# Rules and Algorithms for Objective Construction of Fuzzy Sets

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## Abstract

This paper aims to present objective methods for constructing new fuzzy sets from known fuzzy or classical sets, defined over the elements of a finite universe's superstructure. The paper proposes rules for assigning membership functions to these new fuzzy sets, leading to two important findings. Firstly, the property concerning the cardinality of a power set in classical theory has been extended to the fuzzy setting, whereby the scalar cardinality of a fuzzy set  $\tilde{B}$  defined on the power set of a finite universe of a fuzzy set  $\tilde{A}$  satisfies  $\operatorname{card}(\tilde{B}) = 2^{\operatorname{card}(\tilde{A})}$ . Secondly, the novel algorithms allow for an arbitrary membership value to be objectively achieved and represented by a specific binary sequence.

*Keywords:* Fuzzy sets, Membership functions, Scalar cardinality, Power sets, Binary sequence, ZFC axioms

## 1. Introduction

In classical set theory, an element is either contained in a set or not. Therefore, a discrete indicator function, also known as a characteristic function  $\mathcal{I}_A(x)$ :  $X \to \{0,1\}$ , can be defined to indicate whether an element x belongs to a set A or not:

$$\mathcal{I}_A(x) \coloneqq \begin{cases} 1 & \text{if } x \in A ,\\ 0 & \text{if } x \notin A . \end{cases}$$
(1)

In 1965, Zadeh proposed the concept of a fuzzy set, which extends the classical notion of a set[1]. In fuzzy set theory, the characteristic function is replaced by a membership function, which expresses the degree to which an element belongs to a set in a continuous manner. Specifically, a membership function  $\mu_A(x): U \to [0,1]$  assigns a value between 0 and 1 to an element x in a universe U, indicating the degree to which x belongs to set A.

The membership function is a fundamental concept in fuzzy set theory, as it uniquely characterizes any fuzzy set. Most concepts in fuzzy set theory, such

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as the support of a fuzzy set, alpha-level set, convex fuzzy set, and set-theoretic operations, including the intersection, union, and complement of fuzzy sets, are defined based on membership functions [2]. Thus, it is of great importance to specify the membership function in both the theory and practice of fuzzy sets. Once the membership function is specified, the corresponding fuzzy set is determined. Various approaches have been developed to obtain reasonable membership functions [3–9], including the piecewise linear ones, such as the commonly used triangular and trapezoidal membership functions [10–12]. For applications, appropriate membership functions are usually selected based on the subjective knowledge or perception of the observer [13]. Hence, many researchers believe that the assignment of the membership function for a fuzzy set is subjective [12, 14–17]. This motivates us to develop objective methods to assign membership functions that generate fuzzy sets objectively. Our methods are designed to be context-independent so that they can be applied generally, regardless of the different perceptions of individuals. With our methods, membership functions rely solely on objective computations, enabling fuzzy sets to be used in a broader range of applications without any subjective uncertainty.

Our methods have the advantage of objectivity from a different perspective: they construct new fuzzy sets from existing ones. To achieve this, we can draw from classical set theory and its construction of new sets from old ones. In particular, the ZFC axiomatic system[18] provides a way to construct new sets by defining them in terms of existing ones. ZFC stands for Zermelo–Fraenkel set theory with the axiom of Choice included, and most of its axioms state the existence of specific sets defined in relation to other sets. For instance, the axiom of pairing says that given any two sets X and Y, there is a new set  $\{X,Y\}$  containing exactly X and Y. In the next section, we will explore the ZFC axiomatic system from a fuzzy perspective. In the same way that ZFC constructs new classical sets from existing ones, we will attempt to construct new fuzzy sets from existing ones, and even from classical sets. This approach will enable us to treat some of the axioms of ZFC as special cases of the fuzzy setting, since classical sets are a subset of fuzzy sets with membership functions that are identically equal to 0 or 1.

To achieve the goal outlined above, we will focus on exploring the concept of cardinality. Cardinality is a measure of the size of a set, specifically, it counts the "number of elements" in a set. For classical sets, the cardinality of set A is typically denoted as |A| or card(A). A finite set has a finite number of elements, and its cardinality is a natural number. A set A that has the same cardinality as the set of all natural numbers is referred to as countably infinite, which is denoted as card $(A) = \aleph_0$ . If a set has a cardinality greater than that of a countably infinite set, it is considered uncountable.

The theory of cardinality for fuzzy sets is perhaps one of the most intriguing and mysterious aspects of fuzzy set theory[19]. Various definitions for the cardinality of a fuzzy set have been proposed and studied by many researchers[20–26]. Among these, the concept of scalar cardinality for fuzzy sets was initially introduced by De Luca and Termini to better understand the measurement of information for fuzzy sets[27]. In the 1980s, some researchers suggested that the cardinality of a fuzzy set should be a precise real number, while others believed it should be a fuzzy integer[28–31]. Dubois and Prade attempted to reconcile both definitions by combining them into a single framework[32]. This paper focuses solely on scalar cardinality.

Consider a fuzzy set A, defined on a universe U with a membership function  $\mu_{\tilde{A}}: U \to [0,1]$ . The scalar cardinality of  $\tilde{A}$  is defined as the sum of the membership values of all the elements in U, given by

$$\operatorname{card}(\tilde{A}) = \sum_{x \in U} \mu_{\tilde{A}}(x).$$
<sup>(2)</sup>

Two important concepts still need to be considered: the power set and its representation. The power set of A consists of all subsets of A, including the empty set  $\emptyset$  and A itself, denoted by  $\mathcal{P}(A)$ . Another representation for the power set is  $2^A$  because it includes all possible subsets of A, and the number of subsets can be represented by  $2^{\operatorname{card}(A)}$ .

For instance, if A is a finite set with n elements, then the cardinality of  $\mathcal{P}(A)$  is  $2^n$ , as the number of subsets of A is  $2^n$ .

