Are prime numbers and quadratic residues random?

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Abstract

Appeals to randomness in various number-theoretic constructions appear regularly in modern scientific publications. It is enough to mention such famous names as V.I. Arnold, M. Katz, Yu.G. Sinai and T. Tao. Unfortunately, it all comes down to various, although often very non-trivial and elegant, heuristics. A new analytical approach is proposed to address the issue of randomness/complexity of an individual deterministic sequence. As an application, it demonstrates the expected high complexity of quadratic residues and the unexpectedly low complexity in the case of primes. Technically, the proposed approach is based on a fundamentally new construction of the entropy of a single trajectory of a dynamical system, which in a certain sense occupies an intermediate position between the classical metric Kolmogorov-Sinai entropy and topological entropy.

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1 Introduction

Due to the obvious observable complexity of various number-theoretic constructions and the variety of patterns of numbers that arise, both specialists in this theory and other mathematicians interested in similar problems build purely random models to describe them. Let us note the Cramer's model (see, for example, [12, 16]), the substitution of random numbers in a series that determines the zeta function (see, for example, [31]), the analysis of hidden periodicities of emerging sequences and geometric properties of the Poisson process (see [2]), or construction of a natural invariant measure concentrated on the set of square-free numbers (see [27]). These and other approaches will be discussed in Sections 5,6.

The range of conclusions is also interesting: from Arnold's denial of the randomness of quadratic residues to Cramer's model, which asserts the randomness of prime numbers. It seems that the point here is to highlight some purely specific properties to the detriment of others. In particular, in the case of the Cramer's model, everything is based on the selection of the desired probability distribution, without taking into account the fact that this characteristic alone, although undoubtedly important, does not completely determine the random process.

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It is natural to wonder how to distinguish a truly "complex/random" sequence from a "simple/non-random" one. To do this, consider a typical realization of a Bernoulli random process with equal probabilities $(1/2, 1/2)$ of zeros and ones as the best candidate of the first type. The opposite type of candidate is again the Bernoulli process, but with probabilities $(0, 1)$. Purely deterministic analogues of these processes are trajectories of the doubling map ($x \to 2x \,(\mathrm{mod}\ 1)\,$) and the halving map ($x \to x/2$). As we will see, this is in many ways contrary to the "algorithmic" approach, in which the complexity of a sequence is interpreted through the complexity of the description of the process that generates the sequence (the latter is trivial in both last examples)

The purpose of this article is to offer a quantitative answer to the question of the complexity/randomness of a single purely deterministic series of points of the types discussed above. Of course, such approaches are well known, but they all have serious drawbacks, since they are either non-constructive or completely ad hoc. Perhaps the most beautiful among them was proposed by A.N. Kolmogorov. His main idea was to treat the sequence under study as the trajectory of some dynamical system and reduce the question of the complexity of the sequence to the analysis of the "simplest" dynamical system that generates it.

We will follow the same idea, but remembering that different trajectories of the same dynamical system can exhibit qualitatively different behavior, we will come up with a new concept of "local" dynamical entropy h_{loc} (see Section 3), independent of the choice of the invariant measure (unlike the Kolmogorov-Sinai metric entropy) and making it possible to study even dynamical systems that do not have such a measure (see, for example, [6], for the discussion of such systems).

In what follows, we will refine this concept to analyze the complexity/randomness of individual binary series representing the number theoretic constructions mentioned above.

In order to make a distinction between static and dynamic entropy-like characteristics we use the uppercase letter H in the static setting and the lowercase letter h in the dynamic one. The entropy-like characteristic of randomness $h(\vec{x})$ takes values in $\mathbb{R}_+$. The value $h(\vec{x}) = 0$ is interpreted as a non-random sequence $\vec{x}$, while $h(\vec{x}) > 0$ as a random one. To simplify notation we write $\lim^- := \liminf$, $\lim^+ := \limsup$ and drop the signs if lower and upper limits coincide.

The paper is organized as follows. In Section 2 we recall the classical definitions related to the concept of Shannon entropy of a discrete distribution and demonstrate that from the point of view of small perturbations this functional exhibits a number of unexpected features. In Section 3 we deal with various dynamical versions of the concept of entropy, starting with the famous Kolmogorov-Sinai metric entropy of a dynamical system. To take care about complexity properties of individual trajectories, we introduce here a pair of completely new notions of local and information entropies of a trajectory. By specifying these new entropies for the case of a discrete phase space in Section 4 we introduce new measures of complexity/randomness of sequences of points from finite alphabet and compare them to some known approachers. The remaining part of the paper is devoted to the application of these constructions to prove in Section 5 the nonrandom nature of the set of prime numbers and in Section 6 the randomness of quadratic residues.

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2 Information (Shannon) entropy

Let (Ω, Σ) be a measurable space with the Borel σ-algebra Σ of measurable subsets.

Throughout this section we assume that the space Ω is discrete, i.e., $\Omega := \{\omega_1, \omega_2, \ldots\}$, and $\Sigma := 2^{\Omega}$. Let $\vec{p} := (p_1, p_2, \ldots)$ denote the distribution (non-necessarily probabilistic) on Ω, i.e. $p_i \geq 0\ \forall i$, but $||\vec{p}|| := \sum_i p_i = \vec{p}(\Omega)$ may differ from 1. We also denote $\log(\cdot) := \log_2(\cdot)$, and $H(C) := -C \log C$ for any constant $C \geq 0$, $H(0) = 0$.

Definition 2.1 The (Shannon) *entropy* of $\vec{p}$ is defined as $H(\vec{p}) := -\sum_i p_i \log p_i$.

The following lemma collects several important (but little-known) observations that we will need later.

Lemma 2.1 *Let $r := \#\Omega \leq \infty$. Then*

$$H(\vec{p}) \leq ||\vec{p}|| \log r + H(||\vec{p}||) = ||\vec{p}||(\log r - \log ||\vec{p}||), \tag{1}$$

$$H(\vec{p} + \vec{q}) = H(\vec{p}) + |\log(\inf_i p_i) + 1| \cdot ||\vec{q}|| + o(||\vec{q}|| \cdot |\log(\inf_i p_i)|) \tag{2}$$

$$\begin{aligned}|H(\vec{p} + \vec{q}) - H(\vec{p})| \leq \min\{H(\vec{p}) + H(||\vec{p}||) + H(||\vec{q}||) + ||\vec{q}|| \log(r),\\ 2|\log(\inf_i p_i) + 1| \cdot ||\vec{q}||\}\end{aligned} \tag{3}$$

and this functional reaches its maximum value on the uniform distribution.

