# Coupled Boundary and Volume Integral Equations for Electromagnetic Scattering 

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#### Abstract

We study frequency domain electromagnetic scattering at a bounded, penetrable, and inhomogeneous obstacle $\Omega \subset \mathbb{R}^{3}$. From the Stratton-Chu integral representation, we derive a new representation formula when constant reference coefficients are given for the interior domain. The resulting integral representation contains the usual layer potentials, but also volume potentials on $\Omega$. Then it is possible to follow a single-trace approach to obtain boundary integral equations perturbed by traces of compact volume integral operators with weakly singular kernels. The coupled boundary and volume integral equations are discretized with a Galerkin approach with usual Curl-conforming and Div-conforming finite elements on the boundary and in the volume. Compression techniques and special quadrature rules for singular integrands are required for an efficient and accurate method. Numerical experiments provide evidence that our new formulation enjoys promising properties.


Keywords: volume integral equations, boundary integral equations, electromagnetic scattering

## 1. Introduction

### 1.1. Maxwell Transmission Problem

We are interested in solving the frequency domain electromagnetic wave scattering problem in a medium that is homogeneous outside a bounded region $\Omega_{i} \subset \mathbb{R}^{3}$ (see Figure 11. We denote the exterior domain $\Omega_{o}:=\mathbb{R}^{3} \backslash \bar{\Omega}_{i}$. Material properties are given by functions $\varepsilon \in L^{\infty}\left(\mathbb{R}^{3}\right)$ and $\mu \in L^{\infty}\left(\mathbb{R}^{3}\right)$ where

$$
\begin{equation*}
\varepsilon(\boldsymbol{x}) \equiv \varepsilon_{0}, \quad \mu(\boldsymbol{x}) \equiv \mu_{0} \quad \text { for } \boldsymbol{x} \in \Omega_{o}, \tag{1}
\end{equation*}
$$

and $\varepsilon_{\max }>\varepsilon(\boldsymbol{x})>\varepsilon_{\min }>0, \mu_{\max }>\mu(\boldsymbol{x})>\mu_{\min }>0$ almost everywhere in $\mathbb{R}^{3}$.
The equations governing the problem of finding the total electric field $\boldsymbol{u}:=\boldsymbol{u}^{s}+\boldsymbol{u}^{\text {inc }}$ and total magnetic field $\boldsymbol{v}:=\boldsymbol{v}^{s}+\boldsymbol{v}^{\text {inc }}$ in this inhomogeneous medium are

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}-i \omega \mu(\boldsymbol{x}) \boldsymbol{v}=0, \quad \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon(\boldsymbol{x}) \boldsymbol{u}=0, \quad \text { for } \boldsymbol{x} \in \mathbb{R}^{3}, \tag{2}
\end{equation*}
$$

where $\boldsymbol{u}^{\text {inc }}, \boldsymbol{v}^{\text {inc }}$ are the incident fields satisfying the vacuum Maxwell's equations in the whole space,

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}^{\mathrm{inc}}-i \omega \mu_{0} \boldsymbol{v}^{\mathrm{inc}}=0, \quad \operatorname{curl} \boldsymbol{v}^{\mathrm{inc}}+i \omega \varepsilon_{0} \boldsymbol{u}^{\mathrm{inc}}=0, \quad \text { for } \boldsymbol{x} \in \mathbb{R}^{3}, \tag{3}
\end{equation*}
$$

[^0]

Figure 1: Geometric setting. Inhomogeneous material.
and $\boldsymbol{u}^{s}, \boldsymbol{v}^{s}$ satisfy Silver-Müller radiation conditions [19, Chapter 6]

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \boldsymbol{v}^{s} \times \frac{\boldsymbol{x}}{r}-\boldsymbol{u}^{s}=0, \text { uniformly on } r=|\boldsymbol{x}| \text {. } \tag{4}
\end{equation*}
$$

The problem can be formulated as the following transmission problem:

Find $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}_{\mathrm{loc}}\left(\mathbf{c u r l}, \mathbb{R}^{3} \backslash \Gamma\right)$ such that

$$
\begin{array}{lllll}
\operatorname{curl} \boldsymbol{u}-i \omega \mu_{0} \boldsymbol{v} & =0, & \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{0} \boldsymbol{u} & =0 & \text { in } \Omega_{o}, \\
\operatorname{curl} \boldsymbol{u}-i \omega \mu(\boldsymbol{x}) \boldsymbol{v} & =0, & \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon(\boldsymbol{x}) \boldsymbol{u} & =0 & \text { in } \Omega_{i}, \\
\gamma_{\mathbf{t}}^{+} \boldsymbol{u}-\gamma_{\mathbf{t}}^{-} \boldsymbol{u} & =-\gamma_{\mathbf{t}} \boldsymbol{u}^{\text {inc }}, & \gamma_{\mathbf{t}}^{+} \boldsymbol{v}-\gamma_{\mathbf{t}}^{-} \boldsymbol{v} & =-\gamma_{\mathbf{t}} \boldsymbol{v}^{\text {inc }} & \text { on } \Gamma,  \tag{5}\\
\lim _{r \rightarrow \infty} \boldsymbol{v} \times \frac{\boldsymbol{x}}{r}-\boldsymbol{u}=0, & \text { uniformly on } r=|\boldsymbol{x}|, & &
\end{array}
$$

where $\gamma_{\mathbf{t}}^{ \pm}$denotes the exterior/interior tangential trace operators (see Section 2.1 for details).

### 1.2. VIEs for electromagnetic scattering

In the general setting, it is possible to formulate volume integral equations (VIEs) to solve the transmission problem (5). Depending on the material properties, different formulations can be used [7, 35]. An example is given next:

Find $\boldsymbol{u} \in \mathbf{H}\left(\operatorname{div}, \Omega_{i}\right) \cap \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ such that

$$
\begin{equation*}
\boldsymbol{u}-\operatorname{grad} \operatorname{div} \mathbf{N}_{\Omega_{i}, \kappa_{0}}\left(p_{e} \boldsymbol{u}\right)-\kappa_{0}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{0}}\left(p_{e} \boldsymbol{u}\right)-\operatorname{curl} \mathbf{N}_{\Omega_{i}, \kappa_{0}}\left(q_{m} \operatorname{curl} \boldsymbol{u}\right)=\boldsymbol{u}^{\mathrm{inc}} \tag{6}
\end{equation*}
$$

where $p_{e}(\boldsymbol{x}):=1-\frac{\varepsilon(\boldsymbol{x})}{\varepsilon_{0}}, q_{m}(\boldsymbol{x}):=1-\frac{\mu_{0}}{\mu(\boldsymbol{x})}$, and $\mathbf{N}_{\Omega_{i}, \kappa_{0}}$ is the Newton potential over $\Omega_{i}$ with wavenumber $\kappa_{0}$ (introduced in Section 2.2.

Variants of (6) can be found in [22, 23, 35, 7]. The operators involved in these formulations are not compact in $\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)$ or $\mathbf{H}\left(\operatorname{div}, \Omega_{i}\right)$. Most of the equations include integral operators with strongly singular kernels. Therefore, Fredholm theory can not be used directly, as the operators underlying the VIEs fail to be compact perturbations of the identity. Spectral properties of the volume integral operators (VIOs) have been studied, with results in the continuous setting [22, 23] and numerical experiments for the discrete setting [34]. Well-posedness of their discretizations is not available for the existing formulations. Galerkin discretizations, although widely used in literature, are not guaranteed to be stable or converge in appropriate normed spaces. Galerkin methods for second-kind boundary integral equations in $L^{2}(\Gamma)$ fail to converge for every asymptotically dense sequence of subspaces of $L^{2}(\Gamma)$ [13]. An equivalent result for VIEs remains as an open problem.

### 1.3. BIEs for piecewise-constant coefficients

For the particular case of piecewise-constant material properties, BIEs can be used to obtain stable formulations for the transmission problem. First and second-kind BIEs can be written [12, [17, 18, 46]. In this article we focus on the first-kind single-trace formulation (STF) from [12, Section 7.1], also known in the engineering community as the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) formulation [14, 43, 49]. This formulation can also be extended to the setting of composite scatterers with piecewise-constant material properties. The STF BIEs for (5) with piecewise-constant coefficients have the following structure:

Find $\boldsymbol{\alpha} \in \boldsymbol{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ and $\boldsymbol{\beta} \in \boldsymbol{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ such that

$$
\begin{equation*}
\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}\right)\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}=-\mathbf{M}^{-1}\binom{\gamma_{\mathrm{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\tau} \tilde{\boldsymbol{v}}^{\mathrm{inc}}} \tag{7}
\end{equation*}
$$

in $\boldsymbol{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \times \boldsymbol{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$, where $\mathbf{A}_{\kappa_{\star}}$ is the Maxwell's Calderón operator with wavenumber $\kappa_{\star}$ (see Section 2.4.

For piecewise-constant coefficients, BIEs are arguably the best option as a formulation for the transmission problem. Solving BIE formulations with the boundary-element method (BEM) offers an accurate and efficient approach. Matrix compression techniques such as $\mathcal{H}$ and $\mathcal{H}^{2}$-matrices [2, 4] significantly reduce the cost of storing and solving the dense linear systems arising from a BEM discretization.

### 1.4. FEM-BEM coupling

A widely used approach to the discretization of the transmission problem (5) relies on the coupling of a volume variational formulation in $\Omega_{i}$ with boundary integral equations realizing the Dirichlet-to-Neumann map for $\Omega_{o}$. Subsequent Galerkin finite-element discretization leads to schemes known as FEM-BEM coupling. Different couplings can be obtained depending on the choice of boundary integral equations (BIEs) for the coupling, such as Johnson-Nédélec [30], Bielak-MacCamy [3] or Costabel-Han approaches [20, 29, 27]. Robust formulations with respect to the wavenumber have also been studied in 28. We used solutions produced by FEM-BEM coupling as reference in Section 5.3

### 1.5. STF-VIEs

One drawback of the approaches mentioned in sections 1.2 and 1.4 is that these methods do not benefit from a piecewise-constant material. Neither classical VIEs nor FEM-BEM coupled formulations reduce to pure BIEs when applied in the special case of piecewise-constant coefficients. Our interest is to study an extended formulation based on
boundary and volume integral operators. The approach is similar to 48, and the analysis follows closely the acoustic scattering analog [32, 33], with a few differences that are particular to Maxwell equations. Similar ideas combining BIEs and VIEs can also be found in [39, 40, 42]. Starting from the Stratton-Chu integral representation, we derive a new combined integral representation for the electric and magnetic fields. For the case of piecewise-constant coefficients, the formulation reduces to the simple case of first-kind BIEs (7). The volume integral operators can be shown to be compact, and only supported in the domain of inhomogeneity (i.e. not necessarily the whole domain $\Omega$, see Figure 2 ).


Figure 2: Domains for transmission problems with spatially varying coefficients.

The requirements are established in the following assumption.

Assumption 1.1. The following assumptions will be required

1. $\Omega_{i}$ is a bounded Lipschitz domain with boundary $\Gamma$.
2. The parameters are smooth inside $\Omega_{i}: \varepsilon, \mu \in C^{1}\left(\bar{\Omega}_{i}\right) \cap C^{2}\left(\Omega_{i}\right)$.
3. There are positive constants $\varepsilon_{\min }, \mu_{\min }, \varepsilon_{\max }, \mu_{\max }$ such that

$$
\varepsilon_{\min } \leq \varepsilon(\boldsymbol{x}) \leq \varepsilon_{\max }, \quad \quad \mu_{\min } \leq \mu(\boldsymbol{x}) \leq \mu_{\max }
$$

for all $\boldsymbol{x} \in \Omega_{i}$.
4. Reference coefficients $\varepsilon_{1}, \mu_{1} \in \mathbb{R}$ are chosen such that $\mu_{1}>0, \varepsilon_{1}>0$.

In constrast with the acoustic scattering approach, for Maxwell problems we need different techniques. Problems are no longer coercive, but $T$-coercive [16]. Discrete stability now depends on $h$-uniform inf-sup conditions, equivalent to $T_{h}$-coercivity. First order formulations play a central role, due to the symmetry between electric and magnetic fields. Finally, we observed an interesting problem when discretizing volume integral equations: discrete stability of duality pairings can not be taken for granted as in the scalar case. The required stability estimates are not readily available as in the case of $H^{1}(\Omega)$ and its dual space $\widetilde{H}^{-1}(\Omega)$.

### 1.6. Outline and main results

In Section 2.1 we introduce the preliminaries for the functional setting in which we study our equations. We present the derivation of the representation formula in Section 2.7. Our new representation formula is written in

Section 3 , (62), and we state the variational formulation in Problem 4.1, Section 4
In Section 4 we study the continuous problem using standard techniques: Fredholm theory and T-Coercivity. In Theorem 4.12 we establish the well-posedness of Problem 4.1
Results about the Galerkin discretization are presented in Section 5 Numerical experiments that validate our formulation are shown in Section 5.3.

List of symbols
Symbol
$\varepsilon, \mu$
$C^{\infty}$
$C_{0}^{\infty}$
$C_{\text {comp }}^{\infty}$
$H^{s}, H_{\text {loc }}^{s}, H_{\text {comp }}^{s}$
$\mathbf{H}^{s}, \mathbf{H}_{\text {loc }}^{s}, \mathbf{H}_{\text {comp }}^{s}$
$H^{s}(\Gamma), \mathbf{H}^{s}(\Gamma)$
$H^{-1 / 2}(\Gamma), \mathbf{H}^{-1 / 2}(\Gamma)$
$\gamma, \partial_{n}$
$\gamma_{\mathbf{t}}, \gamma_{\boldsymbol{\tau}}, \gamma_{\mathbf{n}}$
$\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$
$\mathbf{H}^{-1 / 2}\left(\right.$ curl $\left._{\Gamma}, \Gamma\right)$
$G_{j}$
$\mathbf{N}_{j}, \mathbf{N}_{j}$
$\mathbf{N}_{\Omega, j}, \mathbf{N}_{\Omega, j}$
$\boldsymbol{T}_{j}, \mathcal{D}_{j}$
$\boldsymbol{T}_{j}^{\varepsilon, \tilde{\mu}}$
$\mathbf{V}_{j}, \mathbf{K}_{j}, \mathbf{K}_{j}^{\prime}, \mathbf{W}_{j}$
$\mathbf{A}_{j}$
$\varepsilon_{1}, \mu_{1}$
$p_{e}, p_{m}$
$\tilde{\varepsilon}, \tilde{\mu}$
$\mathbf{V}_{j}^{\tilde{\varepsilon}, \tilde{\mu}}, \mathbf{W}_{j}^{\tilde{\mu}, \tilde{\varepsilon}}$
$\mathbf{A}_{j, \tilde{\mu}}^{\varepsilon}$
$\boldsymbol{\Lambda}^{e}, \boldsymbol{\Lambda}^{m}, \mathbf{\Xi}^{e}, \mathbf{\Xi}^{m}$
$\mathbf{J}^{e}, \mathbf{J}^{m}$
$\mathbf{M}$

## Description

Material coefficients varying in space
Spaces of smooth functions
Smooth functions vanishing on the boundary
Compactly supported smooth functions
Scalar Sobolev spaces of order $s$
Vector Sobolev spaces of order $s$
Scalar/vector Sobolev space of order $s$ on $\Gamma$
Dual spaces of $H^{1 / 2}(\Gamma)$ and $\mathbf{H}^{1 / 2}(\Gamma)$
Dirichlet/Neumann/Normal trace operators
Tangential and normal trace operators
Maxwell Trace space $\gamma_{\boldsymbol{\tau}}(\mathbf{H}(\mathbf{c u r l}, \Omega))$
Maxwell Trace space $\gamma_{\mathbf{t}}(\mathbf{H}(\mathbf{c u r l}, \Omega))$
Fundamental solution with wavenumber $\kappa_{j}$
Scalar and vector Newton potential
Scalar and vector Newton potential (local)
Maxwell layer potentials with wavenumber $\kappa_{j}$
Weighted Maxwell single layer potential
BIOs with wavenumber $\kappa_{j}$
Calderón operator
Constant reference coefficients
Contrast functions with reference coefficients
Scaled material coefficients
Weighted BIOs with wavenumber $\kappa_{j}$
Weighted Calderón operator
Volume integral operators (VIOs)
Operators related to traces of VIOs
Diagonal multiplier

## Section

Section (1) (1)
Section 2.1
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Section 2.1
Section 2.1
Section 2.1
Section 2.1
Section 2.2, (9)
Section 2.2, 10
Section 2.2, (12), (16)
Section 2.2, 28), 30)
Section 2.2 64a
Section 2.4
Section 2.4 (40)
Section 2.7, 51
Section 2.7, (51)
Section (3), 61)
Section (36)
Section 3, 70a
Section 3, 63)
Section 3, 70b, (70c)
Section (3, 72)

## 2. Derivation of VIEs

### 2.1. Preliminaries: Function spaces and trace operators

Let $\Omega_{i} \subset \mathbb{R}^{3}$ be a Lipschitz domain, $\Gamma:=\partial \Omega_{i}$ its Lipschitz boundary with outward unit normal $\boldsymbol{n}$. We rely on standard Sobolev spaces $H^{s}\left(\Omega_{i}\right)$ of order $s>0$. We also denote as $\widetilde{H}^{-s}\left(\Omega_{i}\right)$ the dual space of $H^{s}\left(\Omega_{i}\right)$ [36, Section 3]. Spaces of compactly supported (resp. locally integrable) functions will be denoted with a sub-index comp (resp. loc), as in $H_{\text {comp }}^{s}\left(\Omega_{i}\right)$. Sobolev spaces on the boundary $\Gamma$ are denoted as $H^{s+1 / 2}(\Gamma)$. They arise naturally as boundary restrictions of elements of $H^{s+1}\left(\Omega_{i}\right)$ by the interior Dirichlet trace operator

$$
\begin{aligned}
& \gamma^{-}: H^{s+1}\left(\Omega_{i}\right) \rightarrow H^{s+1 / 2}(\Gamma), \quad 0 \leq s \leq \frac{1}{2} \\
& \gamma^{-} u:=\left.u\right|_{\Gamma}, \quad \text { for } u \in C^{\infty}\left(\bar{\Omega}_{i}\right),
\end{aligned}
$$

which is a bounded operator [36] Theorem 3.37]. Note that we use boldface symbols to indicate vector-valued functions and function spaces of vector fields. We define the interior normal (component) trace operator $\gamma_{\mathbf{n}}$ [37, Theorem 3.24]

$$
\begin{aligned}
& \gamma_{\mathbf{n}}^{-}: \mathbf{H}\left(\operatorname{div}, \Omega_{i}\right) \rightarrow H^{-1 / 2}(\Gamma), \\
& \gamma_{\mathbf{n}}^{-} \boldsymbol{u}:=\left.\boldsymbol{u}\right|_{\Gamma} \cdot \boldsymbol{n}, \quad \text { for } \boldsymbol{u} \in\left[C^{\infty}\left(\bar{\Omega}_{i}\right)\right]^{3},
\end{aligned}
$$

where the space $\mathbf{H}\left(\operatorname{div}, \Omega_{i}\right)$ is defined as

$$
\mathbf{H}\left(\operatorname{div}, \Omega_{i}\right):=\left\{\boldsymbol{u} \in\left[L^{2}\left(\Omega_{i}\right)\right]^{3}: \operatorname{div} \boldsymbol{u} \in L^{2}\left(\Omega_{i}\right)\right\}
$$

the interior tangential (component) trace operator $\gamma_{\mathbf{t}}$ [11, Theorem 4.1]

$$
\begin{aligned}
& \gamma_{\mathbf{t}}^{-}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right), \\
& \gamma_{\mathbf{t}}^{-} \boldsymbol{u}:=\boldsymbol{n} \times\left(\left.\boldsymbol{u}\right|_{\Gamma} \times \boldsymbol{n}\right), \quad \text { for } \boldsymbol{u} \in\left[C^{\infty}\left(\bar{\Omega}_{i}\right)\right]^{3},
\end{aligned}
$$

and the rotated tangential (component) trace operator $\gamma_{\boldsymbol{\tau}}$ [11, Theorem 4.1]

$$
\begin{aligned}
& \gamma_{\boldsymbol{\tau}}^{-}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right), \\
& \gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{u}:=\left.\boldsymbol{u}\right|_{\Gamma} \times \boldsymbol{n}, \quad \text { for } \boldsymbol{u} \in\left[C^{\infty}\left(\bar{\Omega}_{i}\right)\right]^{3},
\end{aligned}
$$

where the occurring spaces are defined as

$$
\begin{aligned}
\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) & :=\left\{\boldsymbol{u} \in \mathbf{L}^{2}\left(\Omega_{i}\right): \operatorname{curl} \boldsymbol{u} \in \mathbf{L}^{2}\left(\Omega_{i}\right)\right\}, \\
\mathbf{H}^{1}\left(\operatorname{curl}, \Omega_{i}\right) & :=\left\{\boldsymbol{u} \in \mathbf{H}^{1}\left(\Omega_{i}\right): \operatorname{curl} \boldsymbol{u} \in \mathbf{H}^{1}\left(\Omega_{i}\right)\right\}, \\
\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) & :=\left\{\boldsymbol{\mu} \in \mathbf{H}_{\|}^{-1 / 2}(\Gamma): \operatorname{curl}_{\Gamma} \boldsymbol{\mu} \in H^{-1 / 2}(\Gamma)\right\}, \text { 11, Theorem 4.1] } \\
\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) & :=\left\{\boldsymbol{\mu} \in \mathbf{H}_{\times}^{-1 / 2}(\Gamma): \operatorname{div}_{\Gamma} \boldsymbol{\mu} \in H^{-1 / 2}(\Gamma)\right\},[11, \text { Theorem 4.1] } \\
\mathbf{H}_{\times}^{1 / 2}(\Gamma) & :=\gamma_{\boldsymbol{\tau}}\left(\mathbf{H}^{1}\left(\Omega_{i}\right)\right), \text { 11], Section 2], } \\
\mathbf{H}_{\|}^{1 / 2}(\Gamma) & :=\gamma_{\mathbf{t}}\left(\mathbf{H}^{1}\left(\Omega_{i}\right)\right),[11, \text { Section 2], }
\end{aligned}
$$

and $\mathbf{H}_{\times}^{-1 / 2}(\Gamma):=\left[\mathbf{H}_{\times}^{1 / 2}(\Gamma)\right]^{\prime}, \mathbf{H}_{\|}^{-1 / 2}(\Gamma):=\left[\mathbf{H}_{\|}^{1 / 2}(\Gamma)\right]^{\prime}$.
Differential operators on surfaces of Lipschitz domains are defined according to [11, Section 4]. We also need the isomorphism $\mathrm{R}: \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ given by $\mathrm{R} \boldsymbol{\mu}:=\boldsymbol{n} \times \boldsymbol{\mu}$. In particular [11] Section 2], for
$\boldsymbol{u} \in \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ we have

$$
\begin{equation*}
\mathrm{R}\left(\gamma_{\mathbf{t}} \boldsymbol{u}\right)=-\gamma_{\boldsymbol{\tau}} \boldsymbol{u}, \quad \mathrm{R}\left(\gamma_{\boldsymbol{\tau}} \boldsymbol{u}\right)=\gamma_{\mathbf{t}} \boldsymbol{u} \tag{8}
\end{equation*}
$$

For $u \in H\left(\Delta, \Omega_{i}\right)$, where

$$
H\left(\Delta, \Omega_{i}\right):=\left\{u \in H^{1}\left(\Omega_{i}\right): \Delta u \in L^{2}\left(\Omega_{i}\right)\right\}
$$

we define the Neumann trace operator $\partial_{n}$ as [45, Theorem 2.8.3]

$$
\begin{aligned}
& \partial_{n}^{-}: H\left(\Delta, \Omega_{i}\right) \rightarrow H^{-1 / 2}(\Gamma), \\
& \partial_{n}^{-} u=\left.\operatorname{grad} u\right|_{\Gamma} \cdot \boldsymbol{n}, \quad \text { for } u \in C^{\infty}\left(\bar{\Omega}_{i}\right)
\end{aligned}
$$

Replacing $\Omega_{i}$ by $\Omega_{o}:=\mathbb{R}^{d} \backslash \bar{\Omega}_{i}$ in the previous definitions, we obtain exterior trace operators: $\gamma^{+}, \gamma_{\mathbf{n}}^{+}, \gamma_{\mathbf{t}}^{+}, \gamma_{\tau}^{+}$and $\partial_{n}^{+}$, keeping the normal vector $\boldsymbol{n}$.
We define jump and average trace operators for elements of $H^{1}\left(\mathbb{R}^{d} \backslash \Gamma\right), \mathbf{H}\left(\operatorname{div}, \mathbb{R}^{d} \backslash \Gamma\right)$ and $H\left(\Delta, \mathbb{R}^{d} \backslash \Gamma\right)$ :

$$
\left.\llbracket \gamma \rrbracket:=\left\{\gamma^{+}-\gamma^{-}\right\}, \quad\{\gamma\}\right\}:=\frac{1}{2}\left\{\gamma^{+}+\gamma^{-}\right\},
$$

and similarly for other trace operators. We denote the bilinear inner product in $L^{2}\left(\Omega_{i}\right)$ as $\langle u, v\rangle_{\Omega_{i}}$. It can be extended to a duality pairing between $\widetilde{H}^{-1}\left(\Omega_{i}\right)$ and $H^{1}\left(\Omega_{i}\right)$. Similarly, we define the bilinear dual product for $H^{1 / 2}(\Gamma)$ and its dual $H^{-1 / 2}(\Gamma)$, and denote it as $\langle\cdot, \cdot\rangle_{\Gamma}$. We denote $\langle\boldsymbol{\mu}, \zeta\rangle_{\tau, \Gamma}$ the bilinear duality pairing between $\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ and $\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$.