Another important concept is the superstructure over a set. In mathematics, a universe refers to a collection that includes all the entities relevant to a given situation. In fuzzy set theory, the universe of discourse or universe is used to refer to the reference set. The superstructure over a universe X can be defined through structural recursion, as follows: Let  $S_0(X) = X$ , and  $S_{n+1}(X) =$  $S_n(X) \cup \mathcal{P}(S_n(X))$ . Then the superstructure over X, denoted as S(X), is defined as

$$S(X) \coloneqq \bigcup_{j=0}^{\infty} S_j(X).$$
(3)

The remainder of this paper is structured as follows: In Section 2, we provide the fuzzy interpretations of some ZFC axioms and introduce four rules for constructing new fuzzy sets from existing ones. Building on these rules, we prove an important theorem pertaining to power sets in the fuzzy setting. In Section 3, we introduce two additional rules for constructing new fuzzy sets from classical sets, and we prove a theorem that describes how to represent and generate any membership value using our rules. Section 4 includes numerical examples. Finally, in Section 5, we conclude the paper.

### 2. Constructing Fuzzy Sets from Existing Ones

The ZFC set theory consists of nine axioms. As classical sets can be seen as a special case of fuzzy sets, where the membership function is equal to 0 or 1, certain axioms of ZFC can be reinterpreted using fuzzy set theory. These interpretations serve as criteria for the methods of constructing new fuzzy sets proposed in this paper, since these methods should be compatible with classical cases. The following content discusses these details in depth.

1. The Axiom of Pairing states that for any two sets A and B, there exists a set  $\{A, B\}$  that contains exactly A and B. The fuzzy interpretation of this axiom implies that if we are given two fuzzy sets  $\tilde{A}$  defined on X and  $\tilde{B}$  defined on Y, such that for any  $x \in X$  and  $y \in Y$ , we have  $\mu_{\tilde{A}}(x) = 1$  and  $\mu_{\tilde{B}}(y) = 1$ , then there exists a fuzzy set  $\tilde{C}$  defined on  $\{X, Y\}$  such that

$$\mu_{\tilde{C}}(X) = \mu_{\tilde{C}}(Y) = 1.$$
(4)

This means that the membership values of  $\tilde{C}$  for the sets X and Y are both equal to 1, indicating that  $\tilde{C}$  contains both X and Y.

2. The Axiom of Union states that the union of elements in a set exists. For any set X, there exists a set  $Y = \bigcup X$ . We can interpret this axiom in a more flexible manner. Given a fuzzy set  $\tilde{A}$  defined on X, satisfying  $\mu_{\tilde{A}}(x) = 1$  for any  $x \in X$ , a fuzzy set  $\tilde{B}$  defined on Y exists such that:

$$\mu_{\tilde{B}}(y) = 1 \tag{5}$$

for any  $y \in Y$ . It is clear that  $y \in x$  for some x.

3. The power set axiom expresses the idea that a set exists which contains all the subsets of another set. The corresponding fuzzy interpretation states that if we have a fuzzy set  $\tilde{A}$  defined on X, with  $\mu_{\tilde{A}}(x) = 1$  for every  $x \in X$ , then there exists a fuzzy set  $\tilde{B}$  defined on  $Y = \mathcal{P}(X)$  such that

$$\mu_{\tilde{B}}(y) = 1 \tag{6}$$

for every  $y \in Y$ , and every x belongs to at least one y.

In this section, we will discuss how to construct new fuzzy sets using ZFClike methods. Let  $\tilde{A}$  be a fuzzy set defined on a universe X, and let Y be another universe. We can construct a new fuzzy set, denoted by  $\tilde{B}$ , defined on Y, using the following construction rules:

- **Rule** 1. Set Y must be an element of the superstructure over X, meaning it can be any classical set contained in the superstructure S(X).
- **Rule** 2. The membership value  $\mu_{\tilde{B}}(y)$  equals  $\mu_{\tilde{A}}(x)$  if y = x, but the converse is not always true.
- **Rule** 3. If Y contains the empty set, then the membership function value at the empty set is always equal to 1:  $\mu_{\tilde{B}}(\emptyset) \equiv 1$ .
- **Rule** 4. The membership function of fuzzy set  $\hat{B}$  is defined by

$$\mu_{\tilde{B}}(y) = \prod_{x \in y} \left( 2^{\mu_{\tilde{A}}(x)} - 1 \right), \tag{7}$$

where y is any element contained in Y.

We can interpret the rules described above as follows:

Rule 1 defines the permissive universe of a new fuzzy set constructed from an existing one. For instance, if the fuzzy set  $\tilde{A}$  is defined on the universe  $X = \{x_1, x_2\}$ , we can create another fuzzy set  $\tilde{B}$  from  $\tilde{A}$  on the universe  $Y = \{\emptyset, x_1, \{x_2\}, \{x_1, \{x_1, x_2\}\}\}$ . This is because the element  $x_1$  belongs to  $S_0(X)$ , and the elements  $\emptyset, \{x_2\}, \{x_1, x_2\}$ , and  $\{x_1, \{x_1, x_2\}\}$  belong to  $S_1(X)$ and  $S_2(X)$ , respectively.

Rule 2 indicates that the value of the membership function remains the same for the same element using our method.

Rule 3 is reasonable because the empty set, which is the only set containing no elements, is a subset of every set, and it cannot be divided into parts.

Rule 4 provides the formula for calculating the new membership values. It is evident that Rule 3 is a special case of Rule 4 since it is the case of an empty product. Even if the universe of the new fuzzy set is complex, we can apply formula (7) repeatedly until the information of the original fuzzy set can be directly substituted into this formula.

Next, we demonstrate the compatibility of the aforementioned rules with certain axioms of ZFC, as we mentioned earlier.

