SUBADDITIVITY OF SHIFTS, EILENBERG-ZILBER SHUFFLE PRODUCTS AND HOMOLOGY OF LATTICES

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Abstract. We show that the maximal shifts in the minimal free resolution of the quotients of a polynomial ring by a monomial ideal are subadditive as a function of the homological degree. This answers a question that has received some attention in recent years. To do so, we define and study a new model for the homology of posets, given by the so called synor complex. We also introduce an Eilenberg-Zilber type shuffle product on the simplicial chain complex of lattices. Combining these concepts we prove that the existence of a non-zero homology class for a lattice forces certain non-zero homology classes in lower intervals. This result then translates into properties of the minimal free resolution. In particular, it implies a generalization of the original question.

1. Introduction

Let $\mathfrak{I}$ be a homogeneous ideal $\mathfrak{I}$ in a standard graded $\mathbb{K}$-algebra S for some field $\mathbb{K}$ and $\beta_{ij}(S/\mathfrak{I}) := \dim_{\mathbb{K}} \operatorname{Tor}_i^S(S/\mathfrak{I},\mathbb{K})_j$ the ***graded Betti numbers*** of $S/\mathfrak{I}$. The maximal shift $t_i(S/\mathfrak{I})$ in the ith homological degree is the maximal j such that $\beta_{ij}(S/\mathfrak{I}) \neq 0$. If $\beta_{ij}(S/\mathfrak{I}) = 0$ for all j we set $t_i(S/\mathfrak{I}) = 0$. The following question has been considered (see [ACI15, EHU06]):

Question 1.1. For which classes of homogeneous ideals $\mathfrak{I}$ does

$$t_{i_1+i_2}(S/\mathfrak{I}) \leq t_{i_1}(S/\mathfrak{I}) + t_{i_2}(S/\mathfrak{I})$$

hold for all $0 \leq i_1, i_2$.

Since $t_0(S/\mathfrak{I}) = 0$ the question is interesting only for $i_1, i_2 \geq 1$. It has a negative answer for general homogeneous ideals even in case S is a polynomial ring and counterexamples can be found in [EHU06] or [ACI15].

For the rest of this paper $S = \mathbb{K}[x_1, \ldots, x_n]$ and $\mathfrak{I}$ is a monomial ideal. Indeed as a special case of [ACI15, Conjecture 6.4] it was conjectured that Question 1.1 holds for quadratic monomial ideals. In this case, their equation [ACI15, (6.4)] implies that Question 1.1 holds if $\operatorname{char}(\mathbb{K}) = 0$ and 1 is added on the right hand side. Over the last 10 years numerous results were obtained providing positive evidence that Question 1.1 has a positive answer for all monomial ideals. In [HS13] the case $i_1 = 1$ was settled, with an extension appearing in [Abd22]. Positive results for edge ideals and other

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graph-motivated situations can be found in [AN17, BH17, JK20] and facet ideals of simplicial forests were treated in [Far19]. The case when the monomial ideal has a DGA resolution was settled in [Kat16].

For a monomial ideal $\mathfrak{I}$ with minimal monomial generating set $G(\mathfrak{I})$ the ***LCM-lattice*** $\mathcal{L}(\mathfrak{I})$ is the set of all least common multiples $\mathrm{lcm}_{m\in B} m$ for $B \subseteq G(\mathfrak{I})$ where we set $\mathrm{lcm}_{m\in\emptyset} m = 1$. Ordered by divisibility the set $\mathcal{L}(\mathfrak{I})$ becomes an atomic lattice.

Our approach to answering Question 1.1 leads to the study of LCM-lattices and their topology. For the latter, when referring to the topology or homology of a poset $\mathcal{P}$, we mean the topology or homology of its order complex; that is the simplicial complex of totally ordered subsets of $\mathcal{P}$.

For $p \leq q$ in $\mathcal{P}$ we denote by $[p, q]$ the ***closed interval*** $\{x \in \mathcal{P} \mid p \leq x \leq q\}$. ***Open*** and ***half-open intervals*** in $\mathcal{P}$ are defined analogously.

Our work on Question 1.1 is based on the known relation (see [GPW99]) between graded Betti numbers $\beta_{ij}(S/\mathfrak{I})$ and homology groups of open intervals supported in $\mathcal{L} = \mathcal{L}(\mathfrak{I})$, given by

$$\beta_{ij}(S/\mathfrak{I}) = \sum_{\substack{m\in\mathcal{L}(\mathfrak{I})\\ \deg(m)=j}} \dim_{\mathbb{K}} \widetilde{H}_{i-2}\big((1, m)\big), \tag{1}$$

where the homology is reduced and taken with $\mathbb{K}$-coefficients.

Thus if $1 \leq i_1, i_2$ and $t_{i_1+i_2} > 0$ then $\widetilde{H}_{i_1+i_2-2}\big((1, m)\big) \neq 0$ for some monomial m of degree $t_{i_1+i_2}$. This shows that Question 1.1 has a positive answer when in this situation there are monomials n_1 and n_2 in our lattice such that $\widetilde{H}_{i_1-2}\big((1, n_1)\big) \neq 0$ and $\widetilde{H}_{i_2-2}\big((1, n_2)\big) \neq 0$ and $t_{i_1+i_2}$ is bounded from above by the degree of the lcm of n_1 and n_2. Hence, the question for monomial ideals is implied by the following theorem. In its formulation we write $\hat{0}$ for the unique minimal element of $\mathcal{L}$ and write $n_1 \vee n_2$ for the supremum of two elements of $\mathcal{L}$. We prove:

Theorem 1.2. *Let $1 \leq i_1, i_2$ and let $\mathcal{L}$ be a finite lattice with an element m such that*

$$\widetilde{H}_{i_1+i_2-2}\big((\hat{0}, m)\big) \neq 0.$$

Then there exist elements $n_1, n_2 \in \mathcal{L}$ such that

$$\widetilde{H}_{i_1-2}\big((\hat{0}, n_1)\big) \neq 0,\ \ \widetilde{H}_{i_2-2}\big((\hat{0}, n_2)\big) \neq 0$$

and $n_1 \vee n_2 \geq m$.

Let us note that this statement is immediate for several classes of lattices, such as face lattices of strongly regular CW complexes.

As a consequence of Theorem 1.2 and the arguments preceding it we obtain a positive answer to Question 1.1 for monomial ideals.

Theorem 1.3. *Let $\mathfrak{I}$ be a monomial ideal in S. Then for all $0 \leq i_1, i_2$ we have*

$$t_{i_1+i_2}(S/\mathfrak{I}) \leq t_{i_1}(S/\mathfrak{I}) + t_{i_2}(S/\mathfrak{I}).$$

For the proof of Theorem 1.2 we proceed as follows. In §2 we introduce a shuffle product on the chain complex of the order complex of a finite lattice. Then in §3 we establish the existence of a subcomplex of the chain complex of a poset - a so called synor complex - which encodes the homology of all lower intervals. The synor complex is graded by the synors of the poset, which are the element for which the subposet of elements below is not acyclic. Afterwards in §4 we leverage the shuffle product and the synor complex to prove in Proposition 5.3 a result on the representation of cycles as sums of shuffles of synor chains. As a consequence we can prove Theorem 5.4 which contains Theorem 1.2 as a special case. In §6 we apply Theorem 5.4 to prove in Theorem 6.1 a generalization of Theorem 1.3 as well as well as a bound on the number of distinct multigraded shifts in the minimal free resolution of a monomial ideal. We also show in Theorem 6.3 that the synor complex from §3 can be used to construct a minimal free resolution of a monomial ideal.

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2. Posets, homology and the shuffle product in lattices

We first introduce the concepts for posets and lattices we use in this paper.

For a poset $\mathcal{P}$ we call a multichain $\mathfrak{c} = (c_0 \geq c_1 \geq \cdots \geq c_k)$ of elements of $\mathcal{P}$ an ***order multichain***. We call k the ***length*** of the multichain. Order multichains of length k are simply referred to as ***k-multichains***. We call an order multichain with no repetitions an ***order chain***. Note, that the empty multichain $\mathfrak{c} = ()$ is the unique order multichain of length -1. The ***order complex*** of $\mathcal{P}$ is the simplicial complex of all order chains in $\mathcal{P}$. From now on

all posets and lattices are assumed to be finite.

We adopt the notions of open, half-open and closed intervals in posets, as defined in §1. We call a subset $\mathcal{J} \subseteq \mathcal{P}$ an ***order ideal*** in $\mathcal{P}$ if for $x, y \in \mathcal{P}$, $x \leq y$ and $y \in \mathcal{J}$ imply $x \in \mathcal{J}$. For $x \in \mathcal{P}$ we write $\mathcal{P}_{\leq x}$ for the order ideal of all $y \leq x$ and $\mathcal{P}_{<x}$ for the order ideal $\mathcal{P}_{\leq x} \setminus \{x\}$.

Next we recall some basics about the homology of order complexes of a poset $\mathcal{P}$. We fix a field $\mathbb{K}$ and for $k \geq -1$, we denote by $\widetilde{M}_k(\mathcal{P})$ the $\mathbb{K}$-vector space freely generated by the order multichains of length k. We write $\widetilde{C}_k(\mathcal{P})$ for the subspace spanned by the order chains of length k. For an order multichain $\mathfrak{c} = (c_0 \geq c_1 \geq \cdots \geq c_k)$ of length k and $0 \leq j \leq k$ we set

$$\partial_k^{(j)}(\mathfrak{c}) = (c_0 \geq \cdots \geq c_{j-1} \geq c_{j+1} \geq \cdots \geq c_k)$$

and write

$$\partial_k = \sum_{j=0}^{k} (-1)^j \partial_k^{(j)}$$

for the induced linear map $\partial_k : \widetilde{M}_k(\mathcal{P}) \to \widetilde{M}_{k-1}(\mathcal{P})$. Thus $(\widetilde{M}_*, \partial_*)$ is the standard simplicial chain complex of the simplicial set associated to the order complex of $\mathcal{P}$, i.e., the simplicial complex of chains in $\mathcal{P}$. The complex $(\widetilde{C}_*, \partial_*)$ is the standard simplicial chain complex associated to the order complex of $\mathcal{P}$. By [EM53] we have that $(\widetilde{C}(\mathcal{P}), \partial_*)$ is the normalization of $(\widetilde{M}_*(\mathcal{P}), \partial_*)$ and in particular by the proof of [EM53, Theorem 4.1] we have:

Lemma 2.1. *The projection map $\pi : \widetilde{M}_*(\mathcal{P}) \to \widetilde{C}_*(\mathcal{P})$ which acts identically on order chains and vanishes on multichains that are not order chains, induces a homotopy inverse to the inclusion $(\widetilde{C}_*(\mathcal{P}), \partial_*) \hookrightarrow (\widetilde{M}_*(\mathcal{P}), \partial_*)$.*

Using this lemma we will work with $\widetilde{C}_*(\mathcal{P})$ in the following sections §4 and §6 and ignore multichains which appear in any of the calculations. For this section, §2, we mostly work in $(\widetilde{M}_*(\mathcal{P}), \partial_*)$ since this facilitates the manipulation of the combinatorial product on $(\widetilde{M}_*(\mathcal{P}), \partial_*)$ in case $\mathcal{P}$ is a lattice, which we define below.

