# DeLaM: A Dependent Layered Modal Type Theory for Meta-programming 

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#### Abstract

We scale layered modal type theory to dependent types, introducing DeLAM, dependent layered modal type theory. This type theory is novel in that we have one uniform type theory in which we can not only compose and execute code, but also intensionally analyze the code of types and terms. The latter in particular allows us to write tactics as meta-programs and use regular libraries when writing tactics. DELAM provides a sound foundation for proof assistants to support type-safe tactic mechanism.


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## 1 INTRODUCTION

Hu and Pientka [2024a] develop a layered modal type theory which supports pattern matching on code. A critical idea lying in this system is the layering principle. The layering principle begins with a core language, e.g. simply typed $\lambda$-calculus (STLC). Then we extend the core language with a layer of the $\square$ modality which supports metaprogramming and intensional analysis. Though only two layers are demonstrated by Hu and Pientka [2024a], in principle, this extension can be iterated indefinite number of times, forming an arbitrary $n$-layered modal type theory.

In another dimension, instead of adding more and more layers, we could also increase the expressive power of the core language. One interesting candidate for a core language is Martin-Löf type theory (MLTT). MLTT is the foundation for many type-theory-based proof assistants, including Coq, Agda and Lean. Treating MLTT as the core language and applying the layering principle to it could yield a dependently typed system that allows to meta-program and intensionally analyze code of itself without forgoing the consistency of the overall system. This feature gives a solid foundation for proof assistants to support truly type-safe meta-programming. Due to the layering principle, libraries written for bare MLTT can also be used during meta-programming. For example, we can use the same data structures for natural numbers and lists for both programs and meta-programs. Meanwhile in reality, e.g. in Coq, we have at least four unexchangeable notions natural numbers: the natural numbers defined inductively in Gallina, Ltac's natural numbers, failure levels and hint database's search levels. Therefore, we can foresee that the layering principle also has the additional benefit in engineering.

In this technical report, we first extend the previous layered modal type theory with contextual variables. They are necessary to enable recursion on code. We justify the decidability of conversion checking following Abel et al. [2017]'s reducibility proof. We then scale the setup all the way to MLTT, introducing DeLAM, Dependent Layered Modal type theory. We then scale the reducibility proof to DeLaM and therefore justify its decidability of conversion checking. A corollary is the consistency of DELAM, hence showing that this type theory can be used as a foundation for proof assistants.

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## 2 SUPPORTING CONTEXTUAL VARIABLES

Hu and Pientka [2024a] present a layered modal type theory which supports pattern matching on code and establish a normalization proof via a presheaf model. However, this form of intentional analysis is not in the most desired form: we cannot perform recursion on the structure of code. This limitations comes from two aspects:

- STLC lacks a generic notion of types. When we do recursion on, for example, a $\lambda$ expression, the type of the $\lambda$ and the type of the body necessarily differ. Therefore, we are not able to formulate a recursion principle without type variables as in System F, or a variable of type Set in dependent type theory. This problem naturally goes away if we employ a stronger core type system, so it is not our primary concern.
- However, System F or dependent types does not give us a way to capture local contexts using a variable. Consider again the $\lambda$ case, even if we have type variables, the recursion on the body is in an extended local context, so for the recursion to work, we must be able to capture contextual variables, which capture local contexts as global variables.
In this section, we focus on contextual variables. We develop the syntactic theory for our 2-layered contextual model type theory with contextual variables, and show its consistency via a reducibility predicate argument.


### 2.1 Well-formedness of Contexts and Types

With contextual variables, the type theory becomes "slightly" dependently typed, in that both global and local contexts, and types can depend on contextual variables, so their well-formedness requires dedicated judgments. The syntax of contexts and types is:

| $i$ |  | (Layer, $i \in[0,1])$ <br> $x, y$ <br> $u$ |
| ---: | ---: | ---: |
| $g$ | (Local variables) |  |
| (Global variables) |  |  |
| $S, T$ | $:=\operatorname{Nat}\|\square(\Gamma \vdash T)\| S \longrightarrow T \mid(g: C t x) \Rightarrow T$ | (Contextual variables) |
| $B$ | $:=u:(\Gamma \vdash T) \mid g: C t x$ | (Types, Typ) |
| $\Phi, \Psi:=\cdot \mid \Phi, B$ | (Global bindings) |  |
| $\Gamma, \Delta$ | $:=\cdot\|g\| \Gamma, x: T$ | (Global contexts) |
| (Local contexts) |  |  |

Their well-formedness judgments are:

$$
\begin{aligned}
& \frac{\vdash \Psi}{\Psi \vdash_{i} \mathrm{Nat}} \quad \frac{\Psi \vdash_{i} S \quad \Psi \vdash_{i} T}{\Psi \vdash_{i} S \longrightarrow T} \quad \frac{\Psi \vdash_{0} \Delta \quad \Psi \vdash_{0} T}{\Psi \vdash_{1} \square(\Delta \vdash T)} \quad \frac{\Psi, g: \mathrm{Ctx} \vdash_{1} T}{\Psi \vdash_{1}(g: \mathrm{Ctx}) \Rightarrow T}
\end{aligned}
$$

$\vdash \Psi$ states the well-formedness of a global context $\Psi$. There are two kinds of bindings in a global context: either $x:(\Gamma \vdash T)$ as in previous modal type theory, or $g: C t x$ which is a contextual variable representing a local context. Eventually, $g$ will be substituted by a concrete and well-formed local context. Due to the introduction of contextual variables, the system clearly becomes dependently typed. $\Psi \vdash_{i} \Gamma$ states the well-formedness of a local context $\Gamma$ at layer $i$. Since we are dealing with a 2 -layered system now, we know $i \in[0,1]$. The base case of a local context can either be an empty local context, or be a well-scoped contextual variable. The layer $i$ is propagated to the well-formedness judgment of types $\Psi \vdash_{i} T$, which states that $T$ is well-formed in $\Psi$ at layer $i$.

Notice that the well-formedness of $T$ does not depend on any local context. This judgment essentially combines $T$ core and $T$ type predicates in Hu and Pientka [2024a], with contextual variables taken into consideration. Contextual variables is introduced by a special meta-function type ( $g: \mathrm{Ctx}$ ) $\Rightarrow T$, which pushes a contextual variable $g$ to the global context. This type can be seen as a "meta-function space" in which we define macros or meta-programs, so intuitively, this type and its terms can only live at layer 1 . We prove the presupposition lemma of these three judgments:

Lemma 2.1 (Presupposition).

- If $\Psi \vdash_{i} \Gamma$, then $\vdash \Psi$.
- If $\Psi \vdash_{i} T$, then $\vdash \Psi$.

Lemma 2.2 (Lifting).

- If $\Psi \vdash_{0} \Gamma$, then $\Psi \vdash_{1} \Gamma$.
- If $\Psi \vdash_{0} T$, then $\Psi \vdash_{1} T$.


### 2.2 Weakenings

Similar to previous layered modal type theories, we need two notions of weakenings: a global one and a local one. In this section, due to contextual variables, we change the definition of weakenings based on their counterparts in Hu and Pientka [2024a]:

$$
\begin{aligned}
& \gamma:=\mathrm{id}|q(\gamma)| p(\gamma) \\
& \tau:=\mathrm{id}|q(\tau)| p(\tau)
\end{aligned}
$$

We will use a global weakening $p$ (id) in the typing rule of meta-functions in $\Gamma[p(\mathrm{id})]$ to account for the insertion of $g$ : Ctx to the global context $\Psi$. In the following, we examine the properties of weakenings. First, we define the composition of weakenings. We only define the one for global weakenings and the one for local weakenings is completely identical:

$$
\begin{aligned}
\text { id } \circ \gamma^{\prime}: & : \gamma^{\prime} \\
\gamma \circ \mathrm{id} & :=\gamma \\
p(\gamma) \circ q\left(\gamma^{\prime}\right) & :=p\left(\gamma \circ \gamma^{\prime}\right) \\
q(\gamma) \circ q\left(\gamma^{\prime}\right) & :=q\left(\gamma \circ \gamma^{\prime}\right) \\
\gamma \circ p\left(\gamma^{\prime}\right) & :=p\left(\gamma \circ \gamma^{\prime}\right)
\end{aligned}
$$

Lemma 2.3 (Associativity).

- $\left(\gamma \circ \gamma^{\prime}\right) \circ \gamma^{\prime \prime}=\gamma \circ\left(\gamma^{\prime} \circ \gamma^{\prime \prime}\right)$
$\bullet\left(\tau \circ \tau^{\prime}\right) \circ \tau^{\prime \prime}=\tau \circ\left(\tau^{\prime} \circ \tau^{\prime \prime}\right)$
Then we apply global weakenings to types and contexts:

$$
\begin{aligned}
\operatorname{Nat}[\gamma] & :=\text { Nat } \\
S \longrightarrow T[\gamma] & :=(S[\gamma]) \longrightarrow(T[\gamma]) \\
\square(\Gamma \vdash T)[\gamma] & :=\square(\Gamma[\gamma] \vdash T[\gamma]) \\
(g: \mathrm{Ctx}) \Rightarrow T[\gamma] & :=(g: \mathrm{Ctx}) \Rightarrow(T[q(\gamma)]) \\
\cdot[\gamma] & := \\
g[\gamma] & :=g \quad \text { (properly weakened depending on the name representation) } \\
\Gamma, x: T[\gamma] & :=(\Gamma[\gamma]), x:(T[\gamma])
\end{aligned}
$$

This definition of global weakenings admits the following lemma:
Lemma 2.4 (Algebra of global weakenings).

- $T[i d]=T$
- $\Gamma[i d]=\Gamma$
- $T[\gamma]\left[\gamma^{\prime}\right]=T\left[\gamma \circ \gamma^{\prime}\right]$
- $\Gamma[\gamma]\left[\gamma^{\prime}\right]=\Gamma\left[\gamma \circ \gamma^{\prime}\right]$

The well-formedness of weakenings is given by the following rules:

$$
\begin{aligned}
& \frac{\vdash \Psi}{\mathrm{id}: \Psi \Longrightarrow{ }_{g} \Psi} \\
& \frac{\gamma: \Psi \Longrightarrow g \quad \Psi \quad \Psi \vdash B}{p(\gamma): \Psi, B \Longrightarrow g} \Longrightarrow_{g} \Phi \\
& \frac{\gamma: \Psi \Longrightarrow_{g} \Phi \quad \Phi \vdash B \quad \Psi \vdash B[\gamma]}{q(\gamma): \Psi, B[\gamma] \Longrightarrow_{g} \Phi, B} \\
& \frac{\Psi \vdash_{i} \Gamma}{\text { id }: \Psi ; \Gamma \Longrightarrow_{i} \Gamma} \quad \frac{\tau: \Psi ; \Gamma \not \Longrightarrow_{i} \Delta \quad \Psi \vdash_{i} T}{p(\tau): \Psi ; \Gamma, x: T \Longrightarrow \Longrightarrow_{i} \Delta} \quad \frac{\tau: \Psi ; \Gamma \Longrightarrow{ }_{i} \Delta}{q(\tau): \Psi ; \Gamma, x: T \Longrightarrow \vdash_{i} T}{ }_{i} \Delta, x: T
\end{aligned}
$$

where the well-formedness of global bindings $\Psi \vdash B$ is given as follows:

$$
\frac{\vdash \Psi}{\Psi \vdash g: \operatorname{Ctx}} \quad \frac{\Psi \vdash_{0} \Gamma \quad \Psi \vdash_{0} T}{\Psi \vdash u:(\Gamma \vdash T)}
$$

The identity case for local weakenings is slightly more complex because we must take the contextual variables into consideration.
Then we can prove the following global weakening lemma:
Lemma 2.5 (Global weakenings).

- If $\Phi \vdash_{i} \Gamma$ and $\gamma: \Psi \Longrightarrow g$, then $\Psi \vdash_{i} \Gamma[\gamma]$.
- If $\Phi \vdash_{i} T$ and $\gamma: \Psi \Longrightarrow g$, then $\Psi \vdash_{i} T[\gamma]$.

Proof. Mutual induction on $\Phi \vdash_{i} \Gamma$ and $\Phi \vdash_{i} T$.
The action of global weakenings on a type at layer 0 is no-op:
Lemma 2.6. If $\Psi \vdash_{0} T$, then $T[\gamma]=T$.
Global weakenings do not really affect local weakenings:
Lemma 2.7. If $\gamma: \Psi \Longrightarrow{ }_{g} \Phi$ and $\tau: \Phi ; \Gamma \Longrightarrow{ }_{i} \Delta$, then $\tau: \Psi ; \Gamma[\gamma] \Longrightarrow{ }_{i} \Delta[\gamma]$.
Proof. Induction on $\tau: \Phi ; \Gamma \Longrightarrow{ }_{i} \Delta$.
The actions of local weakenings only affect terms so they will be looked into in the next section, after we consider the syntax of the type theory.

### 2.3 Syntax and Typing

In this section, we define the syntax of the type theory with contextual variables. To isolate concerns, we use letbox for elimination, instead of pattern matching. Nevertheless, pattern matching on code should work with proper adjustments to our development:

$$
\begin{aligned}
m & \\
\delta & :={ }_{g ?}^{m}\left|\mathrm{wk}_{g}^{m}\right| \delta, t / x \\
s, t & :=x \mid u^{\delta}
\end{aligned}
$$

| $\mid$ zero $\mid$ succ $t$ | (natural numbers) |
| :--- | ---: |
| $\mid$ box $t \mid$ letbox $u \leftarrow s$ in $t$ | (box) |
| $\|\lambda x . t\| s t$ | (functions) |
| $\|\Lambda g . t\| t \$ \Gamma$ | (meta-functions) |

Similar to before, we have natural numbers as our base type. To construct and eliminate meta-functions, we have $\Lambda g . t$ and $t \$ \Gamma$ respectively. Since we have contextual types, each global variable must be associated with a local substitution. In a local substitution, due to how a local context is structured, there are also two base cases: it can either be an empty local substitution $\underset{g ?}{m}$ ? or a weakening $\mathrm{wk}_{g}^{m}$ of a contextual variable $g$. The number $m$ associated with both cases are the number of local weakening $p$ 's. Effectively, $m$ equals to the length of the codomain local context in the typing judgment, as we will specify later in this section. This number can be fetched from a local substitution by the following function:

$$
\begin{aligned}
\widehat{\substack{m}} & :=m \\
\widehat{\mathrm{wk}_{g}^{m}} & :=m \\
\widehat{\delta, t / x} & :=\widehat{\delta}
\end{aligned}
$$

In addition, $\cdot$ is optionally associated with a contextual variable $g$. If there is such a $g$, it is the base case of the codomain local context. These information are necessary in order to define the local and global substitution operations. We rely on the following function to return the contextual variable inside of a local substitution, if it exists:

$$
\begin{aligned}
\stackrel{\breve{m}}{\substack{n ?}} & =g ? \\
\overline{\mathrm{wk}_{g}^{m}} & :=g \\
\overline{\delta, t / x} & :=\check{\delta}
\end{aligned}
$$

Next, we define the action of global weakenings on terms and local substitutions:

$$
\begin{aligned}
& x[\gamma]:=x \\
& u^{\delta}[\gamma]:=u^{\delta[\gamma]} \quad \text { (with } u \text { properly weakened depending on name representation) } \\
& \text { zero }[\gamma] \text { := zero } \\
& \operatorname{succ} t[\gamma]:=\operatorname{succ}(t[\gamma]) \\
& \text { box } t[\gamma]:=\operatorname{box}(t[\gamma]) \\
& \text { letbox } u \leftarrow s \text { in } t[\gamma]:=\text { letbox } u \leftarrow s[\gamma] \text { in }(t[q(\gamma)]) \\
& \lambda x . t[\gamma]:=\lambda x .(t[\gamma]) \\
& t s[\gamma]:=(t[\gamma])(s[\gamma]) \\
& \Lambda g \cdot t[\gamma]:=\Lambda g \cdot(t[q(\gamma)]) \\
& t \$ \Gamma[\gamma]:=(t[\gamma]) \$(\Gamma[\gamma]) \\
& \mathrm{wk}_{g}^{m}[\gamma]:=\mathrm{wk}_{g}^{m} \\
& ._{g ?}^{m}[\gamma]:={\underset{g}{m}}_{m} \quad \text { (with } g \text { properly weakened if exists) } \\
& \delta, t / x[\gamma]:=(\delta[\gamma]),(t[\gamma]) / x \\
& \text { (with } g \text { properly weakened) } \\
& \text { (with } g \text { properly weakened if exists) }
\end{aligned}
$$

The application of global weakenings satisfy the following lemma:

Lemma 2.8 (Algebra of global weakenings).

- $t[i d]=t$
- $\delta[i d]=\delta$
- $t[\gamma]\left[\gamma^{\prime}\right]=t\left[\gamma \circ \gamma^{\prime}\right]$
- $\delta[\gamma]\left[\gamma^{\prime}\right]=\delta\left[\gamma \circ \gamma^{\prime}\right]$

Applying local weakenings on terms and local substitutions is defined as follows:

$$
\begin{aligned}
& x[\tau]:=x \\
& \text { (properly weakened) } \\
& u^{\delta}[\tau]:=u^{\delta[\tau]} \\
& \text { zero }[\tau]:=\text { zero } \\
& \operatorname{succ} t[\tau]:=\operatorname{succ}(t[\tau]) \\
& \text { box } t[\tau]:=\text { box } t \\
& \text { letbox } u \leftarrow s \text { in } t[\tau]:=\text { letbox } u \leftarrow s[\tau] \text { in }(t[\tau]) \\
& \lambda x . t[\tau]:=\lambda x .(t[q(\tau)]) \\
& t s[\tau]:=(t[\tau])(s[\tau]) \\
& \text { ภg.t }[\tau]:=\Lambda g .(t[\tau]) \\
& t \$ \Gamma[\tau]:=(t[\tau]) \$ \Gamma \\
& \mathrm{wk}_{g}^{m}[\tau]:=\mathrm{wk}_{g}^{m+m^{\prime}} \quad \text { (where } m^{\prime} \text { is the number of } p \text { constructor in } \tau \text { ) } \\
& { }_{g ?}^{m}[\tau]:={\underset{g}{g} \text { ? }}_{m+m^{\prime}} \quad \text { (where } m^{\prime} \text { is the number of } p \text { constructor in } \tau \text { ) } \\
& \delta, t / x[\tau]:=(\delta[\tau]),(t[\tau]) / x
\end{aligned}
$$

The application of local weakenings satisfy the following lemma:
Lemma 2.9 (Algebra of local weakenings).

- $t[\gamma][\tau]=t[\tau][\gamma]$
- $\delta[\gamma][\tau]=\delta[\tau][\gamma]$
- $t[\tau]\left[\tau^{\prime}\right]=t\left[\tau \circ \tau^{\prime}\right]$
- $\delta[\tau]\left[\tau^{\prime}\right]=\delta\left[\tau \circ \tau^{\prime}\right]$
- $t[\delta][\tau]=t[\delta[\tau]]$
- $\left(\delta \circ \delta^{\prime}\right)[\tau]=\delta \circ\left(\delta^{\prime}[\tau]\right)$

Then the weakenings of dual-contexts are defined as just tuples of global and local weakenings:

$$
\begin{aligned}
& \gamma: \Psi \Longrightarrow_{g} \Phi \quad \tau: \Psi ; \Gamma \Longrightarrow_{i} \Delta[\gamma] \\
& \gamma ; \tau: \Psi ; \Gamma \Longrightarrow_{i} \Phi ; \Delta \\
& t[\gamma ; \tau]:=t[\gamma][\tau] \\
& \delta[\gamma ; \tau]:=\delta[\gamma][\tau]
\end{aligned}
$$

To disambiguate, when we apply a local weakening literal, we usually pair it with an identity global weakening. For example, $t[\mathrm{id} ; p(\mathrm{id})]$ locally weakens $t$ with a local weakening $p(\mathrm{id})$. When we write $t[p(\mathrm{id})]$, we mean that $t$ is globally weakened by $p$ (id). The correctness lemma for these operations can only be proved after defining the typing rules. In order to define the typing rules, we must first give the definition of global substitutions of types and local contexts:

$$
\sigma:=\cdot|\sigma, t / u| \sigma, \Gamma / g
$$

(Global substitutions)

Global substitutions can obviously be globally weakened by iteratively applying a global weakening to the terms and local contexts within. The global substitution operation of types and local contexts is defined as follows:

$$
\begin{aligned}
& \operatorname{Nat}[\sigma]:=\text { Nat } \\
& S \longrightarrow T[\sigma]:=(S[\sigma]) \longrightarrow(T[\sigma]) \\
& \square(\Gamma \vdash T)[\sigma]:=\square(\Gamma[\sigma] \vdash T[\sigma]) \\
&(g: \mathrm{Ctx}) \Rightarrow T[\sigma]:=(g: \mathrm{Ctx}) \Rightarrow(T[\sigma[p(\mathrm{id})], g / g]) \\
& \cdot[\sigma]:=\cdot \\
& g[\sigma]:=\sigma(g) \quad \quad \text { (lookup } g \text { in } \sigma ; \text { undefined if } g \text { is not bound or result is not a local context }) \\
& \Gamma, x: T[\sigma]:=(\Gamma[\sigma]), x:(T[\sigma])
\end{aligned}
$$

For consistency of notations, we often write $q(\sigma)$ for $\sigma[p(i d)], g / g$ or $\sigma[p(i d)], u^{\text {id }} / u$. This notation relates similar operations of weakenings and substitutions. The global substitutions on types satisfy the following lemma:

Lemma 2.10 (Algebra of Global Substitutions).

- $T[\gamma]\left[\gamma^{\prime}\right]=T[\gamma \circ \gamma]$
- $\sigma[\gamma]\left[\gamma^{\prime}\right]=\sigma[\gamma \circ \gamma]$
- $T[\sigma][\gamma]=T[\sigma[\gamma]]$
- $\Gamma[\sigma][\gamma]=\Gamma[\sigma[\gamma]]$
- $t[\sigma][\gamma]=t[\sigma[\gamma]]$
- $\delta[\sigma][\gamma]=\delta[\sigma[\gamma]]$

Next, we give the application operation of local substitutions on terms and composition of local substitutions:

$$
\begin{aligned}
x[\delta] & :=\delta(x) \\
u^{\delta^{\prime}}[\delta] & :=u^{\delta^{\prime} \circ \delta} \\
\text { zero }[\delta] & :=\text { zero } \\
\operatorname{succ} t[\delta] & :=\operatorname{succ}(t[\delta]) \\
\lambda x . t[\delta] & :=\lambda x .(t[\delta[\mathrm{id} ; p(\mathrm{id})], x / x]) \\
t s[\delta] & :=(t[\delta])(s[\delta]) \\
\operatorname{box} t[\delta] & :=\text { box } t \\
\text { letbox } u \leftarrow s \text { in } t[\delta] & :=\operatorname{letbox} u \leftarrow s[\delta] \text { in }(t[\delta[p(\mathrm{id})]]) \\
\Lambda g \cdot t[\delta] & :=\Lambda g \cdot(t[\delta[(p(\mathrm{id}))]]) \\
t \$ \Gamma[\delta] & :=(t[\delta]) \$ \Gamma \\
\mathrm{wk} g_{g}^{m} \circ \delta & :=\mathrm{wk} \widehat{\delta}_{g} \\
\cdot{ }^{m} \circ \delta & :=\widehat{\delta}_{\check{\delta}} \\
._{g}^{m} \circ \delta & :=\widehat{\delta}_{g} \\
\left(\delta^{\prime}, t / x\right) \circ \delta & :=\left(\delta^{\prime} \circ \delta\right), t[\delta] / x
\end{aligned}
$$

(lookup of $x$ in $\delta$ )

Similarly, we might also write $q(\delta)$ for $\delta[\mathrm{id} ; p(\mathrm{id})], x / x$. Notice that in the definition of composition, we make use of the $\widehat{\delta}$ function to fetch the number of weakenings. This number is used in the application operation of global substitutions given below in the application of global substitutions to specify the number of local $p$ weakenings
when a contextual variable is substituted by a concrete context. In the composition of ${ }^{m}$, we use $\check{\delta}$ to query whether $\delta$ 's codomain context starts from a contextual variable. If it does, then we use that contextual variable in the result of the composition. Local substitutions and global weakenings interact in the following way:

Lemma 2.11 (Algebra of Local Substitutions).