The fuzzy interpretation of the axiom of pairing, given by formula (4), can be derived using the rules presented above. Specifically, we have:

$$\mu_{\tilde{C}}(X) = \prod_{x \in X} \left( 2^{\mu_{\tilde{A}}(x)} - 1 \right) = \prod_{x \in X} \left( 2^1 - 1 \right) = 1, \tag{8}$$

and similarly,

$$\mu_{\tilde{C}}(Y) = \prod_{y \in Y} \left( 2^{\mu_{\tilde{B}}(y)} - 1 \right) = \prod_{y \in Y} \left( 2^1 - 1 \right) = 1.$$
(9)

Likewise, we can derive the fuzzy interpretation of the axiom of power set, as given by formula (6):

$$\mu_{\tilde{B}}(y) = \begin{cases} 1, & y = \emptyset; \\ \prod_{x \in y} \left( 2^{\mu_{\tilde{A}}(x)} - 1 \right) = \prod_{x \in y} \left( 2^1 - 1 \right) = 1, & y \neq \emptyset. \end{cases}$$
(10)

The derivation of the axiom of union is slightly different from the previous two axioms. In this case, we use the membership function of the fuzzy set  $\tilde{A}$ , as given by formula (5). Specifically,  $\mu_{\tilde{A}}(x) = 1$  holds if and only if for every  $y \in x$ , we have  $\mu_{\tilde{B}}(y) = 1$ .

Formula (7) is consistent with classical set theory, which holds that an element either belongs to a set or does not. If all x belong to a classical set, denoted as  $\tilde{A}$ , i.e.,  $\mu_{\tilde{A}}(x) \equiv 1$  for all x, then formula (7) implies that  $\mu_{\tilde{B}}(y) \equiv 1$  for any y. On the other hand, if  $\mu_{\tilde{A}}(x) = 0$  for some x, then formula (7) implies that  $\mu_{\tilde{B}}(y) = 0$  for any expression of y that contains x. This result is reasonable since a positive membership value cannot be obtained from null.

**Theorem 1.** Let  $\tilde{A}$  be a fuzzy set defined on a finite set X, where each element  $x \in X$  is paired with its membership degree  $\mu_{\tilde{A}}(x)$ . Let  $\tilde{B}$  be a fuzzy set constructed from  $\tilde{A}$  and defined on the power set of X, that is,

$$\tilde{B} = \left\{ \left( y, \mu_{\tilde{B}}(y) \right) \middle| y \in \mathcal{P}(X) \right\}.$$
(11)

Then the cardinality of the fuzzy set  $\tilde{B}$  is given by:

$$\operatorname{card}(\tilde{B}) = 2^{\operatorname{card}(\tilde{A})}.$$
 (12)

*Proof.* Let  $X_n = \{x_1, \dots, x_n\}$  be a finite set, and let  $\tilde{A}_n$  be a fuzzy set defined on  $X_n$ , where n is the cardinality of  $X_n$ . We will prove this theorem by induction on n.

For the base case, suppose n = 1. Then  $X_1 = \{x_1\}$  is a single-point set, so the power set of  $X_1$  is  $\mathcal{P}(X_1) = \{\emptyset, \{x_1\}\}$ . Using the construction rules, we can create another fuzzy set  $\tilde{B}_1$  on  $\mathcal{P}(X_1)$  as follows:

$$\tilde{B}_1 = \frac{1}{\varnothing} + \frac{\mu_{\tilde{B}_1}(\{x_1\})}{\{x_1\}}.$$
(13)

Thus, the scalar cardinality of fuzzy set  $\tilde{B}_1$  is given by

$$\operatorname{card}(\tilde{B}_1) = 1 + \mu_{\tilde{B}_1}(\{x_1\}) = 1 + 2^{\mu_{\tilde{A}_1}(x_1)} - 1 = 2^{\mu_{\tilde{A}_1}(x_1)} = 2^{\operatorname{card}(\tilde{A}_1)}.$$
 (14)

Now for the induction step, denoting the fuzzy set defined on the power set of  $X_k$  to be  $\tilde{B}_k$ , we assume that  $\operatorname{card}(\tilde{B}_k) = 2^{\operatorname{card}(\tilde{A}_k)}$  if n = k and we must prove that

$$\operatorname{card}(\tilde{B}_{k+1}) = 2^{\operatorname{card}(A_{k+1})}.$$
(15)

Since  $X_{k+1} = X_k \cup \{x_{k+1}\}$ , we have

$$\mathcal{P}(X_{k+1}) = \mathcal{P}(X_k) \cup \{\{x_{k+1}\}\} \cup \{\{x_1, x_{k+1}\}, \{x_2, x_{k+1}\}, \cdots, \{x_k, x_{k+1}\}\} \cup \{\{x_1, x_2, x_{k+1}\}, \{x_1, x_3, x_{k+1}\}, \cdots, \{x_{k-1}, x_k, x_{k+1}\}\} \cup \cdots \cup \{\{x_1, x_2, \cdots, x_k, x_{k+1}\}\}.$$

$$(16)$$

Thus, the cardinality of fuzzy set  $\tilde{B}_{k+1}$  can be calculated as

$$\operatorname{card}(\tilde{B}_{k+1}) = 2^{\operatorname{card}(\tilde{A}_{k})} + 2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1 + (2^{\mu_{\tilde{A}_{k+1}}(x_{1})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1) + (2^{\mu_{\tilde{A}_{k+1}}(x_{2})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1) + \dots + (2^{\mu_{\tilde{A}_{k+1}}(x_{k})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1)$$

$$+ (2^{\mu_{\tilde{A}_{k+1}}(x_{1})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{2})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1) + \cdots + (2^{\mu_{\tilde{A}_{k+1}}(x_{k-1})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1) + \cdots + (2^{\mu_{\tilde{A}_{k+1}}(x_{1})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{2})} - 1)\cdots(2^{\mu_{\tilde{A}_{k+1}}(x_{k})} - 1)(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1) = 2^{\operatorname{card}(\tilde{A}_{k})} + \operatorname{card}(\tilde{B}_{k})(2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} - 1) = 2^{\operatorname{card}(\tilde{A}_{k})}2^{\mu_{\tilde{A}_{k+1}}(x_{k+1})} = 2^{\operatorname{card}(\tilde{A}_{k+1})}.$$

This completes our proof.