Proof. The first claim follows from the well-known fact that in the case of finite Ω, the uniform probabilistic distribution (denoted by $\vec{p}^u$) maximizes entropy. Therefore,

$$\begin{aligned}\log r = H(\vec{p}^u) &\geq H(\vec{p}/||\vec{p}||) = -\sum_i \frac{p_i}{||\vec{p}||} \log(\frac{p_i}{||\vec{p}||})\\ &= -\frac{1}{||\vec{p}||} \sum_i p_i \log p_i + \frac{1}{||\vec{p}||} \sum_i p_i \log(||\vec{p}||)\\ &= \frac{1}{||\vec{p}||} H(\vec{p}) + \log(||\vec{p}||),\end{aligned}$$

which implies (1).

The key point of the proof of the remaining inequalities is the observation that for each $t \in \mathbb{R}_+$ we have

$$H(t + \varepsilon) - H(t) = -\varepsilon(\log t + 1) + o(\varepsilon |\log t|). \tag{4}$$

Applying (4) to the explicit formula for $H(\vec{p})$ we get the result. □

Corollary 2.2 *The functional $H : \ell_1(\Omega) \to \mathbb{R}_+ \cup \{\infty\}$ is continuous on the entire space $\ell_1(\Omega)$ if and only if $\#\Omega < \infty$.*

This simple result is important from both theoretical and practical points of view; in particular, it demonstrates that Khinchin's classical axiomatic entropy construction [17] cannot be extended from finite space to an infinite one. In practice, this makes it possible control the accuracy of entropy calculations when we know the distribution only approximately, which will be very handy in Sections 5,6.

Surprisingly, Lemma 2.1 seems new. At least, I was not able to find results of this sort in numerous publications devoted to the concept of entropy.

In the sequel we will need the following result of perturbation type.

Lemma 2.3 *Let $\vec{p}, \vec{p}^{(N)}$, $N \in \mathbb{Z}_+$ be probability distributions such that $\vec{p}_i^{(N)} = 0\ \forall i > N$, and let $\varepsilon_N := \sum_i |p_i^{(N)} - p_i|$.*

(a) If $\exists C, \alpha \in \mathbb{R}_+ : \quad \varepsilon_N \le CN^{-\alpha} \quad \forall N \gg 1$ then $H(\vec{p}^{(N)}) \overset{N\to\infty}{\longrightarrow} H(\vec{p})$.

(b) If $\exists C \in \mathbb{R}_+ : \quad \varepsilon_N \le C/\log N \quad \forall N \gg 1$ then $|H(\vec{p}^{(N)}) - H(\vec{p})| \le C \quad \forall N \gg 1$. Moreover, the upper bound can be achieved. Therefore, there may be no convergence of entropies, despite the fact that $\varepsilon_N \overset{N\to\infty}{\longrightarrow} 0$.

Proof. Fix some $t > 0$ (to be specified later) and denote by $I^t := (i_1^t, i_2^t, \ldots)$ those indices, for which $p_{i_1^t} \ge t$. We estimate separately the contribution to the difference of entropies coming from the indices $j \in I^t$ and from the others, namely, in the former case we use the inequality (4), while in the latter case we make an estimate from above by zeroing the corresponding p_j. Setting $N_t := \# I^t$, we get

$$\begin{aligned} |H(\vec{p}^{(N)}) - H(\vec{p})| &\le -\varepsilon_N(\log t + 1) + o(\varepsilon_N |\log t|) - \sum_{j \notin I^t} \frac{\varepsilon_N}{N - N_t} \log(\frac{\varepsilon_N}{N - N_t}) \\ &\le 2\varepsilon_N |\log t| - \varepsilon_N \log \varepsilon_N + \varepsilon_N \log(N - N_t) \\ &\le 2CN^{-\alpha}|\log t| - CN^{-\alpha}\log(CN^{-\alpha}) + CN^{-\alpha}\log N \\ &\le N^{-\alpha}\left(2C|\log t| + C\alpha \log(C^{-\alpha}N) + C\log N\right) \overset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

Note that the convergence does not depend on the choice of t.

We prove the second claim for the special situation when $\vec{p}$ is a probabilistic distribution supported by a single element (say ω_1), and hence $H(\vec{p}) = 0$, leaving the general situation for the reader. The reason is that this is exactly the situation which we will need in the sequel.

Consider a sequence of distributions $\vec{p}^{(N)}$ such that $p_i^{(N)} := \frac{C}{N \log N}$ for $i \le N$ and $p_i^{(N)} \equiv 0$ otherwise. Then

$$\begin{aligned} H(\vec{p}^{(N)}) &= -N\frac{C}{N\log N}\log(\frac{C}{N\log N}) \\ &= \frac{C}{\log N}\log(N/C) - \frac{C}{\log N}\log(\frac{C}{\log N}) \\ &= C - \frac{C\log C}{\log N} - \frac{C}{\log N}\log(\frac{C}{\log N}) \overset{N\to\infty}{\longrightarrow} C. \end{aligned}$$

Lemma is proven. □

3 Dynamical entropy in ergodic theory

Let us give a brief account on classical approaches to the construction of entropy-like characteristics of a discrete time dynamical system, defined by a measurable map f from a measurable space (X, Σ, μ) into itself. We start with the Kolmogorov-Sinai construction (see, for example, [9] for details).

Definition 3.1 Given a pair of finite measurable partitions Δ, Δ' of $(X, \mathcal{B}, \mu)$ by their common *refinement* one means $\Delta \bigvee \Delta' := \{\Delta_i \cap \Delta'_j : \ \mu(\Delta_i \cap \Delta'_j) > 0\}$.