### 2.2. Fundamental Solutions and Newton Potential

The fundamental solution for the Helmholtz operator with wavenumber $\kappa \in \mathbb{R}$ is given by $G_{\kappa} \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{3}\right)$ 47, Section 5.4]:

$$
\begin{equation*}
G_{\kappa}(\boldsymbol{x}, \boldsymbol{y}):=\frac{\exp (i \kappa|\boldsymbol{x}-\boldsymbol{y}|)}{4 \pi|\boldsymbol{x}-\boldsymbol{y}|}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{3}, \boldsymbol{x} \neq \boldsymbol{y} \tag{9}
\end{equation*}
$$

The Newton potential $\mathrm{N}_{\kappa}: C_{c}^{\infty}\left(\mathbb{R}^{3}\right) \rightarrow C^{\infty}\left(\mathbb{R}^{3}\right)$ is the mapping defined by [45, Section 3.1.1]

$$
\begin{equation*}
\mathrm{N}_{\kappa} f(\boldsymbol{x}):=\int_{\mathbb{R}^{3}} G_{\kappa}(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \mathrm{d} \boldsymbol{y} \tag{10}
\end{equation*}
$$

The Newton potential can be extended to the following two continuous operators

$$
\begin{align*}
& \mathrm{N}_{\kappa}: H_{\text {comp }}^{-1}\left(\mathbb{R}^{3}\right) \rightarrow H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{3}\right), \\
& \mathrm{N}_{\kappa}: \tag{11}
\end{align*} L_{\text {comp }}^{2}\left(\mathbb{R}^{3}\right) \rightarrow H_{\mathrm{loc}}^{2}\left(\mathbb{R}^{3}\right), ~ l
$$

and more generally, $\mathrm{N}_{\kappa}: H_{\text {comp }}^{s}\left(\mathbb{R}^{3}\right) \rightarrow H_{\mathrm{loc}}^{s+2}\left(\mathbb{R}^{3}\right)$ is continuous for $s \in \mathbb{R}$ [45, Theorem. 3.12].
Similarly, by extension by zero followed by restriction to $\Omega_{i}$, it is possible to consider the Newton potential in a bounded domain $\Omega_{i}$ :

$$
\begin{array}{lll}
\mathrm{N}_{\Omega_{i}, \kappa} & : & L^{2}\left(\Omega_{i}\right) \rightarrow H^{2}\left(\Omega_{i}\right) \\
\mathrm{N}_{\Omega_{i}, \kappa} & : & \widetilde{H}^{-1}\left(\Omega_{i}\right) \rightarrow H^{1}\left(\Omega_{i}\right) \tag{12}
\end{array}
$$

We define the scalar single layer potential as [21, Theorem 1]

$$
\begin{equation*}
\mathrm{S}_{\kappa}:=\mathrm{N}_{\kappa} \circ \gamma^{\prime}: H^{-1 / 2}(\Gamma) \rightarrow H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{3} \backslash \Gamma\right), \tag{13}
\end{equation*}
$$

which for smooth enough densities $\psi \in L^{\infty}(\Gamma)$ has the following integral representations for $\boldsymbol{x} \notin \Gamma$

$$
\begin{equation*}
\left(\mathrm{S}_{\kappa} \psi\right)(\boldsymbol{x})=\int_{\Gamma} G_{\kappa}(\boldsymbol{x}, \boldsymbol{y}) \psi(\boldsymbol{y}) \mathrm{d} s_{y} \tag{14}
\end{equation*}
$$

The following theorem [19, Theorem 8.1] is essential for the derivation of volume integral equations for scattering problems.
Theorem 2.1. The Newton potential defines a solution operator for the Helmholtz equation on $\mathbb{R}^{3}$, i.e. for $f \in$ $L_{\text {comp }}^{2}\left(\mathbb{R}^{3}\right)$ compactly supported in $\Omega_{i}, u:=\mathrm{N}_{\kappa} f$ satisfies

$$
\begin{equation*}
-\Delta u-\kappa^{2} u=f \quad \text { in } \mathbb{R}^{3} \tag{15}
\end{equation*}
$$

and the Sommerfeld radiation conditions.
Both the Newton potential and the single layer potential will also be used with vectorial arguments, for which the following mapping properties hold.

Proposition 2.2. The Newton potential can be extended to vectorial arguments component-wise. We denote it as $\mathbf{N}_{\Omega_{i}, \kappa}$, and it defines a continuous linear operator

$$
\begin{equation*}
\mathbf{N}_{\Omega_{i}, \kappa}: \mathbf{L}^{2}\left(\Omega_{i}\right) \rightarrow \mathbf{H}^{2}\left(\Omega_{i}\right) \tag{16}
\end{equation*}
$$

and it has the integral representation

$$
\mathbf{N}_{\Omega_{i}, \kappa}(\boldsymbol{f}):=\int_{\Omega_{i}} G_{\kappa}(\boldsymbol{x}-\boldsymbol{y}) \boldsymbol{f}(\boldsymbol{y}) d \boldsymbol{y}
$$

for all $\boldsymbol{f} \in \mathbf{L}^{2}\left(\Omega_{i}\right)$.
The single layer potential can also be extended to vectorial arguments component-wise. We denote it $\mathbf{S}_{\kappa}$, and it defines a continuous linear operator

$$
\mathbf{S}_{\kappa}: \mathbf{H}^{-1 / 2+s}(\Gamma) \rightarrow \mathbf{H}_{l o c}^{1+s}\left(\mathbb{R}^{3}\right), \quad-\frac{1}{2}<s<\frac{1}{2}
$$

Proof. We know that the scalar Newton potential satisfies

$$
\begin{equation*}
\mathrm{N}_{\Omega_{i}, \kappa}: L^{2}\left(\Omega_{i}\right) \rightarrow H^{2}\left(\Omega_{i}\right) \tag{17}
\end{equation*}
$$

In particular, for $\boldsymbol{u} \in \mathbf{L}^{2}\left(\Omega_{i}\right)$, using (17) component-wise leads to

$$
\begin{equation*}
\mathbf{N}_{\Omega_{i}, \kappa}: \mathbf{L}^{2}\left(\Omega_{i}\right) \rightarrow \mathbf{H}^{2}\left(\Omega_{i}\right) \tag{18}
\end{equation*}
$$

In a similar way, we know (see [45, Theorem 3.1.16]) that the scalar single-layer potential satisfies

$$
\begin{equation*}
\mathrm{S}_{\kappa}: H^{-1 / 2+s}(\Gamma) \rightarrow H_{\mathrm{loc}}^{1+s}\left(\mathbb{R}^{3}\right), \quad-\frac{1}{2}<s<\frac{1}{2} \tag{19}
\end{equation*}
$$

For any $\boldsymbol{\psi} \in \mathbf{H}^{-1 / 2+s}, s \in\left(-\frac{1}{2}, \frac{1}{2}\right]$, using (19) component-wise, we obtain

$$
\begin{equation*}
\mathbf{S}_{\kappa}: \mathbf{H}^{-1 / 2+s} \rightarrow \mathbf{H}_{\mathrm{loc}}^{1+s}\left(\mathbb{R}^{3}\right), \quad-\frac{1}{2}<s<\frac{1}{2} . \tag{20}
\end{equation*}
$$

Corollary 2.3. The Newton potential defines a continuous linear operator

$$
\mathbf{N}_{\Omega_{i}, \kappa}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}^{2}\left(\Omega_{i}\right) .
$$

Corollary 2.4. The vector-valued single-layer potential $\mathbf{S}_{\kappa}$ defines a continuous linear operator

$$
\mathbf{S}_{\kappa}: \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)
$$

Proof. From Proposition 2.2, since $\mathbf{H}_{\mathrm{loc}}^{1}\left(\mathbb{R}^{3}\right) \subset \mathbf{H}_{\mathrm{loc}}\left(\mathbf{c u r l}, \mathbb{R}^{3}\right)$, and $\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \subset \mathbf{H}_{\times}^{-1 / 2}(\Gamma) \subset \mathbf{H}^{-1 / 2}(\Gamma)=$ $\left[H^{-1 / 2}(\Gamma)\right]^{3}$, we obtain

$$
\begin{equation*}
\mathbf{S}_{\kappa}: \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}_{\mathrm{loc}}\left(\mathbf{c u r l}, \mathbb{R}^{3}\right) . \tag{21}
\end{equation*}
$$

### 2.3. Stratton-Chu Representation Formula

We show an integral representation for arbitrary vector fields, which will be useful for the study of Maxwell solutions [37, Theorem 9.1]

Theorem 2.5 (Stratton-Chu Integral Representation). Let $\boldsymbol{u}, \boldsymbol{v} \in C^{2}\left(\Omega_{i}\right), \varepsilon_{\star}, \mu_{\star}>0$, and $\kappa_{\star}=\omega \sqrt{\mu_{\star} \varepsilon_{\star}}$. Then the following integral representations hold

$$
\begin{align*}
\boldsymbol{u}= & \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{\star}}\left(\operatorname{curl} \boldsymbol{u}-i \omega \mu_{\star} \boldsymbol{v}\right)\right)+i \omega \mu_{\star} \mathbf{N}_{\Omega_{i}, \kappa_{\star}}\left(\operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{\star} \boldsymbol{u}\right) \\
& -\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{\star}}(\operatorname{div} \boldsymbol{u})\right)+\operatorname{curl}\left(\mathbf{S}_{\kappa_{\star}}\left(\gamma_{\boldsymbol{\tau}} \boldsymbol{u}\right)\right)+\operatorname{grad}\left(\mathrm{S}_{\kappa_{\star}}\left(\gamma_{\mathbf{n}} \boldsymbol{u}\right)\right)+i \omega \mu_{\star} \mathbf{S}_{\kappa_{\star}}\left(\gamma_{\boldsymbol{\tau}} \boldsymbol{v}\right), \\
\boldsymbol{v}= & \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{\star}}\left(\operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{\star} \boldsymbol{u}\right)\right)-i \omega \varepsilon_{\star} \mathbf{N}_{\Omega_{i}, \kappa_{\star}}\left(\operatorname{curl} \boldsymbol{u}-i \omega \mu_{\star} \boldsymbol{v}\right)  \tag{22}\\
& -\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{\star}}(\operatorname{div} \boldsymbol{v})\right)+\operatorname{curl}\left(\mathbf{S}_{\kappa_{\star}}\left(\gamma_{\boldsymbol{\tau}} \boldsymbol{v}\right)\right)+\operatorname{grad}\left(\mathrm{S}_{\kappa_{\star}}\left(\gamma_{\mathbf{n}} \boldsymbol{v}\right)\right)-i \omega \varepsilon_{\star} \mathbf{S}_{\kappa_{\star}}\left(\gamma_{\boldsymbol{\tau}} \boldsymbol{u}\right) .
\end{align*}
$$

We introduce the transmission problem with piecewise-constant coefficients $\varepsilon_{0}, \mu_{0}>0$ in $\Omega_{o}, \varepsilon_{1}, \mu_{1}>0$ in $\Omega_{i}$.

$$
\left\{\begin{array}{rlrllll}
\operatorname{curl} \boldsymbol{u}-i \omega \mu_{0} \boldsymbol{v} & = & 0, & \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{0} \boldsymbol{u} & = & 0 &  \tag{23}\\
\operatorname{curl} \boldsymbol{u}-i \omega \mu_{1} \boldsymbol{v} & = & \boldsymbol{f}_{1}, & \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{1} \boldsymbol{u} & = & \boldsymbol{f}_{2} & \\
\text { cur } \Omega_{i}, \\
\gamma_{\mathbf{t}}^{+} \boldsymbol{u}-\gamma_{\mathbf{t}}^{-} \boldsymbol{u} & = & -\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}, & \gamma_{\mathbf{t}}^{+} \boldsymbol{v}-\gamma_{\mathbf{t}}^{-} \boldsymbol{v} & = & -\gamma_{\mathbf{t}} \boldsymbol{v}^{\mathrm{inc}} & \\
\text { on } \Gamma, \\
\lim _{r \rightarrow \infty} \boldsymbol{v} \times & \frac{\boldsymbol{x}}{r}-\boldsymbol{u}=0, & \text { uniformly on } r=|\boldsymbol{x}| & &
\end{array}\right.
$$

where $\boldsymbol{f}_{1}$ and $\boldsymbol{f}_{2}$ are in $\mathbf{H}\left(\operatorname{div}, \Omega_{i}\right)$. For Maxwell solutions, the integral representation takes a different form. If $\boldsymbol{u}, \boldsymbol{v}$ are solutions of (23), it is possible to express $\boldsymbol{u}$ and $\boldsymbol{v}$ in terms of $\gamma_{\mathbf{n}} \boldsymbol{u}, \gamma_{\mathbf{t}} \boldsymbol{u}, \gamma_{\boldsymbol{\tau}} \boldsymbol{v}$. Note that from (23) we have

$$
\begin{array}{llll}
\operatorname{curl} \boldsymbol{u}-i \omega \mu_{1} \boldsymbol{v}=\boldsymbol{f}_{1} \text { in } \Omega \quad \Rightarrow \quad \boldsymbol{v}=\frac{1}{i \omega \mu_{1}}\left(\operatorname{curl} \boldsymbol{u}-\boldsymbol{f}_{1}\right) & \text { in } \Omega_{i}, \\
\operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{1} \boldsymbol{u}=\boldsymbol{f}_{2} \text { in } \Omega \quad \Rightarrow \quad \boldsymbol{u}=\frac{1}{i \omega \varepsilon_{1}}\left(-\operatorname{curl} \boldsymbol{v}+\boldsymbol{f}_{2}\right) & \text { in } \Omega_{i},
\end{array}
$$

and therefore, using the property [11, Section 4]

$$
\begin{equation*}
\gamma_{\mathbf{n}}^{ \pm}(\operatorname{curl} \boldsymbol{F})=\operatorname{div}_{\Gamma}\left(\gamma_{\boldsymbol{\tau}}^{ \pm} \boldsymbol{F}\right), \quad \text { for all } \boldsymbol{F} \in \mathbf{H}\left(\operatorname{curl}, \mathbb{R}^{3} \backslash \Gamma\right), \tag{24}
\end{equation*}
$$

we obtain the following identities in $H^{-1 / 2}(\Gamma)$

$$
\begin{align*}
& \gamma_{\mathbf{n}}^{-} \boldsymbol{u}=\left.\boldsymbol{n} \cdot \boldsymbol{u}\right|_{\Gamma}=\frac{1}{i \omega \varepsilon_{1}}\left(-\left.\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{v}\right|_{\Gamma}+\left.\boldsymbol{n} \cdot \boldsymbol{f}_{2}\right|_{\Gamma}\right)=\frac{1}{i \omega \varepsilon_{1}}\left(-\operatorname{div}_{\Gamma}\left(\gamma_{\tau}^{-} \boldsymbol{v}\right)+\gamma_{\mathbf{n}}^{-} \boldsymbol{f}_{2}\right),  \tag{25a}\\
& \gamma_{\mathbf{n}}^{-} \boldsymbol{v}=\left.\boldsymbol{n} \cdot \boldsymbol{v}\right|_{\Gamma}=\frac{1}{i \omega \mu_{1}}\left(\left.\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{u}\right|_{\Gamma}-\left.\boldsymbol{n} \cdot \boldsymbol{f}_{1}\right|_{\Gamma}\right)=-\frac{1}{i \omega \mu_{1}}\left(\operatorname{curl}_{\Gamma}\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)-\gamma_{\mathbf{n}}^{-} \boldsymbol{f}_{1}\right) . \tag{25b}
\end{align*}
$$

From Theorem 2.5 in $\Omega_{i}$ we have $\left(\kappa_{1}:=\omega \sqrt{\varepsilon_{1} \mu_{1}}\right)$ for the solution $(\boldsymbol{u}, \boldsymbol{v})$ of 23)

$$
\begin{align*}
\boldsymbol{u}= & \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{f}_{1}\right)\right)+i \omega \mu_{1} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{f}_{2}\right)-\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}(\operatorname{div} \boldsymbol{u})\right) \\
& -\operatorname{curl}\left(\mathbf{S}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)\right)+i \omega \mu_{1}\left(\frac{1}{\kappa_{1}^{2}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{1}}\left(\operatorname{div}_{\Gamma}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right)\right)\right)+\mathbf{S}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right)\right),  \tag{26a}\\
& +\frac{1}{i \omega \varepsilon_{1}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{f}_{2}\right)\right), \\
\boldsymbol{v}= & \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{f}_{2}\right)\right)-i \omega \varepsilon_{1} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{f}_{1}\right)-\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}(\operatorname{div} \boldsymbol{v})\right) \\
& +\operatorname{curl}\left(\mathbf{S}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right)\right)+i \omega \varepsilon_{1}\left(\frac{1}{\kappa_{1}^{2}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{1}}\left(\operatorname{curl}_{\Gamma}\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)\right)\right)+\mathbf{S}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)\right)  \tag{26b}\\
& +\frac{1}{i \omega \mu_{1}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{f}_{1}\right)\right) .
\end{align*}
$$

Owing to the vanishing source terms, in $\Omega_{o}$ we find the representation

$$
\begin{align*}
& \boldsymbol{u}=\operatorname{curl}\left(\mathbf{S}_{\kappa_{0}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{+} \boldsymbol{u}\right)\right)-i \omega \mu_{0}\left(\frac{1}{\kappa_{0}^{2}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{0}}\left(\operatorname{div}_{\Gamma}\left(\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}\right)\right)\right)+\mathbf{S}_{\kappa_{0}}\left(\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}\right)\right)  \tag{27a}\\
& \boldsymbol{v}=-\operatorname{curl}\left(\mathbf{S}_{\kappa_{0}}\left(\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}\right)\right)-i \omega \varepsilon_{0}\left(\frac{1}{\kappa_{0}^{2}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{0}}\left(\operatorname{curl}_{\Gamma}\left(\gamma_{\mathbf{t}}^{+} \boldsymbol{u}\right)\right)\right)+\mathbf{S}_{\kappa_{0}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{+} \boldsymbol{u}\right)\right) . \tag{27b}
\end{align*}
$$

Proposition 2.6 (Maxwell Layer Potentials [12, Theorem 5]). We define the Maxwell single layer potential as

$$
\begin{equation*}
\mathcal{T}_{\kappa} \beta:=\frac{1}{\kappa^{2}} \operatorname{grad} \circ \mathrm{~S}_{\kappa} \circ \operatorname{div}_{\Gamma}+\mathbf{S}_{\kappa}, \quad \text { for all } \boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \tag{28}
\end{equation*}
$$

which is a continuous linear operator

$$
\begin{equation*}
\mathcal{T}_{\kappa}: \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}\left(\operatorname{curl}^{2}, \Omega_{i} \cup \Omega_{o}\right) \cap \mathbf{H}\left(\operatorname{div} 0, \Omega_{i} \cup \Omega_{o}\right) \tag{29}
\end{equation*}
$$

We also define the Maxwell double layer potential as

$$
\begin{equation*}
\mathcal{D}_{\kappa} \boldsymbol{\alpha}:=\operatorname{curl} \mathbf{S}_{\kappa}(\mathrm{R} \boldsymbol{\alpha}), \quad \text { for all } \boldsymbol{\alpha} \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \tag{30}
\end{equation*}
$$

which is a continuous linear operator

$$
\begin{equation*}
\mathcal{D}_{\kappa}: \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}\left(\operatorname{curl}^{2}, \Omega_{i} \cup \Omega_{o}\right) \cap \mathbf{H}\left(\operatorname{div} 0, \Omega_{i} \cup \Omega_{o}\right) . \tag{31}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathbf{H}\left(\operatorname{curl}^{2}, \Omega_{i} \cup \Omega_{o}\right) & :=\left\{\boldsymbol{u} \in \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i} \cup \Omega_{o}\right): \operatorname{curl}^{2} \boldsymbol{u} \in \mathbf{L}^{2}\left(\Omega_{i} \cup \Omega_{o}\right)\right\} \\
\mathbf{H}\left(\operatorname{div} 0, \Omega_{i} \cup \Omega_{o}\right) & :=\left\{\boldsymbol{u} \in \mathbf{L}^{2}\left(\Omega_{i} \cup \Omega_{o}\right): \operatorname{div} \boldsymbol{u}=0\right\}
\end{aligned}
$$

Maxwell layer potentials define solutions for the Maxwell's equations complying with the Silver-Müller radiation conditions. Note that

$$
\begin{align*}
& \operatorname{curl} \mathcal{T}_{\kappa}=\operatorname{curl}^{\mathbf{S}_{\kappa}=-\mathcal{D}_{\kappa} \mathrm{R}}  \tag{32a}\\
& \operatorname{curl} \mathcal{D}_{\kappa}=\operatorname{curl}^{2}\left(\mathbf{S}_{\kappa} \mathrm{R}\right)=\left(\operatorname{grad} \operatorname{div}+\kappa^{2}\right) \mathbf{S}_{\kappa} \mathrm{R} \tag{32b}
\end{align*}
$$

The following identity is useful in our computations.
Lemma 2.7 ([12, Lemma 5]). For $\boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ we have $\operatorname{div}\left(\mathbf{S}_{\kappa}(\boldsymbol{\beta})\right)=\mathrm{S}_{\kappa}\left(\operatorname{div}_{\Gamma} \boldsymbol{\beta}\right)$ in $\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)$.
From Lemma 2.7 and 32 b we obtain

$$
\begin{equation*}
\operatorname{curl} \mathcal{D}_{\kappa}=\left(\operatorname{grad} \circ \mathrm{S}_{\kappa} \circ \operatorname{div}_{\Gamma}+\kappa^{2} \mathbf{S}_{\kappa}\right) \mathrm{R}=\kappa^{2} \mathcal{T}_{\kappa} \mathrm{R} \tag{33}
\end{equation*}
$$

The Maxwell layer potentials also satisfy jump relations across a boundary $\Gamma$. This will be useful when deriving boundary integral equations from integral representations.
Proposition 2.8 (Jump relations [12, Theorem 7]). Tangential traces of Maxwell layer potentials are well defined and satisfy

$$
\begin{equation*}
\llbracket \gamma_{\mathbf{t}} \rrbracket \mathcal{T}_{\kappa}=0, \quad \llbracket \gamma_{\mathbf{t}} \rrbracket \mathcal{D}_{\kappa}=-\mathbf{I}, \quad \text { in } \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \tag{34}
\end{equation*}
$$

where $\mathbf{I}$ is the identity operator in $\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$.

