

# Enhancing Valuation of Variable Annuities in Lévy Models with Stochastic Interest Rate

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## Abstract

This paper extends the valuation and optimal surrender framework for variable annuities with guaranteed minimum benefits in a Lévy equity market environment by incorporating a stochastic interest rate described by the Hull-White model. This approach frames a more dynamic and realistic financial setting compared to previous literature. We exploit a robust valuation mechanism employing a hybrid numerical method that merges tree methods for interest rate modeling with finite difference techniques for the underlying asset price. This method is particularly effective for addressing the complexities of variable annuities, where periodic fees and mortality risks are significant factors. Our findings reveal the influence of stochastic interest rates on the strategic decision-making process concerning the surrender of these financial instruments. Through comprehensive numerical experiments, and by comparing our results with those obtained through the Longstaff-Schwartz Monte Carlo method, we illustrate how our refined model can guide insurers in designing contracts that equitably balance the interests of both parties. This is particularly relevant in discouraging premature surrenders while adapting to the realistic fluctuations of financial markets. Lastly, a comparative statics analysis with varying interest rate parameters underscores the impact of interest rates on the cost of the optimal surrender strategy, emphasizing the importance of accurately modeling stochastic interest rates.

*Keywords:* Variable Annuity; Guaranteed Minimum Benefit; stochastic interest rate; Lévy process; tree methods; finite difference;

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# 1 Introduction

In the evolving field of retirement planning and investment strategies, variable annuities, also known as equity-linked or equity-indexed annuities, have emerged as pivotal instruments. These products uniquely blend the growth potential of equity investments with the security of insurance guarantees, notably through features such as the Guaranteed Minimum Accumulation Benefit (GMAB) and the Guaranteed Minimum Death Benefit (GMDB). Specifically, the GMAB ensures that policyholders are guaranteed a minimum return on their investments at the contract's maturity, while the GMDB provides a safety net in the event of the policyholder's untimely demise, ensuring that a predefined benefit is payable to their beneficiaries.

However, accurately valuing these complex financial instruments requires sophisticated mathematical models that can capture the inherent risks and uncertainties of the equity markets. Traditional models often fall short by assuming smooth, continuous market dynamics, thereby overlooking the abrupt, significant shifts that characterize real-world markets. Herein lies the importance of employing jump models, which are more effective at simulating the market's discrete, often volatile nature. Such models are crucial for understanding the pricing and risk management of variable annuities with GMAB and GMDB riders, where market volatility directly influences the contract's value and the insurer's liability. In recent years, the valuation of variable annuities within the framework of Lévy processes has garnered substantial attention, highlighting the complexity and the necessity for advanced modeling techniques in financial mathematics. To this aim, one of the first contributions is due to Jaimungal and Young [14], who pioneered the integration of optimal investment strategies in equity-linked pure endowments contracts under a finite variation Lévy process by investigating stochastic control methods. Ballotta [1] contributes by examining the pricing of ratchet equity-indexed annuities by exploiting Fourier-transform type techniques within Lévy models. Gerber et al. [11] introduce closed-form solutions for valuing equity-linked death benefits in variable annuities under jump diffusion models, significantly advancing actuaries' capabilities in accurately pricing these complex insurance products. Zhou and Wu [25] explore the valuation of equity-linked investment products with a threshold expense when the underlying fund evolves as a jump diffusion process. Hieber [13] advances the computational aspects of pricing of equity-linked life insurance contracts that offer an annually guaranteed minimum return, by employing efficient numerical methods based on the Fourier transform. Cui et al. [8] delved into the design of equity-indexed annuities and their guarantees under various market conditions and develop a novel and efficient transform-based method to price equity-linked annuities. Zhang et al. [23] focus on equity-linked death benefits using a projection method and Fast Fourier Transform within general exponential Lévy models, offering a novel approach for accurate valuations. Dong et al. [9] introduce willow tree algorithms for pricing Guaranteed Minimum Withdrawal Benefits under jump-diffusion and CEV models, significantly reducing computational time while maintaining accuracy, and include optimal dynamic withdrawal strategies. Zhang et al. [24] present an innovative valuation method for equity-linked death benefits within general exponential Lévy models by employing a projection method alongside Fast Fourier Transform.

Recently, Kirkby and Aguilar [16] develop a comprehensive framework for the valuation and optimal surrender of equity-linked variable annuities, considering both guaranteed minimum accumulation and death benefits within a Lévy-driven market model. They introduce discrete-time treatment with periodic premiums and fees, a novel approach that aligns with practical market operations, significantly advancing the understanding and management of surrender behaviors in variable annuity contracts.

Nevertheless, these studies collectively underscore the critical role of incorporating jumps and stochastic components in accurately valuing variable annuities, providing a richer understanding of the underlying risks and market behaviors. Nonetheless, despite these significant advancements in understanding and valuing

equity-linked variable annuities, none of the mentioned authors considers a stochastic interest rate model. This omission represents a critical gap, given the long maturity of the products in question. The evaluation, hedging, and lapse strategy description fundamentally rely on an accurate interest rate model to reflect the long-term financial landscape, making the incorporation of stochastic interest rates an essential direction for future research in this area. Indeed, Kirkby and Aguilar [16] highlight in their work the need for future development to incorporate stochastic interest rates into their model, acknowledging this as a critical aspect for further enhancing the accuracy and relevance of their valuation framework for equity-linked variable annuities, especially given the significant impact of interest rate movements over time. This consideration for future research underscores the importance of developing a comprehensive model that can capture more precisely the complexities and dynamics of financial markets.

Building on the foundational work of Kirkby and Aguilar [16], our study introduces a critical enhancement: the incorporation of a stochastic interest rate model, as described by the Hull-White model. Considering a stochastic interest rate model is crucial for evaluating variable annuity products with long maturities, such as the 25-year products discussed in [16]. Over such an extended period, interest rates are likely to experience significant fluctuations, which can substantially affect the product’s value, its associated guarantees, and the policyholder’s behavior. A stochastic interest rate model captures the uncertainty and dynamics of interest rates over time, allowing for a more accurate assessment of the product’s risks and rewards. It reflects the real financial environment’s complexity, providing a comprehensive framework for evaluating the long-term financial obligations and potential returns of variable annuities. This approach is essential for ensuring that pricing, hedging, and risk management strategies are robust and aligned with the evolving economic landscape.

Recently, some authors have begun to consider the problem of valuing derivative instruments in Lévy stochastic models that also include the stochastic interest rate. Boyarchenko et al. [3] delve into option pricing under Lévy processes without specifying the interest rate model, thus offering a broad framework applicable across different stochastic interest rate environments. Bao and Zhao [2] consider European option pricing in Markov-modulated exponential Lévy models, where stochastic interest rates are modeled by a Markovian regime-switching Hull-White process, showcasing a novel integration of regime switching into the stochastic interest rate context. Tan et al. [22] extend traditional jump-diffusion models to include general Lévy processes with stochastic interest rates, specifically employing the Girsanov theorem and Itô formula for pricing European-style options, without detailing the exact interest rate model used. As far as variable annuities are considered, regime-switching models are investigated by Costabile [7] who propose a lattice-based model to evaluate contract than embody guaranteed minimum withdrawal benefits.

Our approach acknowledges the complex interplay between equity market jumps and interest rate variability, necessitating a robust numerical method capable of handling this multidimensional challenge. To this end, we draw inspiration from the hybrid numerical method developed by Briani et al. [4], originally designed for options pricing under stochastic volatility and stochastic interest rate. Such a method integrates a hybrid approximation approach for option pricing in the Bates model with stochastic interest rate, using a blend of tree methods for volatility and interest rate dimensions, and finite difference techniques for the underlying asset price. This framework is notably flexible and efficient for handling the complexities introduced by stochastic interest rates, as it allows for the accurate modeling of the interest rate’s impact on option pricing through the application of a finite difference scheme to a partial integro-differential equation (PIDE). To address this PIDE, multiple strategies are available, among which is the Implicit-Explicit (IMEX) approach developed by Cont and Voltchkova [5] or the Wiener-Hopf approximated factorization as investigated by Kudryavtsev [17].

This paper aims to bridge the gap between theoretical financial models and the practical exigencies of

the variable annuities market. By integrating a stochastic interest rate into the valuation model introduced by Kirkby and Aguilar [16] and employing an advanced numerical solution, we provide insights that are not only theoretically rigorous but also immensely valuable for practitioners dealing with the intricacies of GMAB and GMDB contracts. Our work underscores the indispensable role of sophisticated modeling and numerical techniques in the actuarial science and financial engineering domains, especially as they pertain to products that play a crucial role in individuals' financial security and retirement planning. Finally, we stress out that the hybrid nature of the method introduced by Briani et al. [4] facilitates the handling of the multi-faceted dynamics present in variable annuities with GMAB and GMDB features, providing a robust tool for their valuation in a market environment where both equity jumps and interest rate fluctuations are critical factors. Moreover, this method is also particularly interesting in that it is very versatile and can be used for various Lévy processes.

In our study, we compare the proposed Hybrid method with a Monte Carlo method based on the Longstaff-Schwarz algorithm, which has been adapted to handle the challenging task of learning the continuation value in scenarios characterized by abrupt value shifts and significant convexity changes. Our analysis spans four Lévy models, namely the Normal Inverse Gaussian, the Variance Gamma, the CGMY, and the Merton Jump Diffusion, revealing that both numerical approaches yield consistent valuations, with the hybrid method demonstrating rapid convergence. Focused tests on calculating the surrender premium, which is critical for understanding lapse dynamics, are supplemented by a qualitative analysis. Through comparative statics analysis, we explore how the surrender premium varies within this stochastic rate model and examine the impact of rate parameters on the surrender premium, offering deeper insights into its behavior and the factors influencing it.

The remainder of the paper is organized as follows. In Section 2 we frame the stochastic model. In Section 3 we describe the insurance contract and the principles useful in its evaluation. In Section 4 we present the pricing algorithms. In Section 5 we discuss numerical results. Finally, in Section 6, we conclude.