# 3. Constructing Fuzzy Sets from Classical Sets

In this section, we focus on addressing a critical issue in the field of fuzzy set theory: how to obtain an arbitrary membership value objectively? Elements in the superstructure over a single point set play a crucial role in this context. To simplify notation, a single element x and the corresponding single point set  $\{x\}$  are denoted by  $\{x\}^{(0)}$  and  $\{x\}^{(1)}$ , respectively. Furthermore, if we have a classical set  $\{y\}$  and let  $x = \{y\}$ , we denote the element y by  $\{x\}^{(-1)}$ . More generally, an element x in nested braces is defined as  $\{x\}^{(n)} \coloneqq \{\{x\}^{(n-1)}\}$  and  $\{x\}^{(-n)} \coloneqq \{\{x\}^{(1-n)}\}^{(-1)}$ , where  $n \ge 1$ . For instance,  $\{x\}^{(3)}$  is shorthand for  $\{\{\{x\}\}\}\}$ , and  $\{x\}^{(-3)}$  stands for  $\{\{\{x\}^{(-1)}\}^{(-1)}\}^{(-1)}$ . Obviously, setting  $y = \{x\}^{(-3)}$ , we obtain  $\{\{\{y\}\}\}\} = x$ .

We have a theorem related to sets that have nested braces:

**Theorem 2.** When constructing new fuzzy sets, the following three expressions can replace one another for any element x in a classical set, where m and n are integers:

$$\{\{x\}^{(m)}\}^{(n)}, \quad \{\{x\}^{(n)}\}^{(m)}, \quad \{x\}^{(m+n)}.$$
(17)

*Proof.* The statement is straightforward if m and n have the same sign, so we only need to prove the case when they are of opposite signs. Without loss of generality, assume that m > 0 and n < 0. Now, let  $\tilde{A}$  be a fuzzy set defined on the universe X, with  $x \in X$  and  $\mu_{\tilde{A}}(x) = u \in [0,1]$ . Another fuzzy set,  $\tilde{B}$ , can be constructed from  $\tilde{A}$  by including  $\{\{x\}^{(m)}\}^{(n)}, \{\{x\}^{(n)}\}^{(m)}, \text{ and } \{x\}^{(m+n)}$  in its universe.

By applying **Rule** 4 and **Rule** 5, we obtain the following equations:

$$\mu_{\tilde{B}}\left(\{\{x\}^{(m)}\}^{(n)}\right) = \log_{2}\left(\log_{2}\left(\cdots \log_{2}\left(\left(\underbrace{2^{2^{\cdots^{2^{u}-1}}}^{\cdots^{1}-1}-1}_{m \text{ times } 2^{(\cdot)}-1}\right)+1\right)\cdots+1\right)+1\right)$$

 $n \, \operatorname{times} \log_2(\cdot) + 1$ 

$$= \begin{cases} u, & \text{if } m+n=0; \\ \underbrace{2^{2^{\cdots^{2^{u-1}}}-1}-1}_{m+n \text{ times } 2^{(\cdot)}-1}, & \text{if } m+n>0; \\ \underbrace{\log_2\left(\log_2\left(\cdots\log_2(u+1)\cdots+1\right)+1\right)}_{|m+n| \text{ times } \log_2(\cdot)+1}, & \text{if } m+n<0 \end{cases}$$

and

$$\mu_{\tilde{B}}\left(\{\{x\}^{(n)}\}^{(m)}\right) \underbrace{\log_{2}\left(\log_{2}\left(\cdots\log_{2}\left(u+1\right)\cdots+1\right)+1\right)}_{n \text{ times}\log_{2}(\cdot)+1} - 1 \\ = 2^{2^{2^{-2^{u-1}}} - 1} - 1 \\ = \begin{cases} u, & \text{if } m+n=0; \\ \underbrace{2^{2^{2^{-2^{u-1}}} - 1} - 1}_{m+n \text{ times } 2^{(\cdot)} - 1}, & \text{if } m+n>0; \\ \underbrace{2^{2^{2^{-2^{u-1}}} - 1} - 1}_{(m+n \text{ times } 2^{(\cdot)} - 1)}, & \text{if } m+n<0. \end{cases}$$

In addition, **Rule** 2 can be applied to yield:

$$\mu_{\tilde{B}}\left(\{x\}^{(m+n)}\right) \\ = \begin{cases} \mu_{\tilde{B}}\left(\{x\}^{(0)}\right) = \mu_{\tilde{B}}(x) = \mu_{\tilde{A}}(x) = u, & \text{if } m = n; \\ \underbrace{2^{2^{\sum^{2^{u-1}} -1} - 1}}_{m+n \text{ times } 2^{(\cdot) - 1}}, & \text{if } m+n > 0; \\ \underbrace{\log_{2}\left(\log_{2}\left(\cdots \log_{2}\left(u+1\right)\cdots + 1\right) + 1\right)}_{|m+n| \text{ times } \log_{2}(\cdot) + 1}, & \text{if } m+n < 0. \end{cases}$$

These results demonstrate the substitution of the three expressions.

We can now explore the process of constructing fuzzy sets from classical sets. To begin with, let us consider a binary sequence consisting of 1's and 0's, represented by the notation:

$$\mathbf{a} = (a_{m^*}, a_{m^*+1}, a_{m^*+2}, \dots); m^* \le 0.$$
(18)

Here, we assume that  $a_{m^*} \equiv 1$ ,  $a_0 \equiv 1$ , and other elements of **a** can be either 0 or 1. For each  $k \ge m^*$ , we construct a classical set  $X_{\mathbf{a}} \in S(x)$  based on **a** as follows:

if  $a_k = 1$ , then the set  $x^{(k)}$  is included in  $X_{\mathbf{a}}$ , while if  $a_k = 0$ , then it is not. It is worth noting that  $X_{\mathbf{a}}$  always includes the element x since  $a_0 \equiv 1$ . To simplify notation, we replace  $a_0$  with a vertical line and use finite binary sequences to represent sequences with infinite 0's. For instance,  $X_{(1,0|1,0,1,1)}$  denotes the set

$$\left\{\{x\}^{(-2)}, x, \{x\}^{(1)}, \{x\}^{(3)}, \{x\}^{(4)}\right\}.$$
(19)

To construct fuzzy sets from classical sets, it is essential to follow another important rule.