### 2.4. Boundary integral operators

Boundary integral operators can be defined by averaging traces of Maxwell layer potentials (28) and (30). First, we define the Maxwell single-layer boundary integral operators (or electric field integral operators) [12, Section 5],

$$
\begin{align*}
\mathbf{V}_{\kappa} & :=\left\{\left\{\gamma_{\mathbf{t}}\right\}\right\} \mathcal{T}_{\kappa}=\frac{1}{\kappa^{2}} \operatorname{grad}_{\Gamma} \circ \mathrm{V}_{\kappa} \circ \operatorname{div}_{\Gamma}+\mathbf{V}_{\kappa}^{\mathbf{t}}  \tag{35}\\
& : \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right), \\
W_{\kappa} & :=-\left\{\left\{\gamma_{\boldsymbol{\tau}}\right\} \operatorname{curl}\left(\mathcal{D}_{\kappa}\right)=-\left\{\left\{\gamma_{\boldsymbol{\tau}}\right\} \kappa^{2} \mathcal{T}_{\kappa} \mathrm{R}=\operatorname{curl}_{\Gamma} \circ \mathrm{V}_{\kappa} \circ \operatorname{curl}_{\Gamma}+\kappa^{2} \mathbf{V}_{\kappa}^{\boldsymbol{\tau}}\right.\right.  \tag{36}\\
& : \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right),
\end{align*}
$$

where

$$
\begin{equation*}
\mathbf{V}_{\kappa}:=\left\{\{\gamma\} \mathbf{S}_{\kappa}, \quad \mathbf{V}_{\kappa}^{\mathbf{t}}:=\left\{\left\{\gamma_{\mathbf{t}}\right\}\right\} \mathbf{S}_{\kappa}, \quad \mathbf{V}_{\kappa}^{\tau}:=\left\{\left\{\gamma_{\boldsymbol{\tau}}\right\}\right\}\left(\mathbf{S}_{\kappa} \mathrm{R}\right) .\right. \tag{37}
\end{equation*}
$$

Proposition 2.9 (Ellipticity of single layer operator [12, Lemma 8]). The operators $\mathrm{V}_{0}, \mathbf{V}_{0}^{\mathbf{t}}$ and $\mathbf{V}_{0}^{\tau}$ are continuous and satisfy

$$
\begin{align*}
\left\langle\mathrm{V}_{0} \psi, \psi\right\rangle_{\Gamma} \geq C\|\psi\|_{H^{-1 / 2}(\Gamma)}^{2} & \text { for all } \psi \in H^{-1 / 2}(\Gamma)  \tag{38a}\\
\left\langle\mathbf{V}_{0}^{\mathbf{t}} \boldsymbol{\beta}, \boldsymbol{\beta}\right\rangle_{\Gamma} \geq C\|\boldsymbol{\beta}\|_{\mathbf{H}^{-1 / 2}(\Gamma)}^{2} & \text { for all } \boldsymbol{\beta} \in \mathbf{H}_{\times}^{-1 / 2}(\Gamma)  \tag{38b}\\
\left\langle\mathbf{V}_{0}^{\tau} \boldsymbol{\alpha}, \boldsymbol{\alpha}\right\rangle_{\Gamma} \geq C\|\boldsymbol{\alpha}\|_{\mathbf{H}^{-1 / 2}(\Gamma)}^{2} & \text { for all } \boldsymbol{\alpha} \in \mathbf{H}_{\|}^{-1 / 2}(\Gamma) \tag{38c}
\end{align*}
$$

with constants $C>0$ only depending on $\Gamma 1$.
We also define the Maxwell double-layer boundary integral operators (or magnetic field integral operators),

$$
\begin{array}{ll}
\mathbf{K}_{\kappa}:=\left\{\left\{\gamma_{\mathbf{t}}\right\}\right\} \mathcal{D}_{\kappa} & : \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right), \\
\mathbf{K}_{\kappa}^{\prime}:=\left\{\left\{\gamma_{\boldsymbol{\tau}}\right\}\right\}\left(\mathcal{D}_{\kappa} \mathrm{R}\right) & : \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \rightarrow \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right), \tag{39}
\end{array}
$$

We collect all of them in the Calderón operator

$$
\mathbf{A}_{\kappa}:=\left(\begin{array}{cc}
-\mathbf{K}_{\kappa} & \mathbf{V}_{\kappa}  \tag{40}\\
\mathbf{W}_{\kappa} & \mathbf{K}_{\kappa}^{\prime}
\end{array}\right): \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)
$$

where we denote

$$
\begin{equation*}
\mathcal{H}(\Gamma):=\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \times \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \tag{41}
\end{equation*}
$$

### 2.5. Calderón Identities

From the jump relations (34) and definitions of BIOs, traces of layer potentials can be written in the form of Calderón identities. We start from the representation formula in (26) for the transmission problem (23), that is the case of piecewise-constant coefficients, and assume that there are no sources $\boldsymbol{f}_{1}$ and $\boldsymbol{f}_{2}$ in $\Omega_{i}$. Then, it follows from the definitions and jump relations

$$
\begin{align*}
& \gamma_{\mathbf{t}}^{-} \boldsymbol{u}=\left(\frac{1}{2} \mathbf{I}-\mathbf{K}_{\kappa_{1}}\right)\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)+i \omega \mu_{1} \mathbf{V}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right),  \tag{42a}\\
& \gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}=\left(\frac{1}{2} \mathbf{I}+\mathbf{K}_{\kappa_{1}}^{\prime}\right)\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right)+\frac{1}{i \omega \mu_{1}} \mathbf{W}_{\kappa_{1}}\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right) . \tag{42b}
\end{align*}
$$

Similarly, for 27a and 27b we have

$$
\begin{align*}
& \gamma_{\mathbf{t}}^{+} \boldsymbol{u}=\left(\frac{1}{2} \mathbf{I}+\mathbf{K}_{\kappa_{0}}\right)\left(\gamma_{\mathbf{t}}^{+} \boldsymbol{u}\right)-i \omega \mu_{0} \mathbf{V}_{\kappa_{0}}\left(\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}\right),  \tag{43a}\\
& \gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}=\left(\frac{1}{2} \mathbf{I}-\mathbf{K}_{\kappa_{0}}^{\prime}\right)\left(\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}\right)-\frac{1}{i \omega \mu_{0}} \mathbf{W}_{\kappa_{0}}\left(\gamma_{\mathbf{t}}^{+} \boldsymbol{u}\right) . \tag{43b}
\end{align*}
$$

We denote

$$
\tilde{v}:= \begin{cases}i \omega \mu_{0} \boldsymbol{v}, & \text { for } \boldsymbol{x} \in \Omega_{o}  \tag{44}\\ i \omega \mu_{1} \boldsymbol{v}, & \text { for } \boldsymbol{x} \in \Omega_{i}\end{cases}
$$

and write (42) and (43) as

$$
\begin{align*}
& \left(\frac{1}{2} \mathbf{I}-\mathbf{A}_{\kappa_{1}}\right)\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}}=0,  \tag{45a}\\
& \left(\frac{1}{2} \mathbf{I}+\mathbf{A}_{\kappa_{0}}\right)\binom{\gamma_{\mathbf{t}}^{+} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{+} \tilde{\boldsymbol{v}}}=0 \tag{45b}
\end{align*}
$$

[^1]
### 2.6. Boundary Integral Formulations for Transmission Problems

The focus is still on the case of piecewise-constant coefficients. From the Calderón identities (45) it is possible to obtain a formulation for transmission problems. So far, we know that $\boldsymbol{u}$ and $\boldsymbol{v}$ are Maxwell solutions, and we have written expressions for their interior and exterior traces. It remains to impose transmission conditions

$$
\binom{\gamma_{\mathbf{t}}^{+} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{+} \tilde{\boldsymbol{v}}}-\left(\begin{array}{cc}
1 & 0  \tag{46}\\
0 & \frac{\mu_{0}}{\mu_{1}}
\end{array}\right)\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}}=\binom{\gamma_{\mathrm{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\tau} \tilde{\boldsymbol{v}}^{\mathrm{inc}}} \quad \text { on } \Gamma .
$$

We denote

$$
\mathbf{M}:=\left(\begin{array}{cc}
1 & 0  \tag{47}\\
0 & \frac{\mu_{0}}{\mu_{1}}
\end{array}\right)
$$

and combine 46 with 45b to obtain

$$
\begin{equation*}
\left(\frac{1}{2} \mathbf{I}+\mathbf{A}_{\kappa_{0}}\right) \mathbf{M}\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}}=-\left(\frac{1}{2} \mathbf{I}+\mathbf{A}_{\kappa_{0}}\right)\binom{\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\boldsymbol{\tau}} \tilde{\boldsymbol{v}}^{\mathrm{inc}}}=-\binom{\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\boldsymbol{\tau}} \tilde{\boldsymbol{v}}^{\mathrm{inc}}}, \tag{48}
\end{equation*}
$$

where we used that $\boldsymbol{u}^{\text {inc }}$ and $\boldsymbol{v}^{\text {inc }}$ are interior Maxwell solutions with wavenumber $\kappa_{0}$. From 48) we get

$$
\begin{equation*}
\left(\frac{1}{2} \mathbf{I}+\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}\right)\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}}=-\mathbf{M}^{-1}\binom{\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\boldsymbol{\tau}} \tilde{\boldsymbol{v}}^{\mathrm{inc}}} \tag{49}
\end{equation*}
$$

Now, subtracting (45a from (49) we obtain the first-kind single-trace formulation [12, Section 7].
Find $\boldsymbol{\alpha} \in \boldsymbol{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ and $\boldsymbol{\beta} \in \boldsymbol{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ such that

$$
\begin{equation*}
\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}\right)\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}=-\mathbf{M}^{-1}\binom{\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\boldsymbol{\tau}} \tilde{\boldsymbol{v}}^{\mathrm{inc}}} \tag{50}
\end{equation*}
$$

in $\boldsymbol{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \times \boldsymbol{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$.

### 2.7. Boundary-Volume Integral Representation

Now we return to the situation where the interior coefficients may not be constant anymore, i.e. may vary in space. We write the transmission problem (5) as follows

$$
\left\{\begin{array}{rlrllll}
\operatorname{curl} \boldsymbol{u}-i \omega \mu_{0} \boldsymbol{v} & =0, & \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{0} \boldsymbol{u} & = & 0 & & \text { in } \Omega_{o},  \tag{51}\\
\operatorname{curl} \boldsymbol{u}-i \omega \mu_{1} \boldsymbol{v} & =\boldsymbol{f}_{1}, & \operatorname{curl} \boldsymbol{v}+i \omega \varepsilon_{1} \boldsymbol{u} & = & \boldsymbol{f}_{2} & & \text { in } \Omega_{i}, \\
\gamma_{\mathbf{t}}^{+} \boldsymbol{u}-\gamma_{\mathbf{t}}^{-} \boldsymbol{u} & = & -\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}, & \gamma_{\mathbf{t}}^{+} \boldsymbol{v}-\gamma_{\mathbf{t}}^{-} \boldsymbol{v} & = & -\gamma_{\mathbf{t}} \boldsymbol{v}^{\mathrm{inc}} & \\
\text { on } \Gamma, \\
\lim _{r \rightarrow \infty} \boldsymbol{v} \times & \frac{\boldsymbol{x}}{r}-\boldsymbol{u}=0, & \text { uniformly on } r=|\boldsymbol{x}|, & &
\end{array}\right.
$$

where

$$
\begin{array}{ll}
f_{1}(\boldsymbol{x}):=-i \omega \mu_{1} p_{m}(\boldsymbol{x}) \boldsymbol{v}(\boldsymbol{x}), & f_{2}(\boldsymbol{x}):=i \omega \varepsilon_{1} p_{e}(\boldsymbol{x}) \boldsymbol{u}(\boldsymbol{x}), \\
p_{m}(\boldsymbol{x}):=1-\frac{\mu(\boldsymbol{x})}{\mu_{1}}, & p_{e}(\boldsymbol{x}):=1-\frac{\varepsilon(\boldsymbol{x})}{\varepsilon_{1}}
\end{array}
$$

for $\boldsymbol{x} \in \Omega_{i}$, and $\varepsilon_{1}, \mu_{1} \in \mathbb{R}_{+}$are conveniently chosen parameters.
Remark 2.10. Note that for smooth parameters $\varepsilon$ and $\mu$ (see Assumption 1.1) and for electric/magnetic fields $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \cap \mathbf{H}\left(\operatorname{div}, \Omega_{i}\right)$, we obtain that $\boldsymbol{f}_{1}{\mid \Omega_{i}}$ and $\left.\boldsymbol{f}_{2}\right|_{\Omega_{i}}$ are in $\mathbf{H}\left(\operatorname{div}, \Omega_{i}\right)$. Therefore, we are in the setting of Section 2.3 .

The representation formula now reads: In $\Omega_{i}$,

$$
\begin{align*}
\boldsymbol{u}= & -i \omega \mu_{1} \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)\right)-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)-\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}(\operatorname{div} \boldsymbol{u})\right)  \tag{52a}\\
& -\operatorname{curl}\left(\mathrm{S}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)\right)-\operatorname{grad}\left(\mathrm{S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{u}\right)\right)+i \omega \mu_{1} \mathbf{S}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right), \\
\boldsymbol{v}= & i \omega \varepsilon_{1} \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)\right)-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)-\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}(\operatorname{div} \boldsymbol{v})\right) \\
& +\operatorname{curl} \mathbf{S}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{u}\right)-\operatorname{grad}\left(\mathrm{S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{v}\right)\right)+i \omega \varepsilon_{1} \mathbf{S}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right) . \tag{52~b}
\end{align*}
$$

The operator $\operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{\star} \cdot\right)\right): \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right) \rightarrow \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)(\star=\{e, m\})$, is only bounded, not compact. This can be seen by an integration by parts result on Newton potentials. For $\mathbf{F} \in \mathbf{L}^{2}\left(\Omega_{i}\right)$,

$$
\begin{equation*}
\operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{\star}}(\boldsymbol{F})\right)=\mathbf{N}_{\Omega_{i}, \kappa_{\star}}(\operatorname{curl} \boldsymbol{F})+\mathbf{S}_{\kappa_{\star}}\left(\gamma_{\tau} \boldsymbol{F}\right) \tag{53}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\operatorname{curl} \mathbf{N}_{\Omega_{i}, \kappa_{1}}(p \boldsymbol{u})=\mathbf{N}_{\Omega_{i}, \kappa_{1}}(\operatorname{curl}(p \boldsymbol{u}))+\mathbf{S}_{\kappa_{1}}\left(p \gamma_{\tau} \boldsymbol{u}\right) \tag{54}
\end{equation*}
$$

where the vector single-layer potential $\mathbf{S}_{\kappa_{1}}$ is only a bounded operator in $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$.
We will repeatedly make use of the product rule

$$
\operatorname{curl}(f \boldsymbol{F})=\operatorname{grad} f \times \boldsymbol{F}+f \operatorname{curl} \boldsymbol{F}, \quad f \in C^{1}\left(\Omega_{i}\right), \boldsymbol{F} \in\left[C^{1}\left(\Omega_{i}\right)\right]^{3} .
$$

Solutions of 51 also satisfy

$$
\begin{array}{lll}
\operatorname{div}(\varepsilon \boldsymbol{u})=\operatorname{grad} \varepsilon \cdot \boldsymbol{u}+\varepsilon \operatorname{div}(\boldsymbol{u})=0 & \Rightarrow \quad \operatorname{div}(\boldsymbol{u})=-\boldsymbol{\tau}_{e} \cdot \boldsymbol{u}, & \text { in } \Omega_{i}, \\
\operatorname{div}(\mu \boldsymbol{v})=\operatorname{grad} \mu \cdot \boldsymbol{v}+\mu \operatorname{div}(\boldsymbol{v})=0 & \Rightarrow \quad \operatorname{div}(\boldsymbol{v})=-\boldsymbol{\tau}_{m} \cdot \boldsymbol{v}, & \text { in } \Omega_{i} \tag{55b}
\end{array}
$$

where we defined

$$
\begin{equation*}
\tau_{e}:=\frac{\operatorname{grad} \varepsilon}{\varepsilon}, \quad \tau_{m}:=\frac{\operatorname{grad} \mu}{\mu} \tag{56}
\end{equation*}
$$

## 3. STF-VIEs

The representation formula from 52 a and 52 b now reads

$$
\begin{align*}
\boldsymbol{u}= & -i \omega \mu_{1} \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)\right)-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)+\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{e} \cdot \boldsymbol{u}\right)\right)  \tag{57}\\
& -\operatorname{curl}\left(\mathbf{S}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)\right)-\operatorname{grad}\left(\mathrm{S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{u}\right)\right)+i \omega \mu_{1} \mathbf{S}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right), \\
\boldsymbol{v}= & i \omega \varepsilon_{1} \operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)\right)-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)+\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{m} \cdot \boldsymbol{v}\right)\right)  \tag{58}\\
& +\operatorname{curl}\left(\mathbf{S}_{\kappa_{1}}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{u}\right)\right)-\operatorname{grad}\left(\mathrm{S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{v}\right)\right)+i \omega \varepsilon_{1} \mathbf{S}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right) .
\end{align*}
$$

The integration by parts result from (53) leads to

$$
\begin{align*}
\operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)\right) & =\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{curl}\left(p_{m} \boldsymbol{v}\right)\right)+\mathbf{S}_{\kappa_{1}}\left(p_{m} \gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right),  \tag{59a}\\
\operatorname{curl}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)\right) & =\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{curl}\left(p_{e} \boldsymbol{u}\right)\right)-\mathbf{S}_{\kappa_{1}}\left(p_{e} \mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right) . \tag{59b}
\end{align*}
$$

From 25a and 25b we obtain

$$
\begin{array}{ll}
\gamma_{\mathbf{n}}^{-} \boldsymbol{u}=-\frac{1}{i \omega \varepsilon_{1} \tilde{\varepsilon}} \operatorname{div}_{\Gamma}\left(\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}\right) & \text { in } H^{-1 / 2}(\Gamma), \\
\gamma_{\mathbf{n}}^{-} \boldsymbol{v}=\frac{1}{i \omega \mu_{1} \tilde{\mu}} \operatorname{curl}_{\Gamma}\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right) & \text { in } H^{-1 / 2}(\Gamma), \tag{60b}
\end{array}
$$

where

$$
\begin{equation*}
\tilde{\varepsilon}(\boldsymbol{x}):=\frac{\varepsilon(\boldsymbol{x})}{\varepsilon_{1}}, \quad \tilde{\mu}(\boldsymbol{x}):=\frac{\mu(\boldsymbol{x})}{\mu_{1}} \tag{61}
\end{equation*}
$$

for all $\boldsymbol{x} \in \Omega_{i}$.
From now on, we denote $\tilde{\boldsymbol{v}}:=i \omega \mu_{1} \boldsymbol{v}$. Combining expressions 59a) (60b into 57 and we obtain a new representation formula for the fields $\boldsymbol{u}, \boldsymbol{v}$ solving (51):

$$
\begin{array}{ll}
\boldsymbol{u}=\boldsymbol{\Xi}^{m} \tilde{\boldsymbol{v}}+\boldsymbol{\Lambda}^{e} \boldsymbol{u}-\mathcal{D}_{\kappa_{1}}\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)+\boldsymbol{\mathcal { T }}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\left(\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}\right), & \text { in } \Omega_{i}, \\
\tilde{\boldsymbol{v}}=\boldsymbol{\Xi}^{e} \boldsymbol{u}+\boldsymbol{\Lambda}^{m} \tilde{\boldsymbol{v}}-\mathcal{D}_{\kappa_{1}}\left(\mathrm{R} \gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}\right)-\kappa_{1}^{2} \mathcal{T}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}}\left(\mathrm{R} \gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right), & \text { in } \Omega_{i}, \tag{62b}
\end{array}
$$

where we defined the volume integral operators

$$
\begin{align*}
\boldsymbol{\Xi}^{m} \boldsymbol{v} & :=-\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{curl}\left(p_{m} \boldsymbol{v}\right)\right),  \tag{63a}\\
\boldsymbol{\Xi}^{e} \boldsymbol{u} & :=-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{curl}\left(p_{e} \boldsymbol{u}\right)\right),  \tag{63b}\\
\Lambda^{m} \boldsymbol{v} & :=-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)+\operatorname{grad} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{m} \cdot \boldsymbol{v}\right),  \tag{63c}\\
\boldsymbol{\Lambda}^{e} \boldsymbol{u} & :=-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)+\operatorname{grad} \mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{e} \cdot \boldsymbol{u}\right), \tag{63~d}
\end{align*}
$$

and the layer potentials

$$
\begin{array}{lr}
\mathcal{T}_{\kappa_{1}}^{a, b}(\boldsymbol{\beta}):=\frac{1}{\kappa_{1}^{2}} \operatorname{grad}\left(\mathrm{~S}_{\kappa_{1}}\left(\frac{1}{a} \operatorname{div}_{\Gamma}(\boldsymbol{\beta})\right)\right)+\mathbf{S}_{\kappa_{1}}(b \boldsymbol{\beta}), & \boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \\
\mathcal{D}_{\kappa_{1}}(\boldsymbol{\alpha}):=\operatorname{curl} \mathbf{S}_{\kappa_{1}}(\operatorname{R} \boldsymbol{\alpha}), & \boldsymbol{\alpha} \in \mathbf{H}^{-1 / 2}\left(\operatorname{cur}_{\Gamma}, \Gamma\right) \tag{64b}
\end{array}
$$

Remark 3.1. We use $\tilde{\boldsymbol{v}}$ as an unknown instead of $\boldsymbol{v}$ in order to avoid scalings in our operators.