- $t[\delta][\gamma]=(t[\gamma])[\delta[\gamma]]$
- $\left(\delta \circ \delta^{\prime}\right)[\gamma]=(\delta[\gamma]) \circ\left(\delta^{\prime}[\gamma]\right)$

Now, we define the application of global substitutions to terms and local substitutions:

$$
\begin{aligned}
& x[\sigma]:=x \\
& u^{\delta}[\sigma]:=\sigma(u)[\delta[\sigma]] \quad \text { (lookup of } u \text { in } \sigma \text { ) } \\
& \text { zero[ } \sigma]:=\text { zero } \\
& \operatorname{succ} t[\sigma]:=\operatorname{succ}(t[\sigma]) \\
& \lambda x . t[\sigma]:=\lambda x .(t[\sigma]) \\
& t s[\sigma]:=(t[\sigma])(s[\sigma]) \\
& \text { box } t[\sigma]:=\operatorname{box}(t[\sigma]) \\
& \text { letbox } u \leftarrow s \text { in } t[\sigma]:=\text { letbox } u \leftarrow s[\sigma] \text { in }\left(t\left[\sigma[p(\mathrm{id})], u^{\text {id }} / u\right]\right) \\
& \Lambda g . t[\sigma]:=\Lambda g .(t[\sigma[(p(\mathrm{id}))], g / g]) \\
& t \$ \Gamma[\sigma]:=(t[\sigma]) \$(\Gamma[\sigma]) \\
& \mathrm{wk}_{g}^{m}[\sigma]:=\mathrm{id}_{\sigma(g)}\left[\mathrm{id} ; p^{m}(\mathrm{id})\right] \quad \text { (defined only when } \sigma(g) \text { is a local context) } \\
& \cdot{ }^{m}[\sigma]:={ }^{m} \\
& { }_{g}^{m}[\sigma]:=.|\Gamma|+m \quad \text { (if } \sigma(g)=\Gamma \text { and } \Gamma \text { ends with a } \cdot \text { ) } \\
& \left.{ }_{g}^{m}[\sigma]:=._{g^{\prime}}^{|\Gamma|+m} \quad \text { (if } \sigma(g)=\Gamma \text { and } \Gamma \text { ends with a } g^{\prime}\right) \\
& (\delta, t / x)[\sigma]:=(\delta[\sigma]), t[\sigma] / x
\end{aligned}
$$

In the definition of global substitution operation, we make use of the local identity substitution, which is defined through local weakening substitutions as below:

$$
\begin{aligned}
\mathrm{wk}_{\cdot}^{m} & :=.^{m} \\
\mathrm{wk}_{g}^{m} & :=\mathrm{wk}_{g}^{m} \\
\mathrm{wk}_{\Gamma, x: T}^{m} & :=\mathrm{wk}_{\Gamma}^{1+m}, x / x
\end{aligned}
$$

The local identity substitution is defined as a special case of the local weakening substitutions by setting $m$ to be 0 :

$$
\mathrm{id}_{\Gamma}:=\mathrm{wk} \mathrm{k}_{\Gamma}^{0}
$$

Though we make heavy use of symbol overloading, but hopefully the exact meanings of symbols should be disambiguated by the surrounding textual contexts. Usually, the symbols are designed such that their behaviors remain the same for their ambiguous readings (e.g. various uses of id and wk ).

At last, we need to define the global identity substitution before giving the typing rules. The global identity substitution is defined in the same principle; it is a special case of the global weakening substitutions:

$$
\mathrm{wk}^{m}:=
$$

$$
\begin{aligned}
\mathrm{wk}_{\Psi, g: \mathrm{Ctx}}^{m} & :=\mathrm{wk}_{\Psi}^{1+m}, g / g \\
\mathrm{wk}_{\Psi, u:(\Gamma \vdash T)}^{m} & :=\mathrm{wk}_{\Psi}^{1+m}, u^{\mathrm{id}_{\Gamma}\left[p^{1+m}(\mathrm{id})\right]} / u
\end{aligned}
$$

Notice that in the cons case for $u:(\Gamma \vdash T)$, the local identity substitution $\mathrm{id}_{\Gamma}$ must be weakened by $p^{1+m}$ (id), because its global typing environment is weakened by $\mathrm{wk}_{\Psi}^{1+m}$, which takes the same effect. This weakening is necessary to make the typing to go through. Then identity is just a special case:

$$
\mathrm{id}_{\Psi}:=\mathrm{wk}_{\Psi}^{0}
$$

We sometimes omit the subscript for different id, as we know their effects on types, local contexts, terms and local substitutions are just identity.

The composition of global substitutions is defined intuitively:

$$
\begin{aligned}
\cdot \circ \sigma^{\prime} & := \\
(\sigma, t / u) \circ \sigma^{\prime} & :=\left(\sigma \circ \sigma^{\prime}\right), t\left[\sigma^{\prime}\right] / u \\
(\sigma, \Gamma / g) \circ \sigma^{\prime} & :=\left(\sigma \circ \sigma^{\prime}\right), \Gamma\left[\sigma^{\prime}\right] / g
\end{aligned}
$$

Essentially, composition just iteratively applies $\sigma^{\prime}$ to all terms and contexts in the first global substitution. Notice that composition can only be defined, after the applications of global substitutions to both terms and local contexts are defined. This definition does lead to some complication when we tries to prove the global substitution lemma of terms and local substitutions in the next section.

At last, the following gives the typing rules of terms and local substitutions.

$$
\begin{aligned}
& \frac{\Psi \vdash_{i} \Gamma \quad x: T \in \Gamma}{\Psi ; \Gamma \vdash_{i} x: T} \quad \frac{\Psi ; \Gamma \vdash_{i} \delta: \Delta \quad u:(\Delta \vdash T) \in \Psi}{\Psi ; \Gamma \vdash_{i} u^{\delta}: T} \quad \frac{\Psi \vdash_{i} \Gamma}{\Psi ; \Gamma \vdash_{i} \text { zero }: \mathrm{Nat}} \quad \frac{\Psi ; \Gamma \vdash_{i} t: \text { Nat }}{\Psi ; \Gamma \vdash_{i} \operatorname{succ} t: \text { Nat }} \\
& \frac{\Psi ; \Gamma, x: S \vdash_{i} t: T}{\Psi ; \Gamma \vdash_{i} \lambda x . t: S \longrightarrow T} \quad \frac{\Psi ; \Gamma \vdash_{i} t: S \longrightarrow T \quad \Psi ; \Gamma \vdash_{i} s: S}{\Psi ; \Gamma \vdash_{i} t s: T} \quad \frac{\Psi \vdash_{1} \Gamma \quad \Psi ; \Delta \vdash_{0} t: T}{\Psi ; \Gamma \vdash_{1} \text { box } t: \square(\Delta \vdash T)} \\
& \begin{array}{lcccc:c|c|c|c|c|c}
\Psi ; \Gamma \vdash_{1} s: \square(\Delta \vdash T) \quad \Psi \vdash_{0} \Delta & \Psi \vdash_{0} T \quad \Psi \vdash_{1} T^{\prime} \quad \Psi, u:(\Delta \vdash T) ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T^{\prime}[p(\mathrm{id})] \\
\Psi ; \Gamma \vdash_{1} \operatorname{letbox} u \leftarrow s \text { in } t: T^{\prime}
\end{array} \\
& \frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T}{\Psi ; \Gamma \vdash_{1} \Lambda g . t:(g: \mathrm{Ctx}) \Rightarrow T} \quad \frac{\Psi ; \Gamma \vdash_{1} t:(g: \mathrm{Ctx}) \Rightarrow T \quad \Psi \vdash_{0} \Delta}{\Psi ; \Gamma \vdash_{1} t \$ \Delta: T\left[\mathrm{id}_{\Psi}, \Delta / g\right]} \\
& \frac{\Psi \vdash_{i} \Gamma \quad \Gamma \text { ends with } \cdot \quad|\Gamma|=m}{\Psi ; \Gamma \vdash_{i}{ }^{m}: \cdot} \quad \frac{\Psi \vdash_{i} \Gamma \quad g: \operatorname{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m}{\Psi ; \Gamma \vdash_{i}{ }_{g}^{m}: \cdot} \\
& \frac{\Psi \vdash_{i} \Gamma \quad g: \mathrm{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m}{\Psi ; \Gamma \vdash_{i} \mathrm{wk}_{g}^{m}: g} \quad \frac{\Psi ; \Gamma \vdash_{i} \delta: \Delta \quad \Psi ; \Gamma \vdash_{i} t: T}{\Psi ; \Gamma \vdash_{i} \delta, t / x: \Delta, x: T}
\end{aligned}
$$

In the typing rules, there are premises highlighted by shades. These shaded premises are necessary to establish the theorem of presupposition or syntactic validity. After establishing presupposition, these premises can be derived from other premises and thus technically can be omitted afterwards. Moreover, in the rule for global variables, the lookup of a global context $\Psi$ must consider the effect of global weakenings as follows:

$$
\overline{u: B[p(\mathrm{id})] \in \Psi, u: B}
$$

$$
\frac{u: B \in \Psi}{u: B[p(\mathrm{id})] \in \Psi, u^{\prime}: B^{\prime}}
$$

The typing rules also support lifting:
Lemma 2.12 (Lifting).

- If $\Psi ; \Gamma \vdash_{0} t: T$, then $\Psi ; \Gamma \vdash_{1} t: T$.
- If $\Psi ; \Gamma \vdash_{0} \delta: \Delta$, then $\Psi ; \Gamma \vdash_{1} \delta: \Delta$.

This lemma ensures that terms at layer 0 are included in layer 1.
Typing rules for global substitutions are defined as follows:

$$
\frac{\vdash \Psi}{\Psi \vdash \cdot:} \quad \frac{\Psi \vdash \sigma: \Phi}{} \quad \Psi \vdash_{0} \Gamma \quad \Psi \vdash_{0} T \quad \Psi ; \Gamma[\sigma] \vdash_{0} t: T[\sigma] \quad \frac{\Psi \vdash \sigma, t / u: \Phi, u:(\Gamma \vdash T)}{\Psi \vdash \sigma, \Gamma / g: \Phi, g: \mathrm{Ctx}}
$$

In the next section, we establish a set of syntactic properties as a basic sanity check of the definitions, which are also useful in later section proving normalization.

### 2.4 Syntactic Properties of 2-layered Modal Type Theory with Contextual Variables

In this section, we list syntactic properties that eventually leads to the substitution lemma of terms and local substitutions for global substitutions. This lemma is our "benchmark" to ensure that the rules for the system make sense. During the process, we must establish other necessary syntactic properties. We elaborate the proofs an important and selected few. Other proofs in this section have been mechanized in Agda.

Lemma 2.13.

- If $n$ is the length of $\sigma^{\prime}$, then $t\left[q^{n}(p(i d))\right]\left[\sigma, t / u, \sigma^{\prime}\right]=t\left[q^{n}(p(i d))\right]\left[\sigma, \Gamma / g, \sigma^{\prime}\right]=t\left[\sigma, \sigma^{\prime}\right]$.
- If $n$ is the length of $\sigma^{\prime}$, then $\delta\left[q^{n}(p(i d))\right]\left[\sigma, t / u, \sigma^{\prime}\right]=\delta\left[q^{n}(p(i d))\right]\left[\sigma, \Gamma / g, \sigma^{\prime}\right]=\delta\left[\sigma, \sigma^{\prime}\right]$.

This lemma allows to skip a binding in the middle of a global substitution according to a global weakening. A similar lemma holds for local substitutions:

Lemma 2.14.

- If $n$ is the length of $\delta^{\prime}$, then $t\left[q^{n}(p(i d))\right]\left[\delta, t / x, \delta^{\prime}\right]=t\left[\delta, \delta^{\prime}\right]$.
- If $n$ is the length of $\delta^{\prime}$, then $\delta\left[q^{n}(p(i d))\right] \circ\left(\delta, t / x, \delta^{\prime}\right)=\delta \circ\left(\delta, \delta^{\prime}\right)$.

Lemma 2.15 (Composition of Global Substitutions).

- $T[\sigma]\left[\sigma^{\prime}\right]=T\left[\sigma \circ \sigma^{\prime}\right]$
- $\Gamma[\sigma]\left[\sigma^{\prime}\right]=\Gamma\left[\sigma \circ \sigma^{\prime}\right]$

Lemma 2.16 (Composition and Associativity of Local Substitutions).

- $t[\delta]\left[\delta^{\prime}\right]=t\left[\delta \circ \delta^{\prime}\right]$
- $\left(\delta \circ \delta^{\prime}\right) \circ \delta^{\prime \prime}=\delta \circ\left(\delta^{\prime} \circ \delta^{\prime \prime}\right)$

Lemma 2.17 (Typing of Local Weakening Substitutions). If $\Psi \vdash_{i} \Delta$, $\Gamma$, then $\Psi ; \Delta, \Gamma \vdash_{i} w k_{\Delta}^{|\Gamma|}: \Delta$.
The corollary is the well-typedness of local identity substitution:
Corollary 2.18. If $\Psi \vdash_{i} \Gamma$, then $\Psi ; \Gamma \vdash_{i} i d_{\Gamma}: \Gamma$.
The next few questions require well-formedness or typing judgments to work:
Lemma 2.19.

- If $\Psi \vdash_{i} T$, then $T\left[w k_{\Psi}^{n}\right]=T\left[p^{n}(i d)\right]$.
- If $\Psi \vdash_{i} \Gamma$, then $\Gamma\left[w k_{\Psi}^{n}\right]=\Gamma\left[p^{n}(i d)\right]$.

This lemma proves that the global weakening substitutions behaves exactly like global weakenings.
Lemma 2.20 (Naturality).

- If $\Psi, g:$ Ctx $^{\prime} \vdash_{i} T$ and $\gamma: \Phi \Longrightarrow g \Psi$, then $T\left[i d_{\Psi}, \Gamma / g\right][\gamma]=T[q(\gamma)]\left[i d_{\Phi}, \Gamma[\gamma] / g\right]$.
- If $\Psi, g: C t x \vdash_{i} \Delta$ and $\gamma: \Phi \Longrightarrow g \Psi$, then $\Delta\left[i d_{\Psi}, \Gamma / g\right][\gamma]=\Delta[q(\gamma)]\left[i d_{\Phi}, \Gamma[\gamma] / g\right]$.

The naturality lemma finds correspondence in the characterization of a presheaf category of a type theory in general, which instructs how $q$ weakenings can be used to swap a global weakening and a global substitution.

Lemma 2.21 (Local Identity).

- If $\Psi ; \Gamma \vdash_{i} t: T$, then $t\left[d_{\Gamma}\right]=t$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $\delta \circ i d_{\Gamma}=\delta$.

This lemma shows that the local identity substitution has no effect on terms and that the right identity property of local substitutions.

Next, we establish the global weakening lemma for typing rules:
Lemma 2.22 (Global weakenings).

- If $\Psi ; \Gamma \vdash_{i} t: T$ and $\gamma: \Psi^{\prime} \Longrightarrow{ }_{g} \Psi$, then $\Psi^{\prime} ; \Gamma[\gamma] \vdash_{i} t[\gamma]: T[\gamma]$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$ and $\gamma: \Psi^{\prime} \Longrightarrow{ }_{g} \Psi$, then $\Psi^{\prime} ; \Gamma[\gamma] \vdash_{i} \delta[\gamma]: \Delta[\gamma]$.

Proof. Mutual induction on $\Phi ; \Gamma \vdash_{i} t: T$ and $\Phi ; \Gamma \vdash_{i} \delta: \Delta$. We only consider a few interesting cases: Case

$$
\frac{\Phi ; \Gamma \vdash_{1} s: \square(\Delta \vdash T) \quad \Phi \vdash_{1} T^{\prime} \quad \Phi, u:(\Delta \vdash T) ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T^{\prime}[p(\mathrm{id})]}{\Phi ; \Gamma \vdash_{1} \text { letbox } u \leftarrow s \text { in } t: T^{\prime}}
$$

| $\Psi ; \Gamma[\gamma] \vdash_{1} s[\gamma]: \square(\Delta \vdash T)[\gamma]$ | (by IH) |
| :--- | ---: |
| $\Psi, u:(\Delta[\gamma] \vdash T[\gamma]) ; \Gamma[p(\mathrm{id})][q(\gamma)] \vdash_{1} t[q(\gamma)]: T^{\prime}[p(\mathrm{id})][q(\gamma)]$ | (by IH) |
| $\Psi, u:(\Delta[\gamma] \vdash T[\gamma]) ; \Gamma[p(\gamma)] \vdash_{1} t[q(\gamma)]: T^{\prime}[p(\gamma)]$ | (by computation) |
| $\Psi, u:(\Delta[\gamma] \vdash T[\gamma]) ; \Gamma[\gamma][p(\mathrm{id})] \vdash_{1} t[q(\gamma)]: T^{\prime}[\gamma][p(\mathrm{id})]$ |  |
| $\Psi ; \Gamma[\gamma] \vdash_{1}$ letbox $u \leftarrow s$ in $t[\gamma]: T^{\prime}[\gamma]$ | (by constructor) |

Case

$$
\frac{\Phi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T}{\Phi ; \Gamma \vdash_{1} \Lambda g \cdot t:(g: \mathrm{Ctx}) \Rightarrow T}
$$

$q(\gamma): \Psi^{\prime}, g: \mathrm{Ctx} \Longrightarrow \quad \Psi, g: \mathrm{Ctx} \quad$ (by typing rules)
$\Psi^{\prime}, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})][q(\gamma)] \vdash_{1} t[q(\gamma)]: T[q(\gamma)] \quad$ (by IH)
$\Psi^{\prime}, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id}) \circ q(\gamma)] \vdash_{1} t[q(\gamma)]: T[q(\gamma)] \quad$ (by algebraic law)
$\Psi^{\prime}, g: \mathrm{Ctx} ; \Gamma[p(\gamma)] \vdash_{1} t[q(\gamma)]: T[q(\gamma)]$
$\Psi^{\prime}, g: \mathrm{Ctx} ; \Gamma[\gamma][p(\mathrm{id})] \vdash_{1} t[q(\gamma)]: T[q(\gamma)]$
$\Psi^{\prime} ; \Gamma[\gamma] \vdash_{1} \Lambda g \cdot(t[q(\gamma)]):(g: \mathrm{Ctx}) \Rightarrow(T[q(\gamma)]) \quad$ (by typing rule)
Case

$$
\frac{\Phi ; \Gamma \vdash_{1} t:(g: \mathrm{Ctx}) \Rightarrow T \quad \Phi \vdash_{0} \Delta}{\Phi ; \Gamma \vdash_{1} t \$ \Delta: T\left[\mathrm{id}_{\Phi}, \Delta / g\right]}
$$

$$
\begin{align*}
& \Psi ; \Gamma[\gamma] \vdash_{1} t[\gamma]:(g: \mathrm{Ctx}) \Rightarrow T[\gamma]  \tag{byIH}\\
& \Psi \vdash_{0} \Delta[\gamma] \\
& \Psi ; \Gamma[\gamma] \vdash_{1}(t[\gamma]) \$(\Delta[\gamma]): T[q(\gamma)]\left[\mathrm{id}_{\Psi}, \Delta[\gamma] / g\right] \\
& \Psi ; \Gamma[\gamma] \vdash_{1}(t[\gamma]) \$(\Delta[\gamma]): T\left[\mathrm{id}_{\Phi}, \Delta / g\right][\gamma]
\end{align*}
$$

(by IH)
(by Lemma 2.5)
(by constructor) (by naturality)

Lemma 2.23 (Global Weakening). If $\Psi \vdash \sigma: \Phi$ and $\gamma: \Psi^{\prime} \Longrightarrow g$, then $\Psi^{\prime} \vdash \sigma[\gamma]: \Phi$.
Lemma 2.24 (Global Weakening Substitutions). If $\vdash \Psi, \Phi$, then $\Psi, \Phi \vdash w k_{\Psi}^{|\Phi|}: \Psi$.
Proof. This lemma is actually requires a bit preliminaries to establish and so this is the earliest point where this lemma can be proven. From $\vdash \Psi$, $\Phi$, we know $\vdash \Psi$, which we do induction on. We consider only one case:

$$
\begin{gathered}
\frac{\Psi \vdash_{0} \Gamma \quad \Psi \vdash_{0} T}{\vdash \Psi, u:(\Gamma \vdash T)} \\
\mathrm{wk}_{\Psi, u:(\Gamma \vdash T)}^{|\Phi|}=\mathrm{wk}_{\Psi}^{1+|\Phi|}, u^{\mathrm{id} \mathrm{~d}_{\Gamma}\left[p^{1+|\Phi|}(\mathrm{id})\right]} / u \\
\Psi, u:(\Gamma \vdash T), \Phi \vdash \mathrm{wk}_{\Psi}^{1+|\Phi|}: \Psi
\end{gathered}
$$

At last, we must prove $\Psi, u:(\Gamma \vdash T), \Phi ; \Gamma\left[\mathrm{wk}_{\Psi}^{1+|\Phi|}\right] \vdash_{0} u^{\mathrm{id}{ }_{\Gamma}\left[p^{1+|\Phi|}(\mathrm{id})\right]}: T\left[\mathrm{wk}_{\Psi}^{1+|\Phi|}\right]$. But we know that this goal is the same as the following due to Lemma 2.19:

$$
\Psi, u:(\Gamma \vdash T), \Phi ; \Gamma\left[p^{1+|\Phi|}(\mathrm{id})\right] \vdash_{0} u^{\mathrm{id}\left[p^{1+|\Phi|}(\mathrm{id})\right]}: T\left[p^{1+|\Phi|}(\mathrm{id})\right]
$$

It remains to prove that the substitution is well-typed:

$$
\Psi, u:(\Gamma \vdash T), \Phi ; \Gamma\left[p^{1+|\Phi|}(\mathrm{id})\right] \vdash_{0} \mathrm{id}_{\Gamma}\left[p^{1+|\Phi|}(\mathrm{id})\right]: \Gamma\left[p^{1+|\Phi|}(\mathrm{id})\right]
$$

This goal is immediate due to Lemmas 2.21 and 2.22.
Corollary 2.25. If $\vdash \Psi$, then $\Psi \vdash i d_{\Psi}: \Psi$.
Finally, we can establish the presupposition of terms and local substitutions:
Lemma 2.26 (Presupposition).