## 2 The Lévy models with stochastic interest rate

Lévy processes represent a fundamental class of stochastic processes with wide-ranging applications in finance, particularly in the modeling of asset returns and the pricing of derivatives. These processes are characterized by their capability to incorporate jumps, thereby capturing the discontinuities and heavy tails often observed in financial data.

A Lévy process is a type of stochastic process characterized by having stationary and independent increments (for general definitions, see, *e.g.*, Sato [21]). Notably, a Lévy process may encompass a Gaussian component, a pure jump component, or both. The latter is characterised by the density of jumps, which is termed the Lévy density. Now, let us consider a filtered risk-neutral probability space  $(\Omega, \mathbb{Q}, \{\mathcal{F}_t\}_{t \geq 0})$  under which valuations are made, and in particular  $\mathbb{E}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot]$ . Then, a Lévy process  $X$  can be completely specified by its characteristic exponent,  $\psi$ , definable from the equality  $\mathbb{E}[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ . The characteristic exponent  $\psi$  is described by the Lévy-Khintchine formula, which reads

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbb{R} \setminus \{0\}} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y| \leq 1}) \nu(dy), \quad (2.1)$$

where  $\sigma^2 \geq 0$  is the variance of the Gaussian component and the Lévy measure  $\nu(dy)$  satisfies the following relation:

$$\int_{\mathbb{R} \setminus \{0\}} \min\{1, y^2\} \nu(dy) < +\infty. \quad (2.2)$$

In our model, we assume that, under a certain risk-neutral measure  $\mathbb{Q}$ , chosen by the market, the dynamics of the underlying process  $S = \{S_t\}_{t \geq 0}$  can be expressed as

$$S_t = e^{Y_t}, \quad Y_t = \int_0^t (r_s - q) ds + X_t, \quad (2.3)$$

where  $X = \{X_t\}_{t \geq 0}$  is a certain Lévy process,  $r = \{r_t\}_{t \geq 0}$  is the stochastic interest rate process and  $q$  the dividend yield. Then, one requires  $\mathbb{E}[e^{X_t}] < +\infty$ , and, therefore,  $\psi$  must admit analytic continuation into a strip  $\Im \xi \in (-1, 0)$  and continuous continuation into the closed strip  $\Im \xi \in [-1, 0]$ .

The infinitesimal generator of  $X$ , denoted by  $L$ , is an integro-differential operator that acts as follows:

$$Lu(x) = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}(x) + \mu \frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty} \left( u(x+y) - u(x) - y \mathbf{1}_{|y| \leq 1} \frac{\partial u}{\partial x}(x) \right) \nu(dy). \quad (2.4)$$

The operator  $L$  can also be expressed as a pseudo-differential operator (PDO) with the symbol  $-\psi(\xi)$ , that is  $L = -\psi(D)$ , where  $D = -i\partial_x$ . Recall that a PDO  $A = a(D)$  acts as follows:

$$A[u(x)] = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi) \hat{u}(\xi) d\xi, \quad (2.5)$$

where  $\hat{u}$  is the Fourier transform of a function  $u$ , that is

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x) dx.$$

It should be noted that the inverse Fourier transform, as specified in (2.5), is classically defined only when both the symbol  $a(\xi)$  and the function  $\hat{u}(\xi)$  exhibit certain desirable properties. For instances, where these conditions are not met, the inverse Fourier transform is more generally defined through the principle of duality.

Moreover, if the underlying stock does not distribute dividends, the discounted price process is required to be a martingale, which implies that the condition  $\psi(-i) = 0$  must be satisfied. This condition facilitates the expression of the drift term  $\mu$  in terms of the other parameters, characterizing the Lévy process, as illustrated in the following equation:

$$\mu = -\frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y \mathbf{1}_{|y| \leq 1}) \nu(dy). \quad (2.6)$$

Hence, the infinitesimal generator may be rewritten as follows:

$$Lu(x) = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}(x) - \frac{\sigma^2}{2} \frac{\partial u}{\partial x}(x) + \int_{\mathbb{R}} \left[ u(x+y) - u(x) - (e^y - 1) \frac{\partial u}{\partial x}(x) \right] \nu(dy). \quad (2.7)$$

The Appendix A shows the values of the  $\psi$ -function for the main Lévy processes considered in the remainder of the paper, under the assumption (2.6).

The stochastic interest rate is another key component of the proposed market model. The Hull-White model is a good choice for modelling the short interest rate  $r$ , due to its flexibility in fitting the current term structure of interest rates, and its capacity to be easily implemented in a tree or lattice, making it practical for valuing interest rate derivatives. Its no-arbitrage framework ensures consistency with market prices, allowing for a realistic modeling of interest rate dynamics over time. The dynamics of the process  $r$ , the short interest rate, reads as follows:

$$dr_t = k_{HW} (\theta_t - r_t) dt + \sigma_{HW} dZ_t^r, \quad r_0 = \bar{r}_0,$$

with  $k_{HW} > 0$  the speed of mean reversion,  $\sigma_{HW} > 0$  the short rate volatility,  $\bar{r}_0$  the initial value and  $Z^r = \{Z_t^r\}_{t \geq 0}$  a Brownian motion. Moreover,  $\theta_t$  is a deterministic function which is completely determined by the market values of the zero-coupon bonds by calibration. Now, let us consider  $P^M(0, T)$ , the market price of the zero-coupon bond at time 0 for the maturity  $T$  and  $f^M(0, T)$ , the market instantaneous forward interest rate which is then defined by

$$f^M(0, T) = -\frac{\partial \ln P^M(0, T)}{\partial T}.$$

In the specified model, the process  $r$  can be decomposed into the sum of a Vašíček process and a deterministic function, delineated as follows:

$$r_t = \sigma_{HW} R_t + \beta(t),$$

with  $R = \{R_t\}_{t \geq 0}$  the stochastic process defined by

$$dR_t = -k_{HW} R_t dt + dZ_t^r, \quad R_0 = 0,$$

and  $\beta(t)$  the function

$$\beta(t) = f^M(0, t) + \frac{\sigma_{HW}^2}{2k^2} (1 - \exp(-k_{HW}t))^2,$$

so obtained through calibration on bond prices.

A scenario referred to as the *flat curve* is considered noteworthy. In this context, we postulate that the market price of a zero-coupon bond is determined by  $P^M(t, T) = e^{-r_0(T-t)}$  and the initial forward rate is uniformly  $f^M(0, T) = r_0$ . Given these presuppositions, the formulas for the coefficients  $\beta(t)$  and  $\theta_t$  are delineated as:

$$\beta(t) = r_0 + \frac{\sigma_{HW}^2}{2k_{HW}^2} (1 - \exp(-k_{HW}t))^2, \quad \text{and} \quad \theta_t = r_0 + \frac{\sigma_{HW}^2}{2k_{HW}^2} (1 - \exp(-2k_{HW}t)).$$

We point out that the flat curve assumption is adopted for the numerical experiments carried out in the dedicated section of this manuscript, as it simplifies the parameter set considered.

### 3 Contract formulation and valuation

In the following, we consider a particular type of variable annuities, called the *equity-linked variable annuities* (ELVAs), which offer policyholders an investment option that combines the elements of conventional annuities with the potential for higher returns through equity market exposure. A key characteristic of ELVAs includes the GMDB and GMAB guarantees, ensuring minimum benefits regardless of market conditions. As explored in the seminal work by Kirkby and Aguilar [16], the surrender option is an essential aspect of ELVAs, which allows policyholders to completely withdraw their policy before maturity under certain conditions. Hereinafter, we provide a brief description of the main features of the ELVA contract and refer the interested reader to the original work for further details.

#### 3.1 The contract formulation

The ELVA contract includes a minimum accumulation clause (GMAB), together with a benefit in the event of the insured's premature death (GMDB). At the initial time  $t = 0$ , the policyholder purchasing the contract pays a premium  $P$  to the insurer. This premium is fully invested in a fund, whose value  $F = \{F_t\}_{t \geq 0}$  is equal to  $P$  at contract inception and it is linked to an underlying asset  $S$  which evolves over time as a Lévy

process. The contract has a pre-determined duration of  $M$  years, but it may be terminated early in the event of the insured's death or early surrender.

When an anniversary is reached, say the  $m$ -th anniversary for  $m \in \{1, \dots, M\}$ , the following actions take place:

1. Updating the value of the fund and withdrawal of fees.

At each anniversary, the value of the fund  $F$  is changed proportionally to the value of the underlying asset  $S$  to which it is connected and is decreased by the fees proportionally to the parameter  $\alpha_m$ . Specifically:

$$F_m = (1 - \alpha_{m-1}) F_{m-1} \cdot S_m / S_{m-1}.$$

Although fees only reduce the value of the fund on an anniversary, as a kind of discrete dividend, since the value of the fund between two anniversaries is irrelevant to the valuation of the contract, it is possible to think of fees paid continuously, as a kind of dividend yield. Specifically, one can replace  $q$  with  $\hat{q} = q - \log(1 - \alpha_m)$  in the dynamics of  $S$  with respect to the time interval  $[m-1, m]$ , and set

$$F_m = F_{m-1} \cdot S_m / S_{m-1}.$$

Within the framework of this particular model, by assuming that the initial stock price  $S_0$  is equivalent to the initial futures price  $F_0$ , one obtains  $F = S$ , and, in particular,  $F_m = S_m$  for all anniversaries dates  $m$ . This equivalence considerably simplifies the notation, a convention that will be maintained throughout the subsequent discourse.

2. Payment of the death benefit in the event of the death of the policyholder.

If the death of the insured occurs between anniversaries  $m-1$  and  $m$ , the insured's heirs benefit from a payment  $DB_m$ , paid at time  $m$  and calculated as follows:

$$DB_m(F_m) = \max \{F_0 e^{gm}, \min \{F_0 e^{cm}, F_m\}\}.$$

Then the contract ends. Please, note that  $g$  and  $c$  are minimal and maximal growth rates respectively, so that the amount paid is equal to the value of the fund, limited between two exponentially increasing values.