Let $\mu \in \mathcal{M}_f$ (the set of all f-invariant measures). Making the refinement of $\{f^{-1}\Delta_i\}$ we again get a finite measurable partition which we denote by $f^{-1}\Delta$. The n-th refinement Δ^n of the partition Δ can be defined inductively

$$\Delta^n := \Delta^{n-1} \bigvee f^{-1}\Delta^{n-1}, \quad \Delta^0 := \Delta.$$

Definition 3.2 The conditional *Kolmogorov-Sinai entropy* of a partition is defined as

$$h_\mu(f|\Delta) := \liminf_{n\to\infty} \frac{1}{n} H_\mu(\Delta^n) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\Delta^n),$$

where $H_\mu(\Delta) := -\sum_i \mu(\Delta_i)\ln\mu(\Delta_i)$ is the entropy of the discrete distribution $\{\mu(\Delta_i)\}$.

Definition 3.3 The *Kolmogorov-Sinai metric entropy* of the dynamical system (f, X, Σ, μ) is $h_\mu(f) := \sup_\Delta h_\mu(f|\Delta)$.

Alternative approaches are known for continuous maps $f \in C^0(X, X)$, where (X, ρ) is a compact metric space.

Definition 3.4 The n-th *Bowen metric* ρ_n on X is defined as $\rho_n(u, v) := \max\left\{\rho\left(f^k(u), f^k(v)\right) : \ k = 0, \dots, n-1\right\}$.

Let $B^n_\varepsilon(x)$ be the open ball of radius ε in the metric ρ_n around x.

Definition 3.5 The *Brin-Katok measure-theoretical entropy* of a measure $\mu \in \mathcal{M}_f(X)$ at a point $u \in X$ is $h_\mu(f, u) := -\lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} \frac{1}{n}\log\mu(B^n_\varepsilon(u))$.

Roughly speaking $h_\mu(f, u)$ measures the exponential rate of decay of the measure of points that stay ε-close to the point u under forward iterates of the map f.

Theorem 3.1 *[7] $h_\mu(f, u)$ is well defined for an ergodic measure μ and does not depend on u for μ-a.e $u \in X$.*

A topological version (independent on the choice of the measure μ) is available in the case of a continuous map f. In this setting $\Delta := \{\Delta_i\}_1^r$ is a *covering* of X by open sets. Define a transition matrix $M := \{m_{ij}\}$, where $m_{ij} = 1$ if $\Delta_i \cap f^{-1}\Delta_j \neq \emptyset$ and $= 0$ otherwise.

Then on the Cantor set X_M (the space of sequences with the alphabet $\mathcal{A} := \{1, 2, \dots, r\}$ with the transition matrix M) the left shift map σ defines a symbolic dynamical system.

$$\vec{x} = (x_1 x_2 \ldots x_k \underbrace{x_{k+1} x_{k+1} \ldots x_{k+n}}_{\vec{w}} x_{k+n+1} \ldots x_N \ldots)$$

Denoting by A_Δ^n the set of all admissible words $\vec{w}$ of length n (i.e. different pieces of length n of all trajectories of (σ, X_M)) and by $\#A_\Delta^n$ – the number of such words, we set

$$h_{\mathrm{top}}(f|\Delta) := \liminf_{n\to\infty} \frac{1}{n} \log(\#A_\Delta^n) = \lim_{n\to\infty} \frac{1}{n} \log(\#A_\Delta^n).$$

Definition 3.6 The *topological entropy* is defined as $h_{\mathrm{top}}(f) := \sup_\Delta h_{\mathrm{top}}(f|\Delta)$.

Note that the construction of Kolmogorov-Sinai metric entropy (as well as of Brin-Katok entropy) are based on the choice of f-invariant measure μ (and depends on it), and the construction of topological entropy makes sense only for continuous maps. On the other hand, a general measurable dynamical system needs not to have even a single invariant measure (not speaking about the assumption on continuity). Discussion of dynamical systems having no invariant measures can be found, for example, in [6].

To overcome these difficulties we propose yet another entropy-like constructions.

Let $\Delta := \{\Delta_i\}$ be a partition (covering) of X by measurable sets. We refer to the indices of Δ_i as an alphabet $\mathcal{A}$, which needs not to be finite. We say that on a starting segment of length N of a given trajectory $\vec{x} := (x_1, x_2, \ldots)$ of our system there is a word $\vec{w} := (w_1, \ldots, w_n)$ composed of the letters $w_i \in \mathcal{A}$, if there is i such that

$$x_{i+j} \in \Delta_{w_j} \quad \forall j < n+1, i+j \le N.$$

Denote by $L(\vec{x}, \vec{w}, N)$ the number of occurrences of a word $\vec{w}$ in the starting piece of length N of the trajectory $\vec{x}$, and let $\vec{p}(\vec{x}, n, N) = (p_1, p_2, \ldots)$ be a distribution (frequency) of all such words of length n.

Definition 3.7 By a conditional *local entropy* of the trajectory $\vec{x}$ we mean

$$h_{\mathrm{loc}}^{\pm}(\vec{x}|\Delta) := \lim\nolimits_{n\to\infty}^{\pm} \lim\nolimits_{N\to\infty}^{\pm} \frac{1}{n} H(\vec{p}(\vec{x}, n, N)).$$

Here ± refers to the upper and lower limits, and

$$H(\vec{p}(\vec{x}, n, N)) := -\sum_{i=1} p_i \log p_i$$

is the entropy of the distribution $\vec{p}(\vec{x}, n, N)$.

It is clear that this construction is something intermediate between metric and topological entropy, but works for any measurable map and provides information about "complexity" of individual trajectories. The "locality" of h_{loc} is trajectory-wise, unlike the Brin-Katok entropy which is point-wise.

In the case of a continuous mapping, a topological type approach to the same problem is known (see [8, 33]), which with very minor modifications can be formulated for the measurable case discussed in this article.

Denote by $L(\vec{x}, n, N)$ the number of different words of length n in the starting piece of length N of the trajectory $\vec{x}$.

Definition 3.8 By conditional *information entropy* of the trajectory $\vec{x}$ we mean

$$h^{\pm}_{\text{info}}(\vec{x}|\Delta) := \lim\nolimits^{\pm}_{n\to\infty} \lim\nolimits^{\pm}_{N\to\infty} \frac{1}{n} \log L(\vec{x}, n, N).$$

Finally, we define the unconditional versions of the entropies under consideration as follows:

$$h^{\pm}_{\text{loc}}(\vec{x}) := \sup_{\Delta} h^{\pm}_{\text{loc}}(\vec{x}|\Delta), \quad h^{\pm}_{\text{info}}(\vec{x}) := \sup_{\Delta} h^{\pm}_{\text{info}}(\vec{x}|\Delta). \tag{5}$$

Definition 3.9 We say that a sequence of points $\vec{x} := (x_1, x_2, \ldots), \;\; x_i \in X$ is *typical* with respect to a probabilistic measure μ, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1_A(x_i) = \mu(A) \quad \forall A \in \Sigma.$$

In other words, the sequence $\vec{x}$ is distributed according to the measure $\mu_{\vec{x}} = \mu$.