Before we can define this product, we set up notation which will be used in the later section for all posets. We will consider an i-multichain $\mathfrak{c}$ in $\mathcal{P}$ as an element of $\mathcal{P}^{i+1}$. For an arbitrary $(i+1)$-tuple $\mathfrak{c}$ in $\mathcal{P}^{i+1}$ and an arbitrary $(j+1)$-tuple $\mathfrak{c}' \in \mathcal{P}^{j+1}$ we write $\mathfrak{c} \cdot \mathfrak{c}'$ for the $(i+j+2)$-tuple in $\mathcal{P}^{i+j+2}$ that is the concatenation of $\mathfrak{c}$ and $\mathfrak{c}'$. In case $\mathfrak{c} \cdot \mathfrak{c}'$ is an order multichain, i.e., both $\mathfrak{c}$ and $\mathfrak{c}'$ are order multichains and the minimal element of $\mathfrak{c}$ is greater or equal to the maximal element of $\mathfrak{c}'$, then we write $\mathfrak{c} * \mathfrak{c}'$ for this order multichain. We write $\min(\mathfrak{c})$ for the minimal element of an order multichain $\mathfrak{c}$. If $\mathfrak{c}$ is an i-multichain, $x = \min(\mathfrak{c})$ and $\gamma = \sum_{\ell=1}^{k} \lambda_\ell \, \mathfrak{c}_\ell \in \widetilde{M}_j(\mathcal{P})$ where $\mathfrak{c}_\ell$ are order multichains supported in $\mathcal{P}_{\leq x}$ then we write $\mathfrak{c} * \gamma$ for the chain $\sum_{\ell=1}^{k} \lambda_\ell \, \mathfrak{c} * \mathfrak{c}_\ell$ in $\widetilde{M}_{i+j+1}(\mathcal{P})$.

Let $\mathcal{L}$ be a lattice. Since by our assumption $\mathcal{L}$ is finite, it has a unique minimal element which we denote by $\hat{0}$ and a unique maximal element which we denote by $\hat{1}$. Note that if $\mathcal{L}$ is the lcm-lattice $\mathrm{lcm}(\mathfrak{I})$ of a monomial ideal $\mathfrak{I}$ then $\hat{0} = 1$, where 1 stands for the monomial $x_1^0 \cdots x_n^0$. For $x, y \in \mathcal{L}$ we write $x \vee y$ for the ***supremum*** or ***join*** of x and y, and $x \wedge y$ for the ***infimum*** or ***meet*** of x and y. In $\mathrm{lcm}(\mathfrak{I})$ the join of two elements is simply their lcm.

We define $\tau : \mathcal{L}^{k+1} \to \widetilde{M}_k(\mathcal{L})$ as the operator which takes the $(k+1)$-tuple $(a_0, \ldots, a_k) \in \mathcal{L}^{k+1}$ to the multichain $(a_0 \vee \cdots \vee a_k \geq a_1 \vee \cdots \vee a_k \geq \cdots \geq a_k)$. For a permutation σ of $\{0, \ldots, k\}$ and a $(k+1)$-tuple $a = (a_0, \ldots, a_k) \in \mathcal{L}^{k+1}$ we write a_σ for $(a_{\sigma(0)}, \ldots, a_{\sigma(n)})$. We again consider an i-multichain $\mathfrak{c}$ in $\mathcal{L}$ as an element of $\mathcal{L}^{i+1}$. We say a permutation σ of $\{0, 1, \ldots, i+j+1\}$ is an (i,j)***-shuffle*** if it preserves the relative order of the first $i+1$ elements and also of the last $j+1$ elements of $\{0, 1, \ldots, i+j+1\}$. Equivalently, σ is an (i,j)-shuffle if it is monotone on the sets $\sigma^{-1}(\{0,1,\dots,i\})$ and $\sigma^{-1}(\{i+1,i+2,\dots,i+j+1\})$. We write $S_{i,j}$ for the set of all (i,j)-shuffles inside the set of permutations of $\{0,1,\dots,i+j+1\}$.

The shuffle operator $\sqcup\!\sqcup : \widetilde{M}_i(\mathcal{L}) \times \widetilde{M}_j(\mathcal{L}) \to \widetilde{M}_{i+j+1}(\mathcal{L})$ is then defined as the linear extension of the map sending an i-multichain $\mathfrak{c}$ and a j-multichain $\mathfrak{c}'$ to

$$\mathfrak{c} \sqcup\!\sqcup \mathfrak{c}' := \sum_{\sigma \in S_{i,j}} \operatorname{sgn}(\sigma)\, \tau\big(\,(\mathfrak{c}\cdot\mathfrak{c}')_\sigma\,\big).$$

Note that if $i=-1$ then $\mathfrak{c} \sqcup\!\sqcup \mathfrak{c}' = \mathfrak{c}'$. Consider the second tensor power

$$(D_*(\mathcal{L}), \delta_*) = (\widetilde{M}_*, \partial_*) \otimes (\widetilde{M}_*, \partial_*)$$

of the chain complex $(\widetilde{M}_*, \partial_*)$, with differential $\delta_n = \bigoplus_{i+j+1=n} \big(\,(\partial_i, \mathrm{id}) + (-1)^{i+1}(\mathrm{id}, \partial_j)\,\big)$.

Since $\widetilde{M}_i(\mathcal{L}) \otimes \widetilde{M}_j(\mathcal{L})$ has a basis consisting of the elementary tensors $\mathfrak{c}\otimes\mathfrak{c}'$ for poset i-multichains $\mathfrak{c}$ and poset j-multichains $\mathfrak{c}'$, it follows that $\mathfrak{c}\otimes\mathfrak{c}' \mapsto \mathfrak{c} \sqcup\!\sqcup \mathfrak{c}'$ induces a map of $\mathbb{K}$-vector spaces $\sqcup\!\sqcup : D_n(\mathcal{L}) \to \widetilde{M}_n(\mathcal{L})$. In Proposition 2.4 we prove that this map is a chain map. This will yield the very helpful boundary formula in Corollary 2.5. Before that, we prove two technical but straightforward combinatorial lemmas.

Let $X^{i,j} = \{0,\dots,i+j+1\} \times S_{i,j}$, and consider the partition $X^{i,j} = \cup_{s=1}^4 A_s^{i,j}$, where

$$\begin{aligned}
A_1^{i,j} &= \left\{ (\ell,\sigma) \in X^{i,j} \,:\, \begin{matrix} \ell \ge 1, \sigma(\ell-1) \in \{0,\dots,i\} \text{ and} \\ \sigma(\ell) \in \{i+1,\dots,i+j+1\} \end{matrix} \right\} \\
A_2^{i,j} &= \left\{ (\ell,\sigma) \in X^{i,j} \,:\, \begin{matrix} \ell \ge 1, \sigma(\ell-1) \in \{i+1,\dots,i+j+1\} \text{ and} \\ \sigma(\ell) \in \{0,\dots,i\} \end{matrix} \right\} \\
A_3^{i,j} &= \{\, (\ell,\sigma) \in X^{i,j} \,:\, \sigma(\ell-1), \sigma(\ell) \in \{0,\dots,i\} \,\} \\
A_4^{i,j} &= \{\, (\ell,\sigma) \in X^{i,j} \,:\, \sigma(\ell-1), \sigma(\ell) \in \{i+1,\dots,i+j+1\} \,\}
\end{aligned}$$

where we interpret the conditions $\sigma(-1) \in \{0,\dots,i\}$ and $\sigma(-1) \in \{i+1,\dots,i+j+1\}$ as true.

Lemma 2.2. *Consider the map $T_{i,j} : A_1^{i,j} \to A_2^{i,j}$ defined by*

$$T_{i,j}(\ell,\sigma) = (\ell, \sigma \circ \tau_\ell),$$

where $\tau_\ell \in S_{i+j+1}$ denotes the transposition $(\ell-1,\ell)$. Then,

(i) *$T_{i,j}$ is a bijection.*
(ii) *For every $\sigma \in S_{i+j+1}$ we have $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma\circ\tau_\ell)$*
(iii) *For every $\mathfrak{c} \in \mathcal{L}^{i+1}, \mathfrak{c}' \in \mathcal{L}^{j+1}$ and $(\ell,\sigma) \in A_1^{i,j}$ we have*

$$\partial_{i+j+1}^{(\ell)}\big[\tau\big(\,(\mathfrak{c}\cdot\mathfrak{c}')_\sigma\,\big)\big] = \partial_{i+j+1}^{(\ell)}\big[\tau\big(\,(\mathfrak{c}\cdot\mathfrak{c}')_{\sigma\circ\tau_\ell}\,\big)\big]$$

Proof. The first two properties follow immediately from the definitions, so we only prove (iii).

Let $\mathfrak{c} = (c_0 \geq \cdots \geq c_i)$ and $\mathfrak{c}' = (c_{i+1} \geq \cdots \geq c_{i+j+1})$ viewed as tuples. For a $\sigma \in S_{i,j}$ we write $\sigma_1(k)$ for $\min\{\{0,\dots,i\} \cap \{\sigma(k),\dots,\sigma(i+j+1)\}\}$ and $\sigma_2(k)$ for $\min\{\{i+1,\dots,i+j+1\} \cap \{\sigma(k),\dots,\sigma(i+j+1)\}\}$ where we consider the minimum over the empty set as $-\infty$.

Setting $c_{-\infty} = c'_{-\infty} = \hat{0}$ we get that

$$\tau\big((\mathfrak{c}\cdot\mathfrak{c}')_\sigma \big) = (c_{\sigma_1(0)} \vee c'_{\sigma_2(0)} \geq \cdots \geq c_{\sigma_1(i+j+1)} \vee c'_{\sigma_2(i+j+1)}).$$

Assuming $(\ell,\sigma) \in A_1^{i,j}$, we note that the sets $\{\sigma(k),\dots,\sigma(i+j+1)\}$ and $\{(\sigma\circ\tau_\ell)(k),\dots,(\sigma\circ\tau_\ell)(i+j+1)\}$ coincide for all $k \neq \ell$. This is because $(\sigma\circ\tau_\ell)(j) = \sigma(j)$ for $j \neq \ell-1,\ell$, while $(\sigma\circ\tau_\ell)(\ell-1) = \sigma(\ell)$ and $(\sigma\circ\tau_\ell)(\ell) = \sigma(\ell-1)$. Consequently, $\sigma_1(k) = (\sigma\circ\tau_\ell)_1(k)$ and $\sigma_2(k) = (\sigma\circ\tau_\ell)_2(k)$ for all $k \neq \ell$. This means that $\tau\big((\mathfrak{c}\cdot\mathfrak{c}')_{\sigma\circ\tau_\ell} \big)$ coincides with $\tau\big((\mathfrak{c}\cdot\mathfrak{c}')_\sigma \big)$, except for the entry corresponding to ℓ. Removing this entry implies the equality in (iii). $\square$

Lemma 2.3. *The map $\Omega_{i,j} : A_3^{i,j} \to \{0,\dots,i\} \times S_{i-1,j}$ defined by*

$$\Omega_{i,j}(\ell,\sigma) = (m,\phi) \text{ iff the pair } (m,\phi) \text{ solves } \phi \circ \partial^{(m)}_{i+j+1} = \partial^{(\ell)}_{i+j+1} \circ \sigma^1 \tag{2}$$

satisfies the following:

(i) $\Omega_{i,j}$ *is a bijection.*

(ii) *For every $(\ell,\sigma) \in A_3^{i,j}$, we have that $(-1)^\ell \operatorname{sgn}(\sigma) = (-1)^m \operatorname{sgn}(\phi)$, where $(m,\phi) = \Omega_{i,j}(\ell,\sigma)$.*

(iii) *For every $\mathfrak{c} \in \mathcal{L}^{i+1}, \mathfrak{c}' \in \mathcal{L}^{j+1}$ and $(\ell,\sigma) \in A_3^{i,j}$ we have that*

$$\partial^{(\ell)}_{i+j+1}\left[\tau\big((\mathfrak{c}\cdot\mathfrak{c}')_\sigma \big)\right] = \tau\big(\partial^{(\ell)}_{i+j+1}\left[(\mathfrak{c}\cdot\mathfrak{c}'))_\sigma\right] \big) = \tau\big((\partial^{(m)}_{i+j+1}(\mathfrak{c}\cdot\mathfrak{c}')_\phi\big)$$

where $(m,\phi) = \Omega_{i,j}(\ell,\sigma)$.