3. Possible surrender (if  $m < M$ ) or contract end (if  $m = M$ ).

On each anniversary before maturity, the alive policyholder may decide to terminate the contract early. In this case, the policyholder receives a surrender benefit equal to the minimum between the value of the fund and the maximum growth factor, reduced by a penalty governed by the parameter  $\gamma_m$ . When the contract matures, payment of the final payoff is automatic and the contract ends. In this case, the surrender benefit is equal to death benefit. Therefore,  $SB_m$ , the surrender benefit at time  $m$ , reads out:

$$SB_m(F_m) = \begin{cases} (1 - \gamma_m) \min \{F_0 e^{cm}, F_m\} & \text{if } m < M \\ DB_m(F_m) & \text{if } m = M. \end{cases}$$

### 3.2 Modeling mortality

Let  $\delta$  represent the random time of death of the policyholder, which is alive and  $\omega$ -years old at contract inception. Following Kirkby and Aguilar [16], such a random variable is assumed to follow a general

probability mass function for an individual of age  $\omega$  at the time of policy purchase. This distribution, represented as  $\{p_m^\omega : m \geq 1\}$ , captures the probability of death occurring within the time interval  $\mathcal{I}_m = (\omega + (m - 1), \omega + m]$ . The model is designed to be neutral under both the risk-neutral measure  $\mathbb{Q}$  and the physical measure  $\mathbb{P}$ , enabling the evaluation of mortality risk in variable annuity contracts. Specifically, we assume that mortality risk is diversifiable. This approach provides a realistic and flexible framework for incorporating mortality risk into the pricing and valuation of life insurance products and annuity contracts.

### 3.3 Early surrender and contract valuation

The contract valuation strongly depends on modeling the surrender strategy. Let  $V(m, F_m, r_m)$  denote the value of the contract which has not yet been terminated at the  $m$ -th anniversary, so the policyholder is still alive and it has not surrendered yet. Such a value is derived through a dynamic programming approach. The optimal surrender strategy is defined as the stopping time  $\tau^S$  that maximizes the expected discounted value of the surrender benefit plus any mortality-related benefits, mathematically represented as:

$$V(0, F_0, r_0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_0^\tau r_t dt} S B_\tau(F_\tau) \mathbb{1}_{\tau < \min(\delta, M)} + e^{-\int_0^{\min(\delta, M)} r_t dt} D B_{\min(\delta, M)}(F_{\min(\delta, M)}) \mathbb{1}_{\tau \geq \min(\delta, M)} \right], \quad (3.1)$$

where  $\mathcal{T}$  denotes the set of permissible surrender times in  $\{1, \dots, M\}$  and  $\mathbb{1}_{(\cdot)}$  is the indicator function. The optimal surrender strategy effectively balances the potential economic gains from early surrender against the contractual benefits of holding the policy until maturity or the policyholder's death.

It is worth noting that if one sets  $\gamma_m = 1$  for all value of  $m$ , then the surrender benefit fades and therefore it is never convenient to exercise in advance as one would be giving up the death benefit, which is always positive. Therefore, to obtain a “no-surrender” version of the product, it is sufficient to set  $\gamma_m = 1$  for all values of  $m$ . In such a case, we obtain

$$V(0, F_0, r_0) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_0^{\min(\delta, M)} r_t dt} D B_{\min(\delta, M)}(F_{\min(\delta, M)}) \right]. \quad (3.2)$$

## 4 The pricing algorithms

We describe the two algorithms used to evaluate the contract discussed in the previous Section, under the assumption that the underlying fund  $F$  follows a Lévy model between two anniversaries and the instantaneous interest rate  $r$  follows the Hull-White model.

### 4.1 The Hybrid method

The Hybrid method for pricing American option under a Lévy process with stochastic interest rate is introduced by Briani et al. [4], where weak convergence and stability of the algorithm are also demonstrated. Specifically, the method is proposed for the Bates-Hull-White model. It is important to note that the Bates model assumes that the underlying evolves as a Merton Jump Diffusion process with stochastic volatility modelled by a Heston process, and the Hybrid method is stitched around these assumptions. However, discarding the stochastic volatility assumption, it can easily be adapted to consider any Lévy process for the underlying.

In the original manuscript, the Hybrid method is based on the development of a hybrid tree/finite difference approach and it employs a binomial trees for discretize the stochastic volatility process and the

stochastic interest rate, combined with a space-continuous approximation for the asset price process. Since we do not consider stochastic volatility here, we only use a tree for the interest rate.

Let us define  $\{\hat{p}_m^\omega : m \geq 1\}$  as the conditional death probabilities, that is  $\hat{p}_m^\omega$  is the probability death occurring within the time interval  $(\omega + (m - 1), \omega + m]$  if it has not occurred yet:

$$\hat{p}_m^\omega = \mathbb{P}(\omega + (m - 1) < \delta \leq \omega + m \mid \delta \geq \omega + (m - 1)).$$

These probabilities can be easily calculated using the following recursive relation:

$$\hat{p}_m^\omega = \frac{p_m^\omega}{1 - p_{m-1}^\omega}.$$

First of all, we assume that fees are paid continuously, so that their effect can be incorporated into the dynamics of the process  $F$ . We observe that, by following risk-neutral valuation principles, during the time between two anniversaries, the contract value  $\mathcal{V}$  evolves as the price of a European option in the Lévy-Hull-White model. Thus, following Briani et al. [4],  $\mathcal{V}$  can be computed as the solution of the following problem:

$$\begin{cases} \frac{\partial v}{\partial t}(t, Y, r) + Lv + (r - q + \log(1 - \alpha_m)) \frac{\partial v}{\partial y} + \frac{\sigma_{HW}^2}{2} r^2 \frac{\partial^2 v}{\partial r^2} + k_{HW}(\theta_t - r) \frac{\partial v}{\partial r} - rv = 0 \\ v(m + 1, Y, r) = \mathcal{V}(m + 1, e^Y, r), \\ \mathcal{V}(m, F_m, r_m) = \hat{p}_m^\omega \cdot DB_m(F_m) + (1 - \hat{p}_m^\omega) \max(SB_m(F_m), v(m, \ln(F_m), r_m)), \end{cases} \quad (4.1)$$

with  $Lu$  defined as in (2.7).

The first PIDE in (4.1) can be solved by means of the Hybrid algorithm. This algorithm is based on the alternating spread of the underlying and the interest rate. The former is handled by solving a one-dimensional PIDE in which the interest rate is frozen, while the latter is handled by using a bivariate tree.

The construction of the interest rate tree proceeds by discretizing the time interval between two anniversaries into sub-intervals of fixed length  $\Delta t$ , so that  $N_T = 1/\Delta t$  is the number of time-steps per year. Here,  $\Delta t$  is assumed to exactly divide the time unity, so one can define a unique tree for all the duration of the contract. In addition, by doing so, anniversary dates are included in the tree structure. For each time step, the tree branches into multiple possible future states, reflecting the possible movements of the interest rate. The rate process  $r$  at each node of the tree is updated based on the discretized form of the Hull-White model's stochastic differential equation, ensuring that the tree is recombining, which means that it converges back to fewer states over time to maintain computational efficiency. Specifically, the Hybrid algorithm exploits the “multiple-jumps” tree introduced in Nelson and Ramaswamy [19]: for  $n = 0, 1, \dots, M \cdot N_T$ , consider the lattice for the process  $R$  defined by

$$\mathcal{R}_n = \{R_j^n\}_{j=0,1,\dots,n} \quad \text{with } R_j^n = (2j - n)\sqrt{\Delta t}.$$

For each fixed  $R_j^n \in \mathcal{R}_n$ , we denote the “up” and “down” jump by  $R_{j_u(n,j)}^{n+1}$  and by  $R_{j_d(n,j)}^{n+1}$ , the indices of which, the jump-indexes  $j_u(n, j), j_d(n, j)$ , are defined as

$$\begin{aligned} j_u(n, j) &= \min \{j^* : j + 1 \leq j^* \leq n + 1 \text{ and } R_j^n + \mu_R(R_j^n) \Delta t \leq R_{j^*}^{n+1}\}, \\ j_d(n, j) &= \max \{j^* : 0 \leq j^* \leq j \text{ and } R_j^n + \mu_R(R_j^n) \Delta t \geq R_{j^*}^{n+1}\}, \end{aligned}$$

where  $\mu_R(R_j^n) = -k_{HW}R_j^n$  is the drift of  $R$ , with the understanding  $j_u(n, k) = n + 1$  and respectively,  $j_d(n, k) = 0$ .

To ensure the tree accurately reflects the no-arbitrage condition and aligns with the observed initial term structure of interest rates, the transition probabilities between nodes are carefully calibrated. This involves

adjusting the probabilities such that the expected value of the tree's future rates matches the market's forward rates, a process that requires iterative adjustments to the model parameters. Starting from node  $(n, j)$ , the probability that the process jumps to  $j_u(n, j)$  and  $j_d(n, j)$  at time-step  $n + 1$  are

$$p_u^R(n, j) = 0 \vee \frac{\mu_R (R_j^n) \Delta t + R_j^n - R_{j_d(n, j)}^{n+1}}{R_{j_u(n, j)}^{n+1} - R_{j_d(n, j)}^{n+1}} \wedge 1 \quad \text{and} \quad p_d^R(n, j) = 1 - p_u^R(n, j).$$

This interest rate tree, once constructed, integrates into the larger framework of the Lévy models for option pricing.