Lemma 3.1 *For a periodic sequence $\vec{x}$ the measure $\mu_{\vec{x}}$ is well defined, and $h_{\text{loc}}(\vec{x}) = h_{\text{info}}(\vec{x}) = 0$.*

Proof. The claim about the measure $\mu_{\vec{x}}$ is obvious and we discuss only statements about the entropies. Consider a ℓ-periodic sequence $\vec{x}$. For each $n > \ell$ there are at most ℓ different sub-words of length n, each with the frequency $1/\ell + o(1/N)$. The local entropy of this sequence is equal to $\frac{1}{n} \log \ell \overset{n\to\infty}{\longrightarrow} 0$. Similarly, $L(\vec{x}, n, N) \le \ell \;\; \forall (n, N)$, which implies the claim about the information entropy. □

An important question is the range of values of the functionals h_{loc} and h_{info}, which is described in the following Lemma, but its proof will be deferred to the next section.

Lemma 3.2 $\forall \alpha, \beta \in [0, \infty] \;\; \exists \vec{x}^{\alpha}, \vec{x}^{\beta}$ *such that* $h_{\text{loc}}(\vec{x}^{\alpha}) = \alpha, \;\; h_{\text{info}}(\vec{x}^{\beta}) = \beta$.

It is known that under reasonably mild assumptions on the dynamical system (f, X, Σ) we have $h_{\text{top}}(f) = \sup_{\mu} h_{\mu}(f)$, where the supremum is taken over all ergodic f-invariant measures. Despite that the calculation of the local entropy does not depend on any free parameter (like an invariant measure in the case of the metric entropy), a similar connection can be established between $h_{\text{info}}(\vec{x})$ and $h_{\text{loc}}(\vec{x})$.

Lemma 3.3 *For any sequence $\vec{x}$ we have*

$$h^{\pm}_{\text{loc}}(\vec{x}) \le h^{\pm}_{\text{info}}(\vec{x}).$$

Proof. For a given pair $n < N$ consider the distribution $\vec{p}(\vec{x}, n, N) = (p_1, p_2, \ldots)$. Since $L(\vec{x}, n, N)$ is the number of different words of length n in the starting piece of length N of the trajectory $\vec{x}$, we deduce that at most $L(\vec{x}, n, N)$ entries of $\vec{p}(\vec{x}, n, N)$ are positive. Using that the entropy of a finite distribution reaches its maximum on the uniform distribution, we get

$$H(\vec{p}(\vec{x}, n, N)) := -\sum_{i=1} p_i \log p_i \le \log L(\vec{x}, n, N),$$

which implies the claim. □

If the measure $\mu_{\vec{x}}$ is well defined, one can explore the connections between our newly defined entropy-like characteristics and more classical approaches. However, this is beyond the scope of the present paper and will be studied in a separate article. Note also that our local entropy can be easily modified to work with bi-infinite trajectories of measurable semi-groups, which will also be studied elsewhere.

In the next section we apply above construction to study "complexity" of sequences of numbers, considered as trajectories of unknown dynamical systems, and later to some number-theoretic constructions.

4 Complexity

One of the main concepts of complexity theory, introduced by A.N. Kolmogorov, was to reduce the question of the complexity of a given sequence of points x to analysis of the complexity of a dynamical system admitting it (i.e. this sequence is the trajectory of such a system). Naturally, there are many dynamical systems admitting a given sequence, and Kolmogorov proposed to take into account the "simplest" among them. To implement this concept, he uses a universal Turing machine to describe any dynamical system living in a finite phase space, and the complexity of such machine is described by the minimum length of the program generating it [19, 33, 8]. The beauty of this approach comes at the cost of being completely non-constructive. In practice, this type of complexity can only be calculated for some toy examples.

Apart from the non-constructiveness of this approach there are two other important issues. First, for a given sequence there might be no dynamical systems, admitting it as a trajectory. Second, different trajectories of the same dynamical system may demonstrate very different qualitative properties.

An alternative (let's call it algorithmic) concept (see [33, 23, 13, 20, 8]) is based on the idea to treat a given sequence as an unordered set of points and boils down to various dimension-like characteristics of this set. A serious disadvantage here is that important information about the order of items is lost.

In what follows we try to make use of Kolmogorov's idea of the "dynamical origin" of the sequence under study.

Let $\mathcal{A} := \{a_1, a_2, \ldots\}$ be at most a countable collection of "letters" (to which we refer as an alphabet), equipped with the complete σ-algebra $\Sigma := 2^{\mathcal{A}}$, and let $\vec{x} := \{a_{k_i}\}_{i \in \mathbb{Z}_+}$ be a sequence composed of the letters from this alphabet.

For a map $f : \mathcal{A} \to \mathcal{A}$, admitting the sequence $\vec{x}$ as a trajectory, we may apply the notions of the local and information entropies, defined by the relations (5). There are two important observations here. First, these notions do not depend on the choice of the map f admitting $\vec{x}$. Second, in the present setting the supremum over all measurable partitions/covers can be easily calculated due to the non-negativity of conditional entropies and the existence of the most fine partition, which coincides with the partition into points. Thus we get

$$h^{\pm}_{\text{loc}}(\vec{x}) := \lim\nolimits^{\pm}_{n\to\infty} \lim\nolimits^{\pm}_{N\to\infty} \frac{1}{n} H(\vec{p}(\vec{x}, n, N)). \tag{6}$$

$$h^{\pm}_{\text{info}}(\vec{x}) := \lim\nolimits^{\pm}_{n\to\infty} \lim\nolimits^{\pm}_{N\to\infty} \frac{1}{n} \log L(\vec{x}, n, N). \tag{7}$$

It is worth noting that a functional qualitatively similar to (6) is known in the literature on information theory under the name "finite-state dimension" (see [13, 23, 20] and further references therein). It was introduced in 2004 as a finite-state version of classical Hausdorff dimension, and it measures the lower asymptotic density of information in an infinite sequence over a finite alphabet. The connection to any version of dynamical entropy has never been discussed in this context.