**Rule** 5. A fuzzy set  $\tilde{A}$  can be constructed from any single point set  $\{x\}$  and any binary sequence **a** generated by (18), defined on  $X_{\mathbf{a}}$ . The scalar cardinality of  $\tilde{A}$  is equal to that of the single point set  $\{x\}$ , which is one.

Rule 5 implies the conservation of cardinality. The explanation is that during the constructing process of fuzzy set  $\tilde{A}$ , no other element except x has been involved, thus essentially only the classical set  $\{x\}$  makes contribution to the cardinality of  $\tilde{A}$ . Therefore, we believe that compared with  $\{x\}$ , the cardinality of fuzzy set  $\tilde{A}$  neither increases nor decreases.

Actually, Rule 5 is more fundamental than other rules because other rules can be adopted only when there already exists a fuzzy set. Thus the most important function of Rule 5 is to construct the first fuzzy set from a classical one. So we claim that a variant version of formula (7) works well in the process of constructing a fuzzy set from a classical single point set:

**Rule** 6. If we construct a fuzzy set  $\tilde{A}$  from a single point set  $\{x\}$  using Rule 5, we can derive the following equations:

$$\mu_{\tilde{A}}\left(\{x\}^{(m)}\right) = 2^{\mu_{\tilde{A}}\left(\{x\}^{(m-1)}\right)} - 1, \tag{20}$$

or equivalently,

$$\mu_{\tilde{A}}(\{x\}^{(m)}) = \log_2(\mu_{\tilde{A}}(\{x\}^{(m+1)}) + 1), \qquad (21)$$

where m is an arbitrary integer.

Formulas (20) and (21) contain an abuse of notation, as  $\{x\}^{(m)}$  may not be included in the universe of the fuzzy set  $\tilde{A}$  for some values of m. However, this does not interfere with our objective, which is to calculate the membership value using recurrence through (20) or (21). The notation  $\{x\}^{(m)}$ , which is not included in the universe, only occurs in the intermediate calculation process.

We will now demonstrate how to represent and calculate an arbitrary membership value using our rules. The specific methods for this calculation are presented in the proof of the following theorem.

**Theorem 3.** Given a binary sequence **a** defined by equation (18), there exists exactly one fuzzy set  $\tilde{A}$  constructed from a single point set  $\{x\}$  and defined on

 $X_{\mathbf{a}}$ . Conversely, for any real number  $0 < w \leq 1$ , there exists a binary sequence **a** such that the fuzzy set  $\tilde{A}$  constructed from  $\{x\}$  and defined on  $X_{\mathbf{a}}$  satisfies  $\mu_{\tilde{A}}(x) = w$ .

*Proof.* Given a fuzzy set  $\tilde{A}$ , which is constructed from a single point set  $\{x\}$ . This fuzzy set  $\tilde{A}$  is defined on  $X_{\mathbf{a}}$ , where  $\mathbf{a} = (a_{m^*}, a_{m^*+1}, a_{m^*+2}, \dots), m^* \leq 0$ . Moreover,  $a_{m^*} \equiv 1, a_0 \equiv 1$ , and the remaining  $a_k$  values are either 0 or 1. To define recursive functions, we set

$$u_{k}(t) = \mu_{\tilde{A}}\left(\{x\}^{(k)}\right), \quad u_{0}(t) = \mu_{\tilde{A}}(x) = t \in [0, 1].$$
(22)

It is evident that  $u_k(t)$  lies between 0 and 1 for any integer k.

(a) Given a binary sequence **a**, the scalar cardinality of the corresponding fuzzy set  $\tilde{A}$  can be computed using the following series:

$$G_{\mathbf{a}}(t) \coloneqq \operatorname{card}(\tilde{A}) = \sum_{k=m^*}^{\infty} a_k u_k(t).$$
(23)

To demonstrate the first part of this theorem, we will divide the proof into three distinct cases.

- I. If  $\mathbf{a} = (|)$ , that is,  $X_{\mathbf{a}} = \{\{x\}^{(0)}\} = \{x\}$ , then we have  $G_{(|)}(t) = t$ . Thus by **Rule 5**, t = 1 is the only real number satisfying the equation  $G_{(|)}(t) = 1$ . Hence  $\tilde{A}$  uniquely exists and degenerates into the classical set  $\{x\}$ .
- II. If **a** is finite, say,  $\mathbf{a} = (a_{m^*}, a_{m^*+1}, \dots, a_N), N \ge \max(m^* + 1, 0)$ . Then the cardinality of fuzzy set  $\tilde{A}$  defined on  $X_{\mathbf{a}}$  is

$$G_{\mathbf{a}}(t) = \sum_{k=m^*}^{N} a_k u_k(t).$$

$$\tag{24}$$

It is easy to verify that

$$(G_{\mathbf{a}}(0) - 1)(G_{\mathbf{a}}(1) - 1) = -M < 0,$$
 (25)

where M is the number of 1's (not counting  $a_0$ ) in binary sequence **a**. Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( G_{\mathbf{a}}(t) - 1 \right) \\
= \begin{cases}
1 + \sum_{k=1}^{N} a_{k} (\ln 2)^{k} \cdot 2^{u_{0}(t) + \dots + u_{k-1}(t)} > 0, & \text{if } m^{*} = 0; \\
1 + \sum_{k=m^{*}}^{-1} a_{k} (\ln 2)^{k} \cdot 2^{-u_{-1}(t) - \dots - u_{k}(t)} > 0, & \text{if } m^{*} \leq -1, N = 0; \\
1 + \sum_{k=m^{*}}^{-1} a_{k} (\ln 2)^{k} \cdot 2^{-u_{-1}(t) - \dots - u_{k}(t)} & \text{if } m^{*} \leq -1, N \geq 1; \\
+ \sum_{k=1}^{N} a_{k} (\ln 2)^{k} \cdot 2^{u_{0}(t) + \dots + u_{k-1}(t)} > 0, & \text{if } m^{*} \leq -1, N \geq 1.
\end{cases}$$
(26)

Hence  $G_{\mathbf{a}}(t) - 1$  is strictly increasing on the interval [0,1]. Together with (25), we see that  $G_{\mathbf{a}}(0) - 1 = 0$  has exactly one root between 0 and 1, which is just the value of  $\mu_{\tilde{A}}(x)$ .