Proof.

(i) To see that this yields a well defined bijection, assume $(\ell,\sigma) \in A_3^{i,j}$ are given and compute

$$(\partial^{(\ell)}_{i+j+1} \circ \sigma)(0,\dots,i+j+1) = (\sigma(0),\dots,\widehat{\sigma(\ell)},\dots,\sigma(i+j+1)).$$

The only element missing is $\sigma(\ell)$, so we must have $m = \sigma(\ell) \in \{0,\dots,i\}$ by the definition of $A_3^{i,j}$. Then, ϕ can be uniquely determined by

$$(0,\dots,\hat{m},\dots,i+j+1)_\phi = (\sigma(0),\dots,\widehat{\sigma(\ell)},\dots,\sigma(i+j+1)) \tag{3}$$

and it is clearly an $(i-1,j)$-shuffle because σ is an (i,j)-shuffle.

Conversely, assume $(m,\phi) \in \{0,\dots,i\} \times S_{i-1,j}$ are given. By the previous step, any solution $(\ell,\sigma) \in A_3^{i,j}$ of (2) satisfies $\sigma(\ell) = m$. To determine the value of ℓ, note that $\sigma(\ell)$ has only one position it can be placed in the given $(0,\dots,\hat{m},\dots,i+j+1)_\phi$ so that the resulting permutation belongs to

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[1] Here, $\partial^{(k)}_{i+j+1}$ acts on a $i+j+1$-tuple by removing its k-th element from the left, for $k = 0,\dots,i+j+1$. Permutations act on tuples as usual.

$A_3^{i,j}$: Exactly after the largest element less than m. Then, σ can be uniquely determined by (3).

(ii) Let $(\ell, \sigma) \in A_3^{i,j}$ and $\Omega_{i,j}(\ell, \sigma) = (m, \phi) \in \{0, \ldots, i\} \times S_{i-1,j}$. We look at the permutation in (3) and place the element $m = \sigma(\ell)$ in the leftmost spot of both sides, thus creating a permutation in S_{i+j+1}. The stated equality then follows by computing the sign of this permutation in two ways: If we count the number of inversions, on the left hand side yields the sign $(-1)^m \operatorname{sgn}(\phi)$, whereas the right hand side yields $(-1)^\ell \operatorname{sgn}(\sigma)$.

(iii) We use the same notation as in the proof of Lemma 2.2: For $(\ell, \sigma) \in A_3^{i,j}$, $\mathfrak{c} \in \mathcal{L}^{i+1}$ and $\mathfrak{c}' \in \mathcal{L}^{j+1}$ we write

$$\tau\big((\mathfrak{c} \cdot \mathfrak{c}')_\sigma \big) = (c_{\sigma_1(0)} \vee c'_{\sigma_2(0)} \geq \cdots \geq c_{\sigma_1(i+j+1)} \vee c'_{\sigma_2(i+j+1)}).$$

We first observe that the element $\sigma(\ell)$ plays no role in the calculation of $\sigma_2(k) = \min\{\{i+1, \ldots, i+j+1\} \cap \{\sigma(k), \ldots, \sigma(i+j+1)\}\}$, since $(\ell, \sigma) \in A_3^{i,j}$ implies $\sigma(\ell) \in \{0, \ldots, i\}$. For $k > \ell$, the same clearly holds for $\sigma_1(k)$, since the sets $\{\sigma(k), \ldots, \sigma(i+j+1)\}$ do not contain $\sigma(\ell)$. For $k < \ell$, the element $\sigma(\ell)$ is again irrelevant to the calculation of $\sigma_1(k)$.

To see this, note that $(\ell, \sigma) \in A_3^{i,j}$ implies $\sigma(\ell-1), \sigma(\ell) \in \{0, \ldots, i\}$ and also $\sigma \in S_{i,j}$. Thus, $\sigma(\ell - 1) < \sigma(\ell)$ by the shuffle condition, which means $\sigma_1(k) \leq \sigma(\ell - 1)$.

All in all, the effect of $\sigma(\ell)$ appears only on the element of the chain $\tau\big((\mathfrak{c} \cdot \mathfrak{c}')_\sigma \big)$ corresponding to the index ℓ. Hence, $\tau\big(\partial_{i+j+1}^{(\ell)} [(\mathfrak{c} \cdot \mathfrak{c}')_\sigma] \big)$ is the same as $\tau\big((\mathfrak{c} \cdot \mathfrak{c}')_\sigma \big)$ with the element corresponding to the index ℓ removed. This is precisely the first equality in (iii). The second equality follows directly from (3) or the definition of $\Omega_{i,j}$ in (2). □

We now go on to prove the main result of this section:

Proposition 2.4. *For $n \geq 0$ and $\alpha \in D_n(\mathcal{L})$ we have*

$$\partial_n(\sqcup(\alpha)) = \sqcup(\delta_n(\alpha)).$$

Proof. It suffices to check the identity for $\alpha = \mathfrak{c} \otimes \mathfrak{c}'$ where $\mathfrak{c}$ is an i-multichain, $\mathfrak{c}'$ a j-multichain and $i + j = n + 1$. Let $\mathfrak{c} = (c_0 \geq \cdots \geq c_i)$ and $\mathfrak{c}' = (c_{i+1} \geq \cdots \geq c_{i+j+1})$.

Then,

$$\mathfrak{c} \sqcup \mathfrak{c}' = \sum_{\sigma \in S_{i,j}} \operatorname{sgn}(\sigma)\, \tau\big((c \cdot c')_\sigma \big)$$

and

$$\partial_{i+j+1}(\mathfrak{c} \sqcup \mathfrak{c}') = \sum_{(\ell,\sigma) \in X^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma)\, \partial_{i+j+1}^{(\ell)} \left[\tau\big((\mathfrak{c} \cdot \mathfrak{c}')_\sigma \big)\right]$$

To make the notation cleaner, we set $\partial_{i+j+1}^{(\ell)} \left[\tau\big((\mathfrak{c} \cdot \mathfrak{c}')_\sigma \big)\right] = g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma)$. We may decompose this sum as

$$\sum_{(\ell,\sigma) \in X^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) = \sum_{s=1}^{4} \sum_{(\ell,\sigma) \in A_s^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma). \tag{4}$$

We now analyze the terms appearing on the right hand side of (4) for $s = 1, 2, 3, 4$:
($s = 1, 2$) The term corresponding to $A_1^{i,j} \cup A_2^{i,j}$ vanishes, meaning that

$$\sum_{s=1}^{2} \sum_{(\ell,\sigma)\in A_s^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) = 0.$$

Indeed, by Lemma 2.2 we have $g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma \circ \tau_\ell) = g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma)$ and $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma \circ \tau_\ell)$ for τ_ℓ the transposition appearing in the definition of the bijection $T_{i,j}$. Thus,

$$\begin{aligned}
&\sum_{s=1}^{2} \sum_{(\ell,\sigma)\in A_s^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) = \\
&= \sum_{(\ell,\sigma)\in A_1^{i,j}} (-1)^\ell \big(\operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) + \operatorname{sgn}(\sigma \circ \tau_\ell) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma \circ \tau_\ell)\big) \\
&= \sum_{(\ell,\sigma)\in A_1^{i,j}} (-1)^\ell \big(\operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) - \operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma)\big) = 0.
\end{aligned}$$

($s = 3$) The term corresponding to $A_3^{i,j}$ coincides with $\partial_i(\mathfrak{c}) \sqcup \mathfrak{c}' = \sqcup(\partial_i(\mathfrak{c}) \otimes \mathfrak{c}')$.

Indeed, by (iii) of Lemma 2.3 we can write

$$\sum_{(\ell,\sigma)\in A_3^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma) g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) = \sum_{\substack{(\ell,\sigma)\in A_3^{i,j} \\ (m,\phi)=\Omega(\ell,\sigma)}} (-1)^\ell \operatorname{sgn}(\sigma) \tau\big((\partial_{i+j+1}^{(m)}(\mathfrak{c}) \cdot \mathfrak{c}')_\phi\big)$$

and then by (i), (ii) of the same Lemma this equals

$$\begin{aligned}
&\sum_{\substack{(\ell,\sigma)\in A_3^{i,j} \\ (m,\phi)=\Omega(\ell,\sigma)}} (-1)^\ell \operatorname{sgn}(\sigma) \tau\big((\partial_{i+j+1}^{(m)}(\mathfrak{c}) \cdot \mathfrak{c}')_\phi\big) \\
&= \sum_{(m,\phi)\in\{0,\dots,i\}\times S_{i-1,j}} (-1)^m \operatorname{sgn}(\phi) \tau\big((\partial_{i+j+1}^{(m)}(\mathfrak{c}) \cdot \mathfrak{c}')_\phi\big).
\end{aligned}$$

This last sum is of course $\partial_i(\mathfrak{c}) \sqcup \mathfrak{c}'$.
($s = 4$) By an argument completely analogous to the case $s = 3$, we get

$$\sum_{(\ell,\sigma)\in A_4^{i,j}} (-1)^\ell \operatorname{sgn}(\sigma)\, g_\ell(\mathfrak{c}, \mathfrak{c}', \sigma) = (-1)^{i+1} \mathfrak{c} \sqcup \partial_j(\mathfrak{c}').$$

Summing all contributions in (4), we finally get $\partial_n(\sqcup(\mathfrak{c} \otimes \mathfrak{c}')) = \sqcup(\delta_n(\mathfrak{c} \otimes \mathfrak{c}'))$.

$\square$

The following boundary formula spells out the exact signs implicit in Proposition 2.4.

Corollary 2.5. *For $\mathfrak{c} \in \widetilde{M}_i(\mathcal{L})$ and $\mathfrak{c}' \in \widetilde{M}_j(\mathcal{L})$ we have*

$$\partial_{i+j+1}(\mathfrak{c} \sqcup \mathfrak{c}') = \partial_i(\mathfrak{c}) \sqcup \mathfrak{c}' + (-1)^{i+1}\, \mathfrak{c} \sqcup \partial_j(\mathfrak{c}').$$

Let $\pi : \widetilde{M}_*(\mathcal{P}) \to \widetilde{C}_*(\mathcal{P})$ be the map from Lemma 2.1. By Lemma 2.1 the map π induces a chain homotopy equivalence and in particular it is a chain map. This implies the following corollary, which further justifies that we ignore actual multichains in our calculations, i.e. we implicitly always apply π to shuffles.

Corollary 2.6. *For $\mathfrak{c} \in \widetilde{C}_i(\mathcal{L})$ and $\mathfrak{c}' \in \widetilde{C}_j(\mathcal{L})$ we have*

$$\partial_{i+j+1}\big(\,\pi(\mathfrak{c} \sqcup \mathfrak{c}')\,\big) = \pi\big(\,\partial_i(\mathfrak{c}) \sqcup \mathfrak{c}'\,\big) + (-1)^{i+1}\,\pi\big(\,\mathfrak{c} \sqcup \partial_j(\mathfrak{c}')\,\big).$$

3. The synor complex

For this section we return to the situation where $\mathcal{P}$ is a poset, and we consider an efficient way to understand the homology of all lower intervals through a subcomplex of its simplicial chain complex. We do so using the following notions.