Lemma 4.1 $\forall \alpha, \beta \in [0,1] \ \ \exists \vec{x}^{\alpha}, \vec{x}^{\beta}$ *such that* $h_{\text{loc}}(\vec{x}^{\alpha}) = \alpha, \ h_{\text{info}}(\vec{x}^{\beta}) = \beta$.

Proof. For a given $q \in [0,1]$ consider the Bernoulli random process $\text{Ber}(q)$, that is a sequence of i.i.d. binary random variables $\{\xi_n\}_{n \in \mathbb{Z}_+}$ with $\text{Prob}(\xi_n = 1) = q$. Denoting by $\vec{p}(\text{Ber}(q), n)$ the distribution of binary sub-words of length n of $\text{Ber}(q)$, we get (see [5, 14])

$$\frac{1}{n} H(\vec{p}(\text{Ber}(q), n)) = -q \log q - (1-\gamma) \log(1-q). \tag{8}$$

Moreover, by Shannon-McMillan-Breiman Theorem (see [5]) for almost every realization $\vec{x}^q$ of $\text{Ber}(q)$ we have

$$\lim_{N \to \infty} \frac{1}{n} H(\vec{p}(\vec{x}^q, n, N)) = \frac{1}{n} H(\vec{p}(\text{Ber}(q), n)),$$

where $\vec{p}(\vec{x}^q, n, N)$ is the distribution of sub-words of length n in the starting piece of length N of $\vec{x}^q$. The observation that the right hand side of (8) depends on q continuously and takes values in $[0,1]$ proves the first claim.

To prove the second claim, note that $h_{\text{info}}(\text{Ber}(q), n) \equiv \log 2 = 1$ a.s. $\forall q \in (0,1)$. Consider instead of the Bernoulli process a topological Markov chain - collection of sequences consisted of letters from a finite alphabet $\mathcal{A}$ defined by a binary transition matrix. Choosing elements of this matrix, one easily controls the number of different admissible sub-words of given length. □

The claims of Lemma 4.1 can be easily generalized for the case of the alphabet with an arbitrary (but finite) number of elements.

Lemma 4.2 *Let* $r : \#\mathcal{A} < \infty$. *Then* $\forall \alpha, \beta \in [0, \log r] \ \ \exists \vec{x}^{\alpha}, \vec{x}^{\beta}$ *such that* $h_{\text{loc}}(\vec{x}^{\alpha}) = \alpha, \ h_{\text{info}}(\vec{x}^{\beta}) = \beta$.

Proof. Observe that for any positive integer n the total number of all different words of length n is equal to r^n. Therefore for any sequence $\vec{x}$ with this alphabet by (1) we get

$$H(\vec{p}(\vec{x}, n, N)) \le \log r^n = n \log r.$$

Using exactly the same arguments as in the proof of Lemma 4.1 we get that each value from the segment $[0, \log r]$ is admissible for our entropies. □

Proof of Lemma 3.2. From the previous result we see that choosing an alphabet $\mathcal{A}$ wiith r symbols we can construct a sequence having entropies in the range $[0, \log r]$. Since r is arbitrary this proves the claim. □

In the sequel we will pay a special attention to binary sequences $\vec{b}$ with only 0 and 1 entries. Let us discuss their entropic properties in some detail.

Denote by $M(\vec{b}, \vec{w}, N)$ the number of occurrences of the word $\vec{w}$ among the first N letters of $\vec{b}$. By a zero word we will mean any word, consisting of zeros only, and $\vec{1}$ stands for the word consisting of a single letter 1.

Lemma 4.3 *Let $Q(\vec{b},n,N)$ be the frequency of zero sub-words of length $n < N$ in the first N letters of the sequence $\vec{b}$. Then*

$$Q(\vec{b},n,N) \geq 1 - \frac{M(\vec{b},\vec{1},N)}{N} \frac{n}{1-n/N}. \tag{9}$$

Proof. The binary sequence of length N, having the smallest number of zero sub-words of length n can be realized as follows: $0\dots01\ 0\dots01\ \dots\ 0\dots01\ 0\dots0$. Each of $M(\vec{b},\vec{1},N)$ blocks $0\dots01$ consists of $(n-1)$ zeros and 1 in the end. Thus the number of zero sub-words of length n is equal to $N - nM(\vec{b},\vec{1},N) - n$, while the total number of sub-words of length n is $N-n$. Therefore the frequency

$$Q(\vec{b},n,N) = \frac{N - nM(\vec{b},\vec{1},N) - n}{N-n} = 1 - \frac{nM(\vec{b},\vec{1},N)}{N-n} = 1 - \frac{M(\vec{b},\vec{1},N)}{N} \frac{n}{1-n/N}.$$

Lemma is proven. □

Lemma 4.4 *(a) If $M(\vec{b},\vec{1},N) \leq CN^{1-\alpha}$, $\alpha \in (0,1)$, then $Q(\vec{b},n,N) \geq 1 - 2CnN^{-\alpha}$ and $h_{\rm loc}(\vec{b}) = 0$.*

(b) If $M(\vec{b},\vec{1},N) \leq C\frac{N}{\log N}$ $\forall N \gg 1$, then $Q(\vec{b},n,N) \geq 1 - C\frac{n}{\log N}$ and $h_{\rm loc}(\vec{b}) \leq C$.

Proof. Both claims follow from the direct application of Lemmas 4.3 and 2.3(a,b) respectively. □

The first claim in this Lemma may be reformulated as follows.

Lemma 4.5 *Let $\vec{p}, \vec{p}^{(N)}$, $N \in \mathbb{Z}_+$ be probability distributions, corresponding to the sequences $\vec{x}, \vec{x}^{(N)}$, $N \in \mathbb{Z}_+$ and let $\varepsilon_N := \sum_i |p_i^{(N)} - p_i| \leq CN^{-\alpha}$ for some $C, \alpha \in \mathbb{R}$. Then $h_{\rm loc}(\vec{x}^{(N)}) \overset{N\to\infty}{\longrightarrow} h_{\rm loc}(\vec{x})$.*

Remark 4.6 The claim of Lemma 4.4(1) can be falsely interpreted as a that zero density of ones in the binary sequence $\vec{b}$ implies that $h_{\rm loc}(\vec{b}) = 0$. To demonstrate that this is not the case, observe that the assumption in Lemma 4.4(2) allows the zero density of ones in $\vec{b}$, but demonstrates that the local entropy in that case might be strictly positive. On the other hand, the high density of ones may lead to the zero local entropy as well (consider the sequence consisting of ones only).