The diffusion of the underlying fund is handled by solving a local PIDE written on the log price  $Y = \log(F)$  which holds between two time steps and it is obtained by assuming a constant interest rate. In detail, consider a node  $R_j^n$  and set  $r_j^n = \sigma_{HW} R_j^n + \beta(n\Delta t)$ . Then, we solve

$$\begin{aligned} \frac{\partial v}{\partial t}(t, Y, r_j^n) + \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial Y^2} + \left( r_j^n - q + \log(1 - \alpha_m) - \frac{\sigma^2}{2} \right) \frac{\partial v}{\partial Y} \\ + \int_{\mathbb{R}} \left[ v(t, Y + x, r_j^n) - v(t, Y, r_j^n) - (e^x - 1) \frac{\partial v}{\partial Y} \right] \nu(dx) - r_j^n v = 0. \end{aligned} \quad (4.2)$$

From an implementation point of view, we proceed by creating the tree structure  $\mathcal{R}$  and define a uniform grid  $\mathcal{Y} = \{y_i, i = 1, \dots, N_Y\}$  representing possible values for the process  $Y$ . In particular, the grid  $\mathcal{Y}$  is defined by following the localization step described in [5] and we term  $dy$  the space step of such a grid. Let  $\tilde{v}_{j,i}^n$  represent the numerical approximation of the function  $v$  at the node  $(R_j^n, y_i)$  at time  $n\Delta t$ .

The initial price is computed by moving backward in time, alternating the interest rate diffusion and the solution of the PIDE (4.2). In particular, we start at the last time-step  $M \cdot N_T$  by assigning to each element  $(R_j^{MN_T}, y_i)$  of the product  $\{R_j^{MN_T}, j = 0, \dots, N_T\} \times \mathcal{Y}$  the value of the final payoff, defined as equal to the death benefit  $DB_M(e^{y_i})$ , that is

$$\tilde{v}_{j,i}^{MN_T} = DB_M(e^{y_i}).$$

Then, we proceed to treat the nodes at second-to-last time-step, that is  $n = MN_T - 1$ . For each node  $(R_j^n, y_i)$ , the interest rate is diffused by calculating the weighted average of the values at future nodes  $(R_{j_u(n, j)}^{n+1}, y_i)$  and  $(R_{j_d(n, j)}^{n+1}, y_i)$ , weighted according to the transition probabilities  $p_u^R(n, j)$  and  $p_d^R(n, j)$ . Then, for each node  $R_j^n$  of the tree at time-step  $n$  the contract values at points  $R_j^n \times \mathcal{Y}$  are updated by solving the PIDE (4.2). This is done by applying a single step of the IMEX scheme proposed by Cont and Voltchkova [5]. Specifically, such a scheme divides the differential equation into stiff and non-stiff components: the stiff parts are treated implicitly, while the non-stiff parts are handled explicitly. Following the approach of Cont and Voltchkova, the boundary conditions chosen for the algorithm are Neumann conditions. In detail, the value of the contract for extreme values of  $\mathcal{Y}$  are set equal to the value of the payoff at maturity at those nodes, discounted to the risk-free rate by spreading the interest rate through the binomial  $\mathcal{R}$ -tree:

$$\begin{aligned} \tilde{v}_{j,1}^n &= e^{-r_j^n \Delta t} \left( p_u^R(n, j) \tilde{v}_{j_u(n, j), 1}^n + p_u^R(d, j) \tilde{v}_{j_d(n, j), 1}^n \right), \\ \tilde{v}_{j, N_Y}^n &= e^{-r_j^n \Delta t} \left( p_u^R(n, j) \tilde{v}_{j_u(n, j), N_Y}^n + p_u^R(d, j) \tilde{v}_{j_d(n, j), N_Y}^n \right). \end{aligned}$$

It is worth noting that the value of the contract outside  $\mathcal{Y}$  is assumed to be constant and equal the closer boundary value, namely  $\tilde{v}_{j,1}^n$  and  $\tilde{v}_{j, N_Y}^n$ . This assumption is necessary to handle the non-local nature of the Lévy operator  $L$ .

From the resolution of the PIDE, we obtain the value of the function  $\tilde{v}$  at time-step  $MN_T - 1$ . By iterating the procedure just described, from  $n = MN_T - 1, \dots, (M - 1)N_T$  it is possible to obtain the value of the function  $\tilde{v}^{(M-1)N_T}$  at all points in  $\left\{R_j^{(M-1)N_T}, j = 0, \dots, (M - 1)N_T\right\} \times \mathcal{Y}$ . Then,  $\tilde{\mathcal{V}}_{j,i}^{(M-1)N_T}$ , the numerical approximation of the contract value  $\mathcal{V}$  at the node  $(R_j^n, y_i)$  at time  $M - 1$ , is computed as:

$$\tilde{\mathcal{V}}_{j,i}^{(M-1)N_T} = \hat{p}_{M-1}^\omega \cdot DB_{M-1}(e^{y_i}) + (1 - \hat{p}_{M-1}^\omega) \max\left(SB_{M-2}(e^{y_i}), \tilde{v}_{j,i}^{(M-1)N_T}\right). \quad (4.3)$$

We stress out that, in the previous Formula (4.3), the death benefit multiplies  $\hat{p}_{M-1}^\omega$ , which is the conditional probability for  $\delta$  to be in the time interval  $\mathcal{I}_{M-1}$ , because at time  $M - 1$  the death benefit is paid out to the heirs of those policyholders that died in  $\mathcal{I}_{M-1}$ . The procedure previously described is repeated iteratively backward in time, considering as starting time a generic starting anniversary  $m$  instead of maturity  $M$ , and replacing the terminal values computed through the payoff function with the values  $\left\{\tilde{\mathcal{V}}_{j,i}^{mN_T}, j = 0, \dots, mN_T \text{ and } i = 1, \dots, N_Y\right\}$  computed at previous step. This routine, should be repeated for up to  $m = 1$ . To finally obtain the numerical estimate of the price at epoch zero, that is for  $m = 0$ , one has to proceed similarly to what has been done so far, but avoid considering any death benefit paid at, and the possibility of a total lapse at contract initiation. Specifically, equation (4.3) is replaced by setting  $\tilde{\mathcal{V}}_{0,i}^0 = \tilde{v}_{0,i}^0$  for all value  $i = 1, \dots, N_Y$ . The initial contract value is finally computed by interpolating the observed values  $\left\{\tilde{\mathcal{V}}_{0,i}^0, i = 1, \dots, N_Y\right\}$  on the grid  $\mathcal{Y}$  at the value  $\log(F_0)$ .

## 4.2 The Longstaff-Schwartz Monte Carlo method

The valuation of the contract by means of a Monte Carlo methodology is relatively simple if one excludes the possibility of early exercise, i.e. one follows the formula (3.2). The crucial element is to have a set of simulations for the underlying process and the rate process. In the literature, there are several established methods for simulating Lévy processes (see e.g. Kienitz and Wetterau [15] or Cools and Nuyens [6]), while for the simulation of the interest rate under the Hull-White model, we refer to the exact method by Ostrovski [20].

The procedure involves simulating the processes  $F$ ,  $r$  and  $I = \{I_t\}_{t \geq 0}$  with  $I_t = \int_0^t r_s ds$ . Specifically, we consider  $N_{MC}$  random paths  $\{(F_m^k, r_m^k, I_m^k), m = 0, \dots, M \text{ and } k = 1, \dots, N_{MC}\}$  and we approximate the initial  $\mathcal{V}(0, F_0, r_0)$  contract price with  $\hat{\mathcal{V}}_0(F_0, r_0)$  defined as

$$\hat{\mathcal{V}}_0(F_0, r_0) = \frac{1}{N_{MC}} \sum_{k=1}^{N_{MC}} \left[ \sum_{m=1}^{M-1} p_m^\omega e^{-I_m^k} DB_m(F_m^k) + \left(1 - \sum_{m=1}^{M-1} p_m^\omega\right) e^{-I_M^k} DB_M(F_M^k) \right].$$

When, on the other hand, the possibility of early surrender is considered, an optimal control strategy must be derived. Assigned  $\tau_m$  a stopping time in  $\{m, m + 1, \dots, M\}$ , we define the continuation value (for an alive policyholder) at the  $m$ -th anniversary as the expected value of discounted cashflows obtained through the application of the stopping time  $\tau_m$ , that is the following function

$$\mathcal{C}_m(a, b) = \mathbb{E}^{\mathbb{Q}} \left[ \sum_{h=m+1}^{\tau-1} p_{h-m}^{\omega+m} e^{-\int_m^h r_s ds} DB_h(F_h) + \left(1 - \sum_{h=m+1}^{\tau_m-1} p_{h-m}^{\omega+m}\right) e^{-\int_m^{\tau_m} r_t dt} SB_{\tau_m}(F_{\tau_m}) \mid F_m = a, r = b \right].$$

It is well known that the optimal strategy is obtained by exercising surrender as soon as the exercise value (i.e. the surrender benefit) exceeds the continuation value (the expected discounted value of the future payoff):

$$\tau_m = \min \{h \in \mathbb{N} \text{ s.t. } h > m \text{ and } SB_h(F_h) \geq \mathcal{C}_h(F_h, r_h)\} \cup \{M\}. \quad (4.4)$$

The determination of the continuation value is the crucial aspect. To estimate it, we apply the well-known Longstaff-Schwartz approach [18], which estimates the continuation value backward in time by means of a polynomial regression of discounted future cashflows, calculated by using the optimal strategy at future anniversaries. Specifically, we aim to compute the optimal polynomial in the least squares sense. This polynomial is intended to regress the set of points

$$\left\{ y_k = \sum_{h=m+1}^{\tau-1} p_{h-m}^{\omega+m} e^{-(I_h^k - I_m^k)} DB_h(F_h^k) + \left( 1 - \sum_{h=m+1}^{\tau_m-1} p_{h-m}^{\omega+m} \right) e^{-(I_{\tau_m}^k - I_m^k)} SB_{\tau_m}(F_{\tau_m}^k), k = 1, \dots, N_{MC} \right\}$$

i.e. the simulated discounted future cash-flows, against the set of points

$$\{(F_m^k, r_m^k), k = 1, \dots, N_{MC}\}.$$

The computation of the least squares bivariate polynomial is straightforward and computational efficient (see e.g. Dyn [10]). Anyway, a direct application of the least squares regression proves to be ineffective in this context since the continuation value of the variable annuity considered here has an S-shape when the price of the underlying fund  $F$  changes, whereas it has a more linear appearance when the interest rate changes. It is therefore difficult to find a polynomial that well approximates such a shape. So, the optimal stop strategy that is generated from such data is underperforming and the price that is obtained is significantly lower than the price associated with an optimal strategy.