We take the trace $\gamma_{\mathbf{t}}^{-}$on 62a) and $\gamma_{\boldsymbol{\tau}}^{-}$on 62b. By the jump relations (34), we obtain

$$
\begin{align*}
& \gamma_{\mathbf{t}}^{-} \boldsymbol{u}=\gamma_{\mathbf{t}}^{-}\left(\boldsymbol{\Xi}^{m} \tilde{\boldsymbol{v}}\right)+\gamma_{\mathbf{t}}^{-}\left(\boldsymbol{\Lambda}^{e} \boldsymbol{u}\right)+\left(\frac{1}{2} \mathbf{I}-\mathbf{K}_{\kappa_{1}}\right)\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)+\mathbf{V}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\left(\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}\right)  \tag{65a}\\
& \gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}=\gamma_{\boldsymbol{\tau}}^{-}\left(\boldsymbol{\Xi}^{e} \boldsymbol{u}\right)+\gamma_{\boldsymbol{\tau}}^{-}\left(\boldsymbol{\Lambda}^{m} \tilde{\boldsymbol{v}}\right)+\mathbf{W}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}}\left(\gamma_{\mathbf{t}}^{-} \boldsymbol{u}\right)+\left(\frac{1}{2} \mathbf{I}+\mathbf{K}_{\kappa_{1}}^{\prime}\right)\left(i \omega \mu_{1} \gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}\right) \tag{65b}
\end{align*}
$$

where we define weighted boundary integral operators as

$$
\begin{align*}
\mathbf{V}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}(\boldsymbol{\beta}) & :=\frac{1}{\kappa_{1}^{2}} \operatorname{grad}_{\Gamma} \mathrm{V}_{\kappa_{1}}\left(\frac{1}{\tilde{\varepsilon}} \operatorname{div}_{\Gamma}(\boldsymbol{\beta})\right)+\mathbf{V}_{\kappa_{1}}^{\mathbf{t}}(\tilde{\mu} \boldsymbol{\beta}), & & \boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right),  \tag{66a}\\
\mathbf{W}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}}(\boldsymbol{\alpha}) & :=\operatorname{curl}_{\Gamma} \mathrm{V}_{\kappa_{1}}\left(\frac{1}{\tilde{\mu}} \operatorname{curl}_{\Gamma}(\boldsymbol{\alpha})\right)+\kappa_{1}^{2} \mathbf{V}_{\kappa_{1}}^{\tau}(\tilde{\varepsilon} \boldsymbol{\alpha}), & & \boldsymbol{\alpha} \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \tag{66b}
\end{align*}
$$

We denote $\boldsymbol{\alpha}^{-}:=\gamma_{\mathbf{t}}^{-} \boldsymbol{u}, \beta^{-}:=\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}$ and rewrite 65a and 65b as

$$
\begin{align*}
& \boldsymbol{\alpha}^{-}=\gamma_{\mathbf{t}}^{-}\left(\boldsymbol{\Xi}^{m} \tilde{\boldsymbol{v}}\right)+\gamma_{\mathbf{t}}^{-}\left(\boldsymbol{\Lambda}^{e} \boldsymbol{u}\right)+\left(\frac{1}{2} \mathbf{I}-\mathbf{K}_{\kappa_{1}}\right)\left(\boldsymbol{\alpha}^{-}\right)+\mathbf{V}_{\kappa_{1}}^{\tilde{\varepsilon_{1}}\left(\boldsymbol{\beta}^{-}\right)}  \tag{67a}\\
& \boldsymbol{\beta}^{-}=\gamma_{\boldsymbol{\tau}}^{-}\left(\boldsymbol{\Xi}^{e} \boldsymbol{u}\right)+\gamma_{\boldsymbol{\tau}}^{-}\left(\boldsymbol{\Lambda}^{m} \tilde{\boldsymbol{v}}\right)+\mathbf{W}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}}\left(\boldsymbol{\alpha}^{-}\right)+\left(\frac{1}{2} \mathbf{I}+\mathbf{K}_{\kappa_{1}}^{\prime}\right)\left(\boldsymbol{\beta}^{-}\right) . \tag{67b}
\end{align*}
$$

We denote $\alpha^{+}:=\gamma_{\mathbf{t}}^{+} \boldsymbol{u}, \beta^{+}:=i \omega \mu_{0} \gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}$. From (27a, 27b) and the jump relations (34) we have

$$
\begin{align*}
\boldsymbol{\alpha}^{+} & =\left(\frac{1}{2} \mathbf{I}+\mathbf{K}_{\kappa_{0}}\right)\left(\boldsymbol{\alpha}^{+}\right)-\mathbf{V}_{\kappa_{0}}\left(\boldsymbol{\beta}^{+}\right)  \tag{68a}\\
\boldsymbol{\beta}^{+} & =-\mathbf{W}_{\kappa_{0}}\left(\boldsymbol{\alpha}^{+}\right)+\left(\frac{1}{2} \mathbf{I}-\mathbf{K}_{\kappa_{0}}^{\prime}\right)\left(\boldsymbol{\beta}^{+}\right) \tag{68b}
\end{align*}
$$

From 67a -68b we can write

$$
\begin{align*}
& \left(\frac{1}{2} \boldsymbol{I}-\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\right)\binom{\boldsymbol{\alpha}^{-}}{\boldsymbol{\beta}^{-}}-\mathbf{J}^{e} \boldsymbol{u}-\mathbf{J}^{m} \tilde{\boldsymbol{v}}=0,  \tag{69a}\\
& \left(\frac{1}{2} \boldsymbol{I}+\mathbf{A}_{\kappa_{0}}\right)\binom{\boldsymbol{\alpha}^{+}}{\boldsymbol{\beta}^{+}}=0, \tag{69b}
\end{align*}
$$

where we defined

$$
\begin{align*}
\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}} & :=\left(\begin{array}{cc}
-\mathbf{K}_{\kappa_{1}} & \mathbf{V}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}} \\
\mathbf{W}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}} & \mathbf{K}_{\kappa_{1}}^{\prime}
\end{array}\right): \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma),  \tag{70a}\\
\mathbf{J}^{e} & :=\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{\Lambda}^{e}}{\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{\Xi}^{e}}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathcal{H}(\Gamma),  \tag{70b}\\
\mathbf{J}^{m} & :=\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{\Xi}^{m}}{\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{\Lambda}^{m}}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathcal{H}(\Gamma) \tag{70c}
\end{align*}
$$

Combining 69a and 69b with the transmission conditions

$$
\left(\begin{array}{cc}
1 & 0  \tag{71}\\
0 & \frac{1}{\mu_{0}}
\end{array}\right)\binom{\boldsymbol{\alpha}^{+}}{\boldsymbol{\beta}^{+}}-\left(\begin{array}{cc}
1 & 0 \\
0 & \frac{1}{\mu_{1}}
\end{array}\right)\binom{\boldsymbol{\alpha}^{-}}{\boldsymbol{\beta}^{-}}=\binom{\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}}{\frac{1}{\mu_{0}} \gamma_{\tau} \tilde{\boldsymbol{v}}^{\mathrm{inc}}}, \quad \tilde{\boldsymbol{v}}^{\mathrm{inc}}:=i \omega \mu_{0} \boldsymbol{v}^{\mathrm{inc}} .
$$

Retaining $\alpha:=\alpha^{-}$and $\beta:=\beta^{-}$as unknown traces on $\Gamma$, we obtain the single-trace boundary-volume integral
equation

$$
\begin{equation*}
\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon} \tilde{\mu}}\right)\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}+\mathbf{J}^{e} \boldsymbol{u}+\mathbf{J}^{m} \tilde{\boldsymbol{v}}=-\mathbf{M}^{-1}\binom{\gamma_{\mathbf{t}} \boldsymbol{u}^{\mathrm{inc}}}{\gamma_{\boldsymbol{\tau}} \tilde{\boldsymbol{v}}^{\mathrm{inc}}} \tag{72}
\end{equation*}
$$

where $\mathbf{M}:=\left(\begin{array}{cc}1 & 0 \\ 0 & \frac{\mu_{0}}{\mu_{1}}\end{array}\right)$.
We summarize the resulting coupled BIEs-VIEs in
Problem 3.2 (STF-VIE). Find fields $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ and traces $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{H}(\Gamma)$ such that
holds in $\mathcal{H}(\Gamma) \times \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right) \times \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$, where we defined the right-hand-side

$$
\boldsymbol{g}_{1}:=\gamma_{\mathbf{t}} \boldsymbol{u}^{i n c}, \quad \boldsymbol{g}_{2}:=\frac{\mu_{1}}{\mu_{0}} \gamma_{\boldsymbol{\tau}} \tilde{\boldsymbol{v}}^{i n c}
$$

## 4. Analysis of STF-VIEs

### 4.1. Variational Formulation

We now present a variational formulation for the coupled system $(3.2)$. We denote by $\langle\cdot, \cdot\rangle_{\Omega_{i}}$ the duality pairing between $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)^{\prime}$ and $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$. Recall that $\langle\cdot, \cdot\rangle_{\boldsymbol{\tau}, \Gamma}$ denotes the duality pairing between $\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ and $\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$. In the trace space $\boldsymbol{\mathcal { H }}(\Gamma)$ we define

$$
\begin{equation*}
\langle\langle\cdot, \cdot\rangle\rangle: \mathcal{H}(\Gamma) \times \mathcal{H}(\Gamma) \rightarrow \mathbb{C}, \quad\langle\langle\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\zeta}}\rangle\rangle:=\left\langle\boldsymbol{\varphi}_{1}, \boldsymbol{\zeta}_{0}\right\rangle_{\boldsymbol{\tau}, \Gamma}+\left\langle\boldsymbol{\zeta}_{1}, \boldsymbol{\varphi}_{0}\right\rangle_{\boldsymbol{\tau}, \Gamma}, \tag{73}
\end{equation*}
$$

for all $\underline{\varphi}=\left(\varphi_{0}, \varphi_{1}\right) \in \mathcal{H}(\Gamma), \underline{\boldsymbol{\zeta}}=\left(\boldsymbol{\zeta}_{0}, \boldsymbol{\zeta}_{1}\right) \in \mathcal{H}(\Gamma)$.
Problem 4.1 (Variational Formulation for STF-VIE). Given $\underline{g} \in \mathcal{H}(\Gamma)$, we seek $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{H}(\Gamma)$ and $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{H}\left(\Omega_{i}\right):=\left[\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)\right]^{2}$ such that the variational formulation

$$
\begin{array}{lllc}
\mathrm{a}((\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{\zeta}, \boldsymbol{\xi})) & +\mathrm{b}((\boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{\zeta}, \boldsymbol{\xi})) & = & \langle\boldsymbol{g},(\boldsymbol{\zeta}, \boldsymbol{\xi})\rangle\rangle \\
\mathrm{c}((\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{w}, \boldsymbol{q})) & +\mathrm{d}((\boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{w}, \boldsymbol{q})) & = & 0
\end{array}
$$

holds for all $(\boldsymbol{\zeta}, \boldsymbol{\xi}) \in \mathcal{H}(\Gamma)$ and $(\boldsymbol{w}, \boldsymbol{q}) \in \mathcal{H}\left(\Omega_{i}\right)^{\prime}$, where we denote the bilinear forms

$$
\begin{align*}
\mathrm{a}((\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{\zeta}, \boldsymbol{\xi})) & :=\left\langle\left\langle\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\right)(\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{\zeta}, \boldsymbol{\xi})\right\rangle\right\rangle,  \tag{74}\\
\mathrm{b}((\boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{\zeta}, \boldsymbol{\xi})) & :=\left\langle\left\langle\mathbf{J}^{e} \boldsymbol{u},(\boldsymbol{\zeta}, \boldsymbol{\xi})\right\rangle\right\rangle+\left\langle\left\langle\mathbf{J}^{m} \boldsymbol{v},(\boldsymbol{\zeta}, \boldsymbol{\xi})\right\rangle\right\rangle  \tag{75}\\
\mathrm{c}((\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{w}, \boldsymbol{q})) & :=\left\langle\left(\boldsymbol{D}_{\kappa_{1}} \boldsymbol{\alpha}-\boldsymbol{\mathcal { T }}_{\kappa_{1}, \tilde{\mu}}^{\tilde{\epsilon}} \boldsymbol{\beta}\right), \boldsymbol{w}\right\rangle_{\Omega_{i}}+\left\langle\left(\boldsymbol{\mathcal { D }}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\beta})+\kappa_{1}^{2} \boldsymbol{\mathcal { T }}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}}(\mathrm{R} \boldsymbol{\alpha})\right), \boldsymbol{q}\right\rangle_{\Omega_{i}},  \tag{76}\\
\mathrm{~d}((\boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{w}, \boldsymbol{q})) & :=\left\langle\boldsymbol{u}-\boldsymbol{\Lambda}^{e} \boldsymbol{u}-\boldsymbol{\Xi}^{m} \boldsymbol{v}, \boldsymbol{w}\right\rangle_{\Omega_{i}}+\left\langle\boldsymbol{v}-\boldsymbol{\Lambda}^{m} \boldsymbol{v}-\boldsymbol{\Xi}^{e} \boldsymbol{u}, \boldsymbol{q}\right\rangle_{\Omega_{i}} \tag{77}
\end{align*}
$$

Proposition 4.2. The bilinear forms defined in (74)-77)

$$
\begin{array}{l:}
\mathrm{a}: \mathcal{H}(\Gamma) \times \mathcal{H}(\Gamma) \rightarrow \mathbb{C}, \\
\mathrm{b}: \\
\mathrm{c}: \mathcal{H}\left(\Omega_{i}\right) \times \mathcal{H}(\Gamma) \rightarrow \mathbb{C}, \\
\mathrm{c}:  \tag{81}\\
\mathrm{d}(\Gamma) \times \mathcal{H}\left(\Omega_{i}\right)^{\prime} \rightarrow \mathbb{C}, \\
\left(\Omega_{i}\right) \times \mathcal{H}\left(\Omega_{i}\right)^{\prime} \rightarrow \mathbb{C},
\end{array}
$$

are all continuous.
Proof. The result follows from

- Continuity of the vector-valued Newton potential (Proposition 2.2).
- Continuity of the Maxwell layer potentials (Proposition 2.6.
- Continuity of trace operators [11, Theorem 4.1].


### 4.2. Coercivity of weak STF-VIEs

Based on the results shown in Section 2.2 we establish mapping properties for the operators defined in Section 3 .
Proposition 4.3. The Newton potentials $\mathrm{N}_{\Omega_{i}, \star}$ and $\mathbf{N}_{\Omega_{i}, \star}$.

$$
\begin{align*}
\mathbf{N}_{\Omega_{i, \star}, \star} & : \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}^{1}\left(\operatorname{curl}, \Omega_{i}\right),  \tag{82a}\\
\mathbf{N}_{\Omega_{i}, \star} \operatorname{curl} & : \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}^{1}\left(\operatorname{curl}, \Omega_{i}\right),  \tag{82b}\\
\operatorname{grad} \mathrm{N}_{\Omega_{i, \star}, \star} & : L^{2}\left(\Omega_{i}\right) \rightarrow \mathbf{H}^{1}\left(\operatorname{curl}, \Omega_{i}\right) . \tag{82c}
\end{align*}
$$

are continuous.
Proof. We know that

$$
\begin{equation*}
\mathbf{N}_{\Omega_{i}, \star}: \mathbf{L}^{2}\left(\Omega_{i}\right) \rightarrow \mathbf{H}^{2}\left(\Omega_{i}\right) \tag{83}
\end{equation*}
$$

We also know the continuous embeddings $\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \subset \mathbf{L}^{2}\left(\Omega_{i}\right)$ and $\mathbf{H}^{2}\left(\Omega_{i}\right) \subset \mathbf{H}^{1}\left(\operatorname{curl}, \Omega_{i}\right)$. Therefore, we conclude 82 a .
We can show 82 b from (83), since curl : $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right) \rightarrow \mathbf{L}^{2}\left(\Omega_{i}\right)$ is bounded.
To show (82c), we consider $f \in L^{2}\left(\Omega_{i}\right)$. We know from 11) that $\mathrm{N}_{\Omega_{i}, \star} f \in H^{2}\left(\Omega_{i}\right)$, and therefore $\operatorname{grad} \mathrm{N}_{\Omega_{i}, \star} f \in$ $\mathbf{H}^{1}\left(\Omega_{i}\right)$. In addition, we know that $\operatorname{curl}\left(\operatorname{grad} \mathrm{N}_{\Omega_{i}, \star} f\right) \equiv 0$. Since both $\mathrm{N}_{\Omega_{i}, \star} f$ and its curl belong to $\mathbf{H}^{1}\left(\Omega_{i}\right)$, we conclude that $\mathrm{N}_{\Omega_{i}, \star}: L^{2}\left(\Omega_{i}\right) \rightarrow \mathbf{H}^{1}\left(\operatorname{curl}, \Omega_{i}\right)$.

Corollary 4.4. Under Assumption 1.1, the operators

$$
\begin{array}{ll}
\boldsymbol{\Lambda}^{e}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right), & \boldsymbol{\Lambda}^{m}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \\
\boldsymbol{\Xi}^{e}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right), & \boldsymbol{\Xi}^{m}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)
\end{array}
$$

defined as in 63a -63b are compact. In particular, the operators

$$
\mathbf{J}^{e}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathcal{H}(\Gamma), \quad \mathbf{J}^{m}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow \mathcal{H}(\Gamma)
$$

defined in 70b and 70c are compact.

Proof. From (63a) and (63b), we observe that $\boldsymbol{\Lambda}^{e}, \boldsymbol{\Lambda}^{m}, \boldsymbol{\Xi}^{e}$ and $\boldsymbol{\Xi}^{m}$ are linear combinations of operators of the form (82a)- 82 c$)$. Multipliers are all bounded and smooth, therefore they map elements of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ to $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$, and to $L^{2}\left(\Omega_{i}\right)$.
The result follows by Rellich's embedding theorem [45. Theorem 2.5.5], which states compact inclusion from $H^{1}\left(\Omega_{i}\right)$ into $L^{2}\left(\Omega_{i}\right)$ (and therefore from $\mathbf{H}^{1}\left(\Omega_{i}\right)$ into $\mathbf{L}^{2}\left(\Omega_{i}\right)$ ).

The left-hand side of the STF-VIE can be decomposed into several operators as suggested by the operator matrix notation in $\boldsymbol{\$}$. An abstract analysis on such block operators is given in Appendix A In particular, we need to establish the coercivity/inf-sup stability of the diagonal operators

$$
\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon} \tilde{\mu}}: \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)
$$

and

$$
\left(\begin{array}{cc}
\mathbf{I}-\boldsymbol{\Lambda}^{e} & -\boldsymbol{\Xi}^{m} \\
-\boldsymbol{\Xi}^{e} & \mathbf{I}-\boldsymbol{\Lambda}^{m}
\end{array}\right): \mathcal{H}\left(\Omega_{i}\right) \rightarrow \mathcal{H}\left(\Omega_{i}\right)
$$

After establishing stability and uniqueness of solutions, from Proposition A. 2 we will be able to infer well-posedness of the continuous problem.

The first step is to show that a generalized Gårding inequality (T-coercivity) holds for the (weighted) Maxwell Calderón operator $\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}$ from 70a. We start with the following result for (weighted) scalar and vector-valued single layer operators.

Lemma 4.5. Let $\chi \in C^{1}\left(\bar{\Omega}_{i}\right)$ be such that

$$
0<\chi_{\min }<\chi(\boldsymbol{x})<\chi_{\max }
$$

for all $\boldsymbol{x} \in \Omega_{i}$. Let $\kappa_{1}>0$ and $\bigvee_{\kappa_{1}}: H^{-1 / 2}(\Gamma) \rightarrow H^{1 / 2}(\Gamma)$ be the scalar single layer boundary integral operator with wavenumber $\kappa_{1}$. Then, there exist a compact operator $\Theta_{\chi}: H^{-1 / 2}(\Gamma) \rightarrow H^{1 / 2}(\Gamma)$ and $c_{\chi}>0$ such that

$$
\begin{equation*}
\operatorname{Re}\left\{\left\langle\mathrm{V}_{\kappa_{1}}(\chi \varphi), \bar{\varphi}\right\rangle_{\Gamma}+\left\langle\Theta_{\chi} \varphi, \bar{\varphi}\right\rangle_{\Gamma}\right\} \geq c_{\chi}\|\varphi\|_{H^{-1 / 2}(\Gamma)}^{2} \tag{84}
\end{equation*}
$$

holds for all $\varphi \in H^{-1 / 2}(\Gamma)$.
The result can also be extended to the vectorial case. There exist a compact operator $\boldsymbol{\Theta}_{\chi}: \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \rightarrow$ $\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ and a constant $C_{\chi}>0$ such that

$$
\begin{equation*}
\operatorname{Re}\left\{\left\langle\mathbf{V}_{\kappa_{1}}^{\mathrm{t}}(\chi \boldsymbol{\beta}), \overline{\boldsymbol{\beta}}\right\rangle_{\boldsymbol{\tau}, \Gamma}+\left\langle\boldsymbol{\Theta}_{\chi} \boldsymbol{\beta}, \overline{\boldsymbol{\beta}}\right\rangle_{\boldsymbol{\tau}, \Gamma}\right\} \geq C_{\chi}\|\boldsymbol{\beta}\|_{\mathbf{H}^{-1 / 2}(\Gamma)}^{2} \tag{85}
\end{equation*}
$$

holds for all $\boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma} 0, \Gamma\right)$.
Proof. Note that $\chi^{1 / 2}$ is well defined and $\chi^{1 / 2} \in C^{1}\left(\bar{\Omega}_{i}\right)$. Then

$$
\begin{aligned}
\left\langle\mathrm{V}_{\kappa_{1}}(\chi \varphi), \bar{\varphi}\right\rangle_{\Gamma} & =\left\langle\mathrm{V}_{\kappa_{1}}\left(\chi^{1 / 2} \chi^{1 / 2} \varphi\right), \bar{\varphi}\right\rangle_{\Gamma} \\
& =\left\langle\chi^{1 / 2} \mathrm{~V}_{\kappa_{1}}\left(\chi^{1 / 2} \varphi\right), \bar{\varphi}\right\rangle_{\Gamma}+\left\langle\left(\mathrm{V}_{\kappa_{1}} \chi-\chi^{1 / 2} \mathrm{~V}_{\kappa_{1}} \chi^{1 / 2}\right) \varphi, \bar{\varphi}\right\rangle_{\Gamma} \\
& =\left\langle\mathrm{V}_{\kappa_{1}}\left(\chi^{1 / 2} \varphi\right), \overline{\left(\chi^{1 / 2} \varphi\right)}\right\rangle_{\Gamma}+\left\langle\left(\mathrm{V}_{\kappa_{1}} \chi-\chi^{1 / 2} \mathrm{~V}_{\kappa_{1}} \chi^{1 / 2}\right) \varphi, \bar{\varphi}\right\rangle_{\Gamma}
\end{aligned}
$$

We define $\Theta_{\chi}:=\mathrm{V}_{\kappa_{1}} \chi-\chi^{1 / 2} \mathrm{~V}_{\kappa_{1}} \chi^{1 / 2}$. This is a compact operator due to the cancellation of singularity at $\boldsymbol{x}=\boldsymbol{y}$ :

$$
G(\boldsymbol{x}, \boldsymbol{y}) \chi(\boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y}) \chi^{1 / 2}(\boldsymbol{x}) \chi^{1 / 2}(\boldsymbol{y})=G(\boldsymbol{x}, \boldsymbol{y}) \chi^{1 / 2}(\boldsymbol{y})\left(\chi^{1 / 2}(\boldsymbol{y})-\chi^{1 / 2}(\boldsymbol{x})\right) .
$$

We also know that there exists a compact operator $\Theta_{\mathrm{V}_{\kappa_{1}}}: H^{-1 / 2}(\Gamma) \rightarrow H^{1 / 2}(\Gamma)$ such that

$$
\operatorname{Re}\left\{\left\langle\mathrm{V}_{\kappa_{1}} \psi, \bar{\psi}\right\rangle_{\Gamma}+\left\langle\Theta_{\mathrm{V}_{\kappa_{1}}} \psi, \bar{\psi}\right\rangle_{\Gamma}\right\} \geq c_{\mathrm{V}_{\kappa_{1}}}\|\psi\|_{H^{-1 / 2}(\Gamma)}^{2}
$$

for every $\psi \in H^{-1 / 2}(\Gamma)$. Note that $\chi^{1 / 2} \varphi \in H^{-1 / 2}(\Gamma)$ for $\chi^{1 / 2}$ smooth and $\varphi \in H^{-1 / 2}(\Gamma)$. We define $\Theta:=$ $\chi^{1 / 2} \Theta_{\mathrm{V}_{\kappa_{1}}} \chi^{1 / 2}-\Theta_{\chi}$ and conclude

$$
\operatorname{Re}\left\{\left\langle\mathrm{V}_{\kappa_{1}}(\chi \varphi), \bar{\varphi}\right\rangle_{\Gamma}+\langle\Theta \varphi, \bar{\varphi}\rangle_{\Gamma}\right\} \geq c \mathrm{~V}_{\kappa_{1}}\left\|\chi^{1 / 2} \varphi\right\|_{H^{-1 / 2}(\Gamma)}^{2} \geq c_{\chi}\|\varphi\|_{H^{-1 / 2}(\Gamma)}^{2}
$$

where $c_{\chi}$ depends on $c_{V_{\kappa_{1}}}$ and $\chi$, and we used Lemma $\quad$ B. 1 in the last inequality.
The result for $\mathbf{V}_{\kappa_{1}}^{\mathbf{t}}$ can be shown by following the same approach and using Lemma B. 1
In the spirit of results valid for the Maxwell Calderón operator [12, Theorem 9], we can state the following proposition.