Consider yet another interesting example. Let $\vec{b}^{\rm nat}$ be the binary sequence obtained by concatenating the binary representations of all natural numbers. This sequence was introduced by D.G. Champernowne [10]. It is of interest, that the number whose fractional part coincides with the sequence $\vec{b}^{\rm nat}$ is transcendental[1] (see [24]).

Lemma 4.7 $h_{\rm loc}(\vec{b}^{\rm nat}) = h_{\rm info}(\vec{b}^{\rm nat}) = 1$.

Proof. It is known (see, for example, [26]) the sequence $\vec{b}^{\rm nat}$, considered as a binary number, is normal in any base[2]. Therefore, the distribution of all binary sub-words of $\vec{b}^{\rm nat}$ of the same length is uniform, which proves the first claim. The second claim follows from the observation that all possible finite binary words are present in this sequence. □

[1] Not the root of a non-zero polynomial of finite degree with rational coefficients.

[2] All its digits of the number represented in the given base follow the uniform distribution.

5 Spatial distribution of prime numbers

In publications on number theory (see, for example, [31, 16, 28, 29, 30, 3]) we often read that random models provide heuristic support for various conjectures and that prime numbers are believed to behave pseudo-randomly in many ways and do not follow any simple pattern. An important example of a purported statement about the pseudo-randomness of primes (known as the Cramer's model) is the Hardy-Littlewood conjecture for k-tuples, namely that the number of occurrences of different patterns in primes can be approximated by treating them as a sequence of random numbers generated by independently counting each $k \in \mathbb{Z}_+$ as "prime" with probability $1/\log k$. For a detailed discussion of this and a number of other examples of this kind, see [30].

Consider the sequence of prime numbers $\vec{\pi} := (1, 2, 3, 5, 7, 11, 13, \ldots)$ and match it to the binary sequence $\vec{b}^{\rm prime} := \{b_i\}_{i\in\mathbb{Z}_+}$, such that $b_{\pi_i} = 1 \ \forall i \in \mathbb{Z}_+$ and $b_j = 0 \ \forall j \notin \vec{\pi}$.

Theorem 5.1 *(a)* $h_{\rm loc}(\vec{b}^{\rm prime}) \le h_{\rm info}(\vec{b}^{\rm prime}) \le \log((1+\sqrt{5})/2) \approx 0.69424191363$*,*

(b) under the validity of Hardy-Littlewood Conjecture (see Conjecture 5.2 below) $h_{\rm info}(\vec{b}^{\rm prime}) = \log((1+\sqrt{5})/2)$*.*

Remark 5.1 The distribution of finite patterns in $\vec{b}^{\rm prime}$ is quite uneven, while the calculation of the upper bound for $h_{\rm loc}(\vec{b}^{\rm prime})$ is based on the uniform one. Therefore, we expect that the true value of local entropy is much smaller and may even be zero. Moreover, below we will show that the local entropy of the Cramer's model of prime numbers is indeed zero.

To prove Theorem 5.1 we need to discuss connections of some known statistics of primes to similar statistics of all finite binary words.

Conjecture 5.2 *[15] Let* $\vec{a} := (a_1, \ldots, a_k)$ *be distinct positive even integers which do not cover all residue classes to any prime modulus. Then the number of integers* $0 < m \le N$ *for which* $m + a_1, \ldots, m + a_k$ *are all primes satisfies the asymptotic formula*

$$L_k(N, \vec{a}) \approx C(k) N \log^{-k} N. \tag{10}$$

This conjecture is a close relative to the Hardy-Littlewood conjectures, but at present only partial results in this direction have been rigorously proven.

Definition 5.1 A pair of consecutive prime numbers, separated by a single composite number are called *prime twins.*

Denote by $L_1(N)$ and $L_2(N)$ the number of primes and prime twins in $1, 2, \ldots, N$ correspondingly.

Theorem 5.2 *[4, 15]*

- $\frac{N}{\ln N - 2} < L_1(N) < \frac{N}{\ln N - 4} \quad \forall N > 54.$
- $L_2(N) < C\frac{N}{\log^2 N}$ *for some* $C < \infty$ *and all* $N \gg 1$*.*

Let us compare these statistics with similar information about general binary sequences.

Lemma 5.3 *Let $Q(n, \vec{w})$ be the number of all binary words of length n, containing the given sub-word $\vec{w}$. Then*
(a) $Q(n+1, 11) = Q(n, 11) + Q(n-1, 11)$,
(b) $Q(n+1, 101) = Q(n, 101) + Q(n-1, 101) + Q(n-2, 101)$.

Proof. We proceed by induction on n. If the new letter on the $(n+1)$-th position is 0 no new sub-word 11 may show up, while if the new letter is 1 the new (in comparison to n) words of length $n+1$, containing 11 show up iff the n-th letter is 1. This proves statement (a). Statement (b) is proved in a similar way. □

Corollary 5.4 *(a) $Q(n, 11)$ are Fibonacci numbers, satisfying the limit relations $z_n/z_{n+1} \to \lambda$, $z_{n+1}/(z_{n+1} + z_n) \to \lambda$, where $\lambda^2 + \lambda = 1$, $\lambda := (1+\sqrt{5})/2 \approx 1.61803398875$, and thus $z_n \approx \lambda^{-n}$.*

(b) $Q(n, 101)$ are tribonacci numbers, satisfying the limit relations $z_n/z_{n+1} \to \nu$, $z_{n+1}/(z_{n+1} + z_n + z_{n-1}) \to \nu$, where $\nu^3 - \nu^2 - \nu = 1$, $\nu \approx 1.839286755$, $z_n \approx \nu^{-n}$.