Instead of addressing the problem of regression as a global problem we propose to use a local regression approach: specifically, the domain is partitioned into several sectors according to the value of the underlying fund  $F$  and a different polynomial in each sector is estimated by using the least squares procedure. A similar technique is used in other contexts by Hainaut and Akbaraly [12]. The domain is divided solely according to the value of the underlying fund considering the following threshold values, based on the minimum and maximum growth rates:

$$b_1/8, b_1/4, b_1/2, b_1, \frac{b_1 + b_2}{2}, b_2, 2b_2, 4b_2, 8b_2, \quad \text{with } b_1 = F_0 e^{g^m} \text{ and } b_2 = F_0 e^{c^m}.$$

Furthermore, if a sector contains more than 20% of the total number of points, then we divide it, determining the threshold value so that half of the points are in the first sub-sector and half in the second. The division of the domain is quite arbitrary, but effective. It responds to the intention of capturing the various moneyness situations (deep out of money, out of money, at the money, in the money, deep in the money) with respect to both bounds, as well as avoiding the excessive concentration of too many points in the same sector. In each sector the regression is carried out using a complete two-variable polynomial. Following a common approach in the Machine Learning field, the degree of the polynomial exploited for regression in each sector is determined according a 80% – 20% criterion. This means that the sample data is randomly divided in two sets: 80% of the data is used for regression, 20% for out-of-sample testing. Starting from a polynomial of degree zero, the degree is increased as long as the mean squared error on the sample group continues to decrease. In this way, the correct polynomial degree is selected with respect to the predictive capacity of the polynomial and overfitting is avoided. Once the best polynomial degree is determined, all the 100% of the data-set is used for the regression.

Once an effective strategy for estimating the continuation value is available, by proceeding backward in time from  $m = M$  to  $m = 1$ , one can approximate the optimal stop strategy  $\tau_0$  as in (4.4) and consequently the initial contract price as

$$\hat{V}_0(F_0, r_0) = \frac{1}{N_{MC}} \sum_{k=1}^{N_{MC}} \left[ \sum_{h=1}^{\tau-1} p_h^\omega e^{-(I_h^k - I_0^k)} DB_h(F_h^k) + \left( 1 - \sum_{h=1}^{\tau_0-1} p_h^\omega \right) e^{-(I_{\tau_0}^k - I_0^k)} SB_{\tau_0}(F_{\tau_0}^k) \right].$$

## 5 Numerical Results

In this Section, we focus on the calculation of the surrender premium, that is the difference in price between the contract under optimal surrender, and the contract that does not allow early surrender. The surrender premium plays a significant role in the pricing and design of ELVA contracts: by assessing the incentive for policyholders to surrender their policies early, insurers can adjust premiums, fees, and benefits to make the contracts more attractive while maintaining profitability. Understanding the surrender premium is essential for designing policies that align with policyholders’ interests. It can be used to structure contracts that disincentivize early surrender, thus encouraging long-term investment, which is beneficial for both the insurer and the policyholder. Regulators require insurers to maintain adequate reserves for their obligations. Accurately calculating the surrender premium is part of ensuring that these reserves are sufficient to cover potential surrenders, thereby complying with regulatory requirements. In the competitive market of insurance products, offering ELVAs with favorable surrender terms can be a differentiator. Accurate calculation of the surrender premium allows insurers to design products that are competitive and appealing to potential customers.

First of all, we calculate the surrender premium, that is the difference between the “surrender” price (optimal surrender available, for  $\gamma_m = 0.02$  for all values of  $m$ ) and the “no-surrender” price of the contract (surrender is not allowed, or is never convenient, that is  $\gamma_m = 1$  for all values of  $m$ ). Then, we focus on a comparative statistics analysis to understand how the surrender premium varies as individual market parameters vary.

In the first analysis, we focus on four Lévy models, while in the second analysis, for the sake of simplicity, we consider only the NIG Lévy model. The parameters of these stochastic models are the same as those employed by Kirkby in other analyses of Levy’s models<sup>1</sup>, while the parameters of the Hull-White model were obtained through a standard calibration procedure<sup>2</sup> from the prices of caplets on Euro (Data downloaded on 02-Feb-2024) and are thus plausible values for real applications, with the exception of  $r_0$  that was chosen equal to 2% as in Kirkby and Aguilar [16]. As for the zero-coupon bond price curve, we choose to use a flat curve in order to facilitate replication of our results.

The algorithms have been implemented in MATLAB and computations have been performed on a server which employs a 2.40 GHz Intel Xenon processor (Gold 6148, Skylake) and 64 GB of RAM. The mortality Table<sup>3</sup> employed herein aligns with that used by Kirkby and Aguilar [16].

### 5.1 Computation of the surrender premium

We advance to assess the surrender benefits within four distinct Lévy models alongside the Hull-White model: namely, the Normal Inverse Gaussian (NIG), Variance Gamma (VG), CGMY, and Merton Jump Diffusion (MJD) models. These evaluations are performed by using both the Hybrid method and the Longstaff-Schwartz Monte Carlo (LSMC) technique, across five unique numerical settings tailored for the pricing algorithms. The initial four settings, labeled A through D, are designed to achieve specific computation run times (4s, 15s, 60s, and 240s, respectively) for both methodologies when applied to the NIG model, with parameters set at  $c = 15\%$  and  $g = 1\%$ . The fifth setting is free from time constraints, serving as a benchmark for comparison. This methodical selection of diverse configurations allows for an in-depth evaluation of the numerical efficiency of each algorithm under review. Table 1 outlines the parameters for the numerical algorithms. It is noteworthy that for configurations A through D, identical simulations were

<sup>1</sup>For further details, see Kirkby’s repository at [https://github.com/jkirkby3/PROJ\\_Option\\_Pricing\\_Matlab](https://github.com/jkirkby3/PROJ_Option_Pricing_Matlab).

<sup>2</sup>See <https://it.mathworks.com/help/fininst/calibrating-hull-white-model-using-market-data.html> for further details.

<sup>3</sup>The mortality Table can be accessed at [https://github.com/jkirkby3/PROJ\\_Option\\_Pricing\\_Matlab](https://github.com/jkirkby3/PROJ_Option_Pricing_Matlab).

Configuration	Target time	LSMC	Hybrid
A	4s	$4.3 \cdot 10^4$	0.015, 7
B	15s	$2.5 \cdot 10^5$	0.010, 10
C	60s	$6.7 \cdot 10^5$	0.008, 15
D	240s	$2.0 \cdot 10^6$	0.005, 22
Benchmarks		$4.0 \cdot 10^7$	0.001, 100

Table 1: parameter for the numerical algorithms. For each configuration, the table reports: the target computational time for the computation of the surrender premium; the number of simulations for the Monte Carlo algorithm; the space increment  $dy$  and the number of time steps per period  $N_T$ .

utilized to ascertain both the exercise strategy and the pricing. For the benchmark, however,  $4.0 \times 10^7$  trajectories were simulated to identify the optimal exercise strategy, and  $1.0 \times 10^7$  out-of-sample trajectories were used for price calculation to mitigate potential benchmark distortion from, even minimal, overfitting. Given the extensive simulation times for the CGMY model, the benchmark simulation count is reduced by a factor of 4; despite this adjustment, the benchmark computation still spans several days.

Tables 2, 3, 4 and 5 report the model parameters, as well as the estimated values of the surrender premium and, for the Longstaff-Schwartz Monte Carlo method, also the 99% confidence intervals. The data shows that the Hybrid benchmark always falls within the confidence interval of the Longstaff-Schwartz Monte Carlo benchmark, so the two benchmarks agree.

The Hybrid method outperforms the Monte Carlo method: in general, in the 4 configurations considered, the values returned by the hybrid method are closer to the benchmark and monotonic convergence is highlighted, unlike the Monte Carlo method. Furthermore, we observe that, if we consider the Hybrid method, already configuration B provides accurate results, so we use it for further testing.

## 5.2 Comparative statics analysis of interest rate parameters on the surrender premium

In this Section, we focus on the NIG model specifically within the framework of numerical configuration B, as previously analyzed. In particular, when not otherwise indicated, we employ the following parameters:  $\alpha_{NIG} = 6$ ,  $\beta_{NIG} = -0.4$ ,  $\delta_{NIG} = 2$ ,  $r_0 = 0.02$ ,  $k_{HW} = 0.2$ ,  $\sigma_{HW} = 0.03$ ,  $q = 0.01$ ,  $M = 25$ ,  $\omega = 30$ ,  $\alpha_m = 0.02$ ,  $\gamma_m = 0.02$  and  $c = 15\%$ . Here, we undertake a graphical exploration to discern how the surrender premium is influenced by varying some of the most interesting market parameters.

In Figure 5.1, we investigate how the surrender benefit varies for different values of  $c$  as we change the value of the parameters  $\sigma_{HW}$  and  $k_{HW}$ . From these plots, we can observe that the surrender benefit has a non-monotonic concave trend: in all the cases analyzed, a maximum value is observed as the value of  $c$  is around 15% – 20%. Moreover, the surrender benefit increases as  $\sigma_{HW}$  increases and decreases as  $k_{HW}$  increases: the greater the volatility of the rate, the greater the incentive to surrender. This phenomenon can be observed in practice: policyholders tend to surrender when interest rates are high because the guarantees offered by the insurance product are less useful in protecting the principal, and these high values are more likely to happen for large values of  $\sigma_{HW}$  and  $k_{HW}$ . Therefore, insurers should be fearful of situations in which interest rates change abruptly and consider introducing some forms of premiums that incentivize customers not to withdraw from the contract when it appears cheaper.