- If $\Psi ; \Gamma \vdash_{i} t: T$, then $\Psi \vdash_{i} \Gamma$ and $\Psi \vdash_{i} T$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $\Psi \vdash_{i} \Gamma$ and $\Psi \vdash_{i} \Delta$.

Proof. We do a mutual induction.
Next, we need a similar lemma to Lemma 2.19 but for terms and local substitutions:
Lemma 2.27.

- If $\Psi ; \Gamma \vdash_{i} t: T$, then $t\left[w k_{\Psi}^{n}\right]=t\left[p^{n}(i d)\right]$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $\delta\left[w k_{\Psi}^{n}\right]=\delta\left[p^{n}(i d)\right]$.

Proof. The proof of this lemma requires an intrigued generalization in order to handle extensions of global contexts due to letbox and $\Lambda$. Details of the generalization are technical and too elaborate to put in this technical report, and thus we choose to leave them in the Agda mechanization for readers' reference.
Next, we should verify the identity rules of composition of global substitutions. Notice that by applying Lemmas 2.19 and 2.27, we obtain the right identity immediately:

Lemma 2.28. If $\Psi \vdash \sigma: \Phi$, then $\sigma \circ i d_{\Psi}=\sigma$.
The left identity, on the other hand, requires certain generalization which must incorporate global weakenings. We again leave the details in the Agda mechanization:

Lemma 2.29. If $\Psi \vdash \sigma: \Phi$, then $i_{\Phi} \circ \sigma=\sigma$.
Another useful equation is that global substitutions and local weakenings commute:
Lemma 2.30 (Commutativity of Global Substitutions and Local Weakenings).

- If $\Psi ; \Gamma \vdash_{i} t: T$ and $\tau: \Psi ; \Delta \Longrightarrow_{i} \Gamma$, then $t[\tau][\sigma]=t[\sigma][\tau]$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$ and $\tau: \Psi ; \Delta \Longrightarrow_{i} \Gamma$, then $\delta[\tau][\sigma]=\delta[\sigma][\tau]$.

Next, we move on to the global substitution lemma for terms and local substitutions. Prior to that, we must first show the local weakening lemma and local substitution lemma:

Lemma 2.31 (Local Weakenings).

- If $\Psi ; \Gamma \vdash_{i} t: T$ and $\tau: \Psi ; \Delta \Longrightarrow_{i} \Gamma$, then $\Psi ; \Delta \vdash_{i} t[\tau]: T$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta^{\prime}$ and $\tau: \Psi ; \Delta \Longrightarrow_{i} \Gamma$, then $\Psi ; \Delta \vdash_{i} \delta[\tau]: \Delta^{\prime}$.

Lemma 2.32 (Local Substitutions).

- If $\Psi ; \Gamma \vdash_{i} t: T$ and $\Psi ; \Delta \vdash_{i} \delta: \Gamma$, then $\Psi ; \Delta \vdash_{i} t[\delta]: T$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$ and $\Psi ; \Gamma^{\prime} \vdash_{i} \delta^{\prime}: \Gamma$, then $\Psi ; \Gamma^{\prime} \vdash_{i} \delta \circ \delta^{\prime}: \Delta$.

Proof. We do a mutual induction. Notice that in this lemma, it is somewhat more cumbersome to establish the proof for local substitutions. When $\delta=\cdot^{m}$, then we must reason about the properties of $\check{\delta^{\prime}}$. If $\check{\delta^{\prime}}=g$ for some $g$, then we must show that both $\Gamma$ and $\Gamma^{\prime}$ start with this $g$. The details are given in the Agda mechanization.

Finally, we give the global substitution lemma:
Lemma 2.33 (Global Substitutions).

- If $\Psi ; \Gamma \vdash_{i} t: T$ and $\Psi^{\prime} \vdash \sigma: \Psi$, then $\Psi^{\prime} ; \Gamma[\sigma] \vdash_{i} t[\sigma]: T[\sigma]$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$ and $\Psi^{\prime} \vdash \sigma: \Psi$, then $\Psi^{\prime} ; \Gamma[\sigma] \vdash_{i} \delta[\sigma]: \Delta[\sigma]$.

Proof. We do a mutual induction. We consider a few interesting cases:

## Case

$$
\frac{\Psi ; \Gamma \vdash_{1} s: \square(\Delta \vdash T) \quad \Psi \vdash_{1} T^{\prime} \quad \Psi, u:(\Delta \vdash T) ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T^{\prime}[p(\mathrm{id})]}{\Psi ; \Gamma \vdash_{1} \text { letbox } u \leftarrow s \text { in } t: T^{\prime}}
$$

$$
\begin{align*}
& \Psi^{\prime} ; \Gamma[\sigma] \vdash_{1} s[\sigma]: \square(\Delta \vdash T)[\sigma]=\square(\Delta[\sigma] \vdash T[\sigma])  \tag{byIH}\\
& \Psi^{\prime}, u:(\Delta[\sigma] \vdash T[\sigma]) \vdash \sigma[p(\mathrm{id})], u^{\mathrm{id} d_{\Delta[\sigma[p(\mathrm{id})]]}} / u: \Psi, u:(\Delta \vdash T) \quad \text { (by IH) } \\
& \Psi^{\prime}, u:(\Delta[\sigma] \vdash T[\sigma]) ; \Gamma[p(\mathrm{id})]\left[\sigma[p(\mathrm{id})], u^{\mathrm{id} \Delta[\sigma[p(\mathrm{id})]]} / u\right] \vdash_{1} t\left[\sigma[p(\mathrm{id})], u^{\mathrm{id} \Delta[\sigma[p(\mathrm{id})]]} / u\right]: T^{\prime}[p(\mathrm{id})]\left[\sigma[p(\mathrm{id})], u^{\left.\mathrm{id}_{\Delta[\sigma[p(\mathrm{id})]]} / u\right]}\right.
\end{align*}
$$

$\Psi^{\prime}, u:(\Delta[\sigma] \vdash T[\sigma]) ; \Gamma[\sigma[p(\mathrm{id})]] \vdash_{1} t\left[\sigma[p(\mathrm{id})], u^{\left.\mathrm{id}_{\Delta[\sigma[p(\mathrm{id})]]} / u\right]: T^{\prime}[\sigma[p(\mathrm{id})]]}\right.$
$\Psi^{\prime}, u:(\Delta[\sigma] \vdash T[\sigma]) ; \Gamma[\sigma][p(\mathrm{id})] \vdash_{1} t\left[\sigma[p(\mathrm{id})], u^{\left.\mathrm{id}_{\Delta[\sigma][p(\mathrm{id})]} / u\right]: T^{\prime}[\sigma][p(\mathrm{id})]}\right.$ (by Lemma 2.13)
$\Psi^{\prime} ; \Gamma[\sigma] \vdash_{1}$ letbox $u \leftarrow s$ in $t: T^{\prime}[\sigma]$ (by algebraic laws) (by typing rule)

Case

$$
\frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T}{\Psi ; \Gamma \vdash_{1} \Lambda g . t:(g: \mathrm{Ctx}) \Rightarrow T}
$$

Basically we do the same as above, but instead of providing a global substitution of terms, we provide a global substitution of a local context.
Case

$$
\begin{aligned}
& \qquad \frac{\Psi ; \Gamma \vdash_{1} t:(g: \mathrm{Ctx}) \Rightarrow T \quad \Psi \vdash_{0} \Delta}{\Psi ; \Gamma \vdash_{1} t \$ \Delta: T[\mathrm{id}, \Delta / g]} \\
& \Psi^{\prime} ; \Gamma[\sigma] \vdash_{1} t[\sigma]:(g: \mathrm{Ctx}) \Rightarrow T[\sigma]=(g: \mathrm{Ctx}) \Rightarrow(T[\sigma[p(\mathrm{id})], g / g]) \\
& \Psi^{\prime} \vdash_{0} \Delta[\sigma] \\
& \Psi^{\prime} ; \Gamma[\sigma] \vdash_{1} t \$ \Delta[\sigma]: T[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id}_{\Psi^{\prime}}, \Delta[\sigma] / g\right]
\end{aligned}
$$

Finally we must show the following equation:

$$
T[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id}_{\Psi}, \Delta[\sigma] / g\right]=T\left[\mathrm{id}_{\Psi}, \Delta / g\right][\sigma]
$$

We reason as follows:

$$
\begin{aligned}
T[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id} \Psi^{\prime}, \Delta[\sigma] / g\right] & =T\left[(\sigma[p(\mathrm{id})], g / g) \circ\left(\mathrm{id} \Psi^{\prime}, \Delta[\sigma] / g\right)\right] & \\
& =T\left[\left(\sigma[p(\mathrm{idd})] \circ\left(\mathrm{id}_{\Psi^{\prime}}, \Delta[\sigma] / g\right)\right), \Delta[\sigma]\right] & \\
& =T\left[\left(\sigma \circ \mathrm{id} \Psi^{\prime}\right), \Delta[\sigma]\right] & \text { (by Lemma 2.13) } \\
& =T[\sigma, \Delta[\sigma]] & \text { (by right identity) }
\end{aligned}
$$

On the right hand side,

$$
T\left[\mathrm{id}_{\Psi}, \Delta / g\right][\sigma]=T\left[\left(\mathrm{id}_{\Psi} \circ \sigma\right), \Delta[\sigma] / g\right]
$$

$$
=T[\sigma, \Delta[\sigma] / g] \quad \text { (by left identity) }
$$

Thus both sides agree.
Case

$$
\frac{\Psi \vdash_{i} \Gamma \quad g: C t x \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m}{\Psi ; \Gamma \vdash_{i} \cdot{ }_{g}^{m}: \cdot}
$$

In this case, we must look up $g$ in $\sigma$, and branch depending on $\sigma(g)$.
Subcase If $\sigma(g)=g^{\prime}, \Delta$ meaning that $\sigma(g)$ ends with a contextual variable $g^{\prime}$, then we must construct a typing judgment for ${ }_{g^{\prime}}^{m^{\prime}}$ for some $m^{\prime}$. In this case, say if $\Gamma=g$, $\Gamma^{\prime}$, then $\Gamma[\sigma]=g^{\prime}, \Delta,\left(\Gamma^{\prime}[\sigma]\right)$. Therefore, $m^{\prime}=$ $|\Delta|+\left|\Gamma^{\prime}[\sigma]\right|=|\Delta|+m$.
Subcase If $\sigma(g)=\cdot, \Delta$, then we proceed similarly except that we must construct a typing judgment for $\cdot{ }^{m^{\prime}}$ instead.

## Lemma 2.34 (Distributivity of Global Substitutions).

- If $\Psi ; \Gamma \vdash_{i} t: T, \Psi ; \Delta \vdash_{i} \delta: \Gamma$ and $\Phi \vdash \sigma: \Psi$, then $t[\delta][\sigma]=(t[\sigma][\delta[\sigma]])$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta, \Psi ; \Gamma^{\prime} \vdash_{i} \delta^{\prime}: \Gamma$ and $\Phi \vdash \sigma: \Psi$, then $\left(\delta \circ \delta^{\prime}\right)[\sigma]=(\delta[\sigma]) \circ\left(\delta^{\prime}[\sigma]\right)$.

Proof. We do a mutual induction on $\Psi ; \Gamma \vdash_{i} t: T$ and $\Psi ; \Gamma \vdash_{i} \delta: \Delta$. The difficulty is coming from the base cases of local substitutions. We must incorporate the shape of $\delta^{\prime}$ depending on the cases of $\delta$. We refer the readers to the Agda mechanization for the detailed proof and how we do case analysis on the base cases.

### 2.5 Equivalence Rules

In this section, we describe the equivalence rules. They follow closely to the equivalence rules by Hu and Pientka [2024a, Sec. 4]. We only show the rules for the newly added constructs.

$$
\begin{array}{cc}
\frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t \approx t^{\prime}: T}{\Psi ; \Gamma \vdash_{1} \Lambda g \cdot t \approx \Lambda g \cdot t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T} & \frac{\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T}{\Psi ; \Gamma \vdash_{1} t \$ \Delta \approx t^{\prime} \$ \Delta: T\left[\mathrm{id}_{\Psi}, \Delta / g\right]} \\
\frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T \quad \Psi \vdash_{0} \Delta}{\Psi ; \Gamma \vdash_{1}(\Lambda g \cdot t) \$ \Delta \approx t\left[\mathrm{id}_{\Psi}, \Delta / g\right]: T\left[\mathrm{id}_{\Psi}, \Delta / g\right]} & \\
\Psi ; \Gamma \vdash_{1} t \approx \Lambda g \cdot(t[p(\mathrm{id})]) \$ g:(g: \mathrm{Ctx}) \Rightarrow T
\end{array}
$$

In the rules above, we specify the congruence for meta-abstraction $\Lambda$ and the meta-application. Moreover, they also have $\beta$ and $\eta$ rules in the expected ways.

We also have a equivalence judgment for local substitutions

$$
\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta
$$

The only rules are the congruence rules determined by all possible constructors. Both equivalence judgments for terms and local substitutions must be mutually defined.

Note that there is no need to define the equivalence for global substitutions. Effectively, the equivalence for global substitutions is defined as the equality. This is because all terms stored in a global substitution are at layer 0 , and thus they do not have meaningful dynamics. Therefore, equivalent global substitutions must also be equal.

We first establish the presupposition lemma for equivalence:
Lemma 2.35 (Presupposition).

- If $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} t: T$ and $\Psi ; \Gamma \vdash_{1} t^{\prime}: T$.
- If $\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash_{1} \delta: \Delta$ and $\Psi ; \Gamma \vdash_{1} \delta^{\prime}: \Delta$.

Proof. We perform a mutual induction. For the $\beta$ and $\eta$ rules, we apply substitution lemmas proved in the previous section.

Lemma 2.36 (Local Weakenings).

- If $\Psi ; \Delta \vdash_{1} t \approx t^{\prime}: T$ and $\tau: \Psi ; \Gamma \Longrightarrow{ }_{1} \Delta$, then $\Psi ; \Gamma \vdash_{1} t[\tau] \approx t^{\prime}[\tau]: T$.
- If $\Psi ; \Delta \vdash_{1} \delta \approx \delta^{\prime}: \Delta^{\prime}$ and $\tau: \Psi ; \Gamma \Longrightarrow{ }_{1} \Delta$, then $\Psi ; \Gamma \vdash_{1} \delta[\tau] \approx \delta^{\prime}[\tau]: \Delta^{\prime}$.

Proof. We follow the local weakening property for typing above.
A counterpart is w.r.t to global weakenings:
Lemma 2.37 (Global Weakenings).

- If $\Phi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$ and $\gamma: \Psi \Longrightarrow g$, then $\Psi ; \Gamma[\gamma] \vdash_{1} t[\gamma] \approx t^{\prime}[\gamma]: T[\gamma]$.
- If $\Phi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$ and $\gamma: \Psi \Longrightarrow g$, then $\Psi ; \Gamma[\gamma] \vdash_{1} \delta[\gamma] \approx \delta^{\prime}[\gamma]: \Delta[\gamma]$.

Proof. Similarly, we follow the global weakening property for typing above.
Lemma 2.38 (Congruence of Local Substitutions).

- If $\Psi ; \Gamma \vdash_{1} t: T$ and $\Psi ; \Delta \vdash_{1} \delta \approx \delta^{\prime}: \Gamma$, then $\Psi ; \Delta \vdash_{1} t[\delta] \approx t\left[\delta^{\prime}\right]: T$.
- If $\Psi ; \Gamma \vdash_{1} \delta: \Delta^{\prime}$ and $\Psi ; \Delta \vdash_{1} \delta^{\prime} \approx \delta^{\prime \prime}: \Gamma$, then $\Psi ; \Delta \vdash_{1} \delta \circ \delta^{\prime} \approx \delta \circ \delta^{\prime \prime}: \Delta^{\prime}$.

Proof. We do a mutual induction on $\Psi ; \Gamma \vdash_{1} t: T$ and $\Psi ; \Gamma \vdash_{1} \delta: \Delta^{\prime}$. We only consider a few interesting cases. Case

$$
\frac{\Psi ; \Gamma \vdash_{1} s: \square\left(\Delta^{\prime} \vdash T\right) \quad \Psi \vdash_{1} T^{\prime} \quad \Psi, u:\left(\Delta^{\prime} \vdash T\right) ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T^{\prime}[p(\mathrm{id})]}{\Psi ; \Gamma \vdash_{1} \text { letbox } u \leftarrow s \text { in } t: T^{\prime}}
$$

In this case, we must use the global weakening lemma above to derive

$$
\Psi ; \Delta[p(\mathrm{id})] \vdash_{1} \delta[p(\mathrm{id})] \approx \delta^{\prime}[p(\mathrm{id})]: \Gamma[p(\mathrm{id})]
$$

and then we apply IH.
Case For all base cases of local substitutions, we realize that given $\Psi ; \Delta \vdash_{1} \delta \approx \delta^{\prime}: \Gamma$, we have
$-\check{\delta}=\check{\delta^{\prime}}$, and
$-\widehat{\delta}=\widehat{\delta^{\prime}}$.
because they characterize $\Delta$, so the exact $\delta$ and $\delta^{\prime}$ are irrelevant.
The target goal follows immediate.

## Lemma 2.39 (Local Substitutions).

- If $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$ and $\Psi ; \Delta \vdash_{1} \delta \approx \delta^{\prime}: \Gamma$, then $\Psi ; \Delta \vdash_{1} t[\delta] \approx t^{\prime}\left[\delta^{\prime}\right]: T$.
- If $\Psi ; \Gamma \vdash_{1} \delta^{\prime \prime} \approx \delta^{\prime \prime \prime}: \Delta^{\prime}$ and $\Psi ; \Delta \vdash_{1} \delta \approx \delta^{\prime}: \Gamma$, then $\Psi ; \Delta \vdash_{1} \delta^{\prime \prime} \circ \delta \approx \delta^{\prime \prime \prime} \circ \delta^{\prime}: \Delta^{\prime}$.

Proof. We proceed by a mutual induction on $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$ and $\Psi ; \Gamma \vdash_{1} \delta^{\prime \prime} \approx \delta^{\prime \prime \prime}: \Delta^{\prime}$. We only look into the $\beta$ and $\eta$ rule for meta-functions, because we cannot apply IH:
Case

$$
\frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T \quad \Psi \vdash_{0} \Delta^{\prime}}{\Psi ; \Gamma \vdash_{1}(\Lambda g . t) \$ \Delta^{\prime} \approx t\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]: T\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]}
$$

$\Psi, g: \mathrm{Ctx} ; \Delta[p(\mathrm{id})] \vdash_{1} t[\delta[p(\mathrm{id})]] \approx t\left[\delta^{\prime}[p(\mathrm{id})]\right]: T \quad$ (by local substitution lemma)

$$
\Psi ; \Delta \vdash_{1}(\Lambda g \cdot(t[\delta[p(\mathrm{id})]])) \$ \Delta^{\prime} \approx t\left[\delta^{\prime}[p(\mathrm{id})]\right]\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]: T\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]
$$

Now we have to align up the right hand side. The target right hand side is

$$
t\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]\left[\delta^{\prime}\right]
$$

We reason as follows

$$
\begin{array}{rlr}
t\left[\delta^{\prime}[p(\mathrm{id})]\right]\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right] & =t\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]\left[\delta^{\prime}[p(\mathrm{id})]\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]\right] \quad \text { (by distributivity of global substitutions) } \\
& =t\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]\left[\delta^{\prime}\right] & \text { (by Lemma 2.13) } \tag{byLemma2.13}
\end{array}
$$

and we have the target goal.
Case

$$
\frac{\Psi ; \Gamma \vdash_{1} t:(g: \mathrm{Ctx}) \Rightarrow T}{\Psi ; \Gamma \vdash_{1} t \approx \Lambda g \cdot(t[p(\mathrm{id})]) \$ g:(g: \mathrm{Ctx}) \Rightarrow T}
$$

We apply the local weakening lemma:

$$
\Psi ; \Delta \vdash_{1} t[\delta] \approx t\left[\delta^{\prime}\right]: T
$$

On the right hand side, we have

$$
\Lambda g \cdot(t[p(\mathrm{id})]) \$ g\left[\delta^{\prime}\right]=\Lambda g \cdot\left(t[p(\mathrm{id})]\left[\delta^{\prime}[p(\mathrm{id})]\right]\right) \$ g
$$

$$
=\Lambda g \cdot\left(t\left[\delta^{\prime}\right][p(\mathrm{id})]\right) \$ g \quad \text { (by algebraic rule) }
$$

Next, we consider the global substitution lemma:
Lemma 2.40 (Global Substitutions).

- If $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$ and $\Phi \vdash \sigma: \Psi$, then $\Phi ; \Gamma[\sigma] \vdash_{1} t[\sigma] \approx t^{\prime}[\sigma]: T[\sigma]$.
- If $\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$ and $\Phi \vdash \sigma: \Psi$, then $\Phi ; \Gamma[\sigma] \vdash_{1} \delta[\sigma] \approx \delta^{\prime}[\sigma]: \Delta[\sigma]$.