III. If **a** is infinite, the cardinality of fuzzy set  $\hat{A}$  defined on  $X_{\mathbf{a}}$  is given by a positive series

$$G_{\mathbf{a}}(t) = \sum_{k=0}^{\infty} a_{n_k} u_{n_k}(t),$$
(27)

where  $n_0 = m^*$  and the subsequence  $a_{n_k}$  consists of all  $a_k = 1$  in **a**. We first need to prove that  $G_{\mathbf{a}}(t)$  is convergent for any real t in the open interval (0, 1).

For fixed  $t \in (0,1)$ , because positive  $u_k(t)$  is strictly monotonically decreasing on k, we have

$$\lim_{k \to \infty} u_{n_k}(t) = \lim_{k \to \infty} u_k(t) = 0.$$
(28)

Applying d'Alembert's ratio test, one computes the limit

$$\lim_{k \to \infty} \frac{u_{k+1}(t)}{u_k(t)} = \ln 2 < 1.$$
(29)

Note that  $G_{\mathbf{a}}(0) \equiv 0$ , thus series  $G_{\mathbf{a}}(t)$ , as a subseries of positive series  $\sum_{k=m^*}^{\infty} u_k(t)$ , converges for any binary sequence **a** and any real  $t \in [0, 1)$ .

Next we show that  $G_{\mathbf{a}}(t)$  is continuous at any point  $t_0$  in the open interval (0, 1). Without loss of generality, assume that  $m^* < 0$ , and observe that

$$|u_{k}(t) - u_{k}(t_{0})| = u'_{k}(c_{k}) \cdot |t - t_{0}|$$

$$\begin{cases} =|t - t_{0}|, & \text{if } k = 0; \\ = 2^{u_{0}(c_{k}) + \dots + u_{k-1}(c_{k})}(\ln 2)^{k} \cdot |t - t_{0}|, & \text{if } k > 0; \\ = 2^{-u_{-1}(d_{k}) - \dots - u_{k}(d_{k})}(\ln 2)^{k} \cdot |t - t_{0}|, & \text{if } k < 0, \end{cases}$$
(30)

where both  $c_k$  and  $d_k$  are between t and  $t_0$ . Let  $c_k^* = u_0(c_k) + \cdots +$ 

 $u_{k-1}(c_k), d_k^* = -u_{-1}(d_k) - \dots - u_k(d_k)$ , then we have

$$\begin{aligned} |G_{\mathbf{a}}(t) - G_{\mathbf{a}}(t_{0})| \\ &= \left| \sum_{k=0}^{\infty} u_{n_{k}}(t) - \sum_{k=0}^{\infty} u_{n_{k}}(t_{0}) \right| \\ &\leq \sum_{k=0}^{\infty} |u_{n_{k}}(t) - u_{n_{k}}(t_{0})| \\ &\leq \sum_{k=m^{*}}^{-1} (\ln 2)^{k} \cdot 2^{d_{k}^{*}} \cdot |t - t_{0}| + |t - t_{0}| + \sum_{k=1}^{\infty} (\ln 2)^{k} \cdot 2^{c_{k}^{*}} \cdot |t - t_{0}| \\ &\leq \left( P + \sum_{k=0}^{\infty} (\ln 2)^{k} \cdot 2^{c^{*}} \right) \cdot |t - t_{0}| \\ &= \left( P + \frac{2^{c^{*}}}{1 - \ln 2} \right) \cdot |t - t_{0}|, \end{aligned}$$

$$(31)$$

where  $P = \sum_{k=m^*}^{-1} (\ln 2)^k \cdot 2^{d_k^*} + 1$  is bounded, and  $c^* = \sup\{c_{n_k}^*\}$ . Hence  $G_{\mathbf{a}}(t)$  is continuous on [0, 1).

Furthermore, the derivative of  $G_{\mathbf{a}}(t)$ 

$$\frac{\mathrm{d}}{\mathrm{d}t}G_{\mathbf{a}}(t) = 1 + \sum_{k=m^*}^{-1} a_k (\ln 2)^k \cdot 2^{-u_{-1}(t) - \dots - u_k(t)} + \sum_{k=1}^{\infty} a_k (\ln 2)^k \cdot 2^{u_0(t) + \dots + u_{k-1}(t)} > 0.$$
(32)

Thus,  $G_{\mathbf{a}}(t)$  is strictly increasing on the interval (0,1).

Now, because  $G_{\mathbf{a}}(0) - 1 \equiv -1$  and  $G_{\mathbf{a}}(t) - 1 \rightarrow +\infty$  as  $t \rightarrow 1$ , we can confirm that for any infinite binary sequence  $\mathbf{a}$ ,  $G_{\mathbf{a}}(t) - 1 = 0$  has exactly one root between 0 and 1, which is just the value of  $\mu_{\tilde{A}}(x)$ .

(b) Conversely, we give an algorithm to show how to find the binary sequence **a** for any given real number  $w \in (0,1]$  such that the fuzzy set  $\tilde{A}$  generated from  $\{x\}$  and defined on  $X_{\mathbf{a}}$  satisfies  $\mu_{\tilde{A}}(x) = w$ .

If w = 1, the special binary sequence (|) is just what we are seeking for.

If 0 < w < 1, the following algorithm gets the binary sequence

- $\mathbf{a} = (a_{m^*}, a_{m^*+1}, a_{m^*+2}, \dots).$ 
  - I. Find the initial nonzero term.
    - Define function  $s_0(k) = u_k(w) + w 1$ ,  $k \neq 0$ . For any fixed  $w \in (0, 1)$ , since  $s_0(k)$  is strictly monotonically decreasing on k with  $\lim_{k\to\infty} s_0(k) = w > 0$  and  $\lim_{k\to\infty} s_0(k) = w - 1 < 0$ , so there exists an integer  $n_0$  such that  $s_0(k) > 0$  if  $k < n_0$  and  $s_0(k) \le 0$  if  $k \ge n_0$ . We set  $m^* = n_0$  if  $n_0 < 0$  and  $m^* = 0$  if  $n_0 > 0$ . If  $s_0(n_0) = 0$ , this algorithm ends with a finite  $\mathbf{a} = (a_{m^*}, \ldots, |)$  if  $n_0 < 0$  or  $\mathbf{a} = (|, \ldots, a_{n_0})$  if  $n_0 > 0$ . In both cases except  $a_{n_0} = 1$  and  $a_0 = 1$ , all other  $a_k$ 's (if there exist) in  $\mathbf{a}$  are set to 0. Otherwise if  $s_0(n_0) < 0$ , go to Step II.