Proof of Theorem 5.1. Comparing the asymptotic number of prime numbers on large intervals $L_1(N)$ with the results of Lemma 4.4, we see that, despite the fact that prime numbers have zero density, they lie exactly on the boundary between sequences with zero local entropy and sequences with positivity entropy. Using part (b) of Lemma 4.4, we can obtain an upper bound that turns out to be quite large. Therefore, we will take a different approach, paying more attention to information entropy.

The general inequality between local and information entropies is proven in Lemma 3.3. Note now that the sequence $\vec{b}^{\rm prime}$ does not contain consecutive ones (except for the first three elements), which from the point of view of asymptotic relations do not play any role. Therefore, we will only consider elements with indices greater than 2, for which, by Corollary 5.4(a) we get

$$L(\vec{b}^{\rm prime}, n, N) \le C((1+\sqrt{5})/2)^{-n}.$$

This proves the upper bound for information entropy.

The second statement about the exact value of information entropy follows from the assumption of the validity of the Hardy-Littlewood Conjecture, which assumes the existence of all finite combinations of prime numbers in odd positions. Thus, the number of different sub-words is equal to the right side of the previous relation. □

5.1 Inhomogeneous random Bernoulli process and Cramer's model

Definition 5.2 *Bernoulli process* $\mathrm{Ber}(\vec{q})$ with the vector-valued parameter $\vec{q} := \{q_k\}_{k=1}^{\infty}$, $q_k \in [0,1]$ is a sequence of independent binary random variables $\{\xi_n\}_{n\in\mathbb{Z}_+}$ with $P(\xi_n = 1) = q_n$, where $P(\cdot)$ is the probability of an event.

Lemma 5.5 $H(\mathrm{Ber}(\vec{q}), n) = \sum_{k=1}^{n} H(\mathrm{Ber}(q_k), 1)$.

Proof. Let $\vec{b}^{(n)}$ be a binary word of length n. Then

$$H(\vec{p}(\mathrm{Ber}(q), n)) := -\sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) \log P(\vec{b}^{(n)}),$$

where $P(\vec{b}^{(n)})$ is the probability that the first n letters of $\mathrm{Ber}(\vec{q})$ coincide with $\vec{b}^{(n)}$. On the other hand, setting $q'_{n+1} := 1 - q_{n+1}$, due to the independence of the elements of the Bernoulli process, we obtain

$$\begin{aligned}
H(\vec{p}(\mathrm{Ber}(\vec{q}), n+1)) &:= -\sum_{\vec{b}^{(n)}} P(b^{(n+1)}) \log P(\vec{b}^{(n+1)}) \\
&= -\sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) q_{n+1} \log(P(\vec{b}^{(n)}) q_{n+1}) - \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) q'_{n+1} \log(P(\vec{b}^{(n)}) q'_{n+1}) \\
&= -\sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) q_{n+1} \log P(\vec{b}^{(n)}) - \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) q_{n+1} \log q_{n+1} \\
&\quad - \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) q'_{n+1} \log P(\vec{b}^{(n)} - \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) q'_{n+1} \log q'_{n+1} \\
&= -(q_{n+1} + q'_{n+1}) \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) \log P(\vec{b}^{(n)}) \\
&\quad - (q_{n+1} \log q_{n+1} + q'_{n+1} \log q'_{n+1}) \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) \\
&= H(\vec{p}(\mathrm{Ber}(\vec{q}), n)) + H(\vec{p}(\mathrm{Ber}(q_{n+1}), 1)),
\end{aligned}$$

since $q_{n+1} + q'_{n+1} = \sum_{\vec{b}^{(n)}} P(\vec{b}^{(n)}) = 1$. □

In 1936, H. Cramer [12] introduced a probabilistic model of primes, where each natural number is selected for inclusion with probability $1/\log n$. From the point of view of the spatial distribution of these random numbers we are getting the inhomogeneous Bernoulli process $\mathrm{Ber}(\vec{q})$ with the vector-valued parameter $\vec{q} := \{q_k := 1/\log k\}_{k=2}^{\infty}$.

Theorem 5.3 *For the Cramer's model* $q_k := \frac{1}{\log k}$ *we have* $h_{\mathrm{loc}}(\mathrm{Ber}(\vec{q})) = 0$.

Proof. By definition,

$$h_{\mathrm{loc}}(\mathrm{Ber}(\vec{q})) := \lim_{n \to \infty} \frac{1}{n} H(\mathrm{Ber}(\vec{q}), n).$$

On the other hand, by Lemma 5.5

$$\frac{1}{n} H(\mathrm{Ber}(\vec{q}), n) = \frac{1}{n} \sum_{k=2}^{n} H(\mathrm{Ber}(q_k), 1) = -\frac{1}{n} \sum_{k=2}^{n} q_k \log q_k - \frac{1}{n} \sum_{k=2}^{n} q'_k \log q'_k.$$

We estimate the last two sums separately.

$$S_n := -\sum_{k=2}^{n} q_k \log q_k = \sum_{k=2}^{n} \frac{\log \log k}{\log k} < \sum_{k=2}^{n} \frac{k}{\log k}.$$

Up to normalization, the last term is the mathematical expectation of the distribution $\{1/\log k\}_{k=2}^{n}$. The function $1/\log k$ monotonously decays, so after normalization, the last term becomes less than or equal to $(n-1)/2$.

To perform normalization, we need to calculate

$$R(n) := \sum_{k=2}^{n} \frac{1}{\log k} = \sum_{k=2}^{n} q_k = \sum_{k=2}^{\frac{n}{\log^2 n}} q_k + \sum_{k=\frac{n}{\log^2 n}}^{n} q_k.$$

Clearly,

$$\sum_{k=2}^{\frac{n}{\log^2 n}} q_k \leq \frac{n}{\log^2 n}.$$

On the other hand,

$$\frac{n}{\log n} < (n - \frac{n}{\log^2 n} + 1)q_n < \sum_{k=\frac{n}{\log^2 n}}^{n} q_k \leq \frac{n(1-q_n)}{\log n - \log\log^2 n} < \frac{n}{\log n} + o(\frac{n}{\log n}).$$

Therefore, $R(n) - \frac{n}{\log n} = o(\frac{n}{\log n})$, and $S_n \leq \frac{n}{2\log n}$.