In Figure 5.2, we observe the surrender benefit as a function of the growth floor,  $g$ , for different settings

	$g = 1\%$			$g = 3\%$		
	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)
$c = 5\%$						
A	$0.1521 \pm 0.0048$	0.1526		$0.0449 \pm 0.0028$	0.0458	
B	$0.1527 \pm 0.0020$	0.1523	0.1520	$0.0452 \pm 0.0012$	0.0457	0.0454
C	$0.1523 \pm 0.0012$	0.1523	$0.1520 \pm 0.0003$	$0.0451 \pm 0.0007$	0.0456	$0.0454 \pm 0.0002$
D	$0.1519 \pm 0.0007$	0.1522		$0.0456 \pm 0.0004$	0.0455	
$c = 15\%$						
A	$0.1961 \pm 0.0170$	0.1888		$0.1390 \pm 0.0153$	0.1305	
B	$0.1893 \pm 0.0075$	0.1888	0.1887	$0.1277 \pm 0.0069$	0.1304	0.1302
C	$0.1895 \pm 0.0046$	0.1888	$0.1889 \pm 0.0012$	$0.1304 \pm 0.0043$	0.1304	$0.1301 \pm 0.0011$
D	$0.1887 \pm 0.0027$	0.1888		$0.1307 \pm 0.0025$	0.1303	
$c = 30\%$						
A	$0.2007 \pm 0.0986$	0.1867		$0.1425 \pm 0.0965$	0.1497	
B	$0.1911 \pm 0.0337$	0.1870	0.1874	$0.1539 \pm 0.0342$	0.1499	0.1502
C	$0.1875 \pm 0.0227$	0.1871	$0.1858 \pm 0.0058$	$0.1509 \pm 0.0227$	0.1500	$0.1472 \pm 0.0059$
D	$0.1866 \pm 0.0128$	0.1873		$0.1485 \pm 0.0126$	0.1501	

Table 2: surrender premium computed for the NIG-HW model with  $\alpha_{NIG} = 6$ ,  $\beta_{NIG} = -0.4$ ,  $\delta_{NIG} = 2$ ,  $r_0 = 0.02$ ,  $k_{HW} = 0.2$ ,  $\sigma_{HW} = 0.03$ ,  $q = 0.01$ ,  $M = 25$ ,  $\omega = 30$ ,  $\alpha_m = 0.02$ ,  $\gamma_m = 0.02$ ,  $g = 1\%, 3\%$ ,  $c = 5\%, 15\%, 30\%$ .

	$g = 1\%$			$g = 3\%$		
	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)
$c = 5\%$						
A	$0.1299 \pm 0.0043$	0.1327		$0.0425 \pm 0.0025$	0.0445	
B	$0.1331 \pm 0.0018$	0.1327	0.1325	$0.0435 \pm 0.0010$	0.0444	0.0441
C	$0.1330 \pm 0.0011$	0.1326	$0.1325 \pm 0.0003$	$0.0441 \pm 0.0006$	0.0443	$0.0441 \pm 0.0002$
D	$0.1325 \pm 0.0006$	0.1326		$0.0440 \pm 0.0004$	0.0442	
$c = 15\%$						
A	$0.1265 \pm 0.0065$	0.1303		$0.0553 \pm 0.0052$	0.0591	
B	$0.1313 \pm 0.0027$	0.1304	0.1307	$0.0588 \pm 0.0022$	0.0591	0.0592
C	$0.1316 \pm 0.0016$	0.1305	$0.1309 \pm 0.0004$	$0.0588 \pm 0.0013$	0.0592	$0.0591 \pm 0.0003$
D	$0.1305 \pm 0.0010$	0.1306		$0.0592 \pm 0.0008$	0.0592	
$c = 30\%$						
A	$0.1357 \pm 0.0069$	0.1383		$0.0596 \pm 0.0056$	0.0649	
B	$0.1392 \pm 0.0029$	0.1385	0.1389	$0.0634 \pm 0.0023$	0.0650	0.0652
C	$0.1399 \pm 0.0017$	0.1386	$0.1387 \pm 0.0005$	$0.0644 \pm 0.0014$	0.0651	$0.0651 \pm 0.0004$
D	$0.1386 \pm 0.0010$	0.1387		$0.0641 \pm 0.0008$	0.0651	

Table 3: surrender premium computed for the VG-HW model with  $\kappa_{VG} = 0.85$   $\theta_{VG} = 0$   $\sigma_{VG} = 0.2$ ,  $r_{0,HW} = 0.02$ ,  $k_{HW} = 0.2$ ,  $\sigma_{HW} = 0.03$ ,  $\Delta = 1$ ,  $M = 25$ ,  $\omega = 30$ ,  $\alpha_m = 0.02$ ,  $\gamma_m = 0.02$ ,  $g = 1\%, 3\%$ ,  $c = 5\%, 15\%, 30\%$ .

	$g = 1\%$			$g = 3\%$		
	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)
$c = 5\%$						
A	$0.1421 \pm 0.0039$	0.1410		$0.0351 \pm 0.0021$	0.0358	
B	$0.1410 \pm 0.0017$	0.1411	0.1413	$0.0355 \pm 0.0009$	0.0358	0.0356
C	$0.1415 \pm 0.0010$	0.1411	$0.1413 \pm 0.0004$	$0.0355 \pm 0.0005$	0.0357	$0.0356 \pm 0.0002$
D	$0.1411 \pm 0.0006$	0.1411		$0.0356 \pm 0.0003$	0.0357	
$c = 15\%$						
A	$0.1432 \pm 0.0040$	0.1416		$0.0360 \pm 0.0022$	0.0369	
B	$0.1422 \pm 0.0017$	0.1418	0.1422	$0.0368 \pm 0.0009$	0.0368	0.0366
C	$0.1425 \pm 0.0010$	0.1420	$0.1424 \pm 0.0004$	$0.0365 \pm 0.0005$	0.0368	$0.0365 \pm 0.0002$
D	$0.1422 \pm 0.0006$	0.1421		$0.0365 \pm 0.0003$	0.0367	
$c = 30\%$						
A	$0.1434 \pm 0.0040$	0.1423		$0.0354 \pm 0.0022$	0.0370	
B	$0.1429 \pm 0.0017$	0.1425	0.1428	$0.0370 \pm 0.0008$	0.0369	0.0367
C	$0.1432 \pm 0.0010$	0.1426	$0.1428 \pm 0.0004$	$0.0365 \pm 0.0005$	0.0368	$0.0367 \pm 0.0002$
D	$0.1427 \pm 0.0006$	0.1427		$0.0366 \pm 0.0003$	0.0368	

Table 4: surrender premium computed for the CGMY-HW model with  $C_{CGMY} = 0.02$ ,  $G_{CGMY} = 5$ ,  $M_{CGMY} = 15$ ,  $Y_{CGMY} = 1.2$ ,  $r_{0,HW} = 0.02$ ,  $k_{HW} = 0.2$ ,  $\sigma_{HW} = 0.03$ ,  $q = 0.01$ ,  $M = 25$ ,  $\omega = 30$ ,  $\alpha_m = 0.02$ ,  $\gamma_m = 0.02$ ,  $g = 1\%, 3\%$ ,  $c = 5\%, 15\%, 30\%$ .

	$g = 1\%$			$g = 3\%$		
	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)	LSMC	Hybrid	Benchmarks (Hybrid, LSMC)
$c = 5\%$						
A	$0.1387 \pm 0.0046$	0.1377		$0.0499 \pm 0.0028$	0.0490	
B	$0.1366 \pm 0.0019$	0.1377	0.1375	$0.0493 \pm 0.0011$	0.0489	0.0488
C	$0.1370 \pm 0.0012$	0.1376	$0.1375 \pm 0.0003$	$0.0487 \pm 0.0007$	0.0489	$0.0487 \pm 0.0002$
D	$0.1371 \pm 0.0007$	0.1376		$0.0484 \pm 0.0004$	0.0488	
$c = 15\%$						
A	$0.1325 \pm 0.0098$	0.1293		$0.0715 \pm 0.0084$	0.0695	
B	$0.1290 \pm 0.0041$	0.1294	0.1298	$0.0699 \pm 0.0036$	0.0696	0.0698
C	$0.1288 \pm 0.0025$	0.1296	$0.1298 \pm 0.0006$	$0.0694 \pm 0.0022$	0.0697	$0.0699 \pm 0.0004$
D	$0.1288 \pm 0.0015$	0.1297		$0.0689 \pm 0.0013$	0.0697	
$c = 30\%$						
A	$0.1480 \pm 0.0108$	0.1428		$0.0802 \pm 0.0097$	0.0809	
B	$0.1428 \pm 0.0048$	0.1431	0.1437	$0.0785 \pm 0.0043$	0.0811	0.0813
C	$0.1431 \pm 0.0029$	0.1433	$0.1438 \pm 0.0008$	$0.0798 \pm 0.0026$	0.0812	$0.0810 \pm 0.0007$
D	$0.1429 \pm 0.0017$	0.1435		$0.0799 \pm 0.0015$	0.0812	

Table 5: surrender premium computed for the MJD-HW model with  $\sigma_{MJD} = 0.25$ ,  $\lambda_{MJD} = 0.6$ ,  $\mu_{MJD}^j = 0.01$ ,  $\sigma_{MJD}^j = 0.13$ ,  $r_{0,HW} = 0.02$ ,  $k_{HW} = 0.2$ ,  $\sigma_{HW} = 0.03$ ,  $q = 0.01$ ,  $M = 25$ ,  $\omega = 30$ ,  $\alpha_m = 0.02$ ,  $\gamma_m = 0.02$ ,  $g = 1\%, 3\%$ ,  $c = 5\%, 15\%, 30\%$ .

of  $\sigma_{HW}$ ,  $k_{HW}$  and  $\alpha_m$ . In all the plots, the surrender premium decreases as  $g$  varies, indicating that the greater the protection on minimum capital growth, the lower the incentive to surrender. The dependence of the surrender premium on the fees applied by the insurer is highlighted in Figure 5.3, where the surrender benefit is plotted against the fee parameter  $\alpha_m$ . In all images, the surrender premium increases as  $\alpha_m$  varies, indicating that the higher the cost of the contract the greater the incentive to surrender.