Proof. As we have noted above, there is no need for an equivalence judgment between two global substitutions. Therefore we directly apply the same $\sigma$ on both sides. Let us consider the $\beta$ and $\eta$ rules for meta-functions: Case

$$
\frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T \quad \Psi \vdash_{0} \Delta^{\prime}}{\Psi ; \Gamma \vdash_{1}(\Lambda g . t) \$ \Delta^{\prime} \approx t\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]: T\left[\mathrm{id}_{\Psi}, \Delta^{\prime} / g\right]}
$$

$\Phi, g: \mathrm{Ctx} \vdash \sigma[p(\mathrm{id})], g / g: \Psi, g: \mathrm{Ctx}$
$\Phi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})][\sigma[p(\mathrm{id})], g / g] \vdash_{1} t[\sigma[p(\mathrm{id})], g / g]: T[\sigma[p(\mathrm{id})], g / g] \quad$ (by global substitution lemma)
$\Phi, g: \mathrm{Ctx} ; \Gamma[\sigma][p(\mathrm{id})] \vdash_{1} t[\sigma[p(\mathrm{id})], g / g]: T[\sigma[p(\mathrm{id})], g / g]$
$\Phi \vdash_{0} \Delta^{\prime}[\sigma]$
$\Phi ; \Gamma[\sigma] \vdash_{1}(\Lambda g .(t[\sigma[p(\mathrm{id})], g / g])) \$\left(\Delta^{\prime}[\sigma]\right) \approx t[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id}_{\Psi}, \Delta^{\prime}[\sigma] / g\right]: T[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id}_{\Psi}, \Delta^{\prime}[\sigma] / g\right]$
We now reason a few equations:

$$
(\Lambda g \cdot(t[\sigma[p(\mathrm{id})], g / g])) \$\left(\Delta^{\prime}[\sigma]\right)=(\Lambda g \cdot t) \$ \Delta^{\prime}[\sigma]
$$

and

$$
\begin{aligned}
t[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id}_{\Psi}, \Delta^{\prime}[\sigma] / g\right] & =t\left[(\sigma[p(\mathrm{id})], g / g) \circ\left(\mathrm{id} \Psi, \Delta^{\prime}[\sigma] / g\right)\right] \\
& =t\left[\sigma[p(\mathrm{id})] \circ\left(\mathrm{id} \Psi, \Delta^{\prime}[\sigma] / g\right), \Delta^{\prime}[\sigma] / g\right] \\
& =t\left[\sigma \circ \mathrm{id}_{\Psi}, \Delta^{\prime}[\sigma] / g\right] \\
& =t\left[\mathrm{id}_{\Phi} \circ \sigma, \Delta^{\prime}[\sigma] / g\right] \\
& =t\left[\mathrm{id}_{\Phi}, \Delta^{\prime} / g\right][\sigma]
\end{aligned}
$$

Similarly, we have

$$
T[\sigma[p(\mathrm{id})], g / g]\left[\mathrm{id}_{\Psi}, \Delta^{\prime}[\sigma] / g\right]=T\left[\mathrm{id}_{\Phi}, \Delta^{\prime} / g\right][\sigma]
$$

This concludes the goal.
Case

$$
\begin{gathered}
\frac{\Psi ; \Gamma \vdash_{1} t:(g: \mathrm{Ctx}) \Rightarrow T}{\Psi ; \Gamma \vdash_{1} t \approx \Lambda g \cdot(t[p(\mathrm{id})]) \$ g:(g: \mathrm{Ctx}) \Rightarrow T} \\
\Phi ; \Gamma[\sigma] \vdash_{1} t[\sigma]:(g: \mathrm{Ctx}) \Rightarrow T[\sigma] \\
\Phi ; \Gamma[\sigma] \vdash_{1} t[\sigma]:(g: \mathrm{Ctx}) \Rightarrow(T[\sigma[p(\mathrm{id})], g / g]) \\
\Phi ; \Gamma[\sigma] \vdash_{1} t[\sigma] \approx \Lambda g \cdot t[\sigma][p(\mathrm{id})] \$ g:(g: \mathrm{Ctx}) \Rightarrow(T[\sigma[p(\mathrm{id})], g / g])
\end{gathered}
$$

We reason the right hand side

$$
\begin{aligned}
\Lambda g \cdot(t[p(\mathrm{id})]) \$ g[\sigma] & =\Lambda g \cdot(t[p(\mathrm{id})][\sigma[p(\mathrm{id})], g / g]) \$ g \\
& =\Lambda g \cdot(t[\sigma[p(\mathrm{id})]]) \$ g \\
& =\Lambda g \cdot(t[\sigma][p(\mathrm{id})]) \$ g
\end{aligned}
$$

and hence establish the goal.

### 2.6 Weak Head Reduction

To compute the decide whether two terms are equivalence based on the typing rules above, we take the approach of reducibility candidates. This approach requires a reduction strategy, and then we use a type directed convertibility checking to determine whether two normal forms after reduction are equivalent. In this approach, it is sufficient to have a term reduced to weak head normal forms, specified as below:

$$
\begin{aligned}
w & :=v \mid \text { zero }|\operatorname{succ} t| \text { box } t|\lambda x . t| \Lambda g . t \\
v & :=x\left|u^{\delta}\right| v t \mid \text { letbox } u \leftarrow v \text { in } t \mid v \$ \Gamma
\end{aligned}
$$

(Weak head normal form ( Nf )
(Neutral form (Ne))
Basically the one-step weak head reduction simply takes the $\beta$ rules from the equivalence relations and adds head reduction:

Head reductions:

$$
\begin{gathered}
\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: S \longrightarrow T \quad \Psi ; \Gamma \vdash_{1} s: S \\
\Psi ; \Gamma \vdash t s \rightsquigarrow t^{\prime} s: T \\
\frac{\Psi ; \Gamma \vdash s \rightsquigarrow s^{\prime}: \square T \quad \Psi, u: T ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T^{\prime}[p(\mathrm{id})]}{\Psi ; \Gamma \vdash \operatorname{letbox} u \leftarrow s \text { in } t \rightsquigarrow \operatorname{letbox} u \leftarrow s^{\prime} \text { in } t: T^{\prime}} \\
\Psi ; \Gamma \vdash t \$ \Delta \rightsquigarrow t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T \quad \Psi \vdash_{0} \Delta \\
\hline \$ \Delta: T\left[\mathrm{id}_{\Psi}, \Delta / g\right]
\end{gathered}
$$

$\beta$ reductions:

$$
\begin{array}{cccc}
\frac{\Psi ; \Gamma, x: S \vdash_{1} t: T}{\Psi ; \Gamma \vdash(\lambda x . t) s \rightsquigarrow t\left[\mathrm{id}_{\Gamma}, s / x\right]: T} & \Psi ; \Gamma \vdash_{1} s: S \\
\hline & & \Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T & \Psi ; \Gamma \vdash(\Lambda g . t) \$ \Delta \rightsquigarrow t\left[\vdash_{0} \Delta\right. \\
\hline & \Psi ; \vdash_{0} s: T & \Psi \vdash_{1} \Gamma & \left.\Psi \vdash_{1} T^{\prime} \quad \Psi, g\right]: T[\mathrm{id} \Psi, \Delta / g] \\
\Psi ; \Gamma ; \Gamma[p(\mathrm{id})] \vdash_{1} t: T^{\prime}[p(\mathrm{id})] \\
\hline
\end{array}
$$

Notice that weak head reduction only occurs at layer 1 , so we do not need to rewrite down the layer explicitly. This is because $\beta$ reduction only occurs at layer 1 and all terms at layer 0 are identified by their exact syntactic structure. We define the reflexive transitive closure $\Psi ; \Gamma \vdash t \rightsquigarrow^{*} t^{\prime}: T$ of weak head reduction in the usual way. In an implementation, we would repeatedly do one-step weak head reduction, until it reaches a normal form. To compare two terms of the same type, we would first compute the weak head normal forms of both sides, and then based on their type, we perform a type-directed convertibility check. This check also performs $\eta$ expansion when necessary, so the resulting algorithm is complete w.r.t. the equivalence relation.
Notice that the weak head reduction relation is a subrelation of the equivalence above. Therefore, several theorems are simply carried over. We omit the proofs and only state the theorems below:

Lemma 2.41 (Local Weakenings). If $\Psi ; \Delta \vdash t \rightsquigarrow t^{\prime}: T$ and $\tau: \Psi ; \Gamma \Longrightarrow{ }_{1} \Delta$, then $\Psi ; \Gamma \vdash t[\tau] \rightsquigarrow t^{\prime}[\tau]: T$.
Lemma 2.42 (Global Weakenings). If $\Phi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: T$ and $\gamma: \Psi \Longrightarrow g$, then $\Psi ; \Gamma[\gamma] \vdash t[\gamma] \rightsquigarrow t^{\prime}[\gamma]: T[\gamma]$.
Theorem 2.43 (Preservation). If $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} t: T$ and $\Psi ; \Gamma \vdash_{1} t^{\prime}: T$.

Lemma 2.44 (Local Substitutions). If $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: T$ and $\Psi ; \Delta \vdash_{1} \delta: \Gamma$, then $\Psi ; \Delta \vdash t[\delta] \rightsquigarrow t^{\prime}[\delta]: T$.
Lemma 2.45 (Globale Substitutions). If $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: T$ and $\Phi \vdash \sigma: \Psi$, then $\Phi ; \Gamma[\sigma] \vdash t[\sigma] \rightsquigarrow t^{\prime}[\sigma]: T[\sigma]$.
Lemma 2.46 (Uniqueness). If $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: T$ and $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime \prime}: T$, then $t^{\prime}=t^{\prime \prime}$.
Proof. Induction on $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime}: T$ and analyze $\Psi ; \Gamma \vdash t \rightsquigarrow t^{\prime \prime}: T$.
These conclusions can be generalized to its reflexive transitive closure.
In the next section, we will first look into the logical relations which establish normalization property, and then we show that this checking strategy indeed is complete.

## 3 NORMALIZATION AND CONVERTIBILITY

In the previous section, we have developed the syntactic theory of contextual variables and the weak head reduction relation. In this section, our goal is to show the termination of the weak head reduction, i.e. normalization, develop a type-directed convertibility checking algorithm, and show that this algorithm is sound and complete w.r.t. the equivalence of terms. In the first step to normalization, we establish the reducibility predicate. This predicate is defined inductively and then recursively in order to describe the semantic well-formedness of types and terms. Then from the reducibility predicate, we prove the semantic soundness of the model and prove the normalization property as a corollary. Next, we give the convertibility algorithm as a judgment. This judgment will be shown equivalent to the equivalence of terms, which wraps up our discussion on simple types.

### 3.1 Generic Equivalence

We follow Abel et al. [2017] to define a modular generic equivalence to ease subsequent proof constructions. Since we do not have type-level computation for this almost simply typed theory, we only have to be concerned about two kinds of equivalence over terms: $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$ describes a generic equivalence between two terms, and $\Psi ; \Gamma \vdash v \sim v^{\prime}: T$ describes a generic equivalence between two neutral terms. Furthermore, $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$ is generalized to $\Psi ; \Gamma \vdash \delta \simeq \delta^{\prime}: \Delta$ by recursion on $\Delta$, which denotes a generic equivalence between local substitutions and the base cases are just congruence. The definition is

$$
\begin{aligned}
& \begin{array}{ccc}
\Psi \vdash_{1} \Gamma \quad \Gamma \text { ends with } \cdot & |\Gamma|=m \\
\Psi ; \Gamma \vdash \cdot^{m} \simeq \cdot^{m}: \cdot & \Psi \vdash_{1} \Gamma \quad g: \operatorname{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m \\
\Psi ; \Gamma \vdash \cdot{ }_{g}^{m} \simeq ._{g}^{m}: \cdot
\end{array} \\
& \frac{\Psi \vdash_{1} \Gamma \quad g: \mathrm{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m}{\Psi ; \Gamma \vdash \mathrm{wk}_{g}^{m} \simeq \mathrm{wk}_{g}^{m}: g} \quad \frac{\Psi ; \Gamma \vdash \delta \simeq \delta^{\prime}: \Delta \quad \Psi ; \Gamma \vdash t \simeq t^{\prime}: T}{\Psi ; \Gamma \vdash \delta, t / x \simeq \delta^{\prime}, t^{\prime} / x: \Delta, x: T}
\end{aligned}
$$

The generic equivalence will be instantiated twice, once for each layer. Following the layering principle, the laws for layer 0 are subsumed by those for layer 1 . The reason why we also need an instantiation for layer 0 is that code from layer 0 can be lifted by letbox and become a program, so that its computation is recovered. Therefore, a logical relation is needed to capture its computational behavior.

These two relations must satisfy the following laws. Law statements for $\square$ and meta-functions only apply when the generic equivalence is indexed by layer 1 .

Law 3.1 (Subsumption).

- If $\Psi ; \Gamma \vdash v \sim v^{\prime}: T$, then $\Psi ; \Gamma \vdash v \simeq v^{\prime}: T$.
- If $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$.

From the subsumption of generic equivalence of terms, we have the subsumption of generic equivalence of local substitutions as a lemma:

Lemma 3.1 (Subsumption). If $\Psi ; \Gamma \vdash \delta \simeq \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$.
LAw 3.2 (PER). Both relations are PERs.
Law 3.3 (Monotonicity). Given $\gamma ; \tau: \Psi ; \Gamma \Longrightarrow \Phi ; \Delta$, if $\Phi ; \Delta \vdash t \simeq t^{\prime}: T$, or $\Phi ; \Delta \vdash v \sim v^{\prime}: T$, then $\Psi ; \Gamma \vdash t[\gamma ; \tau] \simeq t^{\prime}[\gamma ; \tau]: T[\gamma]$, or $\Psi ; \Gamma \vdash v[\gamma ; \tau] \sim v^{\prime}[\gamma ; \tau]: T[\gamma]$, respectively.

As a lemma, we have monotonicity generalized to local substitutions:
Lemma 3.2 (Monotonicity). Given $\gamma ; \tau: \Psi ; \Gamma \Longrightarrow \Phi ; \Delta$, if $\Phi ; \Delta \vdash \delta \simeq \delta^{\prime}: \Delta^{\prime}$, then $\Psi ; \Gamma \vdash \delta[\gamma ; \tau] \simeq \delta^{\prime}[\gamma ; \tau]$ : $\Delta^{\prime}[\gamma]$.

Law 3.4 (Weak head closure). If $\Psi ; \Gamma \vdash t \rightsquigarrow^{*} w: T, \Psi ; \Gamma \vdash t^{\prime} \rightsquigarrow^{*} w^{\prime}: T$ and $\Psi ; \Gamma \vdash w \simeq w^{\prime}: T$, then $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$.

Law 3.5 (Congruence).

- If $\Psi \vdash_{1} \Gamma$, then $\Psi ; \Gamma \vdash$ zero $\simeq$ zero : Nat.
- If $\Psi ; \Gamma \vdash t \simeq t^{\prime}$ : Nat, then $\Psi ; \Gamma \vdash \operatorname{succ} t \simeq \operatorname{succ} t^{\prime}$ : Nat.
- If $\Psi ; \Gamma \vdash_{1} t: S \longrightarrow T, \Psi ; \Gamma \vdash_{1} t^{\prime}: S \longrightarrow T$ and $\Psi ; \Gamma, x: S \vdash t[i d ; p(i d)] x \simeq t^{\prime}[i d ; p(i d)] x: T$, then $\Psi ; \Gamma \vdash t \simeq t^{\prime}: S \longrightarrow T$.
- If $\Psi \vdash_{1} \Gamma$ and $\Psi ; \Delta \vdash_{0} t: T$, then $\Psi ; \Gamma \vdash$ box $t \simeq$ box $t: T$.
- If $\Psi ; \Gamma \vdash_{1} t:(g: C t x) \Rightarrow T, \Psi ; \Gamma \vdash_{1} t^{\prime}:(g: C t x) \Rightarrow T$ and $\Psi, g: C t x ; \Gamma \vdash t[p(i d)] \$ g \simeq t^{\prime}[p(i d)] \$ g: T$, then $\Psi ; \Gamma \vdash t \simeq t^{\prime}:(g: C t x) \Rightarrow T$.

Law 3.6 (Congruence of neutrals).

- If $\Psi ; \Gamma \vdash_{1} x: T$, then $\Psi ; \Gamma \vdash x \sim x: T$.
- If $u:(\Delta \vdash T) \in \Psi$ and $\Psi ; \Gamma \vdash \delta \simeq \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash u^{\delta} \sim u^{\delta^{\prime}}: T$.
- If $\Psi ; \Gamma \vdash v \sim v^{\prime}: S \longrightarrow T$ and $\Psi ; \Gamma \vdash t \simeq t^{\prime}: S$, then $\Psi ; \Gamma \vdash v t \sim v t^{\prime}: T$.
- If $\Psi \vdash_{1} T^{\prime}, \Psi ; \Gamma \vdash v \sim v^{\prime}: \square(\Delta \vdash T)$ and $\Psi, u:(\Delta \vdash T) ; \Gamma \vdash t \simeq t^{\prime}: T^{\prime}[p(i d)]$, then $\Psi ; \Gamma \vdash$ letbox $u \leftarrow$ $v$ in $t \sim$ letbox $u \leftarrow v^{\prime}$ in $t^{\prime}: T^{\prime}$.
- If $\Psi ; \Gamma \vdash v \sim v^{\prime}:(g: C t x) \Rightarrow T$ and $\Psi \vdash_{0} \Delta$, then $\Psi ; \Gamma \vdash v \$ \Delta \sim v^{\prime} \$ \Delta: T\left[i d_{\Psi}, \Delta / g\right]$

From the congruence of local variables, we have that local weakening substitutions, specifically, local identity substitutions, are reflexive in the generic equivalence:

Lemma 3.3 (Reflexivity of Local Weakening Substitutions). If $\Psi \vdash_{1} \Delta, \Gamma$, then $\Psi ; \Delta, \Gamma \vdash w k_{\Delta}^{|\Gamma|} \simeq w k_{\Delta}^{|\Gamma|}: \Delta$.
Lemma 3.4 (Reflexivity of Local Identity Substitutions). If $\Psi \vdash_{1} \Gamma$, then $\Psi ; \Gamma \vdash i d_{\Gamma} \simeq i d_{\Gamma}: \Gamma$.
This further implies
Lemma 3.5 (Congruence of Global Variables). If $\vdash \Psi$ and $u:(\Gamma \vdash T) \in \Psi$, then $\Psi ; \Gamma \vdash u^{i d_{\Gamma}} \sim u^{i d_{\Gamma}}: T$.
We give the first instantiation of both relations as follows:

- $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T:=\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$, and
- $\Psi ; \Gamma \vdash v \sim v^{\prime}: T:=\Psi ; \Gamma \vdash_{1} v \approx v^{\prime}: T$.

The laws are instantiated to appropriate rules. Later, we will instantiate the relations to algorithmic equivalence, showing that the algorithmic rules are complete, following Abel et al. [2017]. Before that, let us first define the reducibility predicates parameterized by the generic equivalence relations for terms.

### 3.2 Reducibility Predicates

Following Abel et al. [2017], we first give the semantic well-formedness of types. The predicates do not need to be defined recursive-inductively, because unlike dependent types, there is no type-level computation here in our system.


Compared to the syntactic well-formedness judgment in Sec. 2.1, the semantic counterpart has extra universal quantifications over global weakenings. These universal quantifications are necessary when we give semantics to terms. Moreover, the semantic well-formedness of types is monotonic w.r.t. global weakenings:

Lemma 3.6 (Monotonicity). If $\gamma: \Phi \Longrightarrow g$ and $\Psi \vDash_{i} T$, then $\Phi \vDash_{i} T[\gamma]$.
Proof. Induction.
The lifting lemma also has a semantic counterpart:
Lemma 3.7 (Lifting). If $\Psi \vDash_{0} T$, then $\Psi \vDash_{1} T$.
Proof. Induction.
Lemma 3.8 (Escape). If $\Psi \vdash_{i} T$, then $\Psi \vdash_{i} T$.
Proof. We do induction on $\Psi \vDash_{i} T$. In the meta-function case, we instantiate the global weakening to $p$ (id) : $\Psi, g:$ Ctx $\Longrightarrow g$ and $\Gamma$ to $g$. Apply IH again to obtain the goal.

Now we move on to defining the reducibility for terms $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$ and local substitutions $\Psi ; \Gamma \vDash_{i}$ $\delta \approx \delta^{\prime}: \Delta$. Both relations are defined first by recursion on the layer $i$. Then, the reducibility predicate for terms $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$ is defined by recursion on $\Psi \vDash_{i} T$. The reducibility predicate for local substitutions $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta$ is defined inductively. The predicates are defined in this way because the layer-1 predicate for terms refers to the predicates at layer 0, as in our presheaf models by Hu and Pientka [2024a]. The definition that we present here does not focus too much on the layered nature to save space and be more modular. However, we imagine that if this model must be put into a proof assistant, then attention must be paid to ensure the wellfoundness. The predicate for terms is a Kripke model, as it is indexed by both global and local contexts. We first define the semantic natural numbers:

| $\Psi ; \Gamma \vdash t \rightsquigarrow^{*} w: N a t \quad \Psi ; \Gamma \vdash t^{\prime} \rightsquigarrow^{*} w^{\prime}: N a t \quad \Psi ; \Gamma \vdash w \simeq w^{\prime}: N a t$ | $\Psi ; \Gamma \vDash^{\mathrm{Nf}} w \approx w^{\prime}: \mathrm{Nat}$ |
| :---: | :---: |
| $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: \mathrm{Nat}$ |  |
| $\overline{\Psi ; \Gamma \vDash^{\mathrm{Nf}} \text { zero } \approx \text { zero }: \mathrm{Nat}} \quad \frac{\Psi ; \Gamma \vDash t \approx t^{\prime}: \mathrm{Nat}}{\Psi ; \Gamma \vDash^{\mathrm{Nf}} \operatorname{succ} t \approx \operatorname{succ} t^{\prime}: \mathrm{Nat}} \quad \frac{\Psi ; \Gamma \vdash v \sim v^{\prime}: \mathrm{Nat}}{\Psi ; \Gamma \vDash^{\mathrm{Nf}} v \approx v^{\prime}: \mathrm{Nat}}$ |  |

Then we let $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}$ : Nat := $\Psi ; \Gamma \vDash t \approx t^{\prime}:$ Nat.
Next, we define the case for function. $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: S \longrightarrow T$ holds iff

- $\Psi ; \Gamma \vdash t \rightsquigarrow * w: S \longrightarrow T$, and
- $\Psi ; \Gamma \vdash t^{\prime} \rightsquigarrow^{*} w^{\prime}: S \longrightarrow T$, and
- $\Psi ; \Gamma \vdash w \simeq w^{\prime}: S \longrightarrow T$, and
- for any $\gamma ; \tau: \Phi ; \Delta \Longrightarrow \Psi ; \Gamma$, and given $\Phi ; \Delta \vDash_{i} s \approx s^{\prime}: S[\gamma]$, then we have $\Phi ; \Delta \vDash_{i} w[\gamma ; \tau] s \approx w^{\prime}[\gamma ; \tau] s^{\prime}:$ $T[\gamma]$.
It means that $t$ reduces to some weak head normal form, and the result of applying this weak head normal form to a reducible term remains reducible.

Next, we define the reducibility for $\square(\Delta \vdash T)$,

$$
\begin{gathered}
\Psi ; \Gamma \vdash t \rightsquigarrow *^{*} w: \square(\Delta \vdash T) \quad \Psi ; \Gamma \vdash t^{\prime} \rightsquigarrow *^{*} w^{\prime}: \square(\Delta \vdash T) \\
\frac{\Psi ; \Gamma \vdash w \simeq w^{\prime}: \square(\Delta \vdash T) \quad \Psi ; \Gamma \vDash_{1}^{\mathrm{Nf}} w \approx w^{\prime}: \square(\Delta \vdash T)}{\Psi ; \Gamma \vDash_{1} t \approx t^{\prime}: \square(\Delta \vdash T)} \\
\frac{\Psi ; \Delta \vdash_{0} t: T \quad \forall \Psi ; \Delta^{\prime} \vDash_{0} \delta \approx \delta^{\prime}: \Delta \cdot \Psi ; \Delta^{\prime} \vDash_{0} t[\delta] \approx t\left[\delta^{\prime}\right]: T}{\Psi ; \Gamma \vDash_{1}^{\mathrm{Nf}} \operatorname{box} t \approx \operatorname{box} t: \square(\Delta \vdash T)} \quad \frac{\Psi ; \Gamma \vdash v \sim v^{\prime}: \square(\Delta \vdash T)}{\Psi ; \Gamma \vDash_{1}^{\mathrm{Nf}} v \approx v^{\prime}: \square(\Delta \vdash T)}
\end{gathered}
$$

Similar to our presheaf models, we must refer to $\Psi ; \Delta \vDash_{0} \delta \approx \delta^{\prime}: \Gamma$ when giving semantics for terms of type $\square(\Delta \vdash T)$, so the predicates at layer 0 must be finished before defining the predicate at layer 1 . To support pattern matching on code, instead of this universal quantification, we must use an inductively defined layer-0 model instead as done in Hu and Pientka [2024a, Sec. 5.4.1].