II. Find the second nonzero term.

Define function  $s_1(k) = s_0(n_0) + u_k(w)$ ,  $k > n_0$  and  $k \neq 0$ . For any fixed  $w \in (0,1)$ ,  $s_1(k)$  is strictly monotonically decreasing on k as  $u_k(w)$  is strictly monotonically decreasing on k. Because

$$s_1(n_0+1) - s_0(n_0-1) = 2^{2^{u_{n_0-1}(w)}-1} + 2^{u_{n_0-1}(w)} - u_{n_0-1}(w) - 2 > 0$$
(33)

holds for any fixed  $w \in (0, 1)$ , we have

$$s_1(n_0+1) > s_0(n_0-1) > 0.$$
(34)

Together with  $\lim_{k\to\infty} s_1(k) = s_0(n_0) < 0$ , so there exists an integer  $n_1$  such that  $s_1(k) > 0$  if  $n_0 < k < n_1$  and  $s_1(k) \le 0$  if  $k \ge n_1$ . Thus we set  $a_k = 0$ ,  $n_0 < k < n_1$  and  $a_{n_1} = 1$ , which is the second (not counting  $a_0$ ) nonzero term in **a**. If  $s_1(n_1) = 0$ , this algorithm ends with a finite **a**, which has one of the following forms depending on the positition of  $a_0$ :

- i.  $(|, \ldots, a_{n_0}, \ldots, a_{n_1}),$
- ii.  $(a_{n_0}, \ldots, |, \ldots, a_{n_1}),$
- iii.  $(a_{n_0}, \ldots, a_{n_1}, \ldots, |).$

Except  $a_{n_0} = 1$ ,  $a_{n_1} = 1$  and  $a_0 = 1$ , all other  $a_k$ 's (if there exist) in **a** are set to 0. Otherwise if  $s_1(n_1) < 0$ , go to Step III.

III. Find the *j*-th,  $j \ge 2$ , nonzero term  $a_{n_j}$ .

Suppose that the (j-1)-th (not counting  $a_0$ ) nonzero term in **a** is  $a_{n_{j-1}}$ and functions  $s_j(k)$  are defined recursively as

$$s_j(k) = s_{j-1}(n_{j-1}) + u_k(w), \quad k > n_{j-1}, \ k \neq 0,$$
(35)

satisfying that  $s_{j-1}(k) > 0$  if  $n_{j-2} < k < n_{j-1}$  and  $s_{j-1}(k) < 0$  if  $k \ge n_{j-1}$ . For any fixed  $w \in (0,1)$ ,  $s_j(k)$  is strictly monotonically decreasing on k as  $u_k(w)$  is strictly monotonically decreasing on k. To lighten notations, we use  $\xi$  to denote  $u_{n_{j-1}-1}(w)$ . Because

$$s_j(n_{j-1}+1) - s_{j-1}(n_{j-1}-1) = 2^{2^{\xi}-1} + 2^{\xi} - \xi - 2 > 0$$
 (36)

holds for any  $\xi \in (0, 1)$ , we have

$$s_j(n_{j-1}+1) > s_{j-1}(n_{j-1}-1) > 0.$$
 (37)

Together with

$$\lim_{k \to \infty} s_j(k) = s_{j-1}(n_{j-1}) < 0, \tag{38}$$

so there exists an integer  $n_j$  such that  $s_j(k) > 0$  if  $n_{j-1} < k < n_j$  and  $s_j(k) \le 0$  if  $k \ge n_j$ . Thus we set  $a_k = 0$ ,  $n_{j-1} < k < n_j$  and  $a_{n_j} = 1$ , which is the *j*-th (not counting  $a_0$ ) nonzero term in **a**. If  $s_j(n_j) = 0$ , this algorithm ends with a finite **a**. Otherwise if  $s_j(n_j) < 0$ , repeat this process to find the (j + 1)-th nonzero term  $a_{n_{j+1}}$  in **a**.

Clearly, if **a** is infinite by this algorithm, then w satisfies the equation  $G_{\mathbf{a}}(w) = 1$  and this completes our proof.

## 4. Numerical Examples

In this section, we will first provide an illustrative example to demonstrate how membership values can be calculated using our construction rules. Then, we will present several numerical examples to explain how to use **Theorem 3** to achieve and represent any desired membership value.

**Example 1.** Consider a fuzzy set  $\tilde{A}$  defined on the universe  $X = \{x_1, x_2, x_3, x_4\}$  as:

$$\tilde{A} = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{1}{x_4}.$$
(39)

We can construct a new fuzzy set  $\tilde{B}$  on the universe

$$Y = \left\{ \left\{ \emptyset, x_1 \right\}, \left\{ \{x_2\}, \{x_3\} \right\}, \left\{ x_1, \left\{ x_2, \{x_3, \{x_4\} \} \right\} \right\} \right\}.$$
(40)

from  $\tilde{A}$  by applying formula (7) repeatedly. The resulting values of  $\mu_{\tilde{B}}(y)$  for each  $y \in Y$  are:

$$\mu_{\tilde{B}}(\{\emptyset, x_1\}) = (2^1 - 1)(2^{\mu_{\tilde{A}}(x_1)} - 1) = 2^{0.2} - 1 \approx 0.1487;$$
(41)

$$\mu_{\tilde{B}}\left(\left\{\{x_{2}\},\{x_{3}\}\right\}\right)$$

$$=\left(2^{\left(2^{\mu_{\tilde{A}}(x_{2})}-1\right)}-1\right)\left(2^{\left(2^{\mu_{\tilde{A}}(x_{3})}-1\right)}-1\right)$$

$$=\left(2^{\left(2^{0.3}-1\right)}-1\right)\left(2^{\left(2^{0.5}-1\right)}-1\right)$$
(42)