An ***i-synor*** is an element x in $\mathcal{P}$ such that $\mathcal{P}_{<x}$ has non-trivial $(i-1)$st (reduced) homology, and a ***synor*** is an element that is an i-synor for some i. It is not hard to see that $\mathcal{P}_{\text{synors}}$, the subposet of synors, and $\mathcal{P}$ are homologically equivalent, that is, the inclusion induces an isomorphism of homology groups (this is a trivial consequence of a homological version of Quillen's Theorem A (see e.g., [BWW05, Corollary 4.3]).)

Let us be more explicit, and construct the synor complex $\mathcal{S}_*(\mathcal{P})$ associated with $\mathcal{P}$. Eventually, we will think of $\mathcal{S}_*(\mathcal{P})$ as a subcomplex of the reduced simplicial chain complex $(\widetilde{C}_*(\mathcal{P}), \partial_*)$ of $\mathcal{P}$. In particular, the boundary operator of the synor complex will coincide with the simplicial boundary operator. However, we first develop the ideas more abstractly:

Let $\mathcal{P}$ be a poset. A ***$\mathcal{P}$-graded complex*** (C_*, ∂_*) is a complex of vector spaces over a field $\mathbb{K}$ such that

- $C_i = \bigoplus_{x\in\mathcal{P}} C_i^{(x)}$ for $i \geq 0$,
- $\partial_i\big(\,C_i^{(x)}\big) \subseteq \bigoplus_{y\leq x} C_{i-1}^{(y)}$ for $i \geq 1$ and
- $C_{-1} = \mathbb{K}$.

We call a $\mathcal{P}$-graded complex ***strictly $\mathcal{P}$-graded*** if $\partial_i\big(\,C_i^{(x)}\,\big) \subseteq \bigoplus_{y<x} C_{i-1}^{(y)}$ for $i \geq 1$.

The reduced simplicial chain complex $(\widetilde{C}_*(\mathcal{P}), \partial_*)$ of a poset $\mathcal{P}$ is $\mathcal{P}$-graded with $\widetilde{C}_i(\mathcal{P})^{(x)}$, $i \geq 0$, being the $\mathbb{K}$-vector space spanned by the order chains of cardinality $i+1$ and largest element x. We also have, $\widetilde{C}(\mathcal{P})_{-1} = \mathbb{K}$. Note that except for trivial cases this chain complex is not strictly $\mathcal{P}$-graded.

Let (C_*, ∂) be a $\mathcal{P}$-graded chain complex and $\mathcal{J} \subseteq \mathcal{P}$ an order ideal. We set $C_i^{\mathcal{J}} = \bigoplus_{x\in\mathcal{J}} C_i^{(x)}$, $i \geq 0$ and $C_{-1}^{J} = \mathbb{K}$. Then the differential ∂_* restricts to a differential on $C_*^{\mathcal{J}}$ and therefore the following holds.

Lemma 3.1. *If (C_*, ∂_*) is a $\mathcal{P}$-graded complex and $\mathcal{J} \subseteq \mathcal{P}$ is an order ideal in $\mathcal{P}$, then $(C_*^{\mathcal{J}}, \partial_*|_{C_*^{\mathcal{J}}})$ is a $\mathcal{J}$-graded complex.*

If (C_*, ∂_*) and (D_*, δ_*) are two $\mathcal{P}$-graded complexes, then a ***$\mathcal{P}$-graded chain map*** $\phi : (C_*, \partial_*) \to (D_*, \delta_*)$ between $\mathcal{P}$-graded complexes is a map ϕ of chain complexes that satisfies $\phi(C_i^{(x)}) \subseteq D_i^{(x)}$ for all $x \in \mathcal{P}, i \geq 0$ and also $\phi(C_{-1}) = D_{-1}$.

A ***synor complex*** $(\mathcal{S}(\mathcal{P})_*, \delta_*)$ for the poset $\mathcal{P}$ is a strictly $\mathcal{P}$-graded complex together with an injective $\mathcal{P}$-graded chain map $\phi : (\mathcal{S}_*(\mathcal{P}), \delta_*) \to (\widetilde{C}_*(\mathcal{P}), \partial_*)$ such that

- (S1) $\dim_{\mathbb{K}} \mathcal{S}_i(\mathcal{P})^{(x)} = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(\mathcal{P}_{<x})$ for every $x \in \mathcal{P}$.
- (S2) for every order ideal J in $\mathcal{P}$ the restriction of ϕ to $\mathcal{S}_*(\mathcal{P})^J$ induces an isomorphism between $H_*(\mathcal{S}(\mathcal{P})^J)$ and $\widetilde{H}_*(J)$.

We first prove the existence of such objects. We should mention that uniqueness is not guaranteed in general.

Proposition 3.2. *For every poset $\mathcal{P}$ there exists a synor complex $(\mathcal{S}_*(\mathcal{P}), \delta_*)$ for $\mathcal{P}$.*

Proof. We prove the claim by induction on $|\mathcal{P}|$.

If $|\mathcal{P}| \leq 1$ then we can take as $(\mathcal{S}_*(\mathcal{P}), \delta_*)$ the reduced simplicial chain complex of $\mathcal{P}$ and set ϕ to be the identity.

Now assume $|\mathcal{P}| \geq 2$. Let x be a maximal element of $\mathcal{P}$. By induction there exists a synor complex $(\mathcal{S}_*(\mathcal{P} - x), \delta_*)$ for $\mathcal{P} - x$ together with a map $\phi : (\mathcal{S}_*(\mathcal{P} - x), \delta_*) \to (\widetilde{C}_*(\mathcal{P} - x), \partial_*)$ inducing an isomorphism in homology.

We now set $\mathcal{S}(\mathcal{P})_i^{(x)}$ to be the $\mathbb{K}$-vector space with basis indexed by symbols $\{x \star \zeta : \zeta \in Z_i\}$, where Z_i denotes a basis of $H_i(\mathcal{S}(\mathcal{P}))$. Thus $\mathcal{S}(\mathcal{P})_i^{(x)}$ satisfies (S1).

We extend ϕ to $\mathcal{S}(\mathcal{P})_i^{(x)}$ by defining $\phi(x \star \zeta)$ as $x * \phi(\zeta)$.

Define $\mathcal{S}_*(\mathcal{P})$ as the sum of $\mathcal{S}_*(\mathcal{P} - x)$ and $\mathcal{S}_*(\mathcal{P})^{(x)}$ and extend the differential by $\delta_i(x \star \zeta) = \zeta \in \mathcal{S}_{i-1}(\mathcal{P} - x)$. It follows that ϕ is a $\mathcal{P}$-graded chain map $(\mathcal{S}_*(\mathcal{P}), \delta_*) \to (\widetilde{C}_*(\mathcal{P}), \partial_*)$.

Now assume that we are given an order ideal J in $\mathcal{P}$. If the order ideal does not contain x then $\mathcal{S}_*(\mathcal{P} - x)^J = \mathcal{S}_*(\mathcal{P})^J$ and (S2) follows by induction.

Let J be an order ideal containing x. Consider the quotient complex $(\mathcal{S}_*(J, J - x), \delta_*)$ with $\mathcal{S}_i(J, J - x) = \mathcal{S}_i(J)/\mathcal{S}_i(J - x) \cong \mathcal{S}_i(\mathcal{P})^{(x)}$. The induced map $\phi' : (\mathcal{S}_*(J, J - x), \delta_*) \to (\widetilde{C}_*(J, J - x), \partial_*)$ defined by

$$\phi'(x \star \zeta + \mathcal{S}_i(J - x)) = \phi(x \star \zeta) + \widetilde{C}_i(J - x) = x * \zeta + \widetilde{C}_i(J - x)$$

for every i, induces an isomorphism in homology: Since the boundary maps of the complexes $(\mathcal{S}_*(J, J - x), \delta_*)$ and $(\widetilde{C}_*(J, J - x), \partial_*)$ are both trivial, this is the same as saying that $\phi' : \mathcal{S}_i(J, J - x) \cong \mathcal{S}_i(\mathcal{P})^{(x)} \to \widetilde{C}_i(J, J - x)$ is an isomorphism for every i. The set Z_i considered as a set of cycles in $\widetilde{C}_i(J - x)$ is a basis of $\widetilde{H}_{i-1}(J - x)$, hence the $x * \zeta + \widetilde{C}_i(J - x)$ form a basis of $\widetilde{H}_i(J, J-x)$. Thus $\phi' : \mathcal{S}_i(J, J-x) \cong \mathcal{S}_i(\mathcal{P})^{(x)} \to \widetilde{C}_i(J, J-x)$ sends a basis to a basis, so it is indeed an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & S_i(J-x) \arrow[r] \arrow[d, "\phi"] & S_i(J) \arrow[r] \arrow[d, "\phi"] & S_i(J, J-x) \arrow[r] \arrow[d, "\phi'"] & 0 \\
0 \arrow[r] & \widetilde{C}_i(J-x) \arrow[r] & \widetilde{C}_i(J) \arrow[r] & \widetilde{C}_i(J, J-x) \arrow[r] & 0
\end{tikzcd}$$

FIGURE 1. Commutative diagram with exact rows

$$\begin{tikzcd}
{} \arrow[r] & H_i(\mathcal{S}(J-x)) \arrow[r] \arrow[d, "\phi"] & H_i(\mathcal{S}(J)) \arrow[r] \arrow[d, "\phi"] & H_i(\mathcal{S}(J, J-x)) \arrow[r] \arrow[d, "\phi'"] & {} \\
{} \arrow[r] & \widetilde{H}_i(J-x) \arrow[r] & \widetilde{H}_i(J) \arrow[r] & \widetilde{H}_i(J, J-x) \arrow[r] & {}
\end{tikzcd}$$

FIGURE 2. Induced commutative diagram of long exact sequences

Figure 1 induces the diagram Figure 2 of long exact sequences. Since the ϕ and ϕ' are isomorphisms, it follows by the five lemma that the middle ϕ is an isomorphism as well. Now (S2) follows. □

The next corollary is now immediate.

Corollary 3.3. *Let $\mathcal{P}$ be a poset and $\mathcal{J} \subseteq \mathcal{P}$ an order ideal. For a synor complex $(\mathcal{S}_*(\mathcal{P}), \delta_*)$ for $\mathcal{P}$ the inclusion of $\mathcal{S}_*(\mathcal{P})$ into $\widetilde{C}_*(\mathcal{P})$ induces an isomorphism in homology of $H_*(\mathcal{S}(\mathcal{P}), \mathcal{S}(\mathcal{J}))$ and $\widetilde{H}_*(\mathcal{P}, \mathcal{J})$.*

Proof. Consider the following commutative diagram of chain groups with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & S_i(\mathcal{J}) \arrow[r] \arrow[d, "\phi"] & S_i(\mathcal{P}) \arrow[r] \arrow[d, "\phi"] & S_i(\mathcal{P}, \mathcal{J}) \arrow[r] \arrow[d, "\phi'"] & 0 \\
0 \arrow[r] & \widetilde{C}_i(J) \arrow[r] & \widetilde{C}_i(\mathcal{P}) \arrow[r] & \widetilde{C}_i(\mathcal{P}, J) \arrow[r] & 0
\end{tikzcd}$$

The two leftmost vertical arrows induce isomorphisms in homology by Proposition 3.2. By the five-lemma this then implies that the rightmost map induces an isomorphism of $H_*(\mathcal{S}(\mathcal{P}), \mathcal{S}(\mathcal{J}))$ and $H_*(C(\mathcal{P}), C(\mathcal{J}))$. □

The following property of a synor complex is obviously true in the case of the synor complex constructed in the proof of Proposition 3.2. But the lemma guarantees that it must in fact hold for any synor complex. To state it, we set up some some notation that will appear frequently from now on.