Next, we define the case for meta-functions. $\Psi ; \Gamma \vDash_{1} t \approx t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T$ holds iff

- $\Psi ; \Gamma \vdash t \rightsquigarrow^{*} w:(g: \mathrm{Ctx}) \Rightarrow T$, and
- $\Psi ; \Gamma \vdash t^{\prime} \rightsquigarrow^{*} w^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T$, and
- $\Psi ; \Gamma \vdash w \simeq w^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T$, and
- for any $\gamma ; \tau: \Phi ; \Delta \Longrightarrow \Psi ; \Gamma$, and given $\Phi \vdash_{0} \Delta^{\prime}$, then we have $\Phi ; \Delta \vDash_{1} w[\gamma ; \tau] \$ \Delta^{\prime} \approx w^{\prime}[\gamma ; \tau] \$ \Delta^{\prime}$ : $T[q(\gamma)]\left[\mathrm{id}_{\Phi}, \Delta^{\prime} / g\right]$.
We generalize the reducibility for terms to local contexts and local substitutions by doing an inductiverecursive definition:

$$
\frac{\vdash \Psi}{\Psi \Vdash_{i}}
$$

Then $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \cdot$ holds by considering the cases for $\Gamma$,

- if $\Gamma$ ends with $\cdot$, then $\delta=\delta^{\prime}=.|\Gamma|$;
- if $\Gamma$ ends with $g$, then $g: \mathrm{Ctx} \in \Psi$ and $\delta=\delta^{\prime}={ }_{g}^{|\Gamma|}$.

$$
\frac{\vdash \Psi \quad g: \mathrm{Ctx} \in \Psi}{\Psi \vDash_{i} g}
$$

Then $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: g$ holds iff $\Gamma$ ends with $g$ and $\delta=\delta^{\prime}=\mathrm{wk}_{g}^{|\Gamma|}$.

$$
\frac{\Psi \vDash_{i} \Delta \quad \Psi \vDash_{i} T}{\Psi \models_{i} \Delta, x: T}
$$

Then $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta, x: T$ holds iff

- $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta$,
- $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$.

Note that one should consider the rules above really give two predicates, one for layer 0 and one for layer 1 . In this technical report, however, we do not really type out the replication.

At this point, we have finished defining reducibility predicates for all types. We further let $\Psi ; \Gamma \vDash_{i} t: T$ be $\Psi ; \Gamma \vDash_{i} t \approx t: T$. This predicate basically means that $t$ can be reduced to some weak head normal form.

By the definition of the predicates, we have the following lemmas:
Lemma 3.9. If $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash s \rightsquigarrow^{*} t: T$, and $\Psi ; \Gamma \vdash s^{\prime} \rightsquigarrow^{*} t^{\prime}: T$, and $\Psi ; \Gamma \vDash_{i} s \approx s^{\prime}: T$.
Proof. By induction on $\Psi \vDash_{i} T$.
Lemma 3.10. If $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash t \rightsquigarrow^{*} w: T$, and $\Psi ; \Gamma \vdash t^{\prime} \rightsquigarrow^{*} w^{\prime}: T$, and $\Psi ; \Gamma \vdash w \simeq w^{\prime}: T$.
Proof. By induction on $\Psi \models_{i} T$ and transitivity of multi-step weak head reduction.
Corollary 3.11. If $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$.
Lemma 3.12. If $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash \delta \simeq \delta^{\prime}: \Delta$.
Proof. Generalization of Corollary 3.11.
Lemma 3.13 (Reducibility of Neutrals). If $\Psi \vDash_{i} T, \Psi ; \Gamma \vdash_{i} v: T$ and $\Psi ; \Gamma \vdash v \sim v^{\prime}: T$, then $\Psi ; \Gamma \vDash_{i} v \approx v^{\prime}: T$.
Proof. Induction by $\Psi \vDash_{i} T$ and apply IHs.
Lemma 3.14 (Reducibility of Weakenings). If $\Psi \vDash_{i} \Gamma, \Delta$, then $\Psi ; \Gamma, \Delta \vDash_{i} w k_{\Gamma}^{|\Delta|} \approx w k_{\Gamma}^{|\Delta|}: \Gamma$.
Proof. A direct consequence of Lemma 3.13.
Corollary 3.15 (Reducibility of Identity). If $\Psi \vDash_{i} \Gamma$, then $\Psi ; \Gamma \vDash_{i} i d_{\Gamma} \approx i d_{\Gamma}: \Gamma$.
Lemma 3.16. If $\Psi \vDash_{0} T$, then $\Psi ; \Gamma \vDash_{0} t \approx t^{\prime}: T$ iff $\Psi ; \Gamma \vDash_{1} t \approx t^{\prime}: T$.
Proof. We only consider the cases for Nat and functions and apply IHs directly.
This is derived from the fact that the predicate for Nat is invariant at different layers.
We verify several important properties for the reducibility predicates:
Lemma 3.17 (Escape). If $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$.
Proof. We do induction on $\Psi \models_{i} T$. Notice that generic equivalence eventually implies syntactic equivalence by the subsumption law.

Lemma 3.18 (Escape). If $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$.
Lemma 3.19 (Monotonicity). If $\Psi \vDash_{i} T$ and $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$, given $\gamma ; \tau: \Phi ; \Delta \Longrightarrow \Psi ; \Gamma$, then $\Phi ; \Delta \vDash_{i} t[\gamma ; \tau] \approx$ $t^{\prime}[\gamma ; \tau]: T[\gamma]$.

Proof. We do induction on $\Psi \vDash_{i} T$.
Case $T=$ Nat, then we have the goal by the monotonicity of multi-step weak head reduction and the generic equivalence. We further do induction on $\Psi ; \Gamma \vDash^{N f} w \approx w^{\prime}:$ Nat.
Case $T=S \longrightarrow T^{\prime}$, then we assume another $\gamma^{\prime} ; \tau^{\prime}: \Phi^{\prime} ; \Delta^{\prime} \Longrightarrow \Phi ; \Delta$ and $\Phi^{\prime} ; \Delta^{\prime} \vDash_{i} s \approx s^{\prime}: S\left[\gamma \circ \gamma^{\prime}\right]$, we must show $\Phi^{\prime} ; \Delta^{\prime} \vDash_{i} t[\gamma ; \tau]\left[\gamma^{\prime} ; \tau^{\prime}\right] s \approx t^{\prime}[\gamma ; \tau]\left[\gamma^{\prime} ; \tau^{\prime}\right] s^{\prime}: S\left[\gamma \circ \gamma^{\prime}\right]$. But this is immediate due to the composition of weakenings.
Case $T=\square\left(\Delta \vdash T^{\prime}\right)$, then it is also immediate after a case analysis of $\Psi ; \Gamma \not \vDash^{N f} w \approx w^{\prime}: \square\left(\Delta \vdash T^{\prime}\right)$. We apply the monotonicity of multi-step weak head reduction and the generic equivalence appropriately.
Case $T=(g: \mathrm{Ctx}) \Rightarrow T^{\prime}$, then this case is very similar to the function case. We assume another $\gamma^{\prime} ; \tau^{\prime}: \Phi^{\prime} ; \Delta^{\prime} \Longrightarrow$ $\Phi ; \Delta$ and $\Phi^{\prime} \vdash_{0} \Gamma^{\prime}$. Our goal is to show

$$
\Phi^{\prime} ; \Delta^{\prime} \vDash_{1} w[\gamma ; \tau]\left[\gamma^{\prime} ; \tau^{\prime}\right] \$ \Gamma^{\prime} \approx w^{\prime}[\gamma ; \tau]\left[\gamma^{\prime} ; \tau^{\prime}\right] \$ \Gamma^{\prime}: T\left[q\left(\gamma \circ \gamma^{\prime}\right)\right]\left[\mathrm{id}_{\Phi^{\prime}}, \Gamma^{\prime} / g\right]
$$

This again has been given by the composition of weakenings and notice that $q\left(\gamma \circ \gamma^{\prime}\right)=q(\gamma) \circ q\left(\gamma^{\prime}\right)$.

Lemma 3.20 (Monotonicity). If $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta^{\prime}$, given $\gamma ; \tau: \Phi ; \Delta \Longrightarrow \Psi ; \Gamma$, then $\Phi ; \Delta \vDash_{i} \delta[\gamma ; \tau] \approx \delta^{\prime}[\gamma ; \tau]$ : $\Delta^{\prime}[\gamma]$.

Lemma 3.21 (PER). The $\Psi ; \Gamma \vDash_{i} t \approx t^{\prime}: T$ relation satisfies symmetry and transitivity.
Proof. Transitivity relies on the uniqueness of weak head reduction.
Lemma 3.22 (PER). The $\Psi ; \Gamma \vDash_{i} \delta \approx \delta^{\prime}: \Delta$ relation satisfies symmetry and transitivity.
The following lemma is the semantic counterpart for the layering principle. It ensures that terms inhabiting types in STLC have the same behaviors at both layers.

Lemma 3.23 (Layering Restriction).

- If $\Psi \vDash_{0} T$, then $\Psi ; \Gamma \vDash_{0} t \approx t^{\prime}: T$ is equivalent to $\Psi ; \Gamma \vDash_{1} t \approx t^{\prime}: T$.
- If $\Psi \vDash_{0} \Delta$, then $\Psi ; \Gamma \vDash_{0} \delta \approx \delta^{\prime}: \Delta$ is equivalent to $\Psi ; \Gamma \vDash_{1} \delta \approx \delta^{\prime}: \Delta$.

This lemma is particularly useful to help treat global variables the same at both layers.
Our goal is then to show the following theorem:
Theorem 3.24 (Completeness).

- If $\Psi \vdash_{i} T$, then $\Psi \vDash_{i} T$.
- If $\Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$.
- If $\Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$.

If the $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$ is the algorithmic convertibility checking algorithm, then we show that syntactic equivalence implies algorithmic equivalence. In other words, algorithmic convertibility is complete w.r.t. syntactic equivalence. However, to arrive at that solution, we must first show that the completeness theorem above holds w.r.t. the generic equivalence relations. Following Abel et al. [2017], however, we still have one step missing to conclude this goal. We must define a set of validity judgments to handle the meta-function case in the semantic well-formedness of types.

### 3.3 Validity Judgments

According to Abel et al. [2017], validity judgments are introduced to characterize the reducible terms that are also closed under substitutions. In the same spirit, we also need the validity judgments to handle the case for meta-functions:

$$
\frac{\Psi, g: \operatorname{Ctx}_{1} T \quad \forall \gamma: \Phi \Longrightarrow g \Psi \text { and } \Phi \vdash_{0} \Gamma \cdot \Phi \vDash_{1} T[q(\gamma)]\left[\mathrm{id}_{\Phi}, \Gamma / g\right]}{\Psi \vDash_{1}(g: \mathrm{Ctx}) \Rightarrow T}
$$

When we attempt to prove the following statement from the completeness theorem:

$$
\text { If } \Psi \vdash_{i} T \text {, then } \Psi \vDash_{1} T \text {. }
$$

This case breaks down, because the IH only provides

$$
\Psi, g: \mathrm{Ctx} \vDash_{1} T
$$

and by monotonicity, we derive

$$
\Phi, g: \mathrm{Ctx}^{\vDash_{1}} T[q(\gamma)]
$$

but then we are stuck. We have no way to prove that the semantic well-formedness of types is closed under substitutions. Following Abel et al. [2017], we define validity judgments which further restrict reducibility predicates to subsets that are closed under substitutions. Since types are only affected by the global contexts, the validity judgments are defined by induction-recursion on global contexts:

$$
\overline{\vDash^{v}}
$$

The validity equivalence for global substitutions $\Phi \vDash^{v} \sigma: \cdot$ is defined as $\vdash \Phi$ and $\sigma=\cdot$.

$$
\frac{\vDash^{v} \Psi}{\models^{v} \Psi, g: \mathrm{Ctx}}
$$

$\Phi \models^{v} \sigma: \Psi, g:$ Ctx is defined as

- $\sigma=\sigma_{1}, \Gamma / g$,
- $\Phi \vdash_{0} \Gamma$, and
- $\Phi \vDash^{v} \sigma_{1}: \Psi$.

$$
\frac{\vDash^{v} \Psi \quad \Psi \models_{0}^{v} \Gamma \quad \Psi \models_{0}^{v} T}{\vDash^{v} \Psi, u:(\Gamma \vdash T)}
$$

$\Phi \vDash^{v} \sigma: \Psi, u:(\Gamma \vdash T)$ is defined as

- $\sigma=\sigma_{1}, t / u$,
- $\Phi \vDash^{v} \sigma_{1}: \Psi$, and
- for all $\Phi ; \Delta \vDash_{0} \delta \approx \delta^{\prime}: \Gamma\left[\sigma_{1}\right]$, we have $\Phi ; \Delta \vDash_{0} t[\delta] \approx t\left[\delta^{\prime}\right]: T\left[\sigma_{1}\right]$.

The validity of types $\Psi \vDash_{i}^{v} T$ is defined as: $\Psi \vDash_{i} T$ and given $\Phi \vDash^{v} \sigma$ : $\Psi$, we have $\Phi \vDash_{i} T[\sigma]$. The validity of local contexts $\Psi \models_{i}^{v} \Gamma$ is defined by applying the validity of types pointwise. The judgments are simplified because there is no need to have an equivalence judgment between global substitutions. In a global substitution, there are only two kinds of components: code of STLC and local contexts. The former is determined to be identified by syntax. Local contexts are also identified by syntax because we are dealing with simple types.

With these definitions ready, we put a universal quantification on top of the reducibility predicates, which specifies the reducible terms that are closed under valid global substitutions:

- $\Psi ; \Gamma \vDash_{i}^{v} t \approx t^{\prime}: T$ iff for any $\Phi \vDash^{v} \sigma: \Psi$ and $\Phi ; \Delta \vDash_{i} \delta \approx \delta^{\prime}: \Gamma[\sigma]$, we have $\Phi ; \Delta \vDash_{i} t[\sigma][\delta] \approx t^{\prime}[\sigma][\delta]:$ $T[\sigma]$.
- $\Psi ; \Gamma \vDash_{i}^{v} \delta \approx \delta^{\prime}: \Delta$ iff for any $\Phi \vDash^{v} \sigma: \Psi$ and $\Phi ; \Delta^{\prime} \vDash_{i} \delta^{\prime \prime} \approx \delta^{\prime \prime \prime}: \Gamma[\sigma]$, we have $\Phi ; \Delta^{\prime} \vDash_{i} \delta[\sigma] \circ \delta^{\prime \prime} \approx$ $\delta^{\prime}[\sigma] \circ \delta^{\prime \prime}: \Delta[\sigma]$.
Now we work out several lemmas:
Lemma 3.25 (Escape). If $\Phi \vDash^{v} \sigma: \Psi$, then $\Phi \vdash \sigma: \Psi$.
Proof. We do induction on $\vDash^{v} \Psi$. In the case for extension of code, we apply the escape lemma for semantically well-formed types and reducible terms.

Lemma 3.26 (Monotonicity). If $\Phi \vDash^{v} \sigma: \Psi$ and $\gamma: \Phi^{\prime} \Longrightarrow_{g} \Phi$, then $\Phi^{\prime} \vDash^{v} \sigma[\gamma]: \Psi$.
Proof. We do induction on $\vDash^{v} \Psi$. Use monotonicity of reducible terms and the algebra of global substitutions.

Lemma 3.27 (Validity of Global Weakening Substitutions). If $\Psi, \Phi$ and $\vDash^{v} \Psi$, then $\Psi, \Phi \vDash^{v} w k_{\Psi}^{|\Phi|}: \Psi$.

Proof. We do induction on $\vDash^{v} \Psi$. The most interesting case is the extension case for code. If $\Psi=\Psi^{\prime}, u:(\Gamma \vdash$ $T$ ), then our goal is given $\Psi^{\prime}, u:(\Gamma \vdash T), \Phi ; \Delta \vDash_{0} \delta \approx \delta^{\prime}: \Gamma\left[w \mathrm{k}_{\Psi^{\prime}}^{1+|\Phi|}\right]$, to prove

$$
\Psi^{\prime}, u:(\Gamma \vdash T), \Phi ; \Delta \models_{0} u^{\mathrm{id} \mathrm{~d}_{\Gamma}\left[p^{1+|\Phi|}(\mathrm{id})\right]}[\delta] \approx u^{\mathrm{id} \mathrm{~d}_{\Gamma}\left[p^{1+|\Phi|}(\mathrm{id})\right]}\left[\delta^{\prime}\right]: T\left[\mathrm{wk}_{\Psi^{\prime}}^{1+|\Phi|}\right]
$$

In the last section, we have established that $\left[w k_{\Psi^{\prime}}^{1+|\Phi|}\right]$ has the same effect as $\left[p^{1+|\Phi|}(\mathrm{id})\right]$, so this goal becomes

$$
\Psi^{\prime}, u:(\Gamma \vdash T), \Phi ; \Delta \models_{0} u^{\mathrm{id}\left[\left[p^{1+|\Phi|}(\mathrm{id})\right]\right.}[\delta] \approx u^{\mathrm{id}\left[\left[p^{1+|\Phi|}(\mathrm{id})\right]\right.}\left[\delta^{\prime}\right]: T\left[p^{1+|\Phi|}(\mathrm{id})\right]
$$

By computation, the goal further becomes:

$$
\Psi^{\prime}, u:(\Gamma \vdash T), \Phi ; \Delta \vDash_{0} u^{\delta} \approx u^{\delta^{\prime}}: T\left[p^{1+|\Phi|}(\mathrm{id})\right]
$$

But this is immediate due to Lemma 3.12, the reducibility of neutrals and congruence of the generic equivalence

$$
\Psi^{\prime}, u:(\Gamma \vdash T), \Phi ; \Delta \vdash u^{\delta} \sim u^{\delta^{\prime}}: T\left[p^{1+|\Phi|}(\mathrm{id})\right]
$$

In particular, it proves that the identity is valid:
Corollary 3.28. If $\vdash \Psi$, then $\Psi \vDash^{v}$ id : $\Psi$.
Theorem 3.29 (Fundamental).

- If $\vdash \Psi$, then $\vDash^{v} \Psi$.
- If $\Psi \vdash_{i} T$, then $\Psi \vDash_{i}^{v} T$.
- If $\Psi \vdash_{i} \Gamma$, then $\Psi \vDash_{i}^{v} \Gamma$.

Proof. We do induction. The cases for global contexts are simple.

## Case

$$
\frac{\vdash \Psi}{\Psi \vdash_{i} N a t}
$$

Assuming $\Phi \vDash^{v} \sigma: \Psi$, by escape, we have $\Phi \vdash \sigma: \Psi$, and then $\vdash \Phi$ by presupposition. We then conclude the goal.
Case

$$
\frac{\Psi \vdash_{i} S \quad \Psi \vdash_{i} T}{\Psi \vdash_{i} S \longrightarrow T}
$$

We assume $\Phi \vDash^{v} \sigma: \Psi$. We now should prove $\Phi \vDash_{i}(S \longrightarrow T)[\sigma]$. This can be concluded from $\Phi \vDash_{i} S[\sigma]$ and $\Phi \vDash_{i} T[\sigma]$. They are immediate from $\Psi \vDash_{i}^{v} S$ and $\Psi \models_{i}^{v} T$ by IH.
Case

$$
\frac{\Psi \vdash_{0} \Delta \quad \Psi \vdash_{0} T}{\Psi \vdash_{1} \square(\Delta \vdash T)}
$$

We assume $\Phi \vDash^{v} \sigma: \Psi$. We now should prove $\Phi \vDash_{i}(\square(\Delta \vdash T))[\sigma]$. We further assume $\gamma: \Phi^{\prime} \Longrightarrow{ }_{g} \Phi$. The goal can be concluded from $\Phi^{\prime} \vDash_{1} T[\sigma][\gamma]$. Since $T[\sigma][\gamma]=T[\sigma[\gamma]]$ and $\Psi \vDash_{0}^{v} T$ from IH, we only need $\Phi^{\prime} \vDash^{v} \sigma[\gamma]: \Psi$, which we get from monotonicity.

Case

$$
\frac{\Psi, g: \mathrm{Ctx} \vdash_{1} T}{\Psi \vdash_{1}(g: \mathrm{Ctx}) \Rightarrow T}
$$

We assume $\Phi \vDash^{v} \sigma: \Psi$. We now should prove $\Phi \vDash_{i}((g: \mathrm{Ctx}) \Rightarrow T)[\sigma]$. We further assume $\gamma: \Phi^{\prime} \Longrightarrow{ }_{g} \Phi$ and $\Phi^{\prime} \vdash_{0} \Gamma$. The goal can be concluded from $\Phi^{\prime} \vDash_{1} T[q(\sigma)][q(\gamma)][\mathrm{id}, \Gamma / g]$. We compute:

$$
\begin{aligned}
T[q(\sigma)][q(\gamma)][\mathrm{id}, \Gamma / g] & =T[q(\sigma)[q(\gamma)]][\mathrm{id}, \Gamma / g] \\
& =T[q(\sigma[\gamma])][\mathrm{id}, \Gamma / g] \\
& =T[\sigma[\gamma][p(\mathrm{id})], g / g][\mathrm{id}, \Gamma / g] \\
& =T[(\sigma[\gamma][p(\mathrm{id})] \circ(\mathrm{id}, \Gamma / g)), g[\mathrm{id}, \Gamma / g] / g] \\
& =T[\sigma[\gamma], \Gamma / g]
\end{aligned}
$$

Therefore the goal becomes to prove $\Phi^{\prime} \vDash_{1} T[\sigma[\gamma], \Gamma / g]$. By IH, we have $\Psi, g: \mathrm{Ctx} \vDash_{1}^{v} T$. We simply need $\Phi^{\prime} \vDash^{v} \sigma[\gamma], \Gamma / g: \Psi, g:$ Ctx. This is immediate from monotonicity and definition.

Theorem 3.30 (Fundamental).

- If $\Psi ; \Gamma \vdash_{i} t: T$, then $\Psi ; \Gamma \vDash_{i}^{v} t: T$.
- If $\Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $\Psi ; \Gamma \vDash_{i}^{v} \delta: \Delta$.
- If $\Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vDash_{i}^{v} t \approx t^{\prime}: T$.
- If $\Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash_{i}^{v} \delta \approx \delta^{\prime}: \Delta$.

Proof. We again do mutual induction. We focus on modal cases.