The 2nd sum boils down to

$$\begin{aligned} S_n' &:= -\sum_{k=2}^{n} q_k' \log q_k' = -\sum_{k=2}^{n} (1 - \frac{1}{\log k}) \log(1 - \frac{1}{\log k}) \\ &= -\sum_{k=2}^{n} \log(1 - \frac{1}{\log k}) + \sum_{k=2}^{n} \frac{1}{\log k} \log(1 - \frac{1}{\log k}) \\ &\leq C \sum_{k=2}^{n} \frac{1}{\log k} \leq C \frac{n}{\log n}. \end{aligned}$$

Finally, collecting above estimates, we get

$$\frac{1}{n} H(\mathrm{Ber}(\vec{q}), n) = \frac{1}{n} C \frac{n}{\log n} \overset{n\to\infty}{\longrightarrow} 0.$$

□

In distinction to the homogeneous case one cannot use Shannon-McMillan-Breiman Theorem and we can claim that statistics of realizations $\frac{1}{n}H(\vec{p}(\vec{x}^{\vec{q}}, n, N))$ converge to $\frac{1}{n}H(\vec{p}(\mathrm{Ber}(\vec{q}), n))$ only in probability. Recall that $\vec{p}(\vec{x}^{\vec{q}}, n, N)$ is the distribution of subwords of length n in the starting piece of length N of $\vec{x}^{\vec{q}}$.

6 Spatial distribution of quadratic residues

Patterns formed by quadratic residues and non-residues modulo a prime have been studied since the 19th century [1] and still they continue to attract attention of contemporary mathematicians [2, 11, 25, 18] from various points of view. For a detailed historical overview of the concept of quadratic residues, we refer the reader to the monograph [32], and to the modern analysis from an algebraic-geometric point of view - to [18].

Probably V.I. Arnold [2] was the first to discuss these matters from the point of view of randomness, albeit at a heuristic level. Arnold's negative answer to this question stands in stark contrast to S. Wright's [32] positive answer, who used a completely different heuristic approach based on the Central Limit Theorem. S. Wright argues also that the positive answer follows from earlier results due to P. Kurlberg and Z. Rudnick [21, 22] about the distribution of spacings between quadratic residues.

Definition 6.1 An integer k is called a *quadratic residue* modulo q if it is congruent to a perfect square modulo q; i.e., if there exists an integer ℓ such that: $\ell^2 \equiv k \pmod{q}$. Otherwise, k is called a quadratic non-residue modulo q.

For $q = 19$, the 9 quadratic residues are $(1, 4, 5, 6, 7, 9, 11, 16, 17)$, while another 9 numbers $(2, 3, 8, 10, 12, 13, 14, 15, 18)$ are quadratic non-residues. A number of other examples with their analysis can be found in [32].

For an odd prime q consider the finite sequence $(1, 2, \ldots, q-1)$. Replacing each number in this sequence with 1 if it is a quadratic residue modulo q, and 0 otherwise, we get the binary word $\vec{b}^{(q)} := (b_1, \ldots, b_{q-1})$.

In this section we will be interested in the "randomness" of the sequence of these binary words growing as $q \to \infty$. Note that longer words here do not include shorter ones, and that none of the complexity-type concepts discussed in the literature capture situations of this kind. Therefore we need a new definition in terms of a scheme of series.

Definition 6.2 Let $\vec{B} := (\vec{b}^{(m)})_{m\in\mathbb{Z}_+}$ be a sequence of binary words with $|\vec{b}^{(m)}| \overset{m\to\infty}{\longrightarrow} \infty$. By the *local entropy* of $\vec{B}$ we mean $h_{\rm loc}(\vec{B}) := \lim_{m\to\infty} h_{\rm loc}((\vec{b})^{m})$, and by the *information entropy* $h_{\rm info}(\vec{B}) := \lim_{m\to\infty} h_{\rm info}(\vec{b}^{(m)})$.

Theorem 6.1 *Let $\vec{B} := (\vec{b}^{(q_m)})_{m\in\mathbb{Z}_+}$ be a sequence of binary words representing quadratic residues, where q_m is the m-th prime number. Then $h_{\rm loc}(\vec{B}) = h_{\rm info}(\vec{B}) = 1$.*

To prove this result we need some information about the statistics of quadratic residues.

Let $L(\vec{w}, \vec{b})$ be the number of occurrences of the binary word $\vec{w}$ in a finite binary sequence $\vec{b}$. The following result on the asymptotic equidistribution of these quantities for quadratic residues was proved in [11] (see also a discussion in [18]).

Theorem 6.2 *[11] For each binary word $\vec{w}$ of length $n := |\vec{w}| \le q$ we have*

$$|L(\vec{w}, \vec{b}^{(q)}) - q2^{-|\vec{w}|}| < (|\vec{w}| - 1)\sqrt{q} + |\vec{w}|/2.$$

Proof of Theorem 6.1. By Theorem 6.2 the distribution of sub-words of the same length in $\vec{b}^{(q_m)}$ is asymptotically uniform with accuracy or order $1/\sqrt{q_m}$. Setting $N := q_m, n := |\vec{w}|$, we see that the point-wise moduli of differences between the theoretical uniform distribution $\vec{p}^u$ of words of length n and the observed distribution $\vec{p}^o$ cannot exceed $CN^{-1/2}$. Thus it seems that we are in a position to apply the result of Lemma 2.3.Unfortunately this is not the case. Indeed, the ℓ_1-norm of the difference of distributions is of order $NN^{-1/2} = N^{1/2} \overset{N\to\infty}{\longrightarrow} \infty$.

Therefore, it is necessary to adjust the approach. Denote by $\{\varepsilon_i := p_i^u - p_i^o\}$ the point-wise differences of distributions. Then

$$|\varepsilon_i| \le CN^{-1/2}, \quad \sum_i \varepsilon_i = 0.$$

The equality above follows from the fact the both distributions are probabilistic.

Using the same arguments as in the proof of Lemma 2.1, we get

$$|H(\vec{p}^u) - H(\vec{p}^o)| = \sum_i \varepsilon_i |\log N| + o(1/N) = o(1/N) \overset{N\to\infty}{\longrightarrow} 0.$$

This proves that

$$h_{\rm loc}(\vec{B}) = \lim_{N\to\infty} H(\vec{p}^{\,u}) = 1.$$

The last claim that $h_{\rm info}(\vec{B}) = 1$ follows from the observation that by Theorem 6.2 all finite binary patterns have positive frequencies. □

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