For $x \in \mathcal{P}$ we call an element of $\mathcal{S}_*(\mathcal{P})^{(x)}$ a ***principal synor chain*** of $\mathcal{P}$. For a principal synor chain $\gamma \in \mathcal{S}_i(\mathcal{P})^{(x)}$ it follows that $\phi(\gamma)$ is a linear combination of order chains $c = (x = x_0 > \cdots > x_i)$. For $0 \le \ell \le i$ we write X_ℓ^γ for the collection of all order chains $(x_0 > \cdots > x_\ell)$ for which there is a chain $c = (x_0 > \cdots > x_\ell > \cdots > x_i)$ in the support of $\phi(\gamma)$. For any chain $c = (x_0 > \cdots > x_i)$ we write $\min(c)$ for x_i.

Lemma 3.4. *Let $x \in \mathcal{P}$ and let $\gamma \in \mathcal{S}_i(\mathcal{P})^{(x)}$ be a principal synor chain. Then for $0 \le \ell \le i$ we can write $\phi(\gamma) = \sum_{\chi \in X_\ell^\gamma} \chi * \phi(\zeta_\chi)$ for cycles $\zeta_\chi \in \bigoplus_{y<min(\chi)} \mathcal{S}_{i-\ell-1}(\mathcal{P})^{(y)}$.*

Proof. We proceed by induction on ℓ. If $\ell = 0$ then by $\phi(\gamma) \in \widetilde{C}_i(\mathcal{P})^{(x)}$ it follows that $\phi(\gamma) = x * \zeta$ and $X_1^\gamma = \{\, (x) \,\}$. We then have

$$\phi(\delta_i(\gamma)) = \partial_i(\phi(\gamma)) = \partial_i(x * \zeta) = \zeta - x * \partial_{i-1}(\zeta).$$

Since the synor complex is strictly $\mathcal{P}$-graded and ϕ is a $\mathcal{P}$-graded chain map, it follows that $\partial_{i-1}(\zeta) = 0$ and thus $\phi(\gamma) = x * \zeta = x * \phi(\delta_i(\gamma))$ is precisely of the desired form, because the cycle $\delta_i(\gamma) = \zeta'_{(x)}$ is clearly in $\bigoplus_{y<x} \mathcal{S}_{i-1}(\mathcal{P})^{(y)}$.

Now assume that for some $\ell < i$ we have $\phi(\gamma) = \sum_{\chi \in X_\ell^\gamma} \chi * \phi(\zeta_\chi)$, where each ζ_χ is a cycle in $\bigoplus_{y<\min(\chi)} \mathcal{S}_{i-\ell-1}(\mathcal{P})^{(y)}$. It follows that $\phi(\zeta_\chi) = \sum_{y<min(\chi)} \phi(\gamma_{\chi,y})$, where each $\gamma_{\chi,y}$ is a principal $(i-\ell-1)$-synor chain in $\mathcal{S}_{i-\ell-1}(\mathcal{P})^{(y)}$. By the induction basis we have that $\phi(\gamma_{\chi,y}) = y * \zeta_{\chi,y}$ for some $(i-\ell-2)$-cycle $\zeta_{\chi,y} \in \bigoplus_{z<y} \mathcal{S}_{i-\ell-2}(\mathcal{P})^{(z)}$. Injectivity of ϕ easily implies that $X_{\ell+1}^\gamma = \left\{\, \chi * y \;\middle|\; \chi \in X_\ell^\gamma, \zeta_{\chi,y} \neq 0 \,\right\}$. But then

$$\phi(\gamma) = \sum_{\chi \in X_\ell^\gamma} \chi * \Big(\sum_{y<\min(\chi)} y * \phi(\zeta_{\chi,y}) \Big).$$

Setting $\zeta_{\chi'} = \zeta_{\chi,y}$ for $\chi' = \chi * y$ then yields

$$\phi(\gamma) = \sum_{\chi' \in X_{\ell+1}^\gamma} \chi' * \phi(\zeta_{\chi'})$$

where the $\zeta_{\chi'}$ are clearly of the desired form. This completes the induction step. ☐

We call a representation $\phi(\gamma) = \sum_{\chi \in X_\ell^\gamma} \chi * \phi(\zeta_\chi)$ of a principal synor chain γ an ***ℓ-representation*** of γ. Shifting our attention away from the embedding ϕ, we may identify the synor complex $(\mathcal{S}_*(\mathcal{P}), \delta_*)$ with its image under ϕ. Thus, from now on we will always consider $(\mathcal{S}_*(\mathcal{P}), \delta_*)$ as a subcomplex of $(\widetilde{C}_*(\mathcal{P}), \partial_*)$. Then, an ℓ-representation of the principal synor chain γ can be written as $\gamma = \sum_{\chi \in X_\ell^\gamma} \chi * \zeta_\chi$ where each ζ_χ is a cycle in $\bigoplus_{y<min(\chi)} \mathcal{S}_{i-\ell-1}(\mathcal{P})^{(y)}$.

4. Shuffling elements into cycles

In this section we introduce a sequence of lemmas for future reference. The first two (Lemma 4.1,Lemma 4.2) concern a vanishing property associated with the synor complex. The third (Lemma 4.3) provides a useful map that sends general order chains to synor chains. The fourth, Lemma 4.4, concerns an obvious result in poset homology.

To initiate the discussion, let $\mathfrak{c} = (x_0 > \cdots > x_m), \mathfrak{c}' = (x'_0 > \cdots > x'_m)$ be two order chains in $\mathcal{P}$. For $0 \le j \le m$ we set $\mathfrak{c} \sim_j \mathfrak{c}'$ if $x_i = x'_i$ for $i \neq j$ and $0 \le i \le m$. For a fixed order chain $\mathfrak{c} = (x_0 > \cdots > x_m)$ in $\mathcal{P}$ and an m-chain $\tau = \sum_{\ell=1}^k \lambda_\ell \mathfrak{c}_\ell \in \widetilde{C}_m(\mathcal{P})$, we write $[\tau : \mathfrak{c}]_j$ for $\displaystyle\sum_{\mathfrak{c}_\ell \sim_j \mathfrak{c}} \lambda_\ell$.

The following property of synor cycles distinguishes them from general simplicial cycles:

Lemma 4.1. *If $\zeta \in \mathcal{S}_m(\mathcal{P})$ is a cycle and $\mathfrak{c} \in \widetilde{C}_m(\mathcal{P})$ then $[\zeta : \mathfrak{c}]_j = 0$ for $j = 0, \ldots, m$.*

Proof. We proceed by induction on m.

If $m = 0$ then $j = 0$ and $\bar{\mathfrak{c}} \sim_j \mathfrak{c}$ for all 0-order chains $\bar{\mathfrak{c}}$. Thus, writing $\zeta = \sum_{\ell=0}^k \lambda_\ell \, (v_\ell)$ yields $0 = \partial_0(\zeta) = \sum_{\ell=0}^k \lambda_\ell = [\zeta : \mathfrak{c}]_0$ as desired.

Assume now that $m > 0$. By the previous section, we can write

$$\zeta = \sum_{x \in \mathcal{P}} x * \zeta_x$$

for $(m-1)$-cycles $\zeta_x \in \bigoplus_{y<x} \mathcal{S}_{m-1}(\mathcal{P})^{(y)}$. Note that for $x \in \mathcal{P} \setminus X_0^\zeta$, the corresponding cycle ζ_x is of course zero.

Set $\mathfrak{c}' = (x_1 > \cdots > x_m)$, where $\mathfrak{c} = (x_0 > \cdots > x_m)$. Consider the case $1 \le j \le m$, for which we have $[\zeta : \mathfrak{c}]_j = [\zeta_{x_0} : \mathfrak{c}']_{j-1}$. Applying the inductive hypothesis to $\zeta_{x_0} \in \mathcal{S}_{m-1}(\mathcal{P})$ and $c' \in \widetilde{C}_{m-1}(\mathcal{P})$, we get $[\zeta_{x_0} : \mathfrak{c}'] = 0$ which implies $[\zeta : \mathfrak{c}]_j = 0$.

It remains to consider the case $j = 0$. Since $\partial_m(x * \zeta_x) = \zeta_x$ and since ζ is a cycle, it follows that $\sum_{x \in \mathcal{P}} \zeta_x = 0$. In particular, if λ_x is the coefficient of the term $\mathfrak{c}'$ appearing in ζ_x, then $\sum_{x \in \mathcal{P}} \lambda_x = 0$. Note that we do not exclude the possibility $\lambda_x = 0$. Observe then that $[x * \zeta_x : \mathfrak{c}]_0 = \lambda_x$, which gives $[\zeta : \mathfrak{c}]_0 = \sum_{x \in \mathcal{P}} \lambda_x = 0$ as desired. □

We now use Lemma 4.1 to prove a surprising vanishing result associated with principal synor chains. To state it, we let $\gamma \in \mathcal{S}_m(\mathcal{P})^{(x)}$ be a principal synor chain with 0-representation $\gamma = x * \zeta$. Consider the equivalence classes $X_\ell^\gamma / \sim_j$ of the relation $\sim_j$. We write $[\chi]$ for the class of $\chi \in X_\ell^\gamma$.

Lemma 4.2. *Let $1 \le j \le \ell \le m$*[2] *and let $\gamma = x * \zeta \in \mathcal{S}_m(\mathcal{P})^{(x)}$ with ℓ-representation*

$$\gamma = \sum_{\chi \in X_\ell^\gamma} \chi * \zeta_\chi.$$

Then, for any $[\chi] \in X_\ell^\gamma / \sim_j$ we have

$$\sum_{\chi' \in [\chi]} \zeta_{\chi'} = 0.$$

[2]Note that the result is clearly false for $j = 0$.

Proof. We can write γ as

$$\gamma = \sum_{\overline{\chi} \in X_m^\gamma} \lambda_{\overline{\chi}} \overline{\chi}$$

where $\lambda_{\overline{\chi}}$ are scalars. We decompose the order chains $\overline{\chi} = (\overline{x_0} > \cdots > \overline{x_m}) \in X_m^\gamma$ into $\overline{\chi} = \chi * \chi_0$, where $\chi = (\overline{x_0} > \cdots > \overline{x_l})$ and $\chi_0 = (\overline{x_{l+1}} > \cdots > \overline{x_m})$. We take two such chain decompositions $\overline{\chi} = \chi * \chi_0$, $\overline{\chi}' = \chi' * \chi_0'$ and note that $j \leq \ell$ implies $\overline{\chi} \sim_j \overline{\chi}'$ if and only if $\chi \sim_j \chi'$ and $\chi_0 = \chi_0'$.

Now $\gamma = x * \zeta$ and ζ is a cycle. Thus any $\overline{\chi} \in X_m^\gamma$ can be written as $x * \partial_m^{(0)}(\overline{\chi})$ where $\partial_m^{(0)}(\overline{\chi}) = \overline{\chi} \setminus \{x\} \in X_{m-1}^\zeta$.