## Case

$$
\frac{\Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta \quad u:(\Delta \vdash T) \in \Psi}{\Psi ; \Gamma \vdash_{i} u^{\delta} \approx u^{\delta^{\prime}}: T}
$$

$H_{0}: \Psi ; \Gamma \vDash_{i}^{v} \delta \approx \delta^{\prime}: \Delta$
(by IH)

$$
H_{1}: \Phi \vDash^{v} \sigma: \Psi
$$

(by assumption)
$\Phi ; \Delta^{\prime} \vDash_{i} \delta^{\prime \prime} \approx \delta^{\prime \prime \prime}: \Gamma[\sigma] \quad$ (by assumption)

$$
H_{2}: \Phi ; \Delta^{\prime} \vDash_{i} \delta[\sigma] \circ \delta^{\prime \prime} \approx \delta^{\prime}[\sigma] \circ \delta^{\prime \prime \prime}: \Delta[\sigma]
$$

$H_{3}: \Phi ; \Delta^{\prime} \vDash_{0} \delta[\sigma] \circ \delta^{\prime \prime} \approx \delta^{\prime}[\sigma] \circ \delta^{\prime \prime \prime}: \Delta[\sigma] \quad$ (by layering restriction and $\Delta$ is well-formed at layer 0 )
$\Phi ; \Delta^{\prime} \vDash_{i} \sigma(u)\left[\delta[\sigma] \circ \delta^{\prime \prime}\right] \approx \sigma(u)\left[\delta^{\prime}[\sigma] \circ \delta^{\prime \prime \prime}\right]: T\left[\sigma\left[p^{1+u}(\mathrm{id})\right]\right] \quad$ (by $H_{1}$ and $\left.H_{3}\right)$
Case

$$
\begin{array}{rr}
\frac{\Psi \vdash_{1} \Gamma \quad \Psi ; \Delta \vdash_{0} t \approx t^{\prime}: T}{\Psi ; \Gamma \vdash_{1} \operatorname{box} t \approx \operatorname{box} t^{\prime}: \square(\Delta \vdash T)} & \\
H_{0}: \Psi ; \Delta \vDash_{0}^{v} t \approx t^{\prime}: T & \text { (by IH) } \\
H_{1}: \Phi \vDash^{v} \sigma: \Psi & \text { (by assumption) } \\
H_{2}: \Phi ; \Delta^{\prime} \vDash_{1} \delta \approx \delta^{\prime}: \Gamma[\sigma] & \text { (by assumption) } \\
\Phi \vdash_{1} \Delta^{\prime} & \text { (by } H_{2}, \text { escape and presupposition) } \\
H_{3}: \Phi ; \Delta^{\prime \prime} \vDash_{0} \delta^{\prime \prime} \approx \delta^{\prime \prime \prime}: \Delta[\sigma] & \text { (by assumption) }
\end{array}
$$

$$
\begin{array}{lr}
t=t^{\prime} & \text { (by static code) } \\
\Phi ; \Delta^{\prime \prime} \vDash_{0} t[\sigma]\left[\delta^{\prime \prime}\right] \approx t^{\prime}[\sigma]\left[\delta^{\prime \prime \prime}\right]: T[\sigma] & \text { (by } \left.H_{0}, H_{1} \text { and } H_{3}\right) \\
\Phi ; \Delta[\sigma] \vDash_{1}^{\mathrm{Nf}} \text { box } t[\sigma] \approx \operatorname{box} t^{\prime}[\sigma]: T[\sigma] & \text { (by definition) } \\
\Phi ; \Delta^{\prime} \vdash_{1} \operatorname{box} t[\sigma][\delta] \approx \operatorname{box} t^{\prime}[\sigma]\left[\delta^{\prime}\right]: \square(\Delta \vdash T)[\sigma] &
\end{array}
$$

In the last step, notice that local substitutions do not propagate into box.
Case

$$
\begin{array}{rr}
\Psi ; \Gamma \vdash_{1} s \approx s^{\prime}: \square(\Delta \vdash T) \\
\hline \Psi \vdash_{0} \Delta & \Psi \vdash_{0} T \quad \Psi \vdash_{1} T^{\prime} \quad \Psi, u:(\Delta \vdash T) ; \Gamma[p(\mathrm{id})] \vdash_{1} t \approx t^{\prime}: T^{\prime}[p(\mathrm{id})] \\
\hline \Psi ; \Gamma \vdash_{1} \operatorname{letbox} u \leftarrow s \text { in } t \approx \operatorname{letbox} u \leftarrow s^{\prime} \text { in } t^{\prime}: T^{\prime} \\
H_{0}: \Psi ; \Gamma \vDash_{1}^{v} s \approx s^{\prime}: \square(\Delta \vdash T) & \\
H_{1}: \Psi, u:(\Delta \vdash T) ; \Gamma[p(\mathrm{id})] \vDash_{1}^{v} t \approx t^{\prime}: T^{\prime}[p(\mathrm{id})] & \text { (by IH) } \\
H_{2}: \Phi \vDash^{v} \sigma: \Psi & \text { (by IH) } \\
H_{3}: \Phi ; \Delta^{\prime} \vDash_{1} \delta \approx \delta^{\prime}: \Gamma[\sigma] & \text { (by assumption) } \\
\Phi ; \Delta^{\prime} \vDash_{0} s[\sigma][\delta] \approx s^{\prime}[\sigma]\left[\delta^{\prime}\right]: \square(\Delta \vdash T)[\sigma] & \text { (by assumption) } \left.H_{0}, H_{2} \text { and } H_{3}\right)
\end{array}
$$

Now, we consider $H_{3}$, where we know for some $w$ and $w^{\prime}$

$$
\begin{aligned}
& \Phi ; \Delta^{\prime} \vdash s[\sigma][\delta] \rightsquigarrow^{*} w: \square(\Delta[\sigma] \vdash(T[\sigma])) \\
& \Phi ; \Delta^{\prime} \vdash s^{\prime}[\sigma][\delta] \rightsquigarrow^{*} w^{\prime}: \square(\Delta[\sigma] \vdash(T[\sigma])) \\
& H_{4}: \Phi ; \Delta^{\prime} \vDash_{1}^{\mathrm{Nf}} w \approx w^{\prime}: \square(\Delta[\sigma] \vdash(T[\sigma]))
\end{aligned}
$$

We then case analyze $H_{4}$ :
Subcase

$$
\frac{\Phi ; \Delta[\sigma] \vdash_{0} t^{\prime \prime}: T[\sigma] \quad \forall \Phi^{\prime} ; \Delta^{\prime \prime} \vDash_{0} \delta^{\prime \prime} \approx \delta^{\prime \prime \prime}: \Delta[\sigma] \cdot \Phi^{\prime} ; \Delta^{\prime \prime} \vDash_{0} t^{\prime \prime}\left[\delta^{\prime \prime}\right] \approx t^{\prime \prime}\left[\delta^{\prime \prime \prime}\right]: T}{\Phi ; \Delta^{\prime} \vDash_{1}^{\mathrm{Nf}} \operatorname{box} t^{\prime \prime} \approx \operatorname{box} t^{\prime \prime}: \square(\Delta \vdash T)[\sigma]}
$$

Then we have

$$
\begin{array}{rlr}
H_{5}: \Phi \vDash^{v} \sigma, t^{\prime \prime} / u: \Psi, u:(\Delta \vdash T) & \text { (by definition) } \\
& \Phi ; \Delta^{\prime} \vDash_{1} t\left[\sigma, t^{\prime \prime} / u\right][\delta] \approx t^{\prime}\left[\sigma, t^{\prime \prime} / u\right]\left[\delta^{\prime}\right]: T^{\prime}[p(\mathrm{id})]\left[\sigma, t^{\prime \prime} / u\right] & \\
& T^{\prime}[p(\mathrm{id})]\left[\sigma, t^{\prime \prime} / u\right]=T^{\prime}[\sigma] & \text { (by computation) } \left.H_{1} \text { and } H_{5}\right) \\
& \text { letbox } u \leftarrow s \text { in } t[\sigma][\delta] & \\
= & \text { letbox } u \leftarrow s[\sigma][\delta] \operatorname{in~}(t[q(\sigma)][\delta[p(\mathrm{id})]]) & \\
\rightsquigarrow^{*} \text { letbox } u \leftarrow \operatorname{box} t^{\prime \prime} \text { in }(t[q(\sigma)][\delta[p(\mathrm{id})]]) & \\
\rightsquigarrow & t[q(\sigma)][\delta[p(\mathrm{id})]]\left[\mathrm{id}, t^{\prime \prime} / u\right] & \\
= & t\left[q(\sigma) \circ\left(\mathrm{id}, t^{\prime \prime} / u\right)\right]\left[\delta[p(\mathrm{id})]\left[\mathrm{id}, t^{\prime \prime} / u\right]\right] & \text { (by computation) } \\
= & t\left[\sigma, t^{\prime \prime} / u\right][\delta] & \text { (similarly) } \\
& l e t b o x u \leftarrow s^{\prime} \text { in } t^{\prime}[\sigma]\left[\delta^{\prime}\right] \rightsquigarrow^{*} t^{\prime}\left[\sigma, t^{\prime \prime} / u\right]\left[\delta^{\prime}\right] & \\
& \Phi ; \Delta^{\prime} \vDash_{1} \text { letbox } u \leftarrow s \text { in } t[\sigma][\delta] \approx \text { letbox } u \leftarrow s^{\prime} \text { in } t^{\prime}[\sigma][\delta]: T^{\prime}[\sigma] &
\end{array}
$$

Subcase

$$
\frac{\Phi ; \Delta^{\prime} \vdash v \sim v^{\prime}: \square(\Delta \vdash T)[\sigma]}{\Phi ; \Delta^{\prime} \vDash_{1}^{\mathrm{Nf}} v \approx v^{\prime}: \square(\Delta \vdash T)[\sigma]}
$$

Then we have
$H_{5}: \Phi, u:(\Delta[\sigma] \vdash(T[\sigma])) \vDash^{v} \sigma[p(\mathrm{id})], u^{\text {id }} / u: \Psi, u:(\Delta \vdash T)$
(by monotonicity and Lemma 3.13)
$H_{6}: \Phi, u:(\Delta[\sigma] \vdash(T[\sigma])) ; \Delta^{\prime}[p(\mathrm{id})] \vDash_{1} \delta[p(\mathrm{id})] \approx \delta^{\prime}[p(\mathrm{id})]: \Gamma[\sigma][p(\mathrm{id})]$
(by monotonicity)
$\Gamma[p(\mathrm{id})]\left[\sigma[p(\mathrm{id})], u^{\mathrm{id}} / u\right]=\Gamma[\sigma[p(\mathrm{id})]]=\Gamma[\sigma][p(\mathrm{id})] \quad$ (by computation)
$T^{\prime}[p(\mathrm{id})]\left[\sigma[p(\mathrm{id})], u^{\mathrm{id}} / u\right]=T^{\prime}[\sigma[p(\mathrm{id})]]=T^{\prime}[\sigma][p(\mathrm{id})] \quad$ (by computation)
$\Phi, u:(\Delta[\sigma] \vdash(T[\sigma])) ; \Delta^{\prime}[p(\mathrm{id})] \vDash_{1} t\left[\sigma[p(\mathrm{id})], u^{\mathrm{id}} / u\right][\delta[p(\mathrm{id})]] \approx t^{\prime}\left[\sigma[p(\mathrm{id})], u^{\mathrm{id}} / u\right]\left[\delta^{\prime}[p(\mathrm{id})]\right]: T^{\prime}[\sigma][p(\mathrm{id})]$ (by $H_{1}, H_{5}$ and $H_{6}$ )
$\Phi, u:(\Delta[\sigma] \vdash(T[\sigma])) ; \Delta^{\prime}[p(\mathrm{id})] \vdash t\left[\sigma[p(\mathrm{id})], u^{\mathrm{id}} / u\right][\delta[p(\mathrm{id})]] \simeq t^{\prime}\left[\sigma[p(\mathrm{id})], u^{\mathrm{id}} / u\right]\left[\delta^{\prime}[p(\mathrm{id})]\right]: T^{\prime}[\sigma][p(\mathrm{id})]$ (by Corollary 3.11)
letbox $u \leftarrow s$ in $t[\sigma][\delta]$
$=$ letbox $u \leftarrow s[\sigma][\delta]$ in $(t[q(\sigma)][\delta[p(\mathrm{id})]])$
$\rightsquigarrow^{*}$ letbox $u \leftarrow v$ in $(t[q(\sigma)][\delta[p(\mathrm{id})]]) \quad$ (by computation)
letbox $u \leftarrow s^{\prime}$ in $t^{\prime}[\sigma]\left[\delta^{\prime}\right] \rightsquigarrow^{*}$ letbox $u \leftarrow v^{\prime}$ in $\left(t^{\prime}[q(\sigma)]\left[\delta^{\prime}[p(\mathrm{id})]\right]\right) \quad$ (similarly)
$\Phi ; \Delta^{\prime} \vdash$ letbox $u \leftarrow v$ in $(t[q(\sigma)][\delta[p(i d)]]) \sim$ letbox $u \leftarrow v^{\prime}$ in $\left(t^{\prime}[q(\sigma)]\left[\delta^{\prime}[p(i d)]\right]\right): T[\sigma]$ (neutral terms)
$\Phi ; \Delta^{\prime} \vDash_{1}$ letbox $u \leftarrow v$ in $(t[q(\sigma)][\delta[p(\mathrm{id})]]) \approx$ letbox $u \leftarrow v^{\prime}$ in $\left(t^{\prime}[q(\sigma)]\left[\delta^{\prime}[p(\mathrm{id})]\right]\right): T[\sigma]$ (by Lemma 3.13)
$\Phi ; \Delta^{\prime} \vDash_{1}$ letbox $u \leftarrow s$ in $t[\sigma][\delta] \approx$ letbox $u \leftarrow s^{\prime}$ in $t^{\prime}[\sigma][\delta]: T^{\prime}[\sigma]$
Case

$$
\frac{\Psi \vdash_{1} \Gamma \quad \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vdash_{1} t \approx t^{\prime}: T}{\Psi ; \Gamma \vdash_{1} \Lambda g \cdot t \approx \Lambda g \cdot t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T}
$$

$$
\begin{array}{lr}
H_{0}: \Psi, g: \mathrm{Ctx} ; \Gamma[p(\mathrm{id})] \vDash_{1}^{v} t \approx t^{\prime}: T & \text { (by IH) } \\
H_{1}: \Phi \vDash^{v} \sigma: \Psi & \text { (by assumption) } \\
H_{2}: \Phi ; \Delta^{\prime} \vDash_{1} \delta \approx \delta^{\prime}: \Gamma[\sigma] & \text { (by assumption) } \\
H_{3}: \gamma ; \tau: \Phi^{\prime} ; \Delta^{\prime \prime} \Longrightarrow \Phi ; \Delta^{\prime} & \text { (by assumption) } \\
H_{4}: \Phi^{\prime} \vdash_{0} \Gamma^{\prime} & \text { (by assumption) } \\
H_{5}: \Phi^{\prime} \vDash^{v} \sigma[\gamma], \Gamma^{\prime} / g: \Psi, g: \mathrm{Ctx} & \text { (by monotonicity) } \\
H_{6}: \Phi^{\prime} ; \Delta^{\prime \prime} \vDash_{1} \delta[\gamma ; \tau] \approx \delta^{\prime}[\gamma ; \tau]: \Gamma[\sigma][\gamma] & \text { (by monotonicity) } \\
\quad \Phi^{\prime} ; \Delta^{\prime \prime} \vDash_{1} t\left[\sigma[\gamma], \Gamma^{\prime} / g\right][\delta[\gamma ; \tau]] \approx t^{\prime}\left[\sigma[\gamma], \Gamma^{\prime} / g\right]\left[\delta^{\prime}[\gamma ; \tau]\right]: T\left[\sigma[\gamma], \Gamma^{\prime} / g\right] & \text { (by } H_{0}, H_{5} \text { and } H_{6} \text { ) } \\
& \Lambda g \cdot t[\sigma][\delta][\gamma ; \tau] \$ \Gamma^{\prime} \\
=\Lambda g \cdot(t[q(\sigma[\gamma])][\delta[\gamma ; \tau][p(\mathrm{id})]]) \$ \Gamma^{\prime} & \\
\rightsquigarrow & t[q(\sigma[\gamma])][\delta[\gamma ; \tau][p(\mathrm{id})]]\left[\mathrm{id}, \Gamma^{\prime} / g\right]
\end{array}
$$

$$
\begin{array}{rlr}
= & t[q(\sigma[\gamma])]\left[\mathrm{id}, \Gamma^{\prime} / g\right]\left[\delta[\gamma ; \tau][p(\mathrm{id})]\left[\mathrm{id}, \Gamma^{\prime} / g\right]\right] \\
= & t\left[\sigma[\gamma], \Gamma^{\prime} / g\right][\delta[\gamma ; \tau]] & \text { (by computation) } \\
& \Lambda g \cdot t^{\prime}[\sigma]\left[\delta^{\prime}\right][\gamma ; \tau] \$ \Gamma^{\prime} \rightsquigarrow t^{\prime}\left[\sigma[\gamma], \Gamma^{\prime} / g\right]\left[\delta^{\prime}[\gamma ; \tau]\right] & \text { (similarly) } \\
& \Phi^{\prime} ; \Delta^{\prime \prime} \vDash_{1} \Lambda g \cdot t[\sigma][\delta][\gamma ; \tau] \$ \Gamma^{\prime} \approx \Lambda g \cdot t^{\prime}[\sigma]\left[\delta^{\prime}\right][\gamma ; \tau] \$ \Gamma^{\prime}: T\left[\sigma[\gamma], \Gamma^{\prime} / g\right] & \text { (by Lemma 3.9) } \\
& \Phi ; \Delta^{\prime} \vDash_{1} \Lambda g \cdot t[\sigma][\delta] \approx \Lambda g \cdot t^{\prime}[\sigma]\left[\delta^{\prime}\right]:(g: \operatorname{Ctx}) \Rightarrow T[\sigma]
\end{array}
$$

Case

$$
\begin{array}{ll}
\qquad \frac{\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T \quad \Psi \vdash_{0} \Delta}{\Psi ; \Gamma \vdash_{1} t \$ \Delta \approx t^{\prime} \$ \Delta: T\left[\mathrm{id}_{\Psi}, \Delta / g\right]} \\
H_{0}: \Psi ; \Gamma \vDash_{1}^{v} t \approx t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T \\
H_{1}: \Phi \vDash^{v} \sigma: \Psi & \text { (by assumption) } \\
H_{2}: \Phi ; \Delta^{\prime} \vDash_{i} \delta \approx \delta^{\prime}: \Gamma[\sigma] \\
\Phi ; \Delta^{\prime} \vDash_{1} t[\sigma][\delta] \approx t^{\prime}[\sigma]\left[\delta^{\prime}\right]:(g: \mathrm{Ctx}) \Rightarrow T[\sigma] \\
\Phi \vdash_{0} \Delta[\sigma] & \text { (by assumption) } \\
\Phi ; \Delta^{\prime} \vDash_{1} t[\sigma][\delta] \$ \Delta[\sigma] \approx t^{\prime}[\sigma]\left[\delta^{\prime}\right] \$ \Delta[\sigma]: T[\sigma][\mathrm{id}, \Delta[\sigma] / g] \\
T[\sigma][\mathrm{id}, \Delta[\sigma] / g]=T[\mathrm{id}, \Delta / g][\sigma] & \text { (by computation) }
\end{array}
$$

The goal is then concluded.
Case

$$
\begin{array}{lll}
\Psi \vdash_{i} \Gamma & \Gamma \text { ends with } \cdot \quad|\Gamma|=m \\
\hline \Psi ; \Gamma \vdash_{i} \cdot{ }^{m} \approx \cdot^{m}: \cdot & \\
& \Phi \vDash^{v} \sigma: \Psi & \text { (by assumption) } \\
\Phi ; \Delta^{\prime} \vDash_{i} \delta \approx \delta^{\prime}: \Gamma[\sigma] & \text { (by assumption) }
\end{array}
$$

We must compare $\cdot{ }^{m}[\sigma] \circ \delta=\cdot \frac{\widehat{\delta}}{\check{\delta}}$ and $\cdot \cdot^{m}[\sigma] \circ \delta^{\prime}=\frac{\widehat{\delta}^{\prime}}{\check{\delta}^{\prime}}$. But they are immediately equal, as we can show that $\Phi ; \Delta^{\prime} \vDash_{i} \delta \approx \delta^{\prime}: \Gamma[\sigma]$ implies $\breve{\delta}=\breve{\delta^{\prime}}$ and $\widehat{\delta}=\widehat{\delta^{\prime}}$.
Case

$$
\begin{array}{cc}
\Psi \vdash_{i} \Gamma \quad g: \mathrm{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m \\
\hline \Psi ; \Gamma \vdash_{i} \stackrel{m}{g}_{m}^{\overbrace{g}^{m}: \cdot} \\
\Phi \vDash^{v} \sigma: \Psi & \\
\Phi ; \Delta^{\prime} \vdash_{i} \delta \approx \delta^{\prime}: \Gamma[\sigma] & \text { (by assumption) } \\
& \text { (by assumption) }
\end{array}
$$

We look up $\sigma(g)$ and consider what it ends with.
Subcase $\sigma(g)$ ends with $\cdot$. Then we must compare ${ }_{g}^{m}[\sigma] \circ \delta=\frac{\widehat{\delta}}{\check{\delta}}$ and $\cdot_{g}^{m}[\sigma] \circ \delta^{\prime}=\frac{\widehat{\delta}^{\prime}}{\delta^{\prime}}$, which we know are equal. Subcase $\sigma(g)$ ends with some $g^{\prime}$. Then we must compare ${ }_{g}^{m}[\sigma] \circ \delta=. \widehat{g^{\prime}}$ and ${ }_{g}^{m}[\sigma] \circ \delta^{\prime}=\frac{._{g^{\prime}}^{\prime}}{\widehat{\prime}}$, which we know are also equal.
Case

$$
\begin{array}{ccc}
\Psi \vdash_{i} \Gamma \quad g: \mathrm{Ctx} \in \Psi \quad \Gamma \text { ends with } g & |\Gamma|=m \\
\hline \Psi ; \Gamma \vdash_{i} \mathrm{wk}_{g}^{m} \approx \mathrm{wk}_{g}^{m}: g &
\end{array}
$$

$$
\begin{array}{rr}
\Phi \vDash^{v} \sigma: \Psi & \text { (by assumption) } \\
H_{0}: \Phi ; \Delta^{\prime} \vDash_{i} \delta \approx \delta^{\prime}: \Gamma[\sigma] & \text { (by assumption) }
\end{array}
$$

Then we have

$$
\mathrm{wk}_{g}^{m}[\sigma] \circ \delta=\mathrm{id}_{\sigma(g)}\left[i d ; p^{m}(\mathrm{id})\right] \circ \delta=\mathrm{wk}_{\sigma(g)}^{m} \circ \delta
$$

and

$$
\mathrm{wk}_{g}^{m}[\sigma] \circ \delta^{\prime}=\mathrm{id}_{\sigma(g)}\left[i d ; p^{m}(\mathrm{id})\right] \circ \delta^{\prime}=\mathrm{wk}_{\sigma(g)}^{m} \circ \delta^{\prime}
$$

We show $\Phi ; \Delta^{\prime} \vDash_{i} \mathrm{wk}_{\sigma(g)}^{m} \circ \delta \approx \mathrm{wk}_{\sigma(g)}^{m} \circ \delta^{\prime}: \sigma(g)$ by unraveling $H_{0} m$ times.