Proposition 4.6 (Generalized Gårding inequality for $\left.\mathbf{A}_{\kappa_{1}}^{\tilde{,}, \tilde{\mu}}\right)$. Let $\varepsilon, \mu \in C^{1}\left(\bar{\Omega}_{i}\right)$ and define arbitrary positive coefficients $\varepsilon_{1}, \mu_{1}, \kappa_{1}$. Let $\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}$ be defined as in 70a), where $\tilde{\varepsilon}=\frac{\varepsilon}{\varepsilon_{1}}$ and $\tilde{\mu}=\frac{\mu}{\mu_{1}}$. Then, there is a compact operator $\Theta_{\mathbf{A}}: \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)$, an isomorphism $\mathbf{X}_{\Gamma}: \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)$ and a constant $C_{\mathbf{A}}>0$ depending on $\varepsilon, \mu, \varepsilon_{1}, \mu_{1}, \kappa_{1}, \Omega_{i}$, such that

$$
\begin{equation*}
\operatorname{Re}\left\{\left\langle\left\langle\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}, \mathbf{X}_{\Gamma}\binom{\overline{\boldsymbol{\alpha}}}{\overline{\boldsymbol{\beta}}}\right\rangle\right\rangle_{\Gamma}+\left\langle\left\langle\boldsymbol{\Theta}_{\mathbf{A}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}},\binom{\overline{\boldsymbol{\alpha}}}{\overline{\boldsymbol{\beta}}}\right\rangle\right\rangle_{\Gamma}\right\} \geq C_{\mathbf{A}}\|(\boldsymbol{\alpha}, \boldsymbol{\beta})\|_{\mathcal{H}(\Gamma)}^{2} \tag{86}
\end{equation*}
$$

Proof. The proof is largely based on the one in [12, Theorem 9], with the difference that the operator $\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}$ is weighted by the strictly positive $C^{1}$-smooth multipliers $\tilde{\varepsilon}$ and $\tilde{\mu}$.
We use the regular decomposition theorem [12, Lemma 2]: $\boldsymbol{\alpha} \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ can be written as

$$
\begin{equation*}
\boldsymbol{\alpha}=\boldsymbol{\alpha}_{\perp}+\boldsymbol{\alpha}_{0}, \quad \boldsymbol{\alpha}_{\perp} \in \mathbf{H}_{\|}^{1 / 2}(\Gamma), \boldsymbol{\alpha}_{0} \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma} 0, \Gamma\right) \tag{87}
\end{equation*}
$$

where

$$
\left\|\boldsymbol{\alpha}_{\perp}\right\|_{\mathbf{H}_{\|}^{1 / 2}(\Gamma)} \leq C\left\|\operatorname{curl}_{\Gamma} \boldsymbol{\alpha}\right\|_{H^{-1 / 2}(\Gamma)}
$$

Similarly, $\boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ can be written as

$$
\begin{equation*}
\boldsymbol{\beta}=\boldsymbol{\beta}_{\perp}+\boldsymbol{\beta}_{0}, \quad \boldsymbol{\beta}_{\perp} \in \mathbf{H}_{\times}^{1 / 2}(\Gamma), \boldsymbol{\beta}_{0} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma} 0, \Gamma\right) \tag{88}
\end{equation*}
$$

where

$$
\left\|\boldsymbol{\beta}_{\perp}\right\|_{\mathbf{H}_{\times}^{1 / 2}(\Gamma)} \leq C\left\|\operatorname{div}_{\Gamma} \boldsymbol{\beta}\right\|_{H^{-1 / 2}(\Gamma)} .
$$

We define

$$
\begin{equation*}
\mathbf{X}_{\Gamma}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}:=\binom{\boldsymbol{\alpha}_{\perp}-\boldsymbol{\alpha}_{0}}{\boldsymbol{\beta}_{\perp}-\boldsymbol{\beta}_{0}}, \quad \text { for all } \boldsymbol{\alpha} \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right), \boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \tag{89}
\end{equation*}
$$

Now we write

$$
\begin{align*}
& +\left\langle\left\langle\mathbf{A}_{\kappa_{1}}^{\tilde{\tilde{\varepsilon}} \tilde{\mu}}\binom{\alpha_{0}}{\boldsymbol{\beta}_{\perp}},\binom{\bar{\alpha}_{\perp}}{-\overline{\boldsymbol{\beta}}_{0}}\right\rangle\right\rangle_{\Gamma} \\
& +\left\langle\left\langle\mathbf{A}_{\kappa_{1}, \tilde{\mu}}^{\tilde{\omega}_{1}}\binom{\boldsymbol{\alpha}_{\perp}}{\boldsymbol{\beta}_{0}},\binom{-\overline{\boldsymbol{\alpha}}_{0}}{\overline{\boldsymbol{\beta}}_{\perp}}\right\rangle\right\rangle_{\Gamma}  \tag{90}\\
& +\left\langle\left\langle\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\binom{\boldsymbol{\alpha}_{0}}{\boldsymbol{\beta}_{\perp}},\binom{-\overline{\boldsymbol{\alpha}}_{0}}{\overline{\boldsymbol{\beta}}_{\perp}}\right\rangle\right\rangle_{\Gamma}
\end{align*}
$$

We study the first term in the right-hand side of 90 ,

$$
\begin{align*}
\left\langle\left\langle\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\binom{\boldsymbol{\alpha}_{\perp}}{\boldsymbol{\beta}_{0}},\binom{\overline{\boldsymbol{\alpha}}_{\perp}}{-\overline{\boldsymbol{\beta}}_{0}}\right\rangle\right\rangle_{\Gamma}= & -\left\langle\mathbf{K}_{\kappa_{1}} \boldsymbol{\alpha}_{\perp}, \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma}-\left\langle\mathbf{V}_{\kappa_{1}}^{\left.\tilde{\varepsilon_{\tilde{\prime}}} \boldsymbol{\beta}_{0}, \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma}}\right.  \tag{91}\\
& +\left\langle\mathbf{W}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}} \boldsymbol{\alpha}_{\perp}, \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma}+\left\langle\mathbf{K}_{\kappa_{1}}^{\prime} \boldsymbol{\beta}_{0}, \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma}
\end{align*}
$$

where

$$
\begin{align*}
\left\langle\mathbf{V}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}} \boldsymbol{\beta}_{0}, \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma} & =\left\langle\mathbf{V}_{\kappa_{1}}^{\mathbf{t}}\left(\tilde{\mu} \boldsymbol{\beta}_{0}\right), \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma},  \tag{92a}\\
\left\langle\mathbf{W}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}} \boldsymbol{\alpha}_{\perp}, \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma} & =\left\langle\mathrm{V}_{\kappa_{1}}\left(\frac{1}{\tilde{\mu}} \operatorname{curl}_{\Gamma} \boldsymbol{\alpha}_{\perp}\right), \operatorname{curl}_{\Gamma} \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma}+\kappa_{1}^{2}\left\langle\mathbf{V}_{\kappa_{1}}^{\boldsymbol{\tau}}\left(\tilde{\varepsilon} \boldsymbol{\alpha}_{\perp}\right), \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma} \tag{92b}
\end{align*}
$$

The second term in 92 b is a compact perturbation since $\boldsymbol{\alpha}_{\perp} \in \mathbf{H}_{\times}^{1 / 2}(\Gamma)$ and $\mathbf{H}_{\times}^{1 / 2}(\Gamma)$ is compactly embedded in $\mathbf{H}_{\times}^{-1 / 2}(\Gamma)$ [12, Corollary 1], while for the first term in 92 b and 92 a we have a coercivity result that follows from Lemma 4.5

$$
\begin{align*}
\operatorname{Re}\left\{\left\langle\mathbf{V}_{\kappa_{1}}^{\mathbf{t}}\left(\tilde{\mu} \boldsymbol{\beta}_{0}\right), \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma}+\mathrm{t}\left(\boldsymbol{\beta}_{0}, \overline{\boldsymbol{\beta}}_{0}\right)\right\} \geq c_{1}\left\|\boldsymbol{\beta}_{0}\right\|_{\mathbf{H}^{-1 / 2}(\Gamma)}^{2}  \tag{93a}\\
\operatorname{Re}\left\{\left\langle\mathrm{~V}_{\kappa_{1}}\left(\frac{1}{\tilde{\varepsilon}} \operatorname{curl}_{\Gamma} \boldsymbol{\alpha}_{\perp}\right), \operatorname{curl}_{\Gamma} \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma}+\mathrm{t}\left(\boldsymbol{\alpha}_{\perp}, \overline{\boldsymbol{\alpha}}_{\perp}\right)\right\} \geq c_{2}\left\|\operatorname{curl}_{\Gamma} \boldsymbol{\alpha}_{\perp}\right\|_{H^{-1 / 2}(\Gamma)}^{2} . \tag{93b}
\end{align*}
$$

On the other hand, from [12, Lemma 6], a symmetry between $\mathbf{K}_{\kappa_{1}}$ and $\mathbf{K}_{\kappa_{1}}^{\prime}$ with respect to the duality pairing $\langle\cdot, \cdot\rangle_{\Gamma}$ implies [12, Theorem 9]

$$
\begin{equation*}
-\left\langle\mathbf{K}_{\kappa_{1}} \boldsymbol{\alpha}_{\perp}, \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma}+\left\langle\mathbf{K}_{\kappa_{1}}^{\prime} \boldsymbol{\beta}_{0}, \overline{\boldsymbol{\alpha}}_{\perp}\right\rangle_{\Gamma}=-2 i \operatorname{Im}\left\{\left\langle\mathbf{K}_{\kappa_{1}} \boldsymbol{\alpha}_{\perp}, \overline{\boldsymbol{\beta}}_{0}\right\rangle_{\Gamma}\right\} . \tag{94}
\end{equation*}
$$

Then, we can establish that there exist a compact perturbation $\boldsymbol{\Theta}_{\mathbf{A}, 1}$ and a constant $c_{\mathbf{A}, 1}>0$ such that

$$
\begin{array}{r}
\operatorname{Re}\left\{\left\langle\left\langle\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\binom{\boldsymbol{\alpha}_{\perp}}{\boldsymbol{\beta}_{0}},\binom{\overline{\boldsymbol{\alpha}}_{\perp}}{-\overline{\boldsymbol{\beta}}_{0}}\right\rangle\right\rangle_{\Gamma}+\left\langle\left\langle\boldsymbol{\Theta}_{\mathbf{A}, 1}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}},\binom{\overline{\boldsymbol{\alpha}}}{\overline{\boldsymbol{\beta}}}\right\rangle\right\rangle_{\Gamma}\right\}  \tag{95}\\
+c_{\mathbf{A}, 1}\left(\left\|\operatorname{curl}_{\Gamma} \boldsymbol{\alpha}_{\perp}\right\|_{H^{-1 / 2}(\Gamma)}^{2}\right. \\
\left.+\left\|\boldsymbol{\beta}_{0}\right\|_{\mathbf{H}^{-1 / 2}(\Gamma)}^{2}\right) .
\end{array}
$$

In a similar way, we study the third and fourth terms in 90 and show

$$
\begin{array}{r}
\operatorname{Re}\left\{\left\langle\left\langle\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\binom{\boldsymbol{\alpha}_{0}}{\boldsymbol{\beta}_{\perp}},\binom{\overline{\boldsymbol{\alpha}}_{0}}{\overline{\boldsymbol{\beta}}_{\perp}}\right\rangle\right\rangle_{\Gamma}+\left\langle\left\langle\boldsymbol{\Theta}_{\mathbf{A}, 2}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}},\binom{\overline{\boldsymbol{\alpha}}}{\overline{\boldsymbol{\beta}}}\right\rangle\right\rangle_{\Gamma}\right\}
\end{array} \begin{gathered}
 \tag{96}\\
c_{\mathbf{A}, 2}\left(\left\|\operatorname{div}_{\Gamma} \boldsymbol{\beta}_{\perp}\right\|_{H^{-1 / 2}(\Gamma)}^{2}\right. \\
\left.+\left\|\boldsymbol{\alpha}_{0}\right\|_{\mathbf{H}^{-1 / 2}(\Gamma)}^{2}\right)
\end{gathered}
$$

Combining (95) and (96), and by the stability of the decomposition in (87) and (88), we conclude that (86) holds.
Corollary 4.7. Let us define $\mathbb{A}:=\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}$. Then, under Assumption 1.1 , there is a compact operator $\Theta_{A}: \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)$, an isomorphism $\mathbb{X}_{\Gamma}: \mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)$ and a constant $C_{\mathbb{A}}>0$ depending on $\varepsilon, \mu, \varepsilon_{1}, \mu_{1}, \varepsilon_{0}, \mu_{0}, \Omega_{i}$, such that

$$
\begin{equation*}
\operatorname{Re}\left\{\left\langle\mathbb{A}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}, \mathbb{X}_{\Gamma}\binom{\overline{\boldsymbol{\alpha}}}{\overline{\boldsymbol{\beta}}}\right\rangle_{\Gamma}+\left\langle\boldsymbol{\Theta}_{\mathbb{A}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}},\binom{\overline{\boldsymbol{\alpha}}}{\overline{\boldsymbol{\beta}}}\right\rangle_{\Gamma}\right\} \geq C_{\mathbb{A}}\|(\boldsymbol{\alpha}, \boldsymbol{\beta})\|_{\mathcal{H}(\Gamma)}^{2} \tag{97}
\end{equation*}
$$

where

$$
\|(\boldsymbol{\alpha}, \boldsymbol{\beta})\|_{\mathcal{H}(\Gamma)}^{2}:=\|\boldsymbol{\alpha}\|_{\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)}^{2}+\|\boldsymbol{\beta}\|_{\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)}^{2},
$$

for all $\boldsymbol{\alpha} \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ and $\boldsymbol{\beta} \in \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$.
Proof. The proof for a Generalized Gårding inequality for $\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}$ follows the same approach as in Proposition 4.6 also usign the isomorphism $\mathbf{X}_{\Gamma}$. The result follows by noting that $\mathbb{A}$ is a linear combination of two operators that satisfy a Generalized Gårding inequality with same isomorphisms $\mathbf{X}_{\Gamma}$ (Proposition 4.6).

Proposition 4.8 (Generalized Gårding inequality for $\mathbf{I}-\boldsymbol{\Lambda}^{\star}$ ). Let $\boldsymbol{\Lambda}^{\star}$, $\star=\{e, m\}$ defined as in 63a) or 63b),

$$
\begin{aligned}
\boldsymbol{\Lambda}^{e} \boldsymbol{u} & :=-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)+\operatorname{grad} \mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{e} \cdot \boldsymbol{u}\right), \\
\boldsymbol{\Lambda}^{m} \boldsymbol{v} & :=-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \boldsymbol{v}\right)+\operatorname{grad} \mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{m} \cdot \boldsymbol{v}\right) .
\end{aligned}
$$

Then, there exist a compact operator $\Theta_{\star}: \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right) \rightarrow \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$, an isomorphism $X_{\star}: \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right) \rightarrow$ $\left(\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)\right)^{\prime}$ and a constant $C_{\star}>0$ such that

$$
\begin{equation*}
\operatorname{Re}\left\{\left\langle\left(\mathbf{I}-\boldsymbol{\Lambda}^{\star}\right) \boldsymbol{u}, X_{\star} \overline{\boldsymbol{u}}\right\rangle_{\Omega_{i}}+\left(\Theta_{\star} \boldsymbol{u}, \overline{\boldsymbol{u}}\right)_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}\right\} \geq C_{\star}\|\boldsymbol{u}\|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}^{2} \tag{98}
\end{equation*}
$$

holds for all $\boldsymbol{u} \in \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$.
Proof. First, we study the duality pairing $\langle\boldsymbol{u}, \boldsymbol{w}\rangle_{\Omega_{i}}$, with $\boldsymbol{w} \in\left(\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)\right)^{\prime}$. Note that a simple choice is $X_{\star}$ such that

$$
\begin{equation*}
\left\langle\boldsymbol{u}, X_{\star} \overline{\boldsymbol{v}}\right\rangle_{\Omega_{i}}:=(\boldsymbol{u}, \overline{\boldsymbol{v}})_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}=(\boldsymbol{u}, \overline{\boldsymbol{v}})_{\Omega_{i}}+(\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \overline{\boldsymbol{v}})_{\Omega_{i}}, \tag{99}
\end{equation*}
$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$, i.e. the inverse Riesz isomorphism.
In that case, $\left\|X_{\star}\right\|=1$ and

$$
\begin{equation*}
\left\langle\boldsymbol{u}, X_{\star} \overline{\boldsymbol{u}}\right\rangle_{\Omega_{i}}=\|\boldsymbol{u}\|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}^{2} . \tag{100}
\end{equation*}
$$

From Proposition 4.4 , we know that $\boldsymbol{\Lambda}^{\star}$ is compact in $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$. Therefore, choosing $\Theta_{\star}=\boldsymbol{\Lambda}^{\star}$ leads to (98).

### 4.3. STF-VIEs: Uniqueness of solutions

The results from this section require an assumption on the material properties $\varepsilon$ and $\mu$.

Assumption 4.9. We assume that the material properties $\varepsilon$ and $\mu$ are constant on the interface $\Gamma$, i.e.

$$
\begin{equation*}
\varepsilon(\boldsymbol{x}) \equiv \varepsilon_{1}, \quad \mu(\boldsymbol{x}) \equiv \mu_{1}, \quad \text { for all } \boldsymbol{x} \in \Gamma . \tag{101}
\end{equation*}
$$

Proposition 4.10. Under Assumption 4.9, there exists a unique solution to Problem 3.2.
Proof. Let us assume that we have a solution $\boldsymbol{u} \in \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right), \tilde{\boldsymbol{v}} \in \mathbf{H}\left(\boldsymbol{\operatorname { c u r l }}, \Omega_{i}\right),(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \times$ $\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ such that

$$
\begin{array}{r}
\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}+\mathbf{J}^{e} \boldsymbol{u}+\mathbf{J}^{m} \tilde{\boldsymbol{v}}=0,\right. \\
\mathcal{D}_{\kappa_{1}}(\boldsymbol{\alpha})-\boldsymbol{\mathcal { T }}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}(\boldsymbol{\beta})+\boldsymbol{u}-\boldsymbol{\Lambda}^{e} \boldsymbol{u}-\boldsymbol{\Xi}^{m} \tilde{\boldsymbol{v}}=0 \\
\kappa_{1}^{2} \boldsymbol{\mathcal { T }}_{\kappa_{1}}^{\tilde{\mu}, \tilde{\varepsilon}}(\mathrm{R} \boldsymbol{\alpha})+\boldsymbol{\mathcal { D }}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\beta})+\tilde{\boldsymbol{v}}-\boldsymbol{\Lambda}^{m} \tilde{\boldsymbol{v}}-\boldsymbol{\Xi}^{e} \boldsymbol{u}=0, \tag{102c}
\end{array}
$$

in $\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right) \times \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right) \times \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \times \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)$.
Because we assume $\varepsilon(\boldsymbol{x}) \equiv \varepsilon_{1}>0$ and $\mu(\boldsymbol{x}) \equiv \mu_{1}>0$ for all $\boldsymbol{x} \in \Gamma$, we can rewrite 102a-102c as

$$
\begin{array}{r}
\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}\right)\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}+\mathbf{J}^{e} \boldsymbol{u}+\mathbf{J}^{m} \tilde{\boldsymbol{v}}=0 \\
\mathcal{D}_{\kappa_{1}}(\boldsymbol{\alpha})-\boldsymbol{\mathcal { T }}_{\kappa_{1}}(\boldsymbol{\beta})+\boldsymbol{u}-\boldsymbol{\Lambda}^{e} \boldsymbol{u}-\boldsymbol{\Xi}^{m} \tilde{\boldsymbol{v}}=0 \\
\kappa_{1}^{2} \boldsymbol{\mathcal { T }}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\alpha})+\mathcal{D}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\beta})+\tilde{\boldsymbol{v}}-\boldsymbol{\Lambda}^{m} \tilde{\boldsymbol{v}}-\boldsymbol{\Xi}^{e} \boldsymbol{u}=0 \tag{103c}
\end{array}
$$

The proof is divided into five parts.