Furthermore, we also studied the surrender premium as the  $\alpha_{NIG}$  parameter varies, finding results similar to those produced by Kirkby and Aguilar [16] in the non-stochastic rate Lévy model.

Finally, in Figure 5.4, we plot the optimal early exercise region, i.e. the values of  $F$  and  $r$  for which it is convenient to surrender. We plot these regions for different value of anniversaries dates, namely  $m = 5, 10, 15$  and  $20$  for both  $g = 1\%$  and  $g = 3\%$ . From the graphs we can observe that in all cases early surrender is convenient when  $F$  takes intermediate values (in-the-money contract), while it is not convenient when it takes extreme values (out-of-the-money contract). Furthermore, the optimal strategy is influenced by the value of the interest rate  $r$ : in general, high values of  $r$  incentivise surrender. Consequently, a rise in rates encourages lapse, creating potential risks for insurers. We can also observe that surrender is less advantageous in the first few years, and more so as time goes by: policyholders are less likely to surrender at the beginning, to benefit from the protection of the contract for a long time, and are more likely to surrender later to stop paying fees. Finally, it is worth noting that the strategy that is most convenient for the policyholder is also the most expensive for the insurer. Therefore, the latter might find this chart useful in determining when an early termination of the policy might be more costly for them, based on current assessments of the fund value and the short-term interest rate.

## 6 Conclusions

In this paper, we have presented a comprehensive framework for the valuation of variable annuities within Lévy models, considering stochastic interest rates. Our approach integrates the robustness of the Hull-White model to account for the dynamical financial environment and allows for the incorporation of realistic market conditions into the valuation process. Through the implementation of advanced numerical methods, namely the Hybrid method and the Longstaff-Schwartz Monte Carlo method, we have demonstrated the impact of stochastic interest rates on the strategic decision-making process related to the optimal surrender of financial instruments.

Our findings highlight the necessity of considering stochastic interest rates for long-term financial products and contribute to the deeper understanding of the risk and return profile of variable annuities. Moreover, we have shown that our enhanced valuation model can significantly influence the structuring of contracts, particularly in ways that disincentivize premature surrender while accommodating realistic market fluctuations.

The numerical results obtained through our experiments confirm the validity of the proposed models and open new avenues for future research. As part of our ongoing work, we aim to explore the broader implications of stochastic rate models in other areas of financial planning and risk assessment.

In summary, the insights gained from our study serve as a critical step towards more sophisticated pricing and risk management strategies for variable annuities, which are essential to ensuring financial security and stability for policyholders in an unpredictable economic landscape.

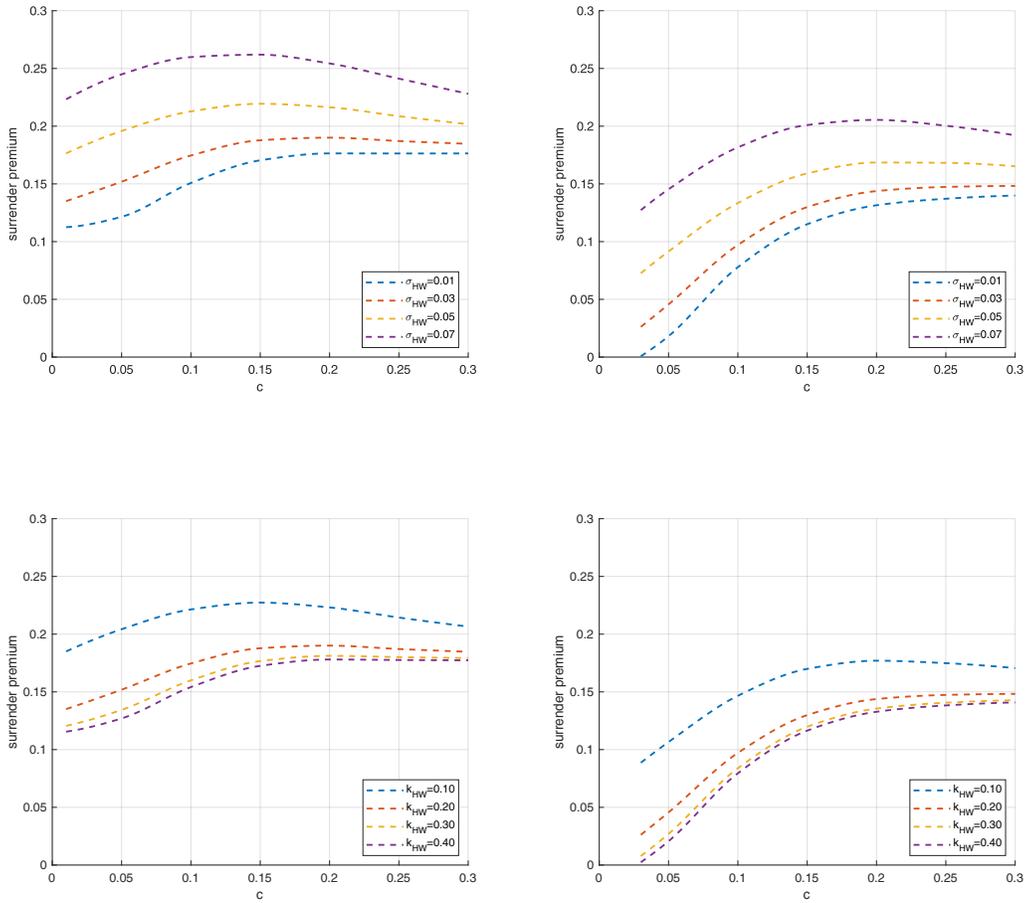


Figure 5.1: surrender premium as a function of  $c$  while changing  $\sigma_{HW}$  (first row) and  $k_{HW}$  (second row) for  $g = 1\%$  (first column) and  $g = 3\%$  (second column).

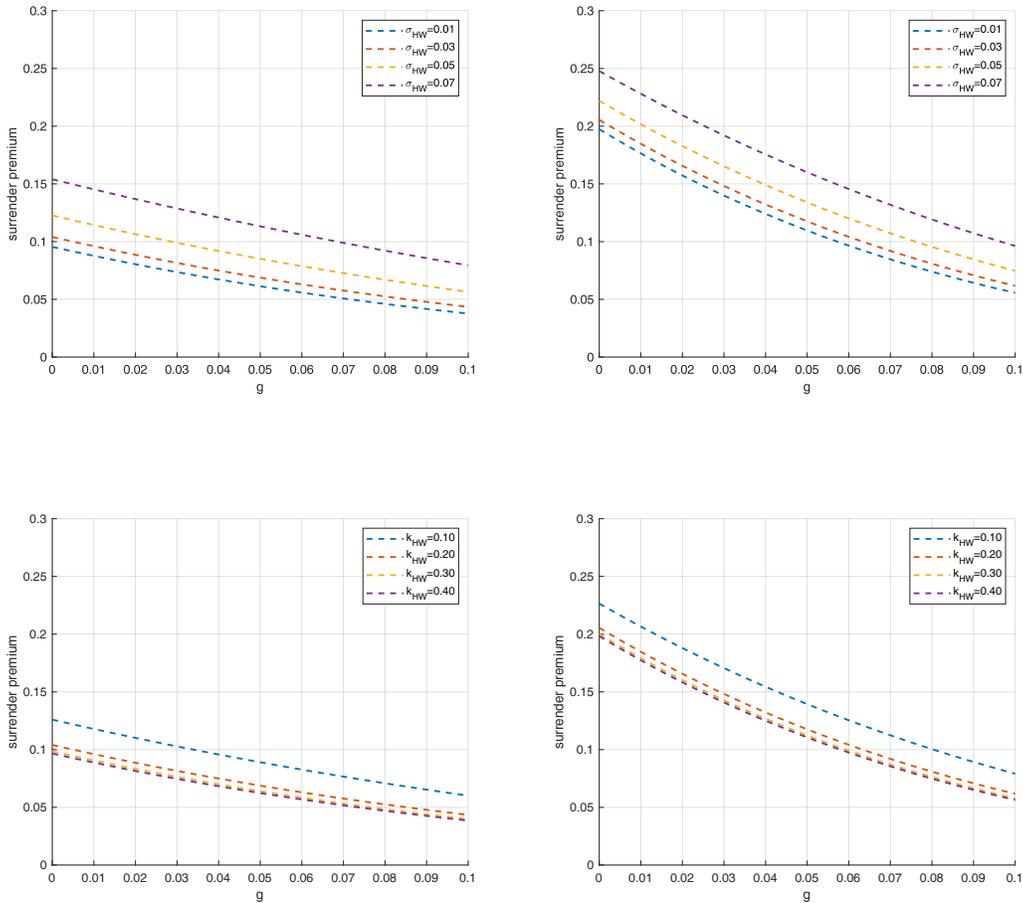


Figure 5.2: surrender premium as a function of  $g$  while changing  $\sigma_{HW}$  (first row) and  $k_{HW}$  (second row) for  $\alpha_m = 0\%$  (first column) and  $\alpha_m = 2\%$  (second column).