It follows by Lemma 4.1 that $0 = [\zeta : \partial_m^{(0)}(\overline{\chi})]_{j-1} = [\gamma : \overline{\chi}]_j$. These facts imply that for any $\overline{\chi} = \chi * \chi_0 \in X_m^\gamma$ with $\chi \in X_\ell^\gamma$ we have that

$$0 = [\gamma : \overline{\chi}]_j = \sum_{\chi' \in [\chi] \in X_\ell^\gamma / \sim_j} \lambda_{\chi' * \chi_0}. \tag{5}$$

On the other hand

$$\zeta_{\chi'} = \sum_{\substack{\chi_0' \\ \chi' * \chi_0' \in X_m^\ell}} \lambda_{\chi' * \chi_0'} \chi_0'.$$

It follows that

$$\begin{aligned}
\sum_{\chi' \in [\chi]} \zeta_{\chi'} &= \sum_{\chi' \in [\chi]} \sum_{\substack{\chi_0' \\ \chi' * \chi_0' \in X_m^\ell}} \lambda_{\chi' * \chi_0'} \chi_0' \\
&= \sum_{\chi_0'} \Big(\sum_{\substack{\chi' \in [\chi] \\ \chi' * \chi_0' \in X_m^\gamma}} \lambda_{\chi' * \chi_0'} \Big) \chi_0' \\
&\overset{(5)}{=} 0
\end{aligned}$$

□

Next, we construct a map $\rho : \widetilde{C}(\mathcal{P}) \to \mathcal{S}(\mathcal{P})$ whose various properties we use freely in the next section.

Lemma 4.3. *Let $\mathcal{P}$ be a poset. For each $k \geq -2$ there is a linear map*

$$\rho_k : \widetilde{C}_k(\mathcal{P}) \longrightarrow \mathcal{S}_k(\mathcal{P})$$

such that

(R0) *ρ_{-1} sends $\emptyset$ to itself, viewed as a (-1)-synor cycle.*
(R1) *For all $k \geq -1$, we have $\rho_{k-1} \circ \partial_k = \partial_k \circ \rho_k$*
(R2) *For all $k \geq 0$ and for all order chains $c = (x_0 > \cdots > x_k)$, we have that $\rho_k(c)$ is supported in $\mathcal{P}_{\leq x_0}$.*

Proof. We define ρ_k on the basis of $\widetilde{C}_k(\mathcal{P})$ given by order chains $c = (x_0 > \cdots > x_k)$ and then extend linearly.

By (R0) the map ρ_{-1} is defined. Since $C_{-2}(\mathcal{P}) = \mathcal{S}_{-2}(\mathcal{P}) = 0$, the map ρ_{-2} must be the obvious map. By $\partial_{-1}(\emptyset) = 0$ it follows that $\rho_{-2} \circ \partial_{-1} = \partial_{-1} \circ \rho_{-1}$ and hence (R1) is satisfied for $k = -1$. (R2) is trivially satisfied for $k = -1$.

Assume ρ_k has been defined for some $k \geq -1$ and (R1) and (R2) hold for k. To define $\rho_{k+1}(c)$ for any order chain $c = (x_0 > \cdots > x_{k+1})$ we compute $\partial_k \circ \rho_k \circ \partial_{k+1}(c) \overset{\text{(R1)}}{=} \rho_{k-1} \circ \partial_k \circ \partial_{k+1}(c) = 0$. Using (R2) it follows that $\rho_k \circ \partial_{k+1}(c)$ is is a synor cycle supported on $\mathcal{P}_{\leq x_0}$. By contractibility of $\mathcal{P}_{\leq x_0}$ we can use Corollary 3.3 to set $\rho_{k+1}(c)$ equal to a synor chain in $\mathcal{P}_{\leq x_0}$ whose boundary is $\rho_k \circ \partial_{k+1}(c)$. Now (R1) holds by $\partial_{k+1} \circ \rho_{k+1}(c) = \rho_k \circ \partial_{k+1}(c)$ and linearity and (R2) holds by construction. $\square$

Finally, the next elementary lemma will allow us to prove certain homologous equivalences by taking boundaries. Formally:

Lemma 4.4. *Let $\mathcal{P}$ be a poset with unique maximal element $\hat{1}$. Let $\gamma, \gamma' \in \widetilde{C}_m(\mathcal{P})$ where $\gamma = \hat{1} * \zeta$ for a cycle $\zeta \in \widetilde{C}_{m-1}(\mathcal{P}_{<\hat{1}})$. Then the following are equivalent:*

(i) *γ and γ' are homologous in $\widetilde{H}_m(\mathcal{P}, \mathcal{P}_{<\hat{1}})$.*

(ii) *$\partial_m(\gamma) = \zeta$ and $\partial_m(\gamma')$ are homologous in $\widetilde{H}_{m-1}(\mathcal{P}_{<\hat{1}})$.*

Proof. (i) $\Rightarrow$ (ii)

First notice that $\gamma' \in \widetilde{H}_m(\mathcal{P}, \mathcal{P}_{<\hat{1}})$ implies that $\partial_m(\gamma')$ is supported in $\mathcal{P}_{<1}$ (whereas the analogous result for γ is clear).

Let $c \in \widetilde{C}_{m+1}(\mathcal{P})$ and $\sigma \in \widetilde{C}_m(\mathcal{P}_{<\hat{1}})$ be such that $\gamma - \gamma' = \partial_{m+1}(c) + \sigma$. Taking boundaries yields $\partial_m(\gamma) - \partial_m(\gamma') = \partial_m(\sigma)$. Since $\partial_m(\sigma)$ is a boundary in $\mathcal{P}_{<\hat{1}}$, this implies the stated homologous equivalence in $\widetilde{H}_{m-1}(\mathcal{P}_{<\hat{1}})$.

(ii) $\Rightarrow$ (i)

Let $\sigma \in \widetilde{C}_m(\mathcal{P}_{<\hat{1}})$ be such that $\partial_m(\gamma) - \partial_m(\gamma') = \partial_m(\sigma)$. Thus, the chain $\gamma - \gamma' - \sigma \in \widetilde{C}_m(\mathcal{P})$ is a cycle. The contractibility of $\mathcal{P}$ implies it is also a boundary: There exists some $c \in \widetilde{C}_{m+1}(\mathcal{P})$ with $\gamma - \gamma' - \sigma = \partial_{m+1}(c)$. Rewriting this as $\gamma - \gamma' = \partial_{m+1}(c) + \sigma$, we get the homologous equivalence in $\widetilde{H}_m(\mathcal{P}, \mathcal{P}_{<\hat{1}})$. $\square$

5. Synor representations and the proof of the main theorem

In this section, we return to the case of a lattice $\mathcal{L}$ with maximal element $\hat{1}$ and minimal element $\hat{0}$. We will often work with the associated posets $\overline{\mathcal{L}} = \mathcal{L} \setminus \{\hat{1}\}$ and $\check{\mathcal{L}} = \mathcal{L} \setminus \{\hat{1}, \hat{0}\}$.

In view of Theorem 5.4, we will assume that the top element $\hat{1}$ is an m-synor of $\overline{\mathcal{L}}$ for some appropriate m. By $\dim_{\mathbb{K}} \mathcal{S}_m(\overline{\mathcal{L}})^{(\hat{1})} = \dim_{\mathbb{K}} \widetilde{H}_{m-1}(\check{\mathcal{L}}) \neq 0$, this allows us to pick a principal m-synor chain $\gamma = 1 * \zeta \in \mathcal{S}_m(\overline{\mathcal{L}})^{(\hat{1})}$. We will modify the ℓ-representations of γ in order to extract some innocent looking but powerful homological information in Proposition 5.3. In doing so, we will need most of the material developed in previous sections.

We begin with an trivial observation about consecutive representations of γ.

Lemma 5.1. *Let $1 \leq \ell \leq m$ and let $\gamma = \hat{1} * \zeta \in \mathcal{S}_m(\overline{\mathcal{L}})^{(\hat{1})}$ be a principal m-synor chain with ℓ- and $(\ell-1)$-representations*

$$\gamma = \sum_{\chi \in X_\ell^\gamma} \chi * \zeta_\chi = \sum_{\chi' \in X_{\ell-1}^\gamma} \chi' * \zeta_{\chi'}.$$

Then

$$\zeta_{\chi'} = \sum_{\substack{\chi \in X_\ell^\gamma \\ \partial_\ell^{(\ell)}(\chi) = \chi'}} \min(\chi) * \zeta_\chi.$$

The following less trivial result prepares us for Proposition 5.3:

Lemma 5.2. *Let $\gamma = \hat{1} * \zeta \in \mathcal{S}_m(\overline{\mathcal{L}})^{(\hat{1})}$ be a principal m-synor chain and for some $1 \leq \ell \leq m$ let $\gamma = \sum_{\chi \in X_\ell^\gamma} \chi * \zeta_\chi$ be its ℓ-representation. Then*

$$\sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \sqcup \zeta_\chi$$

and

$$\sum_{\chi' \in X_{\ell-1}^\gamma} \rho_{\ell-2}(\partial_{\ell-1}^{(0)}(\chi')) \sqcup \zeta_{\chi'}$$

are homologous in $\widetilde{H}_m(\check{\mathcal{L}})$.

Proof. We first need to check that both expressions yield cycles in $\widetilde{C}_m(\check{\mathcal{L}})$. Since the argument is the same in both cases, we confine ourselves to the proof for $\sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \sqcup \zeta_\chi$:

To see that this is supported in $\check{\mathcal{L}}$, we may use (R2) of Lemma 4.3 together with the fact that the shuffle product is supported in joins.

To see that it is a cycle, we first use the boundary formula for the shuffle product, Corollary 2.6, together with (R1) from Lemma 4.3:

$$\begin{aligned}
\partial_{m-1}\Big(\sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \sqcup \zeta_\chi \Big) &= \sum_{\chi \in X_\ell^\gamma} \partial_{\ell-1}\big(\rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \big) \sqcup \zeta_\chi \\
&\overset{(\mathrm{R1})}{=} \sum_{\chi \in X_\ell^\gamma} \rho_{\ell-2}\big(\partial_{\ell-1}(\partial_\ell^{(0)}(\chi)) \big) \sqcup \zeta_\chi \\
&= \sum_{j=0}^{l-1} (-1)^j \sum_{\chi \in X_\ell^\gamma} \rho_{\ell-2}\big(\partial_{\ell-1}^{(j)}(\partial_\ell^{(0)}(\chi)) \big) \sqcup \zeta_\chi
\end{aligned}$$

We claim that for each j, the corresponding term $\sum_{\chi\in X_\ell^\gamma} \rho_{\ell-2}\big(\,\partial_{\ell-1}^{(j)}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup \zeta_\chi$ vanishes. Indeed,

$$\sum_{\chi\in X_\ell^\gamma} \rho_{\ell-2}\big(\,\partial_{\ell-1}^{(j)}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup \zeta_\chi$$
$$= \sum_{[\chi]\in X_\ell^\gamma/\sim_{j+1}} \Big(\rho_{\ell-2}\big(\,\partial_{\ell-1}^{(j)}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup \sum_{\chi'\in[\chi]}\zeta_{\chi'}\Big) = 0,$$

where in the last step we used Lemma 4.2. Importantly, $1 \leq j+1$ makes the application of the Lemma legitimate.