1. From 103b and 103 c , we show that $\boldsymbol{u}$ and $\boldsymbol{v}$ "almost" satisfy Maxwell equations in $\Omega$. We have an extra term, which is the gradient of a volume potential.
2. We show that the extra term from Part 1 is zero, and therefore $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy Maxwell equations in $\Omega_{i}$.
3. Using Maxwell layer potentials (see 28) and (30) and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we define $\boldsymbol{u}_{0}$ and $\boldsymbol{v}_{0}$ such that they satisfy Maxwell equations in $\Omega_{o}$. Then, we compute the jumps $\gamma_{\mathbf{t}}^{+} \boldsymbol{u}_{0}-\gamma_{\mathbf{t}}^{-} \boldsymbol{u}$ and $\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}_{0}-\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}$. Using 103a), we show that the jumps are zero.
4. We conclude that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_{0}$ and $\boldsymbol{v}_{0}$ define solutions for the Maxwell transmission problem with no sources. It follows that all of them are zero. From 103a, we conclude that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are also zero.
Part 1. We take the curl of 103 b . Using 32a, 32b we get

$$
\begin{equation*}
\kappa_{1}^{2} \mathcal{T}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\alpha})+\mathcal{D}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\beta})+\operatorname{curl} \boldsymbol{u}+\kappa_{1}^{2} \operatorname{curl} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)+\operatorname{curl} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{curl}\left(p_{m} \tilde{\boldsymbol{v}}\right)\right)=0 \tag{104}
\end{equation*}
$$

which by integration by parts 53 and Assumption 4.9, can be rewritten as

$$
\begin{equation*}
\kappa_{1}^{2} \mathcal{T}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\alpha})+\mathcal{D}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\beta})+\operatorname{curl} \boldsymbol{u}+\kappa_{1}^{2} \operatorname{curl} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)+\operatorname{curl}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right)=0 \tag{105}
\end{equation*}
$$

Similarly, 103 c can be rewritten as

$$
\begin{equation*}
\kappa_{1}^{2} \mathcal{T}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\alpha})+\mathcal{D}_{\kappa_{1}}(\mathrm{R} \boldsymbol{\beta})+\tilde{\boldsymbol{v}}+\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right)-\operatorname{grad} \mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{m} \cdot \tilde{\boldsymbol{v}}\right)+\kappa_{1}^{2} \operatorname{curl} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{e} \boldsymbol{u}\right)=0 \tag{106}
\end{equation*}
$$

Substracting (106) from we obtain

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}-\tilde{\boldsymbol{v}}+\operatorname{curl}^{2}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right)\right)-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right)+\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\boldsymbol{\tau}_{m} \cdot \tilde{\boldsymbol{v}}\right)\right)=0 \tag{107}
\end{equation*}
$$

Note that, as $\boldsymbol{w}=\mathbf{N}_{\Omega_{i}, \kappa_{1}} \boldsymbol{f}$ defines a solution for the (vector) Helmholtz equation

$$
-\boldsymbol{\Delta} \boldsymbol{w}-\kappa_{1}^{2} \boldsymbol{w}=\boldsymbol{f} \quad \text { in } \Omega_{i},
$$

and

$$
-\boldsymbol{\Delta}=\mathbf{c u r l}^{2}-\mathbf{g r a d} \text { div },
$$

we get

$$
\begin{align*}
\operatorname{curl}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right)-\kappa_{1}^{2} \mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right) & =\operatorname{grad}\left(\operatorname{div}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \tilde{\boldsymbol{v}}\right)\right)\right)+p_{m} \tilde{\boldsymbol{v}}  \tag{108a}\\
& =\operatorname{grad}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{div}\left(p_{m} \tilde{\boldsymbol{v}}\right)\right)\right)+p_{m} \tilde{\boldsymbol{v}} \tag{108b}
\end{align*}
$$

where 108 b is obtained by the integration by parts

$$
\begin{equation*}
\operatorname{div}\left(\mathbf{N}_{\Omega_{i}, \kappa_{1}}(\boldsymbol{f})\right)=\mathbf{N}_{\Omega_{i}, \kappa_{1}}(\operatorname{div} \boldsymbol{f})-\mathrm{S}_{\kappa_{1}}\left(\gamma_{\mathbf{n}}^{-} \boldsymbol{f}\right) \tag{109}
\end{equation*}
$$

and Assumption 4.9
From 107) and 108 we write

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}-\tilde{\boldsymbol{v}}+p_{m} \tilde{\boldsymbol{v}}+\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{div}\left(p_{m} \tilde{\boldsymbol{v}}\right)+\boldsymbol{\tau}_{m} \cdot \tilde{\boldsymbol{v}}\right)\right)=0 \tag{110}
\end{equation*}
$$

Recall that $p_{m}(\boldsymbol{x})=1-\frac{\mu(\boldsymbol{x})}{\mu_{1}}$, so we can write

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}-\frac{\mu}{\mu_{1}} \tilde{\boldsymbol{v}}+\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(\operatorname{div}\left(p_{m} \tilde{\boldsymbol{v}}\right)+\boldsymbol{\tau}_{m} \cdot \tilde{\boldsymbol{v}}\right)\right)=0 . \tag{111}
\end{equation*}
$$

Part 2. In this part, we show that the third term in 111) is zero. First, we do the following computation

$$
\begin{align*}
\operatorname{div}\left(p_{m} \tilde{\boldsymbol{v}}\right)+\boldsymbol{\tau}_{m} \cdot \tilde{\boldsymbol{v}} & =\operatorname{div}\left(p_{m} \tilde{\boldsymbol{v}}\right)+\frac{\operatorname{grad} \mu}{\mu} \cdot \tilde{\boldsymbol{v}} \\
& =\operatorname{div} \tilde{\boldsymbol{v}}-\operatorname{div}\left(\frac{\mu}{\mu_{1}} \tilde{\boldsymbol{v}}\right)+\frac{\operatorname{grad} \mu}{\mu} \cdot \tilde{\boldsymbol{v}} \\
& =\operatorname{div} \tilde{\boldsymbol{v}}-\frac{1}{\mu_{1}}(\operatorname{grad} \mu \cdot \tilde{\boldsymbol{v}}+\mu \operatorname{div}(\tilde{\boldsymbol{v}}))+\frac{\operatorname{grad} \mu}{\mu} \cdot \tilde{\boldsymbol{v}} \\
& =\operatorname{div} \tilde{\boldsymbol{v}}+\left(-\frac{1}{\mu_{1}}+\frac{1}{\mu}\right) \operatorname{grad} \mu \cdot \tilde{\boldsymbol{v}}-\frac{\mu}{\mu_{1}} \operatorname{div}(\tilde{\boldsymbol{v}}) \\
& =\operatorname{div} \tilde{\boldsymbol{v}}+p_{m} \frac{1}{\mu} \operatorname{grad} \mu \cdot \tilde{\boldsymbol{v}}-\frac{\mu}{\mu_{1}} \operatorname{div}(\tilde{\boldsymbol{v}})  \tag{112}\\
& =p_{m} \frac{1}{\mu} \operatorname{grad} \mu \cdot \tilde{\boldsymbol{v}}+p_{m} \operatorname{div}(\tilde{\boldsymbol{v}}) \\
& =p_{m} \frac{1}{\mu}(\operatorname{grad} \mu \cdot \tilde{\boldsymbol{v}}+\mu \operatorname{div}(\tilde{\boldsymbol{v}}))=p_{m} \frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}})
\end{align*}
$$

From (112), we can write (111) as

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}-\frac{\mu}{\mu_{1}} \tilde{\boldsymbol{v}}=-\operatorname{grad}\left(\mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}})\right)\right) \tag{113}
\end{equation*}
$$

We take the divergence of (113) and get

$$
\begin{equation*}
-\frac{1}{\mu_{1}} \operatorname{div}(\mu \tilde{\boldsymbol{v}})=-\Delta \mathrm{N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}})\right)=\kappa_{1}^{2} \mathrm{~N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}})\right)+p_{m} \frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}}), \tag{114}
\end{equation*}
$$

where we used that $w=\mathrm{N}_{\Omega_{i}, \kappa_{1}} f$ defines a solution for the (scalar) Helmholtz equation $-\Delta w-\kappa_{1}^{2} w=f$. Rearranging terms in (114) we get

$$
\begin{equation*}
\left(-\frac{1}{\mu_{1}}-\frac{1}{\mu}+\frac{1}{\mu} \frac{\mu}{\mu_{1}}\right) \operatorname{div}(\mu \tilde{\boldsymbol{v}})=\kappa_{1}^{2} \mathrm{~N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}})\right) . \tag{115}
\end{equation*}
$$

Writing $\eta:=\frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}})$, 114) becomes the Lippmann-Schwinger equation with zero right-hand side [19, Section 8.2]

$$
\begin{equation*}
\eta+\kappa_{1}^{2} \mathrm{~N}_{\Omega_{i}, \kappa_{1}}\left(p_{m} \eta\right)=0 \tag{116}
\end{equation*}
$$

This is an equivalent formulation to an homogeneous Helmholtz transmission problem (see [22, Lemma 7], [19, Theorem 8.3]).
This problem is known to have a unique solution as long as a unique continuation principle holds [19, Section 8.3], which is the case for $p_{m} \in C^{1}\left(\bar{\Omega}_{i}\right)$. [19, Theorem 8.6].
The homogeneous problem has only the trivial solution, and we know

$$
\eta=\frac{1}{\mu} \operatorname{div}(\mu \tilde{\boldsymbol{v}}) \equiv 0
$$

It follows that $\boldsymbol{u}$ and $\tilde{\boldsymbol{v}}$ satisfy

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{u}-i \omega \mu\left(\frac{1}{i \omega \mu_{1}} \tilde{\boldsymbol{v}}\right)=0 \quad \text { in } \Omega_{i} . \tag{117}
\end{equation*}
$$

We denote $\boldsymbol{v}:=\frac{1}{i \omega \mu_{1}} \tilde{\boldsymbol{v}}$. Similar computations show that

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{v}-i \omega \varepsilon \boldsymbol{u}=0 \quad \text { in } \Omega_{i} \tag{118}
\end{equation*}
$$

Part 3. Now, we define an exterior field

$$
\begin{array}{ll}
\boldsymbol{u}_{0}=\boldsymbol{\mathcal { T }}_{\kappa_{0}}\left(\frac{\mu_{0}}{\mu_{1}} \boldsymbol{\beta}\right)+\mathcal{D}_{\kappa_{0}}(\boldsymbol{\alpha}), & \text { in } \mathbb{R}^{3} \backslash \bar{\Omega}_{i} \\
\tilde{\boldsymbol{v}}_{0}=\mathcal{D}_{\kappa_{0}}\left(\frac{\mu_{0}}{\mu_{1}} \mathrm{R} \boldsymbol{\beta}\right)+\kappa_{0}^{2} \boldsymbol{\mathcal { T }}_{\kappa_{0}}(\mathrm{R} \boldsymbol{\alpha}), & \text { in } \mathbb{R}^{3} \backslash \bar{\Omega}_{i} \tag{119b}
\end{array}
$$

with $\tilde{\boldsymbol{v}}_{0}=i \omega \mu_{0} \boldsymbol{v}_{0}$ that satisfies

$$
\begin{align*}
\operatorname{curl} \boldsymbol{u}_{0}-i \omega \mu_{0} \boldsymbol{v}_{0}=0 & \text { in } \mathbb{R}^{3} \backslash \bar{\Omega}_{i},  \tag{120a}\\
\operatorname{curl} \boldsymbol{v}_{0}+i \omega \varepsilon_{0} \boldsymbol{u}_{0}=0 & \text { in } \mathbb{R}^{3} \backslash \bar{\Omega}_{i} . \tag{120b}
\end{align*}
$$

Taking traces on 119a we get

$$
\begin{equation*}
\binom{\gamma_{\mathbf{t}}^{+} \boldsymbol{u}_{0}}{\gamma_{\boldsymbol{\tau}}^{+} \tilde{\boldsymbol{v}}_{0}}=\left(\frac{1}{2} \mathbf{I}-\mathbf{A}_{\kappa_{0}}\right) \mathbf{M}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} . \tag{121}
\end{equation*}
$$

Taking traces on 103b and 103c we obtain

$$
\begin{equation*}
\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{-} \tilde{\boldsymbol{v}}}=\left(\frac{1}{2} \mathbf{I}+\mathbf{A}_{\kappa_{1}}\right)\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}+\mathbf{J}^{e} \boldsymbol{u}+\mathbf{J}^{m} \tilde{\boldsymbol{v}} \tag{122}
\end{equation*}
$$

Combining 121 and 122, from 103a we conclude that

$$
\begin{equation*}
\binom{\gamma_{\mathbf{t}}^{+} \boldsymbol{u}_{0}}{\gamma_{\boldsymbol{\tau}}^{+} \boldsymbol{v}_{0}}-\binom{\gamma_{\mathbf{t}}^{-} \boldsymbol{u}}{\gamma_{\boldsymbol{\tau}}^{-} \boldsymbol{v}}=0 \tag{123}
\end{equation*}
$$

Part 4. We know that $\boldsymbol{u}, \boldsymbol{v}$ satisfy (117) and 118. We also know that $\boldsymbol{u}_{0}$ and $\boldsymbol{v}_{0}$ satisfy 120 . Moreover, the transmission conditions 123 hold. Therefore,

$$
\boldsymbol{U}(\boldsymbol{x}):=\left\{\begin{array}{ll}
\boldsymbol{u}_{0}(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^{3} \backslash \bar{\Omega}_{i}, \\
\boldsymbol{u}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_{i} .
\end{array} \quad \boldsymbol{V}(\boldsymbol{x}):= \begin{cases}\boldsymbol{v}_{0}(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^{3} \backslash \bar{\Omega}_{i}, \\
\boldsymbol{v}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_{i}\end{cases}\right.
$$

are solutions of the homogeneous Maxwell transmission problem. It follows that $\boldsymbol{U} \equiv 0, \boldsymbol{V} \equiv 0$ and $\boldsymbol{u} \equiv 0, \boldsymbol{v} \equiv 0$. We conclude from 103a that

$$
\begin{equation*}
\left(\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}\right)\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}=0 \tag{124}
\end{equation*}
$$

which is known to be an invertible operator [12, Theorem 12]. Therefore, $\boldsymbol{\alpha} \equiv 0$ and $\boldsymbol{\beta} \equiv 0$, which concludes the proof.

Remark 4.11. The assumption of constant coefficients over the boundary $\Gamma$ is essential in two parts of the proof: (1) for obtaining homogeneous right-hand side in (116) and therefore a divergence free field; (2) to ensure injectivity of the single-trace equation in (124). This is similar to what was observed in the Helmholtz transmission problem [32, Section 3.3].

Theorem 4.12 (Well-Posedness of Problem 4.1. Under Assumptions 1.1 and 4.9 , there exists a unique solution

$$
\left(\boldsymbol{\alpha}^{\star}, \boldsymbol{\beta}^{\star}, \boldsymbol{u}^{\star}, \boldsymbol{v}^{\star}\right) \in \mathcal{H}(\Gamma) \times \mathcal{H}\left(\Omega_{i}\right)
$$

to Problem 4.1. which satisfies

$$
\left\|\boldsymbol{\alpha}^{\star}\right\|_{\mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)}+\left\|\boldsymbol{\beta}^{\star}\right\|_{\mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)}+\left\|\boldsymbol{u}^{\star}\right\|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}+\left\|\boldsymbol{v}^{\star}\right\|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)} \leq C\|\underline{\boldsymbol{g}}\|_{\mathcal{H}(\Gamma)} .
$$

Proof. The proof follows Proposition A. 2 and the framework of Appendix A In particular, we have

- Compactness results for $\boldsymbol{\Xi}^{e}, \mathbf{J}^{e}$ and $\boldsymbol{\Xi}^{m}, \mathbf{J}^{m}$, from Corollary 4.4
- T-coercivity (or generalized Gårding inequality) for

$$
\mathbb{A}:=\mathbf{M}^{-1} \mathbf{A}_{\kappa_{0}} \mathbf{M}+\mathbf{A}_{\kappa_{1}}^{\tilde{\varepsilon}, \tilde{\mu}}
$$

from Corollary 4.7

- $T$-coercivity for $\mathbf{I}-\boldsymbol{\Lambda}^{\star}$, from Proposition 4.8
- Uniqueness of solutions, from Proposition 4.10

As the assumptions of Proposition A.2 hold, we obtain well-posedness of Problem 4.1

## 5. Galerkin Discretization

### 5.1. Finite Element and Boundary Element Spaces

Let $\left\{\mathcal{T}_{h}\right\}_{h>0}$ be a globally quasi-uniform and shape-regular family of simplicial meshes of $\Omega_{i}$ (see [47, Section 9]). Let $\left\{\Sigma_{h}\right\}_{h>0}$ be the induced family of meshes on $\Gamma: \Sigma_{h}=\left.\mathcal{T}_{h}\right|_{\Gamma}$. We choose finite element spaces:

- $N_{h}:=N_{h}\left(\mathcal{T}_{h}\right) \subset \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)$ of lowest order Nédélec edge elements (in the volume) [41, [26, Section 3], [6].
- $E_{h}:=E_{h}\left(\Sigma_{h}\right) \subset \mathbf{H}^{-1 / 2}\left(\operatorname{curl}_{\Gamma}, \Gamma\right)$ of lowest order surface edge elements [5, Section 2.2].
- $W_{h}:=W_{h}\left(\Sigma_{h}\right) \subset \mathbf{H}^{-1 / 2}\left(\operatorname{div}_{\Gamma}, \Gamma\right)$ of lowest order rotated surface edge elements, also known as RWG (Rao-Wilton-Glisson) boundary elements in computational engineering [44].

We will denote $N_{h}^{\star}$ a conforming subspace of the dual space of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$.
Remark 5.1. It is important to note that, contrary to what is a standard choice in the literature on volume integral equations [7, 35], using $N_{h}^{\star}=N_{h}$ does not lead to a stable discretization of the duality pairing. We briefly describe why this is not the case in Appendix $\bar{C}$. As it happens with the duality product in the trace space [10], a good approach might be the use of a dual barycentric finite element complex, i.e. the use of face elements on a dual mesh as a subspace of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)^{\prime}$. These claims, although intuitive, remain as an open problem. The generalization of a dual barycentric complex has been studied in different contexts [15].

### 5.2. Asymptotic Quasi-Optimality

In order to obtain a final result on the discretization of Problem 4.1 we need a discrete version of Proposition 4.8 As mentioned in Remark 5.1, this is related to a stable discrete duality pairing in $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$.
Our goal is to have a conforming discretization of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)^{\prime}$ such that the following holds.
Assumption 5.2 (Discrete inf-sup Condition for d). There exists $c_{\mathrm{d}}>0$ such that

$$
c_{\mathrm{d}} \leq \inf _{0 \neq \boldsymbol{u}_{h} \in N_{h}} \sup _{0 \neq \boldsymbol{w}_{h} \in N_{h}^{\star}} \frac{\left\langle\boldsymbol{w}_{h}, \boldsymbol{u}_{h}\right\rangle_{\Omega_{i}}}{\left\|\boldsymbol{u}_{h}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)}\left\|\boldsymbol{w}_{h}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)^{\prime}}} \quad \text { for all } h>0
$$

We have to include this assumption in order to arrive at the following main result on the quasi-optimality of Galerkin solutions for 4.1

Theorem 5.3. Provided that Assumptions 4.9 and 5.2 hold, there are $h_{0}>0$ and a constant $c_{q \circ}>0$ independent of $h$ such that there exists a unique Galerkin solution $\left(\boldsymbol{\alpha}_{h}^{\star}, \boldsymbol{\beta}_{h}^{\star}, \boldsymbol{u}_{h}^{\star}, \boldsymbol{v}_{h}^{\star}\right) \in E_{h} \times W_{h} \times N_{h} \times N_{h}$ of Problem 4.1 for all $h<h_{0}$. The solution satisfies

$$
\left\|\left(\boldsymbol{\alpha}^{\star}, \boldsymbol{\beta}^{\star}, \boldsymbol{u}^{\star}, \boldsymbol{v}^{\star}\right)-\left(\boldsymbol{\alpha}_{h}^{\star}, \boldsymbol{\beta}_{h}^{\star}, \boldsymbol{u}_{h}^{\star}, \boldsymbol{v}_{h}^{\star}\right)\right\| \leq c_{\mathrm{qo}}^{\substack{\left(\boldsymbol{\lambda}_{h}, \boldsymbol{\eta}_{h}\right) \in E_{h} \times W_{h},\left(\boldsymbol{w}_{h}, \boldsymbol{q}_{h}\right) \in N_{h} \times N_{h}}} \inf \left\|\left(\boldsymbol{\alpha}^{\star}, \boldsymbol{\beta}^{\star}, \boldsymbol{u}^{\star}, \boldsymbol{v}^{\star}\right)-\left(\boldsymbol{\lambda}_{h}, \boldsymbol{\eta}_{h}, \boldsymbol{w}_{h}, \boldsymbol{v}_{h}\right)\right\|
$$

Proof. The proof is based on the result from Propositions A.4 and A.6. In particular, we need $T_{h}$-coercivity result (see [16. Theorem 2]) for the bilinear form

$$
\begin{aligned}
\mathrm{m}((\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{\zeta}, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})): & =\mathrm{a}((\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{\zeta}, \boldsymbol{\xi}))+\mathrm{b}((\boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{\zeta}, \boldsymbol{\xi})) \\
& +\mathrm{c}((\boldsymbol{\alpha}, \boldsymbol{\beta}),(\boldsymbol{w}, \boldsymbol{q}))+\mathrm{d}((\boldsymbol{u}, \boldsymbol{v}),(\boldsymbol{w}, \boldsymbol{q}))
\end{aligned}
$$

from Problem 4.1 According to Proposition A.4 we need to verify the

- $T_{h}$-coercivity for a. This result follows by noticing that the regular components in the stable regular decomposition from (87) and (88) are in the domain of local linear interpolation operators [12, Lemma 16]. Therefore, $T$ coercivity translates to $T_{h}$-coercivity simply by local interpolation [12, Section 9].
- $T_{h}$-coercivity for d . This property is supplied by Assumption 5.2

As $T_{h}$-coercivity is equivalent to $h$-uniform inf-sup stability (see [16, Theorem 2]), quasi-optimality follows from m being $h$-uniform inf-sup stable up to compact perturbations.

### 5.3. Numerical Experiments

We show numerical experiments to validate our formulation. We compare our results with highly-resolved solution $\left(\boldsymbol{u}_{h}^{\star}, \boldsymbol{v}_{h}^{\star}\right)$ obtained from a FEM-BEM coupling, also known as the Johnson-Nédélec coupling (see [30]). We study convergence of solutions with respect to the $\mathbf{L}^{2}$-norm

$$
\begin{equation*}
\operatorname{error}_{\mathbf{L}^{2}}:=\frac{\left\|\boldsymbol{u}_{h}^{\star}-\boldsymbol{u}_{h}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)}}{\left\|\boldsymbol{u}_{h}^{\star}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)}}, \quad \operatorname{error}_{\mathbf{L}^{2} \times \mathbf{L}^{2}}:=\frac{\sqrt{\left\|\boldsymbol{u}_{h}^{\star}-\boldsymbol{u}_{h}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|\boldsymbol{v}_{h}^{\star}-\boldsymbol{v}_{h}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)}^{2}}}{\sqrt{\left\|\boldsymbol{u}_{h}^{\star}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|\boldsymbol{v}_{h}^{\star}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)}^{2}}} \tag{125}
\end{equation*}
$$

where $\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)$ is a Galerkin solution of Problem 4.1. In the case of FEM-BEM coupling, we compute $\boldsymbol{v}_{h}^{\star}=\frac{1}{i \omega \mu} \operatorname{curl}\left(\boldsymbol{u}_{h}^{\star}\right)$.
In all of our experiments, we use $N_{h}\left(\mathcal{T}_{h}\right)$ as a finite element space for the dual of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$. Note that, as mentioned in Remark 5.1 this may not lead to a stable discretization of the duality pairing in $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$.