As a corollary of the fundamental theorems, we can prove the completeness theorem:
Proof of Theorem 3.24. Notice that the reducibility predicates are just special cases of the validity judgments.

### 3.4 Convertibility Checking

In this section, we will write down the converibility checking rules and instantiate the generic equivalence with it, proving that equivalence terms can be checked. We define three judgments: $\Psi ; \Gamma \vdash t \stackrel{ }{\Longleftrightarrow} t^{\prime}: T$ checks the convertibility of two terms $t$ and $t^{\prime} . \Psi ; \Gamma \vdash w \Longleftrightarrow w^{\prime}: T$ checks the convertibility of two normal forms $w$ and $w^{\prime}$. This operation is directed by types. $\Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}: T$ checks the convertibility of two neutral forms $v$ and $v^{\prime}$. This operation is structural on the neutral forms. We give all the rules below:

$$
\begin{aligned}
& \begin{array}{lcc|}
\Psi ; \Gamma \vdash t \rightsquigarrow^{*} w: T & \Psi ; \Gamma+t^{\prime} \rightsquigarrow^{*} w^{\prime}: T & \Psi ; \Gamma \vdash w \Longleftrightarrow w^{\prime}: T \\
\Psi ; \Gamma+t \stackrel{t^{\prime}}{\Longleftrightarrow} \quad \frac{\Psi \vdash_{1} \Gamma}{\Psi ; \Gamma \vdash \text { zero }} \Longleftrightarrow \text { zero }: \text { Nat }
\end{array} \\
& \Psi ; \Gamma \vdash t \stackrel{\Longleftrightarrow}{\Longleftrightarrow} t^{\prime}: \mathrm{Nat} \\
& \Psi ; \Gamma \vdash \operatorname{succ} t \Longleftrightarrow \operatorname{succ} t^{\prime}: \text { Nat } \\
& \frac{\Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}: N a t}{\Psi ; \Gamma \vdash v \Longleftrightarrow v^{\prime}: \mathrm{Nat}} \\
& \Psi ; \Gamma, x: S \vdash w[\mathrm{id} ; p(\mathrm{id})] x \Longleftrightarrow w^{\prime}[\mathrm{id} ; p(\mathrm{id})] x: T \\
& \Psi ; \Gamma \vdash w \Longleftrightarrow w^{\prime}: S \longrightarrow T \\
& \frac{\Psi \vdash_{1} \Gamma \quad \Psi ; \Delta \vdash_{0} t: T}{\Psi ; \Gamma \vdash \text { box } t \Longleftrightarrow \text { box } t: \square(\Delta \vdash T)} \\
& \frac{\Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}: \square(\Delta \vdash T)}{\Psi ; \Gamma \vdash v \Longleftrightarrow v^{\prime}: \square(\Delta \vdash T)} \quad \frac{\Psi, g: \mathrm{Ctx} ; \Gamma \vdash w[p(\mathrm{id})] \$ g \Longleftrightarrow w^{\prime}[p(\mathrm{id})] \$ g: T}{\Psi ; \Gamma \vdash w \Longleftrightarrow w^{\prime}:(g: \mathrm{Ctx}) \Rightarrow T} \quad \frac{\Psi \vdash_{1} \Gamma \quad x: T \in \Gamma}{\Psi ; \Gamma \vdash x \longleftrightarrow x: T} \\
& \frac{\Psi ; \Gamma \vdash \delta \Longleftrightarrow \delta^{\prime}: \Delta \quad x:(\Delta \vdash T) \in \Psi}{\Psi ; \Gamma \vdash u^{\delta} \longleftrightarrow u^{\delta^{\prime}}: T} \quad \frac{\Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}: S \longrightarrow T \quad \Psi ; \Gamma \vdash t \Longleftrightarrow t^{\prime}: S}{\Psi ; \Gamma \vdash v t \longleftrightarrow v^{\prime} t^{\prime}: T} \\
& \Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}: \square(\Delta \vdash T) \quad \Psi \vdash_{1} T^{\prime} \\
& \Psi, u:(\Delta \vdash T) ; \Gamma[p(\mathrm{id})] \vdash t \stackrel{\Longleftrightarrow}{\Longleftrightarrow} t^{\prime}: T^{\prime}[p(\mathrm{id})] \quad \Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}:(\mathrm{g}: \mathrm{Ctx}) \Rightarrow T \quad \Psi \vdash_{0} \Delta
\end{aligned}
$$

We then instantiate the generic equivalence. We instantiate $\Psi ; \Gamma \vdash t \simeq t^{\prime}: T$ with $\Psi ; \Gamma \vdash t \Longleftrightarrow t^{\prime}: T$ and $\Psi ; \Gamma \vdash t \sim t^{\prime}: T$ with $\Psi ; \Gamma \vdash t \longleftrightarrow t^{\prime}: T$.

Most laws are immediate. We discuss a few of them.
Lemma 3.31 (PERs). All three relations above are PERs.

Proof. When we prove transitivity of $\Psi ; \Gamma \vdash t \Longleftrightarrow t^{\prime}: T$, we use the uniqueness of multi-step weak head reduction.

Lemma 3.32 (Congruence of box). If $\Psi \vdash_{1} \Gamma, \Psi ; \Delta \vdash_{0} t: T$ and $\Psi ; \Delta \vdash t \stackrel{ }{\Longleftrightarrow} t: T$, then $\Psi ; \Gamma \vdash$ box $t \stackrel{\wedge}{\Longleftrightarrow}$ box $t$ : T.

Proof. Notice that we are almost there, except that we must prove $\Psi ; \Delta \vdash t \Longleftrightarrow t: T$ for $t$ at layer 0 . This premise is met due to our layered model, where we instantiate layer 0 and layer 1 separately, so that the fundamental theorem of layer 0 gives $\Psi ; \Delta \vdash t \Longleftrightarrow t: T$.

A successful instantiation gives us the following desired completeness theorem for converibility checking:
Theorem 3.33 (Completeness).

- If $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$, then $\Psi ; \Gamma \vdash t \Longleftrightarrow t^{\prime}: T$.
- If $\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash \delta \Longleftrightarrow \delta^{\prime}: \Delta$.

Soundness is easy by a simple induction:
Theorem 3.34 (Soundness).

- If $\Psi ; \Gamma \vdash t \stackrel{\wedge}{\Longleftrightarrow} t^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} t \approx t^{\prime}: T$.
- If $\Psi ; \Gamma \vdash w \Longleftrightarrow w^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} w \approx w^{\prime}: T$.
- If $\Psi ; \Gamma \vdash v \longleftrightarrow v^{\prime}: T$, then $\Psi ; \Gamma \vdash_{1} v \approx v^{\prime}: T$.
- If $\Psi ; \Gamma \vdash \delta \Longleftrightarrow \delta^{\prime}: \Delta$, then $\Psi ; \Gamma \vdash_{1} \delta \approx \delta^{\prime}: \Delta$.

This concludes our discussion about contextual variables.

## 4 DEPENDENT LAYERED MODAL TYPE THEORY

In this section, we combine the work by Hu and Pientka [2024a] and what we have built up in the previous sections and scale all the way up to dependent types. We present DeLaM, Dependent Layered Modal Type Theory. With dependent types, we can not only analyze the syntax of programs, but also that of types. This ability, therefore, gives us the power to write tactics that could potentially fill in proof obligations in a proof environment. In particular, this type theory addresses a number of problems that we often see in proof assistants like Coq, Lean, and Agda. In Coq, tactics are written in a separate language, Ltac or Ltac2, where the advantages of dependent types in Gallina, the core language, are lost. Stratifying the tactic language and the core language into two also cause duplications: there are multiple notions for natural numbers, functions, etc.. On the other hand, in Lean and Agda, we use reflection to convert a Lean or an Agda term into an AST and then use the core language to manipulate the AST. An instrumentation in the kernel is responsible for converting this AST back to a valid term, if type-checked. This mechanism superficially provides a uniform way to tactics, but reflection generally fails to guarantee the well-formedness of ASTs, making type malformedness run-time errors and necessitating exception mechanisms exclusively for macro executions.

We believe that this type theory provides an example to address all aforementioned problems. Starting this section, let us dive into dependent types.

### 4.1 Highlights

On a high level, we would continue to apply the layering principle in DeLaM to guide us in the design of this type theory. In particular, we would want the layer for static code to be subsumed by layer for computing programs. Moreover, with contextual variables, we are now able to formulate a recursive elimination principle for code, which was not possible in simple types. However, these two requirements combining together causes some highlevel technical effects to the design of the type theory, which are worth mentioning before presenting the type theory itself.
4.1.1 Dynamic Leaks of Code. Since we are going to introduce elimination principle for terms with dependent types, we must also consider how equivalences are handled for code. For example, if we know a given piece of code has type $(\lambda x . x)$ Nat, should we regard this type the "same" as Nat? Intuitively, the answer should be yes. After all, we only want to capture the syntax of the term, not its type. Effectively, for code of type $\square(\Gamma \vdash T)$, it should also be regarded as an inhabitant of another type $\square\left(\Gamma \vdash T^{\prime}\right)$, as long as $T \approx T^{\prime}$ in $\Gamma$.

In the context of dependent types, however, that causes some problems in the presence of function applications. Consider a function $f: \Pi(x: S) . T$ and an argument $t: S$. Then in general, the type of $f t$ is $T[t / x]$. Now, let us construct this term as code. Even though $t$ is constructed as part of the code, the type of the overall code $\square(\Gamma \vdash(T[t / x]))$ contains $t$ and therefore part of the dynamics of $t$ is in fact leaked in the type. For a more concrete example, if $f: \Pi\left(x: \mathrm{Ty}_{0}\right) . x$ and let the argument be $(\lambda x . x)$ Nat, then box $(f((\lambda x . x)$ Nat $))$ has type $\square(\Gamma \vdash(\lambda x . x)$ Nat $)$, which we agree is just $\square(\Gamma \vdash N a t)$. Clearly, the argument computes and is not purely static code as in simple types. We cannot avoid this phenomenon because of dependent types, so we must handle it with care. This phenomenon is call a dynamic leak.
4.1.2 Non-cumulativity. Due to dynamic leaks, we must permit non-trivial equivalences in types of code. This causes particular problems when we want an elimination principle for code with intensional analyses. When we split code into cases in the elimination form, we must specify in each case how do we construct the original code from its components. Therefore, it is the most convenient, when each term has a unique type, leading to a conclusion of preferring non-cumulative universes. Whereas with cumulative universes, types live in higher universes for free. Cumulativity forms a pre-order of types which cannot be captured solely by equivalence rules and makes the typing rule for the elimination principle extremely difficult to express, if not completely impossible.
4.1.3 Universe Polymorphism. Though it is often omitted in other work, universe levels and universe polymorphism are important ingredients in dependent type theory. They are typically considered as "details" and are not very much paid attention to. However, in this work, we must be explicit about universe polymorphism. Consider some code for function application $t s$. We in general do not know the type of $s$, let alone its universe level (though it must be uniquely determined due to non-cumulativity discussed above). Therefore, the elimination principle for code must work for all universe levels, leading to a formalism of universe polymorphism.
4.1.4 Tarski-style Universes. Another ingredient to consider when approaching an elimination principle for code with dependent types and intensional analyses is the separation between types and terms. Consider Russell-style universes where types and terms are not distinguished naturally. It would probably suffice to say that $\square(\Gamma \vdash T)$ can represent code for both types and terms. In particular, code for some types just has type $\square\left(\Gamma \vdash T y_{0}\right)$, for example. Unfortunately, this thought is too naive. When we consider $\square(\Gamma \vdash T)$ as the type of code, we are considering this type with two indices, the (local) context $\Gamma$ and the type of the code body $T$. But what is the type of $T$ ? Well, it is $\square\left(\Gamma \vdash T y_{l}\right)$ for some $l$, which is just a special case of $\square(\Gamma \vdash T)$ ! A type clearly should not be indexed by a special case of itself. The intertwine between types and terms in Russell-style universes seems to even prevent a proper statement of indices of types for code. However, Tarski-style universes, where types and terms are clearly distinguished, introduce mutually inductive relations between types and terms, and safely bail us out of this problem, as we will see very soon.

### 4.2 Syntax

Let us start with the syntax of DeLaM. Since we employ Tarski-style universes, the syntax of terms and types are separate. Due to non-cumulativity, certain constructs must remember universe levels. Due to the elimination principle of code, some constructs must include additional sub-structures so that the elimination eventually
checks out. Let us begin with the subset that is basically just Martin-Löf type theory (MLTT).

$$
\begin{array}{rlr}
x, y & \begin{array}{r}
\text { (Local variables) } \\
\ell \\
l:=
\end{array} & \ell \mid \text { zero }|\operatorname{succ} l| l \sqcup l^{\prime} \mid \omega \\
M, S, T:= & \text { Nat }\left|\Pi^{l, l^{\prime}}(x: S) . T\right| \mathrm{Ty}_{l}\left|\vec{\ell} \Rightarrow^{l} T\right| \mathrm{El}^{l} t & \text { (Universe variables) } \\
s, t:= & \text { (Typerse levels) } \\
& \mid \text { Nat }\left|\Pi^{l, l^{\prime}}(x: s) . t\right| \mathrm{Ty}_{l} & \text { (Terms, Exp) } \\
& \mid \text { zero }|\operatorname{succ} t| \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t & \text { (encoding of types) }  \tag{Terms,Exp}\\
& \left|\lambda^{l, l^{\prime}}(x: S) . t\right|\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) s & \text { (natural numbers } N \text { ) } \\
& \left|\Lambda^{l} \vec{\ell} \cdot t\right| t \$ \vec{l} & \text { (dependent functions) } \\
\Gamma, \Delta:= & |g| \Gamma, x: T @ l & \text { (universe polymorphic functions) } \\
L:= & \mid L, \ell & \text { (Local contexts, Ctx) } \\
\text { (Universe contexts) }
\end{array}
$$

Due to three kinds of contexts and the scale of the system, we omit the discussion of weakenings, which we diligently kept track of in the previous sections. We take various weakenings for granted. Nevertheless, they will appear in the semantics. Weakenings in general are pretty obvious, as we permit arbitrary lookups for all variables. Since we now must deal with universe levels, we use $\ell$ to range over variables for universe levels. The syntax for universe levels follows Agda's conventions. Universe levels form an idempotent commutative monoid, the laws of which we will show in the next subsection. Here we use $\sqcup$ to denote taking the max of two universe levels. The ability to take maximum between two levels induces a partial order:

$$
l \leq l^{\prime}:=l^{\prime} \approx l \sqcup l^{\prime}
$$

where we use $\approx$ to express the equivalence between universe levels. Thus, with $\leq$, universe levels form a bounded inf-lattice. A strict order is given by requiring the pre-order to hold for the successor of $l$ :

$$
l<l^{\prime}:=l^{\prime} \approx \operatorname{succ} l \sqcup l^{\prime}
$$

This strict order, as to be shown later, is well-founded, based on which we will give semantics to universes. Due to universe polymorphism, we must also include an $\omega$ level, which will be used to represent the universe level of a universe-polymorphic function. The $\omega$ level must not appear in any program, does not participate in the bounded inf-lattice specified above, is only used in type-checking, and therefore can be ignored most of the time. The formalization of universe polymorphism here follows Bezem et al. [2023] tightly. We use $\vec{\ell} \Rightarrow^{l} T$ to denote the type of a universe-polymorphic function. It introduce a non-empty list of universe variables at once, and lives at $\mathrm{Ty}_{\omega}$. The type $T$ lives at $\mathrm{Ty}_{l}$, where $l$ may refer to all variables from $\vec{l}$. Since $l$ cannot be $\omega$, we must have all universe variables introduced in one go. The introduction form is $\Lambda^{l} \vec{\ell} . t$, which also introduces a non-empty list of universe variables first and then the function body as expected. The elimination form $t \$ \vec{l}$ symmetrically eliminates a universe-polymorphic function with the same number of universe level expressions.

The rest of the expressions are pretty much standard from MLTT. We have natural numbers (Nat), their introduction forms and a recursion principle. We always use $M$ to exclusively represent the motives of a recursion principle. For regular dependent functions $\Pi^{l, l^{\prime}}(x: S) . T$, we must specify the universe levels of $S$ and $T$, following Pujet and Tabareau [2023]. We might omit the universe levels if they are not important in the discussion. The function abstraction $\lambda^{l, l^{\prime}}(x: S) . t$ is standard; we might omit $l$ and $S$ if they are not important or can be
inferred from the context. The function application $\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) s$ is arguably more complex. We explicitly specify the type of the function to prepare for a better formulation of the elimination principle for code later in the section. By requiring explicit type annotations in elimination forms, types that are usually hidden in the core syntax become sub-structures in the elimination form for code. This verbosity has no negative impact for programmers: after all, we are discussing a core theory, and we can let a type-inferring front-end to fill in these types for the users if they choose so. Following conventions, we may simply write $s t$ if the types are not important.

Since we are employing Tarski-style universes, as we have specified in the syntax, types and terms are separated. As terms, we use encodings of types, i.e. the overloaded $N a t, \Pi^{l, l^{\prime}}(x: s) . t$ and $\mathrm{Ty}_{l}$, which are members of some universes. They are decoded into actual types through $\mathrm{El}^{l} t$, converting the encodings to actual corresponding types. This part is basically identical to Palmgren [1998]. For simplicity, we have omitted the type lifter, which is responsible for raising the universe levels explicitly, similar to Lift in Agda's standard library. According Palmgren [1998], the type lifter bears additional equivalences and therefore we omit them here for conciseness, as lifting of the levels is an orthogonal issue here.

### 4.3 Universe Levels

In the syntax, we deliberately group all the universe variables into a separate context. This is beneficial as both local and global contexts (to be discussed later) will need to refer to universe variables. It is also helpful for the future work of extending DeLaM to more layers, by simply inserting more contexts after $L$. In this section, we state the well-formedness and equivalence judgments for universe levels. Note that all syntactically valid universe contexts are already well-formed as they only contain universe variables.

$$
\frac{\ell \in L}{L \vdash \ell: \text { Level }} \quad \frac{L \vdash l: \text { Level }}{L \vdash \text { zero : Level }} \quad \frac{L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level }}{L \vdash \operatorname{succ} l: \text { Level }}
$$

Notice that $\omega$ is not well-formed. Indeed, the judgment $L \vdash l$ : Level only captures the well-formed universe levels that can be written by a programmer. The level $\omega$, on the other hand, only appears during type-checking to denote the type of universe-polymorphic functions.

As discussed above, universe levels themselves form an idempotent commutative monoid. Hence they have the following equivalence rules:

$$
\frac{L \vdash l: \text { Level }}{L+l \approx l: \text { Level }} \quad \frac{L+l \approx l^{\prime}: \text { Level }}{L+l^{\prime} \approx l: \text { Level }} \quad \frac{L \vdash l \approx l^{\prime}: \text { Level } \quad L \vdash l^{\prime} \approx l^{\prime \prime}: \text { Level }}{L \vdash l \approx l^{\prime \prime}: \text { Level }}
$$

First we specify the basic PER rules. Then we have congruence rules:

$$
\frac{L \vdash l \approx l^{\prime}: \text { Level }}{L \vdash \operatorname{succ} l \approx \operatorname{succ} l^{\prime}: \text { Level }} \quad \frac{L \vdash l \approx l^{\prime}: \text { Level } \quad L \vdash l^{\prime \prime} \approx l^{\prime \prime \prime}: \text { Level }}{L \vdash l \sqcup l^{\prime \prime} \approx l^{\prime} \sqcup l^{\prime \prime \prime}: \text { Level }}
$$

Finally we have the algebraic rules:

| $\frac{L \vdash l: \text { Level }}{L \vdash l \sqcup \text { zero } \approx l: \text { Level }}$ | $\frac{L \vdash l: \text { Level } L \vdash l^{\prime}: \text { Level } L \vdash l^{\prime \prime}: \text { Level }}{L \vdash\left(l \sqcup l^{\prime}\right) \sqcup l^{\prime \prime} \approx l \sqcup\left(l^{\prime} \sqcup l^{\prime \prime}\right): \text { Level }}$ | $\frac{L \vdash l: \text { Level }}{L \vdash l \sqcup l^{\prime}: \text { Level }}$ |
| :---: | :---: | :---: | :---: |
| $\frac{L \vdash l: \text { Level }}{L \vdash l \sqcup l \approx l: \text { Level }}$ | $\frac{L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level }}{L \vdash \operatorname{succ}\left(l \sqcup l^{\prime}\right) \approx \operatorname{succ} l \sqcup \operatorname{succ} l^{\prime}: \text { Level }}$ | $\frac{\ell \in L}{L \vdash \ell \sqcup \operatorname{succ} \ell \approx \operatorname{succ} \ell: \text { Level }}$ |

The second last rule is distributivity of succ over $\sqcup$. The last rule is absorption of succ over $\sqcup$. The equivalence judgment confirms the well-formedness of both components:

Lemma 4.1 (Presupposition). If $L \vdash l \approx l^{\prime}:$ Level, then $L \vdash l:$ Level and $L \vdash l^{\prime}:$ Level.
Proof. Induction.
We can prove a more general absorption rule by doing a few inductions.
Lemma 4.2 (Absorption). If $L \vdash l$ : Level, then $L \vdash l \sqcup$ succ $l \approx$ succ $l$ : Level.
Proof. Induction. Only the following case is interesting:

$$
\frac{L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level }}{L \vdash l \sqcup l^{\prime}: \text { Level }}
$$

We reason as follows:

$$
\begin{align*}
l \sqcup l^{\prime} \sqcup \operatorname{succ} l \sqcup l^{\prime} & \approx(l \sqcup \operatorname{succ} l) \sqcup\left(l^{\prime} \sqcup \operatorname{succ} l^{\prime}\right) \\
& \approx \operatorname{succ} l \sqcup \operatorname{succ} l^{\prime}  \tag{byIH}\\
& \approx \operatorname{succ} l \sqcup l^{\prime}
\end{align*}
$$

Hence the proof is complete.
Then we generalize further:
Lemma 4.3 (Absorption). If $L \vdash l:$ Level, then for any natural number $n, L \vdash l \sqcup \operatorname{succ}^{n} l \approx \operatorname{succ}^{n} l$ : Level.
Proof. We proceed by induction on $n$. The cases for $n=0$ and $n=1$ are simple. We consider the step case, where $n=1+n^{\prime}$ and we know $L \vdash l \sqcup \operatorname{succ}^{n^{\prime}} l \approx \operatorname{succ}^{n^{\prime}} l$ : Level. We reason as follows:

$$
\begin{align*}
l \sqcup \operatorname{succ}^{1+n^{\prime}} l & \approx l \sqcup \operatorname{succ}\left(\operatorname{succ}^{n^{\prime}} l\right) \\
& \approx l \sqcup \operatorname{succ}\left(l \sqcup \operatorname{succ}^{n^{\prime}} l\right)  \tag{byIH}\\
& \approx l \sqcup \operatorname{succ} l \sqcup \operatorname{succ}^{1+n^{\prime}} l \\
& \approx \operatorname{succ} l \sqcup \operatorname{succ}^{1+n^{\prime}} l \\
& \approx \operatorname{succ} l \sqcup \operatorname{succ}^{n^{\prime}}(\operatorname{succ} l) \\
& \approx \operatorname{succ}^{n^{\prime}}(\operatorname{succ} l) \\
& \approx \operatorname{succ}^{n} l
\end{align*} \quad \text { (by absorption) }
$$

The proof is complete.
As readers might have noticed, the theory for universe levels are self-contained and their equivalence is decidable, as per implemented by Agda's type-checker. For this reason, in the remainder of the discussion, we undermine the importance of well-formedness and equivalence of universe levels, unless it is essential in the surrounding context.