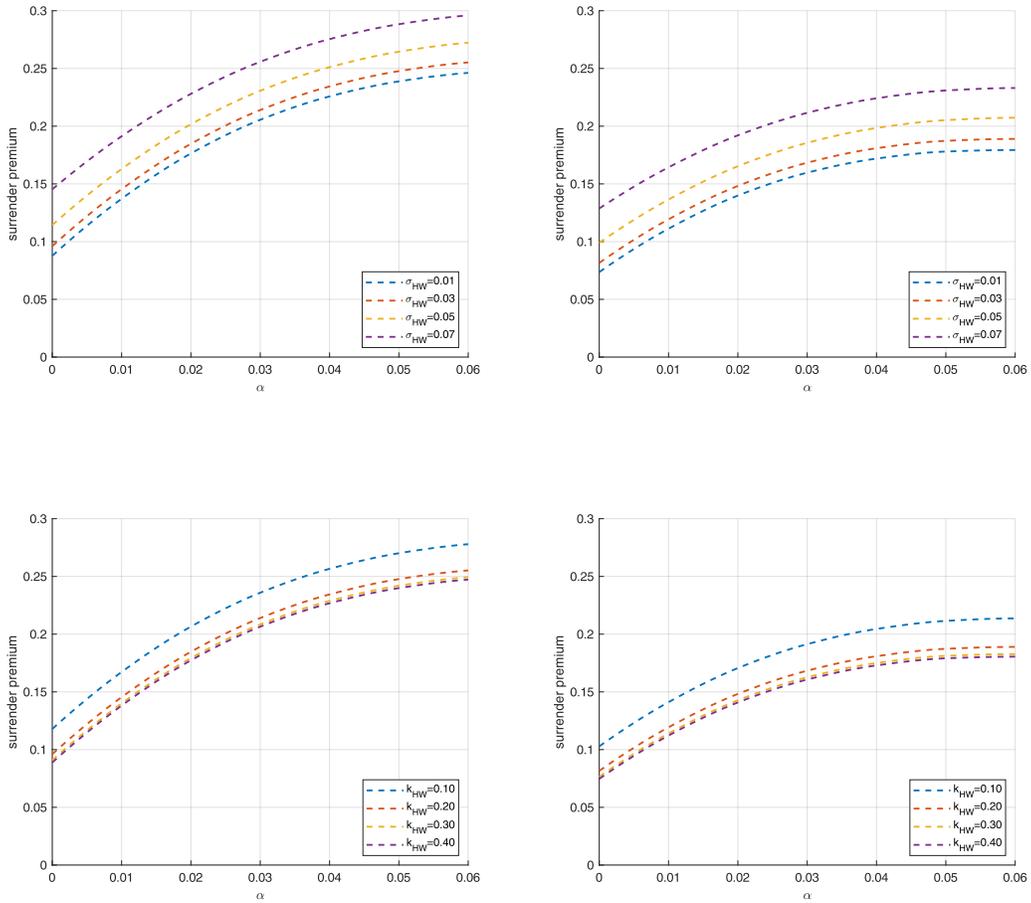
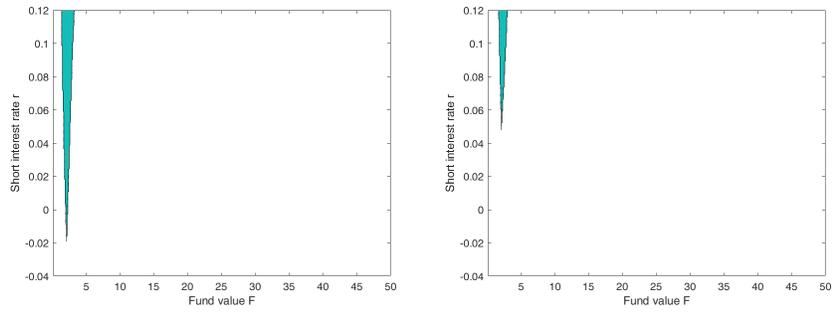
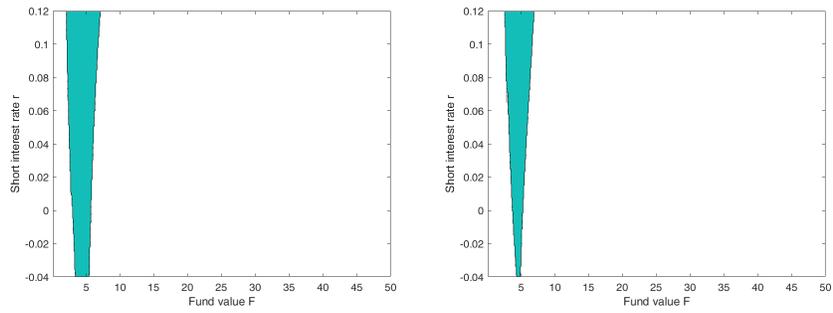


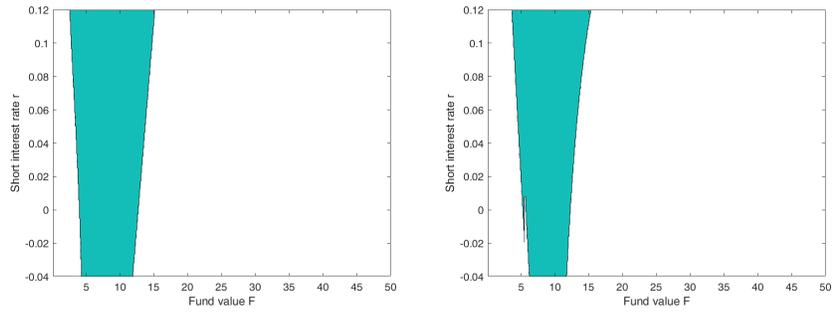
Figure 5.3: surrender premium as a function of  $\alpha_m$  while changing  $\sigma_{HW}$  (first row) and  $k_{HW}$  (second row) for  $g = 1\%$  (first column) and  $g = 3\%$  (second column).



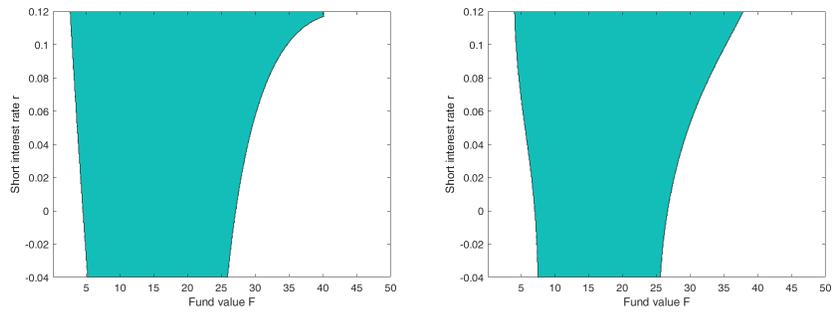
(a)  $m = 5$ .



(b)  $m = 10$ .



(c)  $m = 15$ .



(d)  $m = 20$ .

Figure 5.4: optimal surrender strategy at different anniversaries for  $c = 15\%$  and for  $g = 1\%$  (first column) and  $g = 3\%$  (second column). The green area (darker) indicates where it is optimal to surrender.

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## A Characteristic exponent of Lévy processes

In this Appendix, we fix the notation for the characteristic exponents of some of the principal Lévy processes used throughout this document. The characteristic exponent of a Lévy process, denoted by  $\psi(\xi)$ , plays a crucial role in the analysis of these processes.

### A.1 Normal Inverse Gaussian (NIG)

The Normal Inverse Gaussian process is defined by its characteristic exponent:

$$\psi_{\text{NIG}}(\xi) = \delta_{\text{NIG}} \left( \sqrt{\alpha_{\text{NIG}}^2 - (\beta_{\text{NIG}} + i\xi)^2} - \sqrt{\alpha_{\text{NIG}}^2 - \beta_{\text{NIG}}^2} \right),$$

where  $\alpha_{\text{NIG}} > 0$ ,  $|\beta_{\text{NIG}}| < \alpha_{\text{NIG}}$ , and  $\delta_{\text{NIG}} > 0$  are parameters of the process.

### A.2 Variance Gamma (VG)

The characteristic exponent of the Variance Gamma process is given by:

$$\psi_{\text{VG}}(\xi) = -\frac{1}{\kappa_{\text{VG}}} \log \left( 1 - i\xi\theta_{\text{VG}}\kappa_{\text{VG}} + \frac{1}{2}\kappa_{\text{VG}}\sigma_{\text{VG}}^2\xi^2 \right),$$

where  $\kappa_{\text{VG}}, \sigma_{\text{VG}} > 0$  and  $\theta_{\text{VG}} \in \mathbb{R}$  are the process parameters.

### A.3 CGMY

The CGMY process has the characteristic exponent:

$$\psi_{\text{CGMY}}(\xi) = C_{\text{CGMY}} \cdot \Gamma(-Y) \left[ G_{\text{CGMY}}^{Y_{\text{CGMY}}} - (G_{\text{CGMY}} + i\xi)^{Y_{\text{CGMY}}} + M_{\text{CGMY}}^{Y_{\text{CGMY}}} - (M_{\text{CGMY}} - i\xi)^{Y_{\text{CGMY}}} \right],$$

with  $C_{\text{CGMY}} > 0$ ,  $G_{\text{CGMY}} > 0$ ,  $M_{\text{CGMY}} > 0$ , and  $Y_{\text{CGMY}} < 2$ .

### A.4 Merton Jump Diffusion (MJD)

The characteristic exponent for the Merton Jump Diffusion process is:

$$\psi_{\text{MJD}}(\xi) = \frac{1}{2} (\sigma_{\text{MJD}} \xi)^2 - \lambda_{\text{MJD}} \left( \exp \left( i\mu_{\text{MJD}}^J \xi - \frac{1}{2} (\sigma_{\text{MJD}}^J \xi)^2 \right) - 1 \right),$$

where  $\sigma_{\text{MJD}}$ ,  $\lambda_{\text{MJD}}$ ,  $\mu_{\text{MJD}}^J$ , and  $\sigma_{\text{MJD}}^J$  are the parameters of the MJD process.