The proof of the homologous equivalence is similar:

Clearly,

$$\sum_{\chi\in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi))\sqcup\zeta_\chi = \sum_{\chi\in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi))\sqcup\partial_{m-\ell+1}\big(\,\min(\chi)*\zeta_\chi\,\big)$$
$$= (-1)^\ell\partial_m\big(\sum_{\chi\in X_\ell^\gamma}\rho_{\ell-1}\big(\,\partial_\ell^{(0)}(\chi)\,\big)\sqcup(\min(\chi)*\zeta_\chi)\,\big)+ \tag{6}$$
$$(-1)^{\ell+1}\sum_{\chi\in X_\ell^\gamma}\partial_{\ell-1}\big(\,\rho_{\ell-1}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup(\min(\chi)*\zeta_\chi) \tag{7}$$

Arguing as before, we get that the first term on the right hand side of (6) is a boundary in $\check{\mathcal{L}}$. Now consider the second term (7): We have

$$(-1)^{\ell+1}\sum_{\chi\in X_\ell^\gamma}\partial_{\ell-1}\big(\,\rho_{\ell-1}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup(\min(\chi)*\zeta_\chi)$$
$$\overset{\text{(R1)}}{=} (-1)^{\ell+1}\sum_{\chi\in X_\ell^\gamma}\rho_{\ell-2}\big(\,\partial_{\ell-1}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup(\min(\chi)*\zeta_\chi)$$
$$= \sum_{j=0}^{\ell-1}(-1)^{\ell+1+j}\sum_{\chi\in X_\ell^\gamma}\rho\big(\,\partial_{\ell-1}^{(j)}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup(\min(\chi)*\zeta_\chi).$$

We treat the summands for $j<\ell-1$ and $j=\ell-1$ separately.

For $j<\ell-1$, each term vanishes as before, since we can again make the terms $\sum_{\chi'\in[\chi]}\zeta_{\chi'}$ appear and apply Lemma 4.2.

For $j=\ell-1$, we obtain by Lemma 5.1 that

$$(-1)^{\ell+1+\ell-1}\sum_{\chi\in X_\ell^\gamma}\rho_{\ell-2}\big(\,\partial_{\ell-1}^{(\ell-1)}(\partial_\ell^{(0)}(\chi))\,\big)\sqcup(\min(\chi)*\zeta_\chi)$$
$$= \sum_{\chi'\in X_{\ell-1}^\gamma}\rho_{\ell-2}(\partial_{\ell-1}^{(0)}(\chi'))\sqcup\zeta_{\chi'}.$$

Thus, we have managed to express the difference of cycles

$$\sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \sqcup \zeta_\chi - \sum_{\chi' \in X_{\ell-1}^\gamma} \rho_{\ell-2}(\partial_{\ell-1}^{(0)}(\chi')) \sqcup \zeta_{\chi'}$$

as a term (6), which is a boundary in $\check{\mathcal{L}}$.

□

Now to the relative version:

Proposition 5.3. *Let $\gamma = \hat{1} * \zeta \in \mathcal{S}_m(\overline{\mathcal{L}})^{(\hat{1})}$ be a principal m-synor chain and for some $1 \le \ell \le m$ let $\gamma = \sum_{\chi \in X_\ell^\gamma} \chi * \zeta_\chi$ be its ℓ-representation.*

Then γ is homologous to

$$\sum_{\chi \in X_\ell^\gamma} \rho_\ell(\chi) \sqcup \zeta_\chi$$

in $\widetilde{H}_m(\overline{\mathcal{L}}, \check{\mathcal{L}})$.

Proof. Let $\gamma' = \sum_{\chi \in X_\ell^\gamma} \rho_\ell(\chi) \sqcup \zeta_\chi$. By Lemma 4.4 it suffices to show that $\partial_m(\gamma)$ and $\partial_m(\gamma')$ are homologous as cycles in $\widetilde{H}_{m-1}(\check{\mathcal{L}})$. We first check that both boundary terms are supported in $\check{\mathcal{L}}$.

For $\partial_m(\gamma) = \zeta$ this is clear.

For $\partial_m(\gamma')$ we follow a similar route to the proof of Lemma 5.2: We compute

$$\begin{aligned}
\partial_m\Big(\sum_{\chi \in X_\ell^\gamma} \rho_\ell((\chi)) \sqcup \zeta_\chi\Big) &= \sum_{\chi \in X_\ell^\gamma} \partial_\ell(\rho_\ell(\chi)) \sqcup \zeta_\chi \\
&\overset{\text{(R2)}}{=} \sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell(\chi)) \sqcup \zeta_\chi \\
&= \sum_{j=0}^{\ell} (-1)^j \sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(j)}(\chi)) \sqcup \zeta_\chi.
\end{aligned}$$

For each $j > 0$, the corresponding term

$$\sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(j)}(\chi)) \sqcup \zeta_\chi = \sum_{[\chi] \in X_\ell^\gamma/\sim_j} \Big(\rho_{\ell-1}\big((\partial_\ell^{(j)}(\chi))\big) \sqcup \sum_{\chi' \in [\chi]} \zeta_{\chi'}\Big)$$

vanishes by Lemma 4.2.

Then the remaining term $\sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \sqcup \zeta_\chi$ is supported in $\check{\mathcal{L}}$ by (R2) of Lemma 4.3.

Now let us prove the homologous equivalence. The above argument shows that

$$\partial_m(\gamma') = \sum_{\chi \in X_\ell^\gamma} \rho_{\ell-1}(\partial_\ell^{(0)}(\chi)) \sqcup \zeta_\chi.$$

Applying Lemma 4.2 iteratively, namely for $\ell' = \ell, \ell-1, \dots, 1$, we get that the terms

$$\sum_{\chi \in X^{\gamma}_{\ell-s}} \rho_{\ell-s-1}(\partial^{(0)}_{\ell-s}(\chi)) \sqcup \zeta_\chi$$

belong to the same homology class in $\widetilde{H}_{m-1}(\check{\mathcal{L}})$, for $s = 0, 1, \dots, \ell$. Observe that the term corresponding to $s = 0$ is precisely $\partial_m(\gamma')$, whereas the term corresponding to $s = \ell$ is $\partial_m(\gamma)$: This is because, for $s = \ell$ we have $X^{\gamma}_{\ell-s} = X^{\gamma}_0 = \{\hat{1}\}$, $\zeta_{\hat{1}} = \zeta$ and so by (R1) of Lemma 4.3:

$$\sum_{\chi \in X^{\gamma}_{\ell-s}} \rho_{\ell-s-1}(\partial^{(0)}_{\ell-s}(\chi)) \sqcup \zeta_\chi = \rho_{-1}(\partial^{(0)}_0(\hat{1})) \sqcup \zeta_{\hat{1}} = \rho_{-1}(\emptyset) \sqcup \zeta = \zeta = \partial_m(\gamma).$$

This of course proves the stated homologous equivalence.

□

We are now in position to formulate and prove our main result on lattice homology.

Theorem 5.4. *Let $\mathcal{L}$ be a lattice and let $1 \leq i_1, i_2$ and $0 \leq k \leq \min\{i_1, i_2\}$ be integers. If $\hat{1} \in \overline{\mathcal{L}}$ is an (i_1+i_2-k-1)-synor, then there is an (i_1-1)-synor x of $\overline{\mathcal{L}}$ and an (i_2-1)-synor y of $\overline{\mathcal{L}}$ such that $\hat{1} = x \vee y$.*

Proof. Since $\hat{1}$ is an (i_1+i_2-k-1)-synor, we have $\widetilde{H}_{i_1+i_2-k-1}(\overline{\mathcal{L}}, \check{\mathcal{L}}) \cong \widetilde{H}_{i_1+i_2-k-2}(\check{\mathcal{L}}) \neq 0$. By Corollary 3.3 we deduce $H_{i_1+i_2-k-1}(\mathcal{S}(\overline{\mathcal{L}}), \mathcal{S}(\check{\mathcal{L}})) \neq 0$, so we may choose a principal synor chain $\gamma = \hat{1} * \zeta \in \mathcal{S}_{i_1+i_2-k-1}(\overline{\mathcal{L}})^{(\hat{1})}$ which represents a non-trivial homology class in $H_{i_1+i_2-k-1}(\mathcal{S}(\overline{\mathcal{L}}), \mathcal{S}(\check{\mathcal{L}}))$. Corollary 3.3 (for the inclusion map) implies that γ, viewed as a cycle in $\widetilde{H}_{i_1+i_2-k-1}(\overline{\mathcal{L}}, \check{\mathcal{L}})$ is also non-trivial.

Let

$$\gamma = \sum_{\chi \in X^{\gamma}_{i_1-1}} \chi \sqcup \zeta_\chi$$

be the (i_1-1)-representation of γ. Note that to carry out this representation legitimately, we need $k \leq i_2$, for otherwise the terms ζ_χ do not make sense. By Proposition 5.3, γ is homologous to

$$\sum_{\chi \in X^{\gamma}_{i_1-1}} \rho_{i_1-1}(\chi) \sqcup \zeta_\chi \tag{8}$$

in $\widetilde{H}_{i_1+i_2-k-1}(\overline{\mathcal{L}}, \check{\mathcal{L}})$.

Since γ is non-trivial, it follows that (8) must also represent a non-zero element in $\widetilde{H}_{i_1+i_2-k-1}(\overline{\mathcal{L}}, \check{\mathcal{L}})$. Thus there must be an order chain $\chi \in X^{\gamma}_{i_1-1}$ such that $\hat{1}$ appears in the summand $\rho_{i_1-1}(\chi) \sqcup \zeta_\chi$ of (8).

By $\rho_{i_1-1}(\chi) \in \mathcal{S}_{i_1-1}(\mathcal{L})$ of Lemma 4.3 we know that the maximal elements of order chains appearing in $\rho_{i_1-1}(\chi)$ are (i_1-1)-synors. Analogously, the maximal elements of order chains appearing in ζ_χ are (i_2-k-1)-synors. Since the computation of the shuffle product $\rho_{i_1-1}(\chi) \sqcup \zeta_\chi$ requires taking joins, there must be an (i_1-1)-synor x appearing in $\rho_{i_1-1}(\chi)$ and an (i_2-k-1)-synor y appearing in ζ_χ such that $x \vee y = \hat{1}$.

We are almost finished: We just need to find an (i_2-1)-synor z that satisfies $y \le z$; then, $x \vee z = \hat{1}$ will be the required decomposition by the monotonicity of the join. In case when $k=0$, we can simply choose $z := y$. Otherwise, we look into χ: the whole of χ lies above ζ_χ and thus above y. By construction the s-th element of χ, for $s = 0, \ldots, i_1-1$, is an $(i_1+i_2-k-1-s)$-synor. Thus we can find a t-synor in χ for every $i_2-k \le t \le i_1+i_2-k-1$. As $1 \le k \le i_1$, the number i_2-1 falls in this interval, and hence the corresponding (i_2-1)-synor z appearing in χ yields the desired element. □

The proof of Theorem 1.2 is now immediate.

Proof of Theorem 1.2: Apply Theorem 5.4 in the case $k=0$. □

6. Applications

Let us give two more applications of Theorem 5.4 to the Betti table $\beta_{ij}(S/\mathfrak{I})$ of a monomial ideal $\mathfrak{I}$. First we introduce multigraded Betti numbers $\beta_{i,m}(S/\mathfrak{I}) = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(S/\mathfrak{I}, \mathbb{K})_m$ for $m \in \overline{\mathrm{lcm}(\mathfrak{I})}$. Recall that by [GPW99] we have $\beta_{i,m}(S/\mathfrak{I}) = \dim_{\mathbb{K}} \widetilde{H}_{i-2}\big(\,(1,m)\,\big)$. We set

$$a_i(S/\mathfrak{I}) = \big|\{m \mid \beta_{i,m}(S/\mathfrak{I}) \neq 0\}\big|.$$

Theorem 6.1. *Let $\mathfrak{I}$ be a monomial ideal in S and $0 \le i_1, i_2, k$ numbers such that $k \le \min\{i_1, i_2\}$.*

If $1 \le i_1, i_2$ then for any $m \in \mathrm{lcm}(\mathfrak{I})$ of degree $t_{i_1+i_2-k}$ there are $n_1, n_2 \le m$ in $\mathrm{lcm}(\mathfrak{I})$ such that $m = \mathrm{lcm}(n_1, n_2)$, n_1 is of degree t_{i_1} and n_2 of degree t_{i_2} such that $\beta_{i_1,n_1}(S/\mathfrak{I}) \neq 0$ and $\beta_{i_2,n_2}(S/\mathfrak{I}) \neq 0$.