 $\approx 0.0364;$ 

$$\mu_{\tilde{B}} \left( \left\{ x_{1}, \left\{ x_{2}, \left\{ x_{3}, \left\{ x_{4} \right\} \right\} \right\} \right) \right)$$

$$= \left( 2^{\mu_{\tilde{A}}(x_{1})} - 1 \right) \left( 2^{\left( 2^{\mu_{\tilde{A}}(x_{2})} - 1 \right) \left( 2^{\left( 2^{\mu_{\tilde{A}}(x_{3})} - 1 \right) \left( 2^{\left( 2^{\mu_{\tilde{A}}(x_{3})} - 1 \right) \left( 2^{\left( 2^{\mu_{\tilde{A}}(x_{3})} - 1 \right) \right) - 1} \right) - 1 \right)$$

$$= \left( 2^{0.2} - 1 \right) \left( 2^{\left( 2^{0.3} - 1 \right) \left( 2^{\left( 2^{0.5} - 1 \right) - 1} \right) - 1} \right)$$

$$\approx 0.0081.$$

$$(43)$$

Therefore, we can represent  $\tilde{B}$  as:

$$\tilde{B} = \frac{0.1487}{\{\emptyset, x_1\}} + \frac{0.0364}{\{\{x_2\}, \{x_3\}\}} + \frac{0.0081}{\{x_1, \{x_2, \{x_3, \{x_4\}\}\}\}}.$$
(44)

**Example 2.** Given two finite binary sequences,  $\mathbf{a} = (10|01)$  and  $\mathbf{b} = (|01001)$ , we are tasked with finding the corresponding fuzzy sets,  $\tilde{A}$  and  $\tilde{B}$ , both of which are constructed from a single point set  $\{x\}$  and defined on  $X_{\mathbf{a}}$  and  $X_{\mathbf{b}}$ , respectively.

To find the values of  $\mu_{\tilde{A}}(x)$  and  $\mu_{\tilde{B}}(x)$ , we need to solve the following two equations:

$$\log_2(\log_2(u_a+1)+1) + u_a + 2^{2^{u_a}-1} - 1 = 1,$$
(45)

$$u_b + 2^{2^{u_b} - 1} - 1 + 2^{2^{2^{2^{u_b} - 1} - 1} - 1} - 1 = 1,$$
(46)

where  $u_a$  and  $u_b$  represent the values of  $\mu_{\tilde{A}}(x)$  and  $\mu_{\tilde{B}}(x)$ , respectively.

Solving these equations gives us  $\mu_{\tilde{A}}(x) \approx 0.3222$  and  $\mu_{\tilde{B}}(x) \approx 0.5087$ . Therefore, the two fuzzy sets are:

$$\tilde{A} = \frac{\log_2(\log_2(u_a+1)+1)}{\{x\}^{(-2)}} + \frac{u_a}{x} + \frac{2^{2^{u_a}-1}-1}{\{x\}^{(2)}}$$
$$= \frac{0.4884}{\{x\}^{(-2)}} + \frac{0.3222}{x} + \frac{0.1894}{\{x\}^{(2)}}$$
(47)

and

$$\tilde{B} = \frac{u_b}{x} + \frac{2^{2^{u_b}-1}-1}{\{x\}^{(2)}} + \frac{2^{2^{2^{2^{u_b}-1}-1}-1}-1}{\{x\}^{(5)}} = \frac{0.5087}{x} + \frac{0.3405}{\{x\}^{(2)}} + \frac{0.1508}{\{x\}^{(5)}}.$$
(48)

It is easy to verify that both  $\tilde{A}$  and  $\tilde{B}$  have a cardinality of 1.

**Example 3.** Given two real numbers,  $w_a$  and  $w_b$ , where  $w_a = 0.3$  and  $w_b = 0.8$ , we need to find binary sequences **a** and **b** that correspond to fuzzy sets  $\tilde{A}$  and  $\tilde{B}$ , respectively. These fuzzy sets are constructed from the set  $\{x\}$  and are defined on  $X_{\mathbf{a}}$  and  $X_{\mathbf{b}}$ , respectively. Furthermore, we want the fuzzy sets to satisfy  $\mu_{\tilde{A}}(x) = w_a$  and  $\mu_{\tilde{B}}(x) = w_b$ .

To determine the initial nonzero term of  $\mathbf{a}$ , we first evaluate the expression:

$$\log_2(\log_2(\log_2(\log_2(0.3+1)+1)+1)+1) + 0.3 - 1 \approx 0.0061 > 0$$
(49)

$$\log_2(\log_2(\log_2(\log_2(0.3+1)+1)+1)+1) + 0.3 - 1 \approx -0.0686 < 0, \tag{50}$$

which indicates that the initial nonzero term in **a** is  $a_{-4}$ . We repeat the same steps for subsequent terms to obtain  $a_5$ ,  $a_{15}$ ,  $a_{20}$ , and so on. Therefore, the binary sequence **a** can be expanded as:

$$\mathbf{a} = (1,000, |,000,010,000,000,001,000,010,\dots)$$
(51)

Similarly, for  $w_b = 0.8$ , the binary sequence **b** can be expanded as:

#### 5. Conclusion

In this contribution, we propose novel methods for constructing new fuzzy sets from existing ones or classical sets. These methods consist of six interrelated rules that are compatible with classical sets. Our methods are objective, meaning that both the essential problems of constructing new fuzzy sets and achieving the values of membership functions are obtained through objective calculations or by solving objectively constructed equations, without any subjective assumptions.

The first result of our methods generalizes an important property of the power set from classical settings to fuzzy settings. In addition, the second result reveals deep connections between fuzzy sets and binary sequences, along with a new algorithm. By studying the corresponding binary sequences, it becomes possible to gain a more profound knowledge of fuzzy sets. This will be the subject of our future research.

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