The implementation was carried out in $\mathrm{C}++$, by extending the BemToo ${ }^{2}$ library for BEM computations to the case of VIEs. Numerical integration of singular integrals is computed in terms of a Duffy transformation [24, 25, 38] and tensorized Gauss quadrature rules. Matrix compression with $\mathcal{H}$-matrices is done with the Castor library [1], a C++ header-only library for linear algebra computations. We have made our code available in a Github repository ${ }^{3}$.

### 5.4. Scattering at a dielectric cube

We study the electromagnetic scattering problem at a unit cube

$$
\Omega_{i}:=\left\{\boldsymbol{x}=(x, y, z) \in \mathbb{R}^{3}: 0 \leq x, y, z \leq 1\right\}
$$

Material properties are given by

$$
\varepsilon(\boldsymbol{x})= \begin{cases}2+4 x y z(1-x)(1-y)(1-z), & \text { for } \boldsymbol{x} \in \Omega_{i} \\ 1, & \text { for } \boldsymbol{x} \in \mathbb{R}^{3} \backslash \bar{\Omega}_{i}\end{cases}
$$

and $\mu(\boldsymbol{x}) \equiv 1$ in $\mathbb{R}^{3}$. Note that material properties are constant at the boundary $\Gamma$.
The incident wave is given by

$$
\boldsymbol{u}^{\mathrm{inc}}(\boldsymbol{x})=\boldsymbol{e}_{0} \exp \left(i \kappa_{0} \boldsymbol{x} \cdot \boldsymbol{x}_{0}\right)
$$

where $\kappa_{0}=1, \boldsymbol{x}_{0}=(0,1,0)$ and $\boldsymbol{e}_{0}=(1,0,0)$.
The meshes used for our computations are described in Table 1 The reference solution is obtained by FEM-BEM coupling computed on the finest mesh. Convergence results are shown in Figure 4. We observe $\mathcal{O}(h)$ convergence,

[^2]which is the best we can expect for this setting, because the approximation spaces merely contain the full space of piecewise-constant functions. Apparently a potential violation of Assumption 5.2 does not affect convergence in the $\mathbf{L}^{2}$-norm in this case.

| Meshes |  |  |  |
| :---: | :---: | :---: | :---: |
| Elements | Nodes | Edges | Mesh size |
| 24 | 14 | 49 | $1 / 2$ |
| 192 | 63 | 302 | $1 / 4$ |
| 1536 | 365 | 2092 | $1 / 8$ |
| 12288 | 2457 | 15512 | $1 / 16$ |
| 98304 | 17969 | 119344 | $1 / 32$ |

Table 1: Meshes used in Section 5.4 generated by uniform regular refinement


Figure 3: Mesh with 12288 elements.


Figure 4: Scattering at a cube: problem of Section 5.4 Error norms 125 as functions of $h$.

### 5.5. Scattering at a tetrahedron

Now we study the problem with $\Omega$ being the tetrahedron with vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$.

| Meshes |  |  |  |
| :---: | :---: | :---: | :--- |
| Elements | Nodes | Edges | Mesh size |
| 4 | 7 | 15 | 0.346681 |
| 32 | 22 | 73 | 0.173340 |
| 256 | 95 | 430 | 0.0866702 |
| 2048 | 525 | 2892 | 0.0433351 |
| 16384 | 3417 | 21080 | 0.0216676 |

Table 2: Meshes used in Section 5.5 generated by uniform regular refinement.


Figure 5: Mesh with 2048 elements.

Material properties are given by

$$
\begin{aligned}
& \varepsilon(\boldsymbol{x})= \begin{cases}2+4 x y z(1-x)(1-y)(1-z), & \text { for } \boldsymbol{x} \in \Omega_{i}, \\
1, & \text { for } \boldsymbol{x} \in \mathbb{R}^{3} \backslash \bar{\Omega}_{i} .\end{cases} \\
& \mu(\boldsymbol{x})= \begin{cases}2+4 x y z(1-x)(1-y)(1-z), & \text { for } \boldsymbol{x} \in \Omega_{i} \\
1, & \text { for } \boldsymbol{x} \in \mathbb{R}^{3} \backslash \bar{\Omega}_{i} .\end{cases}
\end{aligned}
$$

Note that in this case, material properties are not homogeneous over the whole bondary $\Gamma$. Convergence results are shown in Figure 6 Again, we observe $\mathcal{O}(h)$ convergence, although this case does not satisfy the assumptions of Proposition 4.10, nor Assumption 5.2

## 6. Conclusion

We presented a new formulation coupling boundary and volume integral equations. Under assumptions on the material properties, we are able to show well-posedness of continuous and discrete settings. Uniqueness of solutions in a general setting remains an open problem. Our numerical experiments show optimal convergence of Galerkin discretizations. The use of a conforming subspace of the dual of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ that ensures a stable discretization remains an open problem.

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## References

[1] M. Aussal, M. Bakry, L. Series, Castor: A C++ library to code "à la Matlab", Journal of Open Source Software 7 (71) (2022) 3965.
[2] M. Bebendorf, Hierarchical matrices, Springer, 2008.
[3] J. Bielak, R. C. MacCamy, An exterior interface problem in two-dimensional elastodynamics, Quarterly of Applied Mathematics 41 (1) (1983) 143-159.


Figure 6: Scattering at a Tetrahedron: problem of Section 5.5 Error norms 125 as functions of $h$.
[4] S. Börm, Efficient numerical methods for non-local operators: $\mathcal{H}^{2}$-matrix compression, algorithms and analysis, Vol. 14, European Mathematical Society, 2010.
[5] S. Börm, J. Ostrowski, Fast evaluation of boundary integral operators arising from an eddy current problem, Journal of Computational Physics 193 (1) (2004) 67-85.
[6] A. Bossavit, A rationale for'edge-elements' in 3-D fields computations, IEEE Transactions on Magnetics 24 (1) (1988) 74-79.
[7] M. M. Botha, Solving the volume integral equations of electromagnetic scattering, Journal of Computational Physics 218 (1) (2006) 141-158.
[8] J. H. Bramble, J. E. Pasciak, O. Steinbach, On the stability of the $L^{2}$ projection in $H^{1}(\Omega)$, Mathematics of Computation 71 (237) (2002) 147-156.
[9] J. H. Bramble, J. Xu, Some estimates for a weighted $L^{2}$ projection, Mathematics of computation 56 (194) (1991) 463-476.
[10] A. Buffa, S. Christiansen, A dual finite element complex on the barycentric refinement, Mathematics of computation 76 (260) (2007) 1743-1769.
[11] A. Buffa, M. Costabel, D. Sheen, On traces for $\mathrm{H}(\operatorname{curl}, \Omega)$ in Lipschitz domains, Journal of Mathematical Analysis and Applications 276 (2) (2002) 845-867.
[12] A. Buffa, R. Hiptmair, Galerkin boundary element methods for electromagnetic scattering, in: Topics in computational wave propagation, Springer, 2003, pp. 83-124.
[13] S. N. Chandler-Wilde, E. A. Spence, Coercivity, essential norms, and the Galerkin method for second-kind integral equations on polyhedral and Lipschitz domains, Numerische Mathematik 150 (2) (2022) 299-371.
[14] Y. Chang, R. Harrington, A surface formulation for characteristic modes of material bodies, IEEE transactions on antennas and propagation 25 (6) (1977) 789-795.
[15] S. H. Christiansen, A construction of spaces of compatible differential forms on cellular complexes, Mathematical Models and Methods in Applied Sciences 18 (05) (2008) 739-757.
[16] P. Ciarlet Jr, T-coercivity: Application to the discretization of Helmholtz-like problems, Computers \& Mathematics with Applications 64 (1) (2012) 22-34.
[17] X. Claeys, R. Hiptmair, Electromagnetic scattering at composite objects: a novel multi-trace boundary integral formulation, ESAIM: Mathematical Modelling and Numerical Analysis 46 (6) (2012) 1421-1445.
[18] X. Claeys, R. Hiptmair, E. Spindler, Second-kind boundary integral equations for electromagnetic scattering at composite objects, Computers \& Mathematics with Applications 74 (11) (2017) 2650-2670.
[19] D. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Vol. 93, Springer Science \& Business Media, 2012.
[20] M. Costabel, Symmetric methods for the coupling of finite elements and boundary elements, in: Boundary elements IX, Vol. 1 (Stuttgart, 1987), Comput. Mech., Southampton, 1987, pp. 411-420.
[21] M. Costabel, Boundary integral operators on Lipschitz domains: elementary results, SIAM Journal on Mathematical Analysis 19 (3) (1988) 613-626.
[22] M. Costabel, E. Darrigrand, E.-H. Koné, Volume and surface integral equations for electromagnetic scattering by a dielectric body, Journal of Computational and Applied Mathematics 234 (6) (2010) 1817-1825.
[23] M. Costabel, E. Darrigrand, H. Sakly, The essential spectrum of the volume integral operator in electromagnetic scattering by a homogeneous body, Comptes Rendus. Mathématique 350 (3-4) (2012) 193-197.
[24] B. Feist, Efficient numerical treamtent of the fractional laplacian in three dimensions, Ph.D. thesis, University of Bayreuth (2023).
[25] B. Feist, M. Bebendorf, Fractional Laplacian-Quadrature Rules for Singular Double Integrals in 3D, Computational Methods in Applied Mathematics 23 (3) (2023) 623-645.
[26] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numerica 11 (2002) 237-339.
[27] R. Hiptmair, Coupling of finite elements and boundary elements in electromagnetic scattering, SIAM Journal on Numerical Analysis 41 (3) (2003) 919-944.
[28] R. Hiptmair, P. Meury, Stabilized FEM-BEM coupling for Maxwell transmission problems, in: Modeling and Computations in Electromagnetics: A Volume Dedicated to Jean-Claude Nédélec, Springer, 2008, pp. 1-38.
[29] H. Houde, A new class of variational formulations for the coupling of finite and boundary element methods, Journal of Computational Mathematics 8 (3) (1990) 223-232.
[30] C. Johnson, J.-C. Nédélec, On the coupling of boundary integral and finite element methods, Mathematics of computation 35 (152) (1980) 1063-1079.
[31] M. Karkulik, D. Pavlicek, D. Praetorius, On 2D newest vertex bisection: optimality of mesh-closure and $H^{1}$-stability of $L^{2}$-projection, Constructive Approximation 38 (2013) 213-234.
[32] I. Labarca, R. Hiptmair, Volume integral equations and single-trace formulations for acoustic wave scattering in an inhomogeneous medium, Computational Methods in Applied Mathematics 24 (1) (2024) 119-139.
[33] I. Labarca-Figueroa, Coupled Boundary-Volume Integral Equations for Wave Propagation, Ph.D. thesis, ETH Zurich (2024).
[34] J. Markkanen, P. Ylä-Oijala, Numerical comparison of spectral properties of volume-integral-equation formulations, Journal of Quantitative Spectroscopy and Radiative Transfer 178 (2016) 269-275.
[35] J. Markkanen, P. Ylä-Oijala, New trends in frequency-domain volume integral equations, in: New Trends in Computational Electromagnetics, Institution of Engineering and Technology, 2020, pp. 161-205.
[36] W. McLean, Strongly Elliptic systems and Boundary Integral Equations, Cambridge University Press, 2000.
[37] P. Monk, Finite element methods for Maxwell's equations, Oxford University Press, 2003.
[38] C. Münger, K. Cools, Efficient Numerical Evaluation of Singular Integrals in Volume Integral Equations, IEEE Journal on Multiscale and Multiphysics Computational Techniques 7 (2022) 168-175.
[39] C. Münger, K. Cools, Multi-trace formulation of internally combined volume-surface integral equations, in: 2023 IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting, IEEE, 2023, pp. 1839-1840.
[40] C. Münger, K. Cools, Single source volume-surface integral equations for scattering on nonuniform structures, in: 2023 IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting, IEEE, 2023, pp. 1827-1828.
[41] J.-C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^{3}$, Numerische Mathematik 50 (1986) 57-81.
[42] P. Olyslager, C. Münger, H. Rogier, K. Cools, Volume-Surface Integral Equation Solver For Chiral Media, in: 2023 International Conference on Electromagnetics in Advanced Applications (ICEAA), IEEE, 2023, pp. 541-544.
[43] A. J. Poggio, E. K. Miller, Integral equation solutions of three-dimensional scattering problems, Pergamon, 1973.
[44] S. Rao, D. Wilton, A. Glisson, Electromagnetic scattering by surfaces of arbitrary shape, IEEE Transactions on antennas and propagation 30 (3) (1982) 409-418.
[45] S. A. Sauter, C. Schwab, Boundary Element Methods, Springer, 2010.
[46] E. Spindler, Second kind single-trace boundary integral formulations for scattering at composite objects, Ph.D. thesis, ETH Zurich (2016).
[47] O. Steinbach, Numerical approximation methods for elliptic boundary value problems: finite and boundary elements, Springer, 2007.
[48] B. C. Usner, K. Sertel, M. A. Carr, J. L. Volakis, Generalized volume-surface integral equation for modeling inhomogeneities within high contrast composite structures, IEEE transactions on antennas and propagation 54 (1) (2006) 68-75.
[49] T.-K. Wu, L. L. Tsai, Scattering from arbitrarily-shaped lossy dielectric bodies of revolution, Radio Science 12 (5) (1977) 709-718.

## A. Block Operators

In this section we present results from [32, Appendix A] that cover a particular case of block operators. We show what is required to obtain inf-sup conditions in the continuous and discrete setting. The theoretical results from this appendix are used to establish well-posedness of the variational STF-VIE problem in Sections 4 and 5 .

## A.1. Fredholm Equation

Let $X, \Pi$ be Hilbert spaces and $X^{\prime}, \Pi^{\prime}$ their duals. Consider the operators
$A: X \rightarrow X^{\prime}$,
$B: \Pi \rightarrow X^{\prime}$,
$C: X \rightarrow \Pi^{\prime}$,
$D: \Pi \rightarrow \Pi^{\prime}$,
all of them bounded linear operators. We study the block operator equation

$$
\left(\begin{array}{ll}
A & B  \tag{A.1}\\
C & D
\end{array}\right)\binom{u}{p}=\binom{f}{0}, f \in X^{\prime}
$$

Assumption A.1. The operator

$$
\mathbf{T}=\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right): X \times \Pi \rightarrow X^{\prime} \times \Pi^{\prime}
$$

is injective. Moreover, $A$ and $D$ are coercive operators. $B$ is a compact operator.
Proposition A. 2 ([32, Proposition A.2]). Under assumption A.1, there exists a unique solution $\left(u^{\star}, p^{\star}\right) \in X \times \Pi$ to the system in A.1. Moreover, the solution satisfies

$$
\left\|u^{\star}\right\|_{X}+\left\|p^{\star}\right\|_{\Pi} \leq C\|f\|_{X^{\prime}}
$$

## A.2. Galerkin Discretization

Next, we consider the Galerkin discretization of A.1. Choose finite dimensional subspaces $X_{h} \subset X$ and $\Pi_{h} \subset \Pi$. We study the following variational problem: find $\left(u_{h}, p_{h}\right) \in X_{h} \times \Pi_{h}$ such that

$$
\begin{aligned}
\left\langle A u_{h}, v_{h}\right\rangle_{X}+\left\langle B p_{h}, v_{h}\right\rangle_{X}=\left\langle f, v_{h}\right\rangle, & & \text { for all } v_{h} \in X_{h}, \\
\left\langle C u_{h}, q_{h}\right\rangle_{\Pi}+\left\langle D p_{h}, q_{h}\right\rangle_{\Pi}=0, & & \text { for all } q_{h} \in \Pi_{h},
\end{aligned}
$$

which can be rewritten as

$$
\begin{equation*}
\mathrm{t}\left(\left(u_{h}, p_{h}\right),\left(v_{h}, q_{h}\right)\right)=\left\langle f, v_{h}\right\rangle, \quad \text { for all } v_{h} \in X_{h}, q_{h} \in \Pi_{h}, \tag{A.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{t}\left(\left(u_{h}, p_{h}\right),\left(v_{h}, q_{h}\right)\right)=\left\langle A u_{h}, v_{h}\right\rangle_{X}+\left\langle B p_{h}, v_{h}\right\rangle_{X}+\left\langle C u_{h}, q_{h}\right\rangle_{\Pi}+\left\langle D p_{h}, q_{h}\right\rangle_{\Pi} \tag{A.3}
\end{equation*}
$$

Proposition A. 3 (Inf-sup condition, [32] Proposition A.3]). Let $A_{0}: X \rightarrow X^{\prime}$ and $D_{0}: \Pi \rightarrow \Pi^{\prime}$ be elliptic operators, and $C: X \rightarrow \Pi^{\prime}$ a bounded operator. The bilinear form $\mathrm{t}_{0}:(X \times \Pi) \times(X \times \Pi) \rightarrow \mathbb{C}$ given by

$$
\mathrm{t}_{0}((u, p),(v, q))=\left\langle A_{0} u, v\right\rangle_{X}+\langle C u, q\rangle_{\Pi}+\left\langle D_{0} p, q\right\rangle_{\Pi}
$$

satisfies the $h$-uniform discrete inf-sup condition

$$
\begin{equation*}
c_{1}^{\mathrm{t}_{0}} \leq \inf _{0 \neq\left(u_{h}, p_{h}\right) \in X_{h} \times \Pi_{h}} \sup _{0 \neq\left(v_{h}, q_{h}\right) \in X_{h} \times \Pi_{h}} \frac{\operatorname{Re}\left\{\mathrm{t}_{0}\left(\left(u_{h}, p_{h}\right),\left(v_{h}, q_{h}\right)\right)\right\}}{\left\|\left(u_{h}, p_{h}\right)\right\|_{X \times \Pi}\left\|\left(v_{h}, q_{h}\right)\right\|_{X \times \Pi}}, \quad \text { for all } h>0 \tag{A.4}
\end{equation*}
$$

For the sake of simplicity we have stated Proposition A.3 assuming elliptic operators $A_{0}$ and $D_{0}$. However, in Section 4.2 we face the situation that $D_{0}$ merely satisfies an inf-sup condition. This case is addressed by the following extended version of Proposition A. 3 .

Proposition A. 4 (inf-sup condition, weakened assumptions, [32, Proposition A.4]). In the setting of Section A, let $\widetilde{\Pi}$ be another Hilbert space and $\widetilde{\Pi}_{h} \subset \widetilde{\Pi}$ a finite dimensional subspace. Let $C: X \rightarrow \widetilde{\Pi}^{\prime}$ be bounded and let $D_{0}: \Pi \rightarrow \widetilde{\Pi}^{\prime}$ be a bounded operator that satisfies an h-uniform discrete inf-sup condition

$$
\begin{equation*}
c_{1}^{\mathrm{d}_{0}} \leq \inf _{0 \neq p_{h} \in \Pi_{h}} \sup _{0 \neq q_{h} \in \widetilde{\Pi}_{h}} \frac{\operatorname{Re}\left\{\left\langle D_{0} p_{h}, q_{h}\right\rangle_{\widetilde{\Pi}}\right\}}{\left\|p_{h}\right\|_{\Pi}\left\|q_{h}\right\|_{\widetilde{\Pi}}} \quad \text { for all } h>0 . \tag{A.5}
\end{equation*}
$$

Then, the bilinear form $\mathrm{t}_{0}:(X \times \Pi) \times(X \times \widetilde{\Pi}) \rightarrow \mathbb{C}$ given by

$$
\mathrm{t}_{0}((u, p),(v, q))=\left\langle A_{0} u, v\right\rangle_{X}+\langle C u, q\rangle_{\widetilde{\Pi}}+\left\langle D_{0} p, q\right\rangle_{\widetilde{\Pi}}
$$

satisfies the $h$-uniform discrete inf-sup condition

$$
\begin{equation*}
c_{1}^{\mathrm{t}_{0}} \leq \inf _{0 \neq\left(u_{h}, p_{h}\right) \in X_{h} \times \Pi_{h}} \sup _{0 \neq\left(v_{h}, q_{h}\right) \in X_{h} \times \widetilde{\Pi}_{h}} \frac{\operatorname{Re}\left\{\mathrm{t}_{0}\left(\left(u_{h}, p_{h}\right),\left(v_{h}, q_{h}\right)\right)\right\}}{\left\|\left(u_{h}, p_{h}\right)\right\|_{X \times \Pi}\left\|\left(v_{h}, q_{h}\right)\right\|_{X \times \widetilde{\Pi}}}, \quad \text { for all } h>0 \text {. } \tag{A.6}
\end{equation*}
$$

Proposition A. 5 ([33, Proposition A.2.6]). Let $V$ and $W$ be Hilbert spaces, $\left\{V_{h}\right\}_{h>0}$ and $\left\{W_{h}\right\}_{h>0}$ asymptotically dense families of finite dimensional subspaces of $V$ and $H$ respectively. Consider a bounded sesquilinear form $\mathrm{t}: V \times W \rightarrow \mathbb{C}$ such that $\mathrm{t}=\mathrm{t}_{0}+\mathrm{t}_{\mathbf{K}}$. We assume the following

1. The operator $\mathbf{A}: V \rightarrow W^{\prime}$ induced by the sesquilinear form t is injective.
2. The operator $\mathbf{K}: V \rightarrow W^{\prime}$ induced by the sesquilinear form $\mathbf{t}_{\mathbf{K}}$ is compact.
3. The sesquilinear form $\mathrm{t}_{0}$ satisfies an inf-sup condition on $V \times W$.
4. The sesquilinear form $\mathrm{t}_{0}$ satisfies an $h$-uniform discrete inf-sup condition on $V_{h} \times W_{h}$.