In particular, we have for all $0 \le i_1, i_2$

(i)
$$t_{i_1+i_2-k}(S/\mathfrak{I}) \le t_{i_1}(S/\mathfrak{I}) + t_{i_2}(S/\mathfrak{I}).$$

(ii)
$$a_{i_1+i_2}(S/\mathfrak{I}) \le a_{i_1}(S/\mathfrak{I}) \cdot a_{i_2}(S/\mathfrak{I}).$$

Proof. We may assume that $t_{i_1+i_2-k}(S/\mathfrak{I}) > 0$. Then there is $m \in \mathrm{lcm}(\mathfrak{I})$ of degree $t_{i_1+i_2-k}(S/\mathfrak{I})$ such that

$$\beta_{i,m}(S/\mathfrak{I}) = \widetilde{H}_{i_1+i_2-k-2}\big(\,(1,m)\,\big) \neq 0.$$

Now apply Theorem 5.4 to $\mathcal{L} = [1,m]$ to obtain an (i_1-1)-synor n_1 and an (i_2-1)-synor n_2 in $\mathcal{L}$ such that $m = n_1 \vee n_2 = \mathrm{lcm}(n_1, n_2)$. Thus $\beta_{i_1,n_1}(S/\mathfrak{I}) = \widetilde{H}_{i_1-2}\big(\,(1,n_1)\,\big) \neq 0$ and $\beta_{i_2,n_2}(S/\mathfrak{I}) = \widetilde{H}_{i_2-2}\big(\,(1,n_2)\,\big) \neq 0$. This concludes the first part of the proof.

It remains to verify (i) and (ii). If $i_1 = 0$ or $i_2 = 0$ then both are trivial. Assume $1 \le i_1, i_2$. For assertion (i) observe that $\deg(n_1) \le t_{i_1}(S/\mathfrak{I})$ and $\deg(n_2) \le t_{i_2}(S/\mathfrak{I})$. This concludes the proof of (i) by

$$t_{i_1+i_2-k}(S/\mathfrak{I}) = \deg(m) = \deg(\mathrm{lcm}(n_1,n_2)) \le \deg(n_1) + \deg(n_2)$$

$$\leq t_{i_1}(S/\mathfrak{I}) + t_{i_2}(S/\mathfrak{I}).$$

For assertion (ii) set $k = 0$. Then observe that an pair n_1, n_2 for which $\beta_{i_1,n_1}(S/\mathfrak{I}) \neq 0$ and $\beta_{i_1,n_1}(S/\mathfrak{I}) \neq 0$ can only contribute the single $m = \mathrm{lcm}(n_1, n_2)$ to the count of $a_{i_1+i_2}(S/\mathfrak{I})$. Thus we obtain the desired inequality $a_{i_1}(S/\mathfrak{I}) \geq a_{i_2}(S/\mathfrak{I}) \geq a_{i_1+i_2}(S/\mathfrak{I})$.

□

Now Theorem 1.3 becomes a special case of Theorem 6.1(i).

Proof of Theorem 1.3: Apply Theorem 6.1(i) in the case $k = 0$. □

We note that subadditivity is a sharp inequality. For example for a natural number $a \geq 1$ the ideals $\mathfrak{I}_a = \left(x_1^a, \ldots, x_n^a\right)$ in S satisfy $t_k(S/\mathfrak{I}) = ak$ for all $0 \leq k \leq \mathrm{pd}(S/\mathfrak{I}_a)$.

On the other hand the following example shows that Theorem 6.1(i) is indeed stronger than Theorem 1.3.

Example 6.2. Consider the ideal $I = \big(af, bf, cf, df, ef, abcde\big)$ in the polynomial ring $\mathbb{K}[a, b, c, d, e, f]$. Then Macaulay2 will spit out the following Betti-table (which can also be easily verified using (1))

	0	1	2	3	4	5
o5 = total:	1	6	11	10	5	1
0:	1	.	.	.	.	.
1:	.	5	10	10	5	1
2:	.	.	.	.	.	.
3:	.	.	.	.	.	.
4:	.	1	1	.	.	.

from which we obtain $t_0 = 0$, $t_1 = 5$, $t_2 = 6$, $t_3 = 4$, $t_4 = 5$, $t_5 = 6$. Theorem 1.3 for $i_1 + i_2 = 4$ yields

$$5 = t_4 \leq \min\{\, t_3 + t_1, t_2 + t_2 \,\} = \min\{\, 12, 16 \,\} = 12$$

whereas by Theorem 6.1(i) for $i_i = i_2 = 3$ and $k = 2$ we get

$$5 = t_4 \leq t_3 + t_3 = 10.$$

Let us take a look into a class of ideals that extends the above example. We start by gluing a hollow p-simplex to a hollow q-simplex along a vertex, where $p > q \geq 2$. The resulting simplicial complex $K_{p,q}$ yields a face lattice which is of course atomic. By [Phan06], there is a corresponding monomial ideal $I_{p,q}$ whose LCM-lattice coincides with the face lattice of $K_{p,q}$. Carrying out the correspondence, it turns out that $I_{p,q}$ can be taken to be an ideal living in $K[x_0, x_1, \ldots, x_p, y_0, y_1, \ldots, y_q]$ and generated by the following families of monomials:

- $x_0 y_0$
- $Y \cdot x_i$, for $i = 1, \ldots, p$
- $X \cdot y_j$, for $j = 1, \ldots, q$

where $X = \prod_{i=0}^{p} x_i$ and $Y = \prod_{j=0}^{q} y_j$.

The sequence of maximal shifts of $I_{p,q}$ is then $t_0 = 0, t_1 = p+2, t_2 = p+3, \ldots, t_{q+1} = p+q+2, t_{q+2} = 2q+3, t_{q+3} = 2q+4, \ldots t_{p+1} = p+q+2$ where the computation is an easy application of (1). One should note that the two maxima in the sequence correspond exactly to the dimensions of the holes of the two simplices.

By Theorem 6.1, we get the inequality $t_{q+3} \leq t_{q+2} + t_{q+2} = 4q+6$, while the best inequality attained by Theorem 1.3 is $t_{q+3} \leq p + 2q + 5$. Hence, by fixing the value of q and increasing p, we see that generalized subadditivity gives a constant bound, while the $k = 0$ case gives a bound growing linearly in p.

Next we show that a synor complex of $\overline{\mathrm{lcm}(\mathfrak{I})}$ can be used to construct a minimal free resolution of $\mathfrak{I}$.

Consider $\mathrm{lcm}(\mathfrak{I})$ be the LCM-lattice of the monomial ideal $\mathfrak{I}$ in the polynomial ring S and $\big(\mathcal{S}_*(\overline{\mathrm{lcm}(\mathfrak{I})}), \partial_*\big)$ a synor-complex for $\overline{\mathrm{lcm}(\mathfrak{I})} = \mathrm{lcm}(\mathfrak{I}) \setminus \{\hat{0}\} = \mathrm{lcm}(\mathfrak{I}) \setminus \{1\}$ where as usual we consider 1 as the monomial $1 = x_1^0 \cdots x_n^0$. For a number $i \geq -1$ and a monomial $m \in \overline{\mathrm{lcm}(\mathfrak{I})}$ set

$$F_i = \bigoplus_{m \in \overline{\mathrm{lcm}(\mathfrak{I})}} \mathcal{S}_{i-1}\big(\overline{\mathrm{lcm}(\mathfrak{I})}\big)^{(m)} \otimes S(-m).$$

Here we consider $\mathcal{S}_{-1}\big(\overline{\mathrm{lcm}(\mathfrak{I})}\big)$ as concentrated in multidegree $1 = x_1^0 \cdots x_n^0$ and write $S(-m)$ for the multigraded rank 1 free S-module where the multidegree of a monomial m' is mm'. For $i \geq 1$ we define $\delta_i : F_i \to F_{i-1}$ in the following way. Let $\zeta \in \mathcal{S}_i\big(\overline{\mathrm{lcm}(\mathfrak{I})}\big)^{(m)}$ be a synor chain. Since a synor complex of $\overline{\mathrm{lcm}(\mathfrak{I})}$ is strictly $\overline{\mathrm{lcm}(\mathfrak{I})}$-graded we have that

$$\partial_i(\zeta) = \sum_{m' \in \overline{\mathrm{lcm}(\mathfrak{I})}_{<m}} \zeta_{m'}$$

for chains $\zeta_{m'} \in \mathcal{S}_{i-1}\big(\overline{\mathrm{lcm}(\mathfrak{I})}\big)^{(m')}$. We then set

$$\delta_i(\zeta \otimes 1) = \sum_{m' \in \overline{\mathrm{lcm}(\mathfrak{I})}_{<m}} \frac{m}{m'}\, \zeta_{m'} \otimes 1.$$

Note, the fact that $\big(\mathcal{S}_*(\overline{\mathrm{lcm}(\mathfrak{I})}), \partial_*\big)$ is strictly $\overline{\mathrm{lcm}(\mathfrak{I})}$-graded implies that δ_* is a well defined homomorphism of multigraded free S-modules.

We call (F_*, δ_*) a ***synor-resolution*** of $S/\mathfrak{I}$.

Theorem 6.3. *A synor-resolution of a monomial ideal $\mathfrak{I}$ is a minimal free resolution of $S/\mathfrak{I}$.*

Proof. First we must show that the synor-resolution (F_*, δ_*) is a resolution of $\mathfrak{I}$.

We have seen that (F_*, δ_*) is a sequence of homomorphisms of multigraded free S-modules. To prove that it indeed is an exact complex of multigraded modules it suffices to verify for each monomial, or equivalently mulidegree,

m, that the m-graded part $(F_*^{(m)}, \delta_*|_{F_*^{(m)}})$ of (F_*, δ_*) is an exact sequence of homomorphisms of vector spaces. From $F_i^{(m)} \cong \bigoplus_{m' \in \overline{\mathrm{lcm}(\mathfrak{I})}_{\leq m'}} \mathcal{S}_{i-1}(\overline{\mathrm{lcm}(\mathfrak{I})})^{(m)}$ and the construction of δ_* we deduce that $(F_*^{(m)}, \delta)$ is an exact sequence if any only if for $J = \overline{\mathrm{lcm}(\mathfrak{I})}_{\leq m}$ we have that $(\mathcal{S}_*(\mathcal{L})^J, \partial_*)$ is exact. The the latter is satisfied by (S2) and the fact that $\widetilde{H}_*(J) = 0$ since J is a cone over m.

Is remains to verify that the cokernel of δ_0 is $S/\mathfrak{J}$. Let $m_1, \ldots, m_r$ be the minimal monomial generators of I. These then are also the minimal elements of $\overline{\mathrm{lcm}(\mathfrak{I})}$. It follows that $\mathcal{S}_0(\mathcal{L})^{(m)} = \mathbb{K}$ if $m = m_i$ for some i and 0 otherwise. For $1 \in \mathcal{S}_0(m)$ we have $\partial_1(1) = 1$ and hence $\delta_1(1) = m(1 \otimes 1)$. It follows that the image of δ_1 in $F_0 = \mathcal{S}_{-1}(\mathcal{L}) \otimes S(-1) \cong S$ is $\mathfrak{I}$. This completes the proof. $\square$

The preceding proof uses arguments very similar to the proof of Proposition 1.2 in [BS98]. On the other hand the setting here is purely homological and there does not have to be a cellular complex whose cellular chain complex is the synor complex.

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