

Remark 6.4. There are various incarnations of event structures, see e.g. [11].¹ In the sense used in [24, Subsec. 2.1.2], an *event structure* is a partially ordered set (E, \leq) together with a family Con of finite subsets of E , the *consistency relation*, such that Con contains all singletons, Con is closed by taking subsets and condition (Mon_Δ) holds with respect to the converse order. In other words, considering the converse order, an event structure in the sense from [24, Subsec. 2.1.2] is a multicontact poset in which one takes off the 0 element. An additional assumption in the definition of an event structure is that \leq is downward finite, namely, for every $e \in E$ the set $\{e' \in E \mid e' \leq e\}$ is finite.

A simpler notion of an *event structure with binary conflict* [25, Section 8] is the analogue of a poset with a weak contact relation, again considering the converse order, discarding 0 and assuming downward finiteness. In this case the binary relation taken into account is called *conflict* and corresponds to the binary version of *the negation* of the consistency relation. See [15, Remark 7] for more details.

The author reports there are no competing interests to declare.

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¹Here we are concerned with event structures in the sense used in computer science; the notion used in linguistics and cognition theory apparently bears no connection to that. Notice that, on the other hand, possible connections between event structures and causality in physics have been analyzed [20].

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