Then, there exist $h_{0}>0$ and $c_{\mathrm{t}}>0$ such that

$$
\begin{equation*}
0<c_{\mathrm{t}} \leq \inf _{0 \neq v_{h} \in V_{h}} \sup _{0 \neq w_{h} \in W_{h}} \frac{\operatorname{Re}\left\{\mathrm{t}\left(v_{h}, w_{h}\right)\right\}}{\left\|v_{h}\right\|_{V}\left\|w_{h}\right\|_{W}}, \quad \text { for all } h<h_{0} . \tag{A.7}
\end{equation*}
$$

Proof. We recall that $h$-uniform inf-sup conditions are equivalent to $T_{h}$-coercivity (see [16, Theorem 2]): let $\left\{T_{h}\right\}_{h>0}$ be the family of bounded linear operators $T_{h}: V_{h} \rightarrow W_{h}$ such that

$$
\begin{equation*}
\left\|T_{h}\right\| \leq C \quad \text { for all } h>0 \tag{A.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Re}\left\{\mathrm{t}_{0}\left(v_{h}, T_{h} v_{h}\right)\right\} \geq c_{\mathrm{t}_{0}}^{\prime}\left\|v_{h}\right\|_{V}^{2} \quad \text { for all } v_{h} \in V_{h} \tag{A.9}
\end{equation*}
$$

where $c_{\mathrm{t}_{0}}^{\prime}>0$ is independent of $h$.
We define an operator $\mathbf{X}: V_{h} \rightarrow W$ such that given $v_{h} \in V_{h}$,

$$
\begin{equation*}
\mathrm{t}\left(q, \mathbf{X} v_{h}\right)=-\mathrm{t}_{\mathbf{K}}\left(q, T_{h} v_{h}\right) \quad \text { for all } q \in V \tag{A.10}
\end{equation*}
$$

which means that $\mathbf{X}=-\left(\mathbf{A}^{\prime}\right)^{-1} \mathbf{K}^{\prime} T_{h}$. This operator is well defined since $\mathbf{A}$ is invertible due to Fredholm alternative and injectivity. Moreover, $\mathbf{X}$ is a compact operator, since $\mathbf{K}$ is compact. We choose conveniently

$$
\begin{equation*}
w_{h}^{\star}=T_{h} v_{h}+P_{h} \mathbf{X} v_{h}, \tag{A.11}
\end{equation*}
$$

where $P_{h}: W \rightarrow W_{h}$ is the $W$-orthogonal projection. Now, we compute

$$
\begin{align*}
\mathrm{t}\left(v_{h}, w_{h}^{\star}\right) & =\mathrm{t}\left(v_{h}, T_{h} v_{h}\right)+\mathrm{t}\left(v_{h}, P_{h} \mathbf{X} v_{h}\right) \\
& =\mathrm{t}\left(v_{h}, T_{h} v_{h}\right)+\mathrm{t}\left(v_{h}, \mathbf{X} v_{h}\right)+\mathrm{t}\left(v_{h},\left(P_{h}-\mathrm{Id}\right) \mathbf{X} v_{h}\right) \\
& =\mathrm{t}\left(v_{h}, T_{h} v_{h}\right)-\mathrm{t}_{\mathbf{K}}\left(v_{h}, T_{h} v_{h}\right)+\mathrm{t}\left(v_{h},\left(P_{h}-\mathrm{Id}\right) \mathbf{X} v_{h}\right)  \tag{A.12}\\
& =\mathrm{t}_{0}\left(v_{h}, T_{h} v_{h}\right)+\mathrm{t}\left(v_{h},\left(P_{h}-\mathrm{Id}\right) \mathbf{X} v_{h}\right)
\end{align*}
$$

From A.12 we obtain

$$
\begin{align*}
\left|\mathrm{t}\left(v_{h}, w_{h}^{\star}\right)\right| & \geq\left|\mathrm{t}_{0}\left(v_{h}, T_{h} v_{h}\right)\right|-\left|\mathrm{t}\left(v_{h},\left(P_{h}-\mathrm{Id}\right) \mathbf{X} v_{h}\right)\right| \\
& \geq c_{\mathrm{t}_{0}}\left\|v_{h}\right\|_{V}^{2}-\|\mathbf{A}\|\left\|v_{h}\right\|_{V}^{2}\left\|\left(P_{h}-\mathrm{Id}\right) \mathbf{X}\right\|, \tag{A.13}
\end{align*}
$$

where $\left\|\left(P_{h}-\mathrm{Id}\right) \mathbf{X}\right\| \rightarrow 0$ uniformly as $h \rightarrow 0$, due to $\mathbf{X}$ being a compact operator. Therefore, there exists $h_{0}>0$ such that

$$
\begin{equation*}
\left|\mathrm{t}\left(v_{h}, w_{h}^{\star}\right)\right|=\left\lvert\,\left\{\mathrm{t}\left(v_{h},\left(T_{h}+P_{h} \mathbf{X}\right) v_{h}\right) \left\lvert\, \geq \frac{1}{2} c_{\mathrm{t}_{0}}\left\|v_{h}\right\|_{V}^{2}\right.\right.\right. \tag{A.14}
\end{equation*}
$$

This corresponds to $T_{h}$-coercivity with a family of operators $\left\{\tilde{T}_{h}\right\}_{h<h_{0}}$, where

$$
\tilde{T}_{h}:=T_{h}+P_{h} \mathbf{X}, \quad\left\|\tilde{T}_{h}\right\| \leq\left\|T_{h}\right\|+\left\|P_{h}\right\|\|\mathbf{X}\| \leq C^{\prime}
$$

with $C^{\prime}>0$ independent of $h$. This result is equivalent to an $h$-uniform inf-sup condition for t , for all $h<h_{0}$ (see [16] Theorem 2]).
Proposition A. 6 (Asymptotic quasi-optimality, 32, Proposition A.5]). Provided that Assumption A. 1 holds, there is $h_{0}>0$ and a constant $c_{\mathrm{qo}}>0$ independent of $h$ such that there exists a unique Galerkin solution $\left(u_{h}, p_{h}\right) \in X_{h} \times \Pi_{h}$ of A.2 for all $h<h_{0}$. The solution satisfies

$$
\begin{equation*}
\left\|(u, p)-\left(u_{h}, p_{h}\right)\right\|_{X \times \Pi} \leq c_{\mathrm{q} \mathrm{\circ}} \inf _{\left(\eta_{h}, \tau_{h}\right) \in X_{h} \times \Pi_{h}}\left\|(u, p)-\left(\eta_{h}, \tau_{h}\right)\right\|_{X \times \Pi} . \tag{A.15}
\end{equation*}
$$

## B. Norm equivalence

The coercivity results in Section 4.2 depend on a norm equivalence in trace spaces. This will be important for the subsequent analysis.

Lemma B.1. Let $\chi \in C^{1}\left(\bar{\Omega}_{i}\right)$ be such that

$$
0<\chi_{\min }<\chi(\boldsymbol{x})<\chi_{\max }
$$

for all $\boldsymbol{x} \in \bar{\Omega}_{i}$. Then

$$
\begin{equation*}
c_{1, \chi}\|\varphi\|_{H^{1 / 2}(\Gamma)} \leq\left\|\chi^{1 / 2} \varphi\right\|_{H^{1 / 2}(\Gamma)} \leq c_{2, \chi}\|\varphi\|_{H^{1 / 2}(\Gamma)}, \quad \text { for all } \varphi \in H^{1 / 2}(\Gamma) \tag{B.1}
\end{equation*}
$$

with constants $c_{1, \chi}, c_{2, \chi}$ depending on $\chi$ and $\Gamma$.

By duality, the result also holds for $H^{-1 / 2}(\Gamma)$

$$
\begin{equation*}
c_{1, \chi}\|\psi\|_{H^{-1 / 2}(\Gamma)} \leq\left\|\chi^{1 / 2} \psi\right\|_{H^{-1 / 2}(\Gamma)} \leq c_{2, \chi}\|\psi\|_{H^{-1 / 2}(\Gamma)}, \quad \text { for all } \psi \in H^{-1 / 2}(\Gamma) \tag{B.2}
\end{equation*}
$$

The result also extends component-wise to the vectorial case, to $\mathbf{H}^{1 / 2}(\Gamma)$ and its dual $\mathbf{H}^{-1 / 2}(\Gamma)$.
Proof. We start by recalling that for any $\varphi \in H^{1 / 2}(\Gamma)$, we can write [47, Section 2.5]

$$
\begin{equation*}
\|\varphi\|_{H^{1 / 2}(\Gamma)}^{2}=\|\varphi\|_{L^{2}(\Gamma)}^{2}+\int_{\Gamma} \int_{\Gamma} \frac{|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})|^{2}}{|\boldsymbol{x}-\boldsymbol{y}|^{3}} \mathrm{~d} s_{y} \mathrm{~d} s_{x} \tag{B.3}
\end{equation*}
$$

Then, we compute

$$
\left\|\chi^{1 / 2} \varphi\right\|_{H^{1 / 2}(\Gamma)}^{2}=\left\|\chi^{1 / 2} \varphi\right\|_{L^{2}(\Gamma)}^{2}+\int_{\Gamma} \int_{\Gamma} \frac{\left|\chi^{1 / 2}(\boldsymbol{x}) \varphi(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y}) \varphi(\boldsymbol{y})\right|^{2}}{|\boldsymbol{x}-\boldsymbol{y}|^{3}} \mathrm{~d} s_{y} \mathrm{~d} s_{x}
$$

We denote

$$
\begin{equation*}
I_{\varphi}^{(1)}:=\int_{\Gamma} \int_{\Gamma} \frac{\left|\chi^{1 / 2}(\boldsymbol{x}) \varphi(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y}) \varphi(\boldsymbol{y})\right|^{2}}{|\boldsymbol{x}-\boldsymbol{y}|^{3}} \mathrm{~d} s_{y} \mathrm{~d} s_{x} \tag{B.4}
\end{equation*}
$$

By adding zero,

$$
\begin{align*}
\left|\chi^{1 / 2}(\boldsymbol{x}) \varphi(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y}) \varphi(\boldsymbol{y})\right| & =\mid \chi^{1 / 2}(\boldsymbol{x}) \varphi(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y}) \varphi(\boldsymbol{x}) \\
& +\chi^{1 / 2}(\boldsymbol{y}) \varphi(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y}) \varphi(\boldsymbol{y}) \mid \\
& \leq\left|\chi^{1 / 2}(\boldsymbol{y})\right||\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})|  \tag{B.5}\\
& +\left|\varphi(\boldsymbol{x}) \| \chi^{1 / 2}(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y})\right|
\end{align*}
$$

Since $\chi \in C^{1}\left(\bar{\Omega}_{i}\right)$ and positively bounded from below, we know that $\chi^{1 / 2} \in C^{1}\left(\bar{\Omega}_{i}\right)$. Therefore, since $\Omega$ is bounded,

$$
\begin{equation*}
\left|\chi^{1 / 2}(\boldsymbol{x})-\chi^{1 / 2}(\boldsymbol{y})\right| \leq C_{\chi}|\boldsymbol{x}-\boldsymbol{y}|, \quad \text { for all } \boldsymbol{x}, \boldsymbol{y} \in \Gamma \tag{B.6}
\end{equation*}
$$

Combining B.5 and B.6 into B.4, we obtain

$$
\begin{equation*}
I_{\varphi}^{(1)} \leq C\left(\chi_{\max }|\varphi|_{H^{1 / 2}(\Gamma)}^{2}+C_{\chi}^{2} I_{\varphi}^{(2)}\right) \tag{B.7}
\end{equation*}
$$

where

$$
\begin{align*}
I_{\varphi}^{(2)} & :=\int_{\Gamma} \int_{\Gamma} \frac{|\varphi(\boldsymbol{x})|^{2}}{|\boldsymbol{x}-\boldsymbol{y}|} \mathrm{d} s_{y} \mathrm{~d} s_{x}  \tag{B.8a}\\
& \leq C \int_{\Gamma}|\varphi(\boldsymbol{x})|^{2}\left(\int_{\Gamma} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \mathrm{d} s_{y}\right) \mathrm{d} s_{x}  \tag{B.8b}\\
& \leq C^{\prime}\|\varphi\|_{L^{2}(\Gamma)}^{2} . \tag{B.8c}
\end{align*}
$$

due to the integral in B.8b being finite, since $\Gamma$ is compact and Lipschitz.

From B.7 and B.8a we conclude that there exists a constant $c_{2, \chi}>0$ such that

$$
\begin{equation*}
\left\|\chi^{1 / 2} \varphi\right\|_{H^{1 / 2}(\Gamma)} \leq c_{2, \chi}\|\varphi\|_{H^{1 / 2}(\Gamma)} . \tag{B.9}
\end{equation*}
$$

Using (B.9) with $\chi^{\prime}=\chi^{-1}$ and $\varphi^{\prime}=\chi^{1 / 2} \varphi$, we obtain

$$
\begin{equation*}
c_{1, \chi}\|\varphi\|_{H^{1 / 2}(\Gamma)} \leq\left\|\chi^{1 / 2} \varphi\right\|_{H^{1 / 2}(\Gamma)} \leq c_{2, \chi}\|\varphi\|_{H^{1 / 2}(\Gamma)} \tag{B.10}
\end{equation*}
$$

The proof for $\psi \in H^{-1 / 2}(\Gamma)$ follows a duality argument. Note that

$$
\begin{align*}
\left\|\chi^{1 / 2} \psi\right\|_{H^{-1 / 2}(\Gamma)} & =\sup _{\varphi \in H^{1 / 2}(\Gamma) \backslash\{0\}} \frac{\left\langle\chi^{1 / 2} \psi, \varphi\right\rangle_{\Gamma}}{\|\varphi\|_{H^{1 / 2}(\Gamma)}}  \tag{B.11a}\\
& =\sup _{\varphi \in H^{1 / 2}(\Gamma) \backslash\{0\}} \frac{\left\langle\psi, \chi^{1 / 2} \varphi\right\rangle_{\Gamma}}{\|\varphi\|_{H^{1 / 2}(\Gamma)}}  \tag{B.11b}\\
& \leq \sup _{\varphi \in H^{1 / 2}(\Gamma) \backslash\{0\}}\|\psi\|_{H^{-1 / 2}(\Gamma)} \frac{\left\|\chi^{1 / 2} \varphi\right\|_{H^{1 / 2}(\Gamma)}}{\|\varphi\|_{H^{1 / 2}(\Gamma)}}  \tag{B.11c}\\
& =c_{2, \chi}\|\psi\|_{H^{-1 / 2}(\Gamma)} . \tag{B.11d}
\end{align*}
$$

Repeating the argument with $\chi^{\prime}=\chi^{-1}$ and $\psi^{\prime}=\chi^{1 / 2} \psi$, we conclude

$$
c_{1, \chi}\|\psi\|_{H^{-1 / 2}(\Gamma)} \leq\left\|\chi^{1 / 2} \psi\right\|_{H^{-1 / 2}(\Gamma)} \leq c_{2, \chi}\|\psi\|_{H^{-1 / 2}(\Gamma)}
$$

## C. $\mathbf{L}^{2}-$ Projection in $\mathbf{H}\left(\operatorname{curl}, \boldsymbol{\Omega}_{i}\right)$

In the scalar case, there is a $h$-uniform discrete inf-sup condition for the dual product between $H^{1}\left(\Omega_{i}\right)$ and $\widetilde{H}^{-1}\left(\Omega_{i}\right)$, discretized with the finite dimensional space of piecewise-linear continuous functions $P_{h}^{1}$ [8]. The result is based on the $H^{1}$-stability of the $L^{2}$-projection $Q_{h}: L^{2}\left(\Omega_{i}\right) \rightarrow P_{h}^{1}$ defined as

$$
\begin{equation*}
\left\langle Q_{h} u, v_{h}\right\rangle_{\Omega_{i}}=\left\langle u, v_{h}\right\rangle, \quad \text { for all } v_{h} \in P_{h}^{1}, u \in L^{2}\left(\Omega_{i}\right) . \tag{C.1}
\end{equation*}
$$

We know $Q_{h}$ satisfies (see [8, Theorem 4.1],[31, Theorem 3],[9, Section 3])

$$
\begin{equation*}
\left\|Q_{h} u\right\|_{H^{1}\left(\Omega_{i}\right)} \leq c_{Q}\|u\|_{H^{1}\left(\Omega_{i}\right)} \quad \text { for all } u \in H^{1}\left(\Omega_{i}\right) \tag{C.2}
\end{equation*}
$$

The result in C.2 is proven by using a quasi-interpolation operator $\widetilde{I}_{h}$, known to be stable in $H^{1}\left(\Omega_{i}\right)$ and for which some approximation properties can be shown [8, Section 3]:

$$
\begin{align*}
\left\|\widetilde{I}_{h} u\right\|_{H^{1}\left(\Omega_{i}\right)} & \leq c_{\tilde{I}}\|u\|_{H^{1}\left(\Omega_{i}\right)},  \tag{C.3}\\
\left\|u-\widetilde{I}_{h} u\right\|_{L^{2}\left(\Omega_{i}\right)} & \leq c_{I I} h|u|_{H^{1}\left(\Omega_{i}\right)}, \tag{C.4}
\end{align*}
$$

for all $u \in H^{1}\left(\Omega_{i}\right)$.
Assuming a quasi-uniform and shape-regular family of meshes, it follows that

$$
\begin{aligned}
\left\|Q_{h} u\right\|_{H^{1}\left(\Omega_{i}\right)} & \leq\left\|Q_{h} u-\widetilde{I}_{h} u+\widetilde{I}_{h} u\right\|_{H^{1}\left(\Omega_{i}\right)} \\
& \leq\left\|\left(Q_{h}-\widetilde{I}_{h}\right) u\right\|_{H^{1}\left(\Omega_{i}\right)}+\left\|\widetilde{I}_{h} u\right\|_{H^{1}\left(\Omega_{i}\right)} \\
& \leq \frac{1}{h}\left\|\left(Q_{h}-\widetilde{I}_{h}\right) u\right\|_{L^{2}\left(\Omega_{i}\right)}+\left\|\widetilde{I}_{h} u\right\|_{H^{1}\left(\Omega_{i}\right)} \\
& \leq \frac{1}{h}\left\|u-Q_{h} u\right\|_{L^{2}\left(\Omega_{i}\right)}+\frac{1}{h}\left\|u-\widetilde{I}_{h} u\right\|_{L^{2}\left(\Omega_{i}\right)}+c_{\tilde{I}}\|u\|_{H^{1}\left(\Omega_{i}\right)} \\
& \leq C|u|_{H^{1}\left(\Omega_{i}\right)}+c_{I I}|u|_{H^{1}\left(\Omega_{i}\right)}+c_{\tilde{I}}\|u\|_{H^{1}\left(\Omega_{i}\right)} \leq c_{Q}\|u\|_{H^{1}\left(\Omega_{i}\right)}
\end{aligned}
$$

The fundamental step in this proof is: being able to bound the $L^{2}$-error with the $H^{1}$-seminorm.
Is it possible to have a similar result in $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ with its seminorm? The answer is no. Consider $\boldsymbol{w} \in \mathbf{H}\left(\mathbf{c u r l} 0, \Omega_{i}\right)$ and $\mathbf{Q}_{h}: \mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right) \rightarrow N_{h} \subset \mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ the standard $\mathbf{L}^{2}$-projection into Nédélec edge elements. Then, such a result requires

$$
\begin{equation*}
\left\|\boldsymbol{w}-\mathbf{Q}_{h} \boldsymbol{w}\right\|_{\mathbf{L}^{2}\left(\Omega_{i}\right)} \leq C|\boldsymbol{w}|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}=0 \tag{C.5}
\end{equation*}
$$

which can only be true for constants or polynomials in $N_{h}$, but as we know, $\mathbf{H}\left(\mathbf{c u r l} 0, \Omega_{i}\right)$ is an infinite dimensional subspace of $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$. Therefore, such a proof is not valid for the standard $\mathbf{L}^{2}$-projection $\mathbf{Q}_{h}$.
Some numerical evidence of this issue and its implications will be shown in Appendix C. 1 .

## C.1. Numerical experiment

In this section, we study the convergence in the $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ norm of the $\mathbf{L}^{2}$-projection $\mathbf{Q}_{h}$, defined as

$$
\begin{equation*}
\left\langle\mathbf{Q}_{h} \boldsymbol{u}, \boldsymbol{v}_{h}\right\rangle_{\Omega_{i}}=\left\langle\boldsymbol{u}, \boldsymbol{v}_{h}\right\rangle_{\Omega_{i}} \quad \text { for all } \boldsymbol{v}_{h} \in N_{h}, \boldsymbol{u} \in \mathbf{L}^{2}\left(\Omega_{i}\right), \tag{C.6}
\end{equation*}
$$

where $N_{h}=N_{h}\left(\mathcal{T}_{h}\right)$ is the finite dimensional space of Nédélec edge functions in a tetrahedral mesh $\mathcal{T}_{h}$ of $\Omega_{i}$.
In particular, we consider

$$
\Omega_{i}:=\left\{\boldsymbol{x}=(x, y, z) \in \mathbb{R}^{3}: 0 \leq x, y, z \leq 1\right\}
$$

and

$$
\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{u}^{\star}(\boldsymbol{x}):=\boldsymbol{e}_{0} \exp \left(i \kappa_{0} \boldsymbol{x} \cdot \boldsymbol{x}_{0}\right)
$$

where $\kappa_{0}=2, \boldsymbol{x}_{0}=(0,1,0)$ and $\boldsymbol{e}_{0}=(1,0,0)$. We can observe in Figure C. 7 the errors of the projections $\mathbf{Q}_{h} \boldsymbol{u}^{\star}$ in the $\mathbf{L}^{2}\left(\Omega_{i}\right)$ and $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ norms.

It is a well known result that the approximation error for $\mathbf{L}^{2}\left(\Omega_{i}\right)$ and $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ in formulations that are stable in $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ has the same convergence rate, due to the interpolation estimates being the same [26, Remark 10].

Assume $\left\|\mathbf{Q}_{h} \boldsymbol{u}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)} \leq C_{S}\|\boldsymbol{u}\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)}$. Then it follows

$$
\begin{align*}
\left\|\boldsymbol{u}-\mathbf{Q}_{h} \boldsymbol{u}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)} & =\left\|\boldsymbol{u}-\mathbf{Q}_{h} \boldsymbol{u}-\boldsymbol{v}_{h}+\mathbf{Q}_{h} \boldsymbol{v}_{h}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)} \\
& =\left\|\left(\mathbf{I}-\mathbf{Q}_{h}\right)\left(\boldsymbol{u}-\boldsymbol{v}_{h}\right)\right\|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)}  \tag{C.7}\\
& \leq\left(1+C_{S}\right)\left\|\boldsymbol{u}-\boldsymbol{v}_{h}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)}
\end{align*}
$$



Figure C.7: $\mathbf{L}^{2}$-projection $\mathbf{Q}_{h}$, Section C.1 Error norms 125 as functions of $h$.
for all $\boldsymbol{v}_{h} \in N_{h}$. From C.7) we obtain that for smooth vector fields $\boldsymbol{u} \in \mathbf{H}^{1}\left(\mathbf{c u r l}, \Omega_{i}\right)$ it holds

$$
\left\|\boldsymbol{u}-\mathbf{Q}_{h} \boldsymbol{u}\right\|_{\mathbf{H}\left(\operatorname{curl}, \Omega_{i}\right)} \leq\left(1+C_{S}\right) \inf _{\boldsymbol{v}_{h} \in N_{h}}\left\|\boldsymbol{u}-\boldsymbol{v}_{h}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)}=\mathcal{O}(h)
$$

on shape-regular and quasi-uniform families of meshes.
As we observe in Figure C.7 there is a reduced order of convergence of the $\mathbf{L}^{2}$-projection in the $\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)$ norm. We conclude that

$$
\begin{equation*}
\frac{\left\|\mathbf{Q}_{h} \boldsymbol{u}\right\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)}}{\|\boldsymbol{u}\|_{\mathbf{H}\left(\mathbf{c u r l}, \Omega_{i}\right)}} \text { is not bounded uniformly in } h \text {. } \tag{C.8}
\end{equation*}
$$


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[^1]:    ${ }^{1}$ We write $C$ for a positive generic constant. The value of $C$ may be different at different occurrences.

[^2]:    ${ }^{2}$ https://github.com/xclaeys/BemTool
    3 https://github.com/ijlabarca/CoupledBVIE