Next, we define substitutions for universe levels:

$$
\begin{array}{rr}
\phi:=\cdot \mid \phi, l / \ell & \quad \text { (Subs } \\
\frac{L \vdash \phi: L^{\prime} \quad L \vdash l: \text { Level }}{L \vdash \cdot} & \frac{L \vdash, l / \ell: L^{\prime}, \ell}{}
\end{array}
$$

(Substitutions for universe levels)

Applying substitutions is intuitive:

$$
\ell[\phi]:=\phi(\ell)
$$

$$
\begin{aligned}
\operatorname{zero}[\phi] & :=\text { zero } \\
\operatorname{succ} l[\phi] & :=\operatorname{succ}(l[\phi]) \\
l \sqcup l^{\prime}[\phi] & :=(l[\phi]) \sqcup\left(l^{\prime}[\phi]\right) \\
\omega[\phi] & :=\omega
\end{aligned}
$$

We then have the following lemmas:
Lemma 4.4. If $L \vdash l:$ Level and $L^{\prime} \vdash \phi: L$, then $L^{\prime} \vdash l[\phi]:$ Level.
Proof. Induction.
Lemma 4.5. If $L \vdash l \approx l^{\prime}:$ Level and $L^{\prime} \vdash \phi: L$, then $L^{\prime} \vdash l[\phi] \approx l^{\prime}[\phi]:$ Level.
Proof. Induction.
The well-foundedness of the strict order < is intuitive. The only bottom element is zero. We simply keep removing succ from all components of $\sqcup$, and we must eventually stop. Thus, the simplest way to argue the well-foundedness of $<$ is to define a measure based on the number of succ's that can be removed from an $l$. This number is defined over all $L \vdash l$ : Level recursively as follows:

$$
\begin{aligned}
\operatorname{count}(\ell) & :=\{\ell \mapsto 0, \text { zero } \mapsto 0\} \\
\operatorname{count}(\text { zero }) & :=\{\text { zero } \mapsto 0\} \\
\operatorname{count}(\operatorname{succ} l) & :=\operatorname{merge}(\{\ell \mapsto 1+n \mid \ell \mapsto n \in \operatorname{count}(l)\},\{\text { zero } \mapsto 1+\operatorname{count}(l)(\text { zero })\}) \\
\operatorname{count}\left(l \sqcup l^{\prime}\right) & :=\operatorname{merge}\left(\operatorname{count}(l), \operatorname{count}\left(l^{\prime}\right)\right)
\end{aligned}
$$

Here count returns a map that counts the number of succ's over all universe variables and zero. The function merge merges two maps and takes the maximum in a conflict. The following definition makes sure equivalent universe levels to have the same representation as maps:

$$
\operatorname{adjust}(m):= \begin{cases}\{\ell \mapsto n \mid m(\ell) \mapsto n\} & \text { if } m(\text { zero }) \leq \max _{\ell \mapsto n \in m} n \\ m & \text { otherwise }\end{cases}
$$

In the first branch, we check if there is a variable which has a higher universe level than the constant. If so, we drop the constant completely. For example, in succ (succ zero) $\sqcup$ succ (succ $\ell$ ), succ (succ zero) is redundant, as we know succ ( $\operatorname{succ} \ell$ ) is at least as large as succ (succ zero). On the other hand, in succ (succ zero) $\sqcup$ succ $\ell$, it is possible for succ $\ell$ to be smaller than succ (succ zero) when we take $\ell$ as zero. Therefore, in this case, we must keep succ (succ zero). Thus, the finiteness of decreasing steps of the universe levels can be seen as taking some finite steps by removing all succ from maps returned by adjust( $\operatorname{count}(l))$. Then we can just take away variables until we can no longer descend. In fact, adjust (count $(l))$ should be considered a normalization algorithm for universe levels. We can simply compare equality between maps computed as such decide whether two universe levels are equivalent. The correctness is as follows:

Lemma 4.6. If $L \vdash l \approx l^{\prime}:$ Level, then $\operatorname{adjust}(\operatorname{count}(l))=\operatorname{adjust}\left(\operatorname{count}\left(l^{\prime}\right)\right)$.
Proof. Induction. Take advantage of the idempotent commutative monoidal nature of maximum.
This lemma ensures that the procedure respects equivalence between universe levels. The other direction is seen by providing a "decoding" function, which converts a map to a universe level. We give one possible function that converts a map to a Level.

$$
\text { flatten }_{L}(m):= \begin{cases}\bigsqcup_{\ell \mapsto n \in m} \operatorname{succ}^{n} \ell & \text { if zero is not in } m \\ \operatorname{succ}^{m(\text { zero })} \text { zero } \sqcup\left(\bigsqcup_{\ell \mapsto n \in m} \operatorname{succ}^{n} \ell\right) & \text { otherwise }\end{cases}
$$

where the order of $\ell$ 's respects their order in $L$ and all $\sqcup$ are right associative. These requirements give a syntactically unique flattening of a map. Then we prove

Lemma 4.7. If $L \vdash l:$ Level, then $\Gamma \vdash l \approx \operatorname{flatten}_{L}(\operatorname{adjust}(\operatorname{count}(l))):$ Level.
Proof. We proceed by induction. It is rather immediate. For the succ case, we use its distributivity to propagate it inwards. For the $\sqcup$ case, we use commutativity to rearrange levels within and absorption to eliminate small levels whenever necessary.

### 4.4 Typing and Equivalence Judgments

In this section, we introduce the typing and equivalence judgments, only for the MLTT part. We will consider the modal part next altogether. The typing and equivalence judgments are defined mutually as usual. All the related judgments are:

- $L \vdash \Psi$ denotes the well-formedness of the global context $\Psi$ under $L$.
- $L \mid \Psi \vdash_{i} \Gamma$ denotes the well-formedness of $\Gamma$ given the universe context $l$ and the global context $\Phi$ at layer $i$. In this section, we are not very concerned about layers yet as most parts about meta-programming and intensional analysis come in a later section (Sec. 4.5).
- $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$ denotes the equivalence between two local contexts $\Gamma$ and $\Delta$.
- $L \mid \Psi ; \Gamma \vdash_{i} T @ l$ denotes the well-formedness of the type $T$ living in the universe level $l$ at layer $i$ in the given contexts.
- $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$ denotes the well-typedness of $t$ of type $T$, which is in the universe level $l$ at layer $i$. In the special occasion of $T$ being some Ty, we might write $L \mid \Psi ; \Gamma \vdash_{i} T: \mathrm{Ty}_{l}$ (@ succ $l$ ) to simultaneously denote two judgments at the same time to save space.
- $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ denotes the equivalence between types $T$ and $T^{\prime}$ living in the universe level $l$ at layer $i$ in the given contexts.
- $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$ denotes the equivalence between $t$ and $t^{\prime}$ of type $T$, which is in the universe level $l$ at layer $i$. The shorthand $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime}: \mathrm{Ty}_{l}$ (@ succ $l$ ) has a meaning similar to above.
- $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$ denotes the well-formedness of a local substitution $\delta$ which substitute all local variables in $\Delta$ into terms referring to $\Gamma$. We will introduce this judgment and the next when we discuss the modal part.
- $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ denotes the equivalence between local substitutions $\delta$ and $\delta^{\prime}$.

The judgments for MLTT are rather routine. Many are just generalization of the judgments in Sec. 2. Let us first consider the well-formedness and equivalence of local contexts:

$$
\begin{aligned}
& L \mid \Psi \vdash_{i} \Gamma
\end{aligned}
$$

$$
\begin{aligned}
& L\left|\Psi \vdash_{i} \Gamma \approx \Delta \quad L\right| \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l \\
& \frac{L\left|\Psi ; \Delta \vdash_{i} T \approx T^{\prime} @ l \quad L\right| \Psi ; \Gamma \vdash_{i} T @ l \quad L \mid \Psi ; \Delta \vdash_{i} T^{\prime} @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi \vdash_{i} \Gamma, x: T @ l \approx \Delta, x: T^{\prime} @ l^{\prime}}
\end{aligned}
$$

The well-formedness of types are also immediate, following Pujet and Tabareau [2023]. When we encounter El, we resort that to the typing judgment of terms. Overlapping rules for well-typedness of encoding as terms are
also listed:


Note that all type constructors with explicitly specified universe levels must not refer to $\omega$. Indeed, $\omega$ level only appears when we validate a universe-polymorphic function and nowhere else. Nor can we pass around a universe-polymorphic function. Moreover, universe-polymorphic functions are only available at the highest layer, which is the layer with capability to do meta-programming and recursive intensional analysis. This is because that universe variables must also be visible by the bindings in global contexts. In the well-formedness rule for Nat, we use a function typeof $(i)$ which alters the layer, in which local contexts live. This treatment is necessary to accommodate dynamic leaks and permit computation in the local contexts and on the type level. We give the actual definition of typeof $(i)$ in Sec. 4.5.

The equivalence between types is composed of three parts. The first part is the PER rules.

$$
\frac{L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} \approx T @ l} \quad \frac{L\left|\Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l \quad L\right| \Psi ; \Gamma \vdash_{i} T^{\prime} \approx T^{\prime \prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime \prime} @ l}
$$

Then we have the congruence rules, which simply straightforwardly propagate equivalence downwards:


Finally, we have a number of computation rules that decode terms into types:

$$
\begin{array}{cc}
\frac{L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Nat} \approx \mathrm{El} \mathrm{l}^{\text {zero }} \mathrm{Nat} \mathrm{@} \mathrm{zero}} & \frac{L \vdash l: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Ty}_{l} \approx \mathrm{El}^{\operatorname{succ} l} \mathrm{Ty}_{l} @ \operatorname{succ} l} \\
\left.\frac{L \vdash l: \text { Level }}{L \vdash l^{\prime}: \text { Level }} L \right\rvert\, \Psi ; \Gamma \vdash_{i} s: \mathrm{Ty}_{l} @ \operatorname{succ} l & L \mid \Psi ; \Gamma, x: \mathrm{El} s @ l \vdash_{i} t: \mathrm{Ty}_{l^{\prime}} @ \operatorname{succ} l^{\prime} \\
L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}\left(x: \mathrm{El}^{l} s\right) . \mathrm{El}^{l^{\prime}} t \approx \mathrm{El}^{l \sqcup l^{\prime}} \Pi^{l, l^{\prime}}(x: s) . t @ l \sqcup l^{\prime}
\end{array}
$$

We do not have an encoding for universe-polymorphic functions, so there is not a decoding rule for them. Then we move on to defining the typing rules for terms. They are pretty much straightforward:

The equivalence rules for terms are also composed of three parts. The PER rules are immediate:

$$
\frac{L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l}{L \mid \Psi ; \Gamma \vdash_{i} t^{\prime} \approx t: T @ l} \quad \frac{L\left|\Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l \quad L\right| \Psi ; \Gamma \vdash_{i} t^{\prime} \approx t^{\prime \prime}: T @ l}{L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime \prime}: T @ l}
$$

$$
\begin{aligned}
& \frac{L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma \quad x: T @ l \in \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} x: T @ l} \quad \frac{L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} \text { zero : Nat @ zero }} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} t: \text { Nat @ zero }}{L \mid \Psi ; \Gamma \vdash_{i} \text { succ } t: \text { Nat @ zero }} \\
& L \vdash l \text { : Level } \quad L \mid \Psi ; \Gamma, x \text { : Nat @ zero } \vdash_{i} M @ l \quad L \mid \Psi ; \Gamma \vdash_{i} s: M[z e r o / x] @ l \\
& L \mid \Psi ; \Gamma, x \text { : Nat @ zero, } y: M @ l \vdash_{i} s^{\prime}: M[\operatorname{succ} x / x] @ l \quad L \mid \Psi ; \Gamma \vdash_{i} t \text { :Nat @ zero } \\
& L \mid \Psi ; \Gamma \vdash_{i} \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right) t: M[t / x] @ l \\
& \frac{L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level } L\left|\Psi ; \Gamma \vdash_{i} S @ l \quad L\right| \Psi ; \Gamma, x: S @ l \vdash_{i} t: T @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} \lambda^{l, l^{\prime}}(x: S) . t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}} \\
& L \vdash l \text { : Level } \quad L \vdash l^{\prime} \text { : Level } \quad L \mid \Psi ; \Gamma \vdash_{i} S @ l \\
& L\left|\Psi ; \Gamma, x: S @ l \vdash_{i} T @ l^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{i} t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{i} s: S @ l \\
& L \mid \Psi ; \Gamma \vdash_{i}\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) s: T[s / x] @ l^{\prime}
\end{aligned}
$$

The congruence rules are naturally induced by the typing rules:

$$
\begin{aligned}
& \frac{L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma \quad x: T @ l \in \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} x \approx x: T @ l} \\
& \frac{L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} \text { zero } \approx \text { zero : Nat @ zero }} \\
& L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: \text { Nat @ zero } \\
& \overline{L \mid \Psi ; \Gamma \vdash_{i} \operatorname{succ} t \approx \operatorname{succ} t^{\prime}: \text { Nat @ zero }} \\
& L \vdash l \approx l^{\prime} \text { : Level } \quad L \mid \Psi ; \Gamma, x \text { : Nat @ zero } \vdash_{i} M \approx M^{\prime} @ l \quad L \mid \Psi ; \Gamma \vdash_{i} s_{1} \approx s_{3}: M[\text { zero } / x] @ l \\
& L \mid \Psi ; \Gamma, x: \text { Nat @ zero, } y: M @ l \vdash_{i} s_{2} \approx s_{4}: M[\operatorname{succ} x / x] @ l \quad L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime} \text { : Nat @ zero } \\
& L \mid \Psi ; \Gamma \vdash_{i} \operatorname{elim}_{\text {Nat }}^{l}(x . M) s_{1}\left(x, y . s_{2}\right) t \approx \operatorname{elim}_{\text {Nat }}^{l^{\prime}}\left(x . M^{\prime}\right) s_{3}\left(x, y . s_{4}\right) t^{\prime}: M[t / x] @ l \\
& L \vdash l_{1} \approx l_{3} \text { : Level } \\
& \frac{L \vdash l_{2} \approx l_{4}: \text { Level } \quad L\left|\Psi ; \Gamma \vdash_{i} S @ l_{1} \quad L\right| \Psi ; \Gamma \vdash_{i} S \approx S^{\prime} @ l_{1} \quad L \mid \Psi ; \Gamma, x: S @ l_{1} \vdash_{i} t \approx t^{\prime}: T @ l_{2}}{L \mid \Psi ; \Gamma \vdash_{i} \lambda^{l_{1}, l_{2}}(x: S) . t \approx \lambda^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot t^{\prime}: \Pi^{l, l^{\prime}}(x: S) \cdot T @ l_{1} \sqcup l_{2}} \\
& L \vdash l_{1} \approx l_{3} \text { : Level } \quad L \vdash l_{2} \approx l_{4} \text { : Level } L\left|\Psi ; \Gamma \vdash_{i} S @ l_{1} \quad L\right| \Psi ; \Gamma \vdash_{i} S \approx S^{\prime} @ l_{1} \\
& \frac{L\left|\Psi ; \Gamma, x: S @ l_{1} \vdash_{i} T \approx T^{\prime} @ l_{2} \quad L\right| \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: \Pi^{l_{1}, l_{2}}(x: S) . T @ l_{1} \sqcup l_{2} \quad L \mid \Psi ; \Gamma \vdash_{i} s \approx s^{\prime}: S @ l_{1}}{L \mid \Psi ; \Gamma \vdash_{i}\left(t: \Pi^{l_{1}, l_{2}}(x: S) . T\right) s \approx\left(t^{\prime}: \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) . T^{\prime}\right) s^{\prime}: T[s / x] @ l_{2}} \\
& \frac{L, \vec{\ell}\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}: T @ l \quad\right| \vec{\ell} \mid>0 \quad L, \vec{\ell} \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} \vec{\ell} \cdot t \approx \Lambda^{l^{\prime}} \vec{l} \cdot t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega} \\
& L, \vec{\ell} \mid \Psi ; \Gamma \vdash_{m} T @ l \\
& L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega \quad\right| \vec{\ell}\left|=|\vec{l}|=\left|\vec{l}^{\prime}\right|>0 \quad \forall 0 \leq n<|\vec{l}| \cdot L \vdash \vec{l}(n) \approx \vec{l}^{\prime}(n):\right. \text { Level } \\
& L \mid \Psi ; \Gamma \vdash_{m} t \$ \vec{l} \approx t^{\prime} \$ \vec{l}^{\prime}: T[\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}] \\
& \frac{L\left|\Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T^{\prime} @ l \quad L\right| \Psi ; \Gamma \vdash_{\text {typeof }(i)} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l^{\prime} L+l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l}
\end{aligned}
$$

### 4.5 Meta-programming Modalities

In this part, we introduce the modalities for meta-programming and intensional analysis. We use the $\square$ modality to represent the type of code and we use layers to control the computational behaviors of the type theory. However, there are two points that we need to pay attention to:
(1) Previously, we have discussed dynamic leaks. Dynamic leaks imply that we must permit computation of code on the type level. This further implies that we must introduce an intermediate layer between layer for code and that for meta-programs, which restricts the language to still MLTT but permits computation.
(2) Due to Tarski-style universes, we must introduce two kinds of contextual types, one for types and one for terms. As seen in the syntax of types and terms, they are mutually defined, so the recursive principles for code of types and terms must also be mutual.
Having set up the basic theme, let us begin with the syntax of the extension to MLTT:

$$
\begin{align*}
& i:=v|c| p \mid m  \tag{Layers}\\
& U
\end{align*}
$$

(Global variables as types)

| Layer | $v$ | $c$ | $p$ | $m$ |
| :--- | :--- | :--- | :--- | :--- |
| Language | Variables only | MLTT | MLTT | MLTT extended with meta－programming |
| Computation | No | No | Yes | Yes |
| Meta－programming | No | No | No | Yes |
| Layer of types | $p$ | $p$ | $p$ | $m$ |

Table 1．Features at each layer

$$
\begin{aligned}
& u \\
& k \\
& \delta:={ }_{g}^{k}\left|\mathrm{wk}_{g}^{k}\right| \delta, t / x \\
& B:=g: C t x\left|U:\left(\Gamma \vdash_{i} @ l\right)\right| u:\left(\Gamma \vdash_{i} T @ l\right) \\
& \Phi, \Psi:=\cdot \mid \Phi, B \\
& \text { (Global variables as terms) } \\
& \text { (Natural numbers, } \mathbb{N} \text { ) } \\
& \text { (Local substitutions) } \\
& \text { (Global bindings) } \\
& \text { (Global contexts) } \\
& S, T:=\cdots\left|U^{\delta}\right|(g: \mathrm{Ctx}) \Rightarrow^{l} T\left|\left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T\right| \square\left(\Gamma \vdash_{c} @ l\right) \mid \square\left(\Gamma \vdash_{c} T @ l\right) \\
& s, t:=\cdots\left|u^{\delta}\right| \Lambda^{l} g . t|t \$ \Gamma| \Lambda_{p}^{l, l^{\prime}} U . t\left|t \$_{p} T\right| \text { box } T \mid \text { box } t \\
& \mid \text { letbox }{ }_{\text {Typ }}^{l^{\prime}} l \Gamma\left(x_{T} \cdot M\right)\left(U . t^{\prime}\right) t \mid \text { letbox }_{\text {Trm }}^{l^{\prime}} l \Gamma T\left(x_{t} \cdot M\right)\left(u . t^{\prime}\right) t \\
& \left|\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma t\right| \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma T t \\
& \vec{M}:=\left(\ell, g, x_{T} \cdot M\right)\left(\ell, g, U_{T}, x_{t} \cdot M^{\prime}\right) \\
& \text { (Two motives for mutual recursion of code) } \\
& \vec{b}:=\vec{b}_{\text {Typ }} \vec{b}_{\text {Trm }} \\
& \text { (Branches for mutual recursion of code) } \\
& b_{\mathrm{Typ}}:=\left(g . t_{\mathrm{Nat}}\right)\left|\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, x_{S}, x_{T} . t_{\Pi}\right)\right|\left(\ell, g . t_{\mathrm{Ty}}\right) \mid\left(\ell, g, u_{t}, x_{t} . t_{\mathrm{El}}\right) \quad \text { (Branches for code of types) } \\
& b_{\text {Trm }}:=\left(\ell, g, U_{T}, u_{x} \cdot t_{x}\right)\left|\left(g . t_{\text {Nat }}^{\prime}\right)\right|\left(\ell, \ell^{\prime}, g, u_{s}, u_{t}, x_{s}, x_{t} \cdot t_{\Pi}^{\prime}\right) \mid\left(\ell, g . t_{T y}^{\prime}\right) \quad \text { (Branches for code of terms) } \\
& \left|\left(g . t_{\text {zero }}\right)\right|\left(g, u_{t}, x_{t} \cdot t_{\text {succ }}\right) \mid\left(\ell, g, U_{M}, u_{s}, u_{s^{\prime}}, u_{t}, x_{M}, x_{s}, x_{s^{\prime}}, x_{t} . t_{\text {elim }}^{\text {Nat }} ⿵ 冂\right. \\
& \left|\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, x_{S}, x_{t} . t_{\lambda}\right)\right|\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{s}, x_{S}, x_{T}, x_{t}, x_{s} . t_{\mathrm{app}}\right)
\end{aligned}
$$

Following the layering principle before，we index our judgments with an layer index $i$ ．We include four layers and these layers are summarized in Table 1．To elaborate，we begin with layer $v$ ，which is the layer that contains only variables．This layer is needed to describe the base case of the recursive principles for code when a local variable is hit．Layer $v$（for variables）is not available for most rules given in Sec． 4.4 other than the local variable rule and its congruence．We will follow this convention for the rest of this report，unless we specifically state that layer $v$ is available for particular rules．Layer $c$（for code）is the layer for code of MLTT．This layer is akin to layer 0 in Sec． 2 and 3 ，where static code resides and no computation is allowed．However，in order to capture dynamic leaks，we must introduce another layer，$p$（for programs），to permit computation of MLTT programs in local contexts and on the type level．This layer is especially crucial in the recursive principle for terms for the argument $T$ where dynamic leaks are implicitly handled by definitional equivalence．However，no meta－programs are allowed at layer $p$ ；in other words，the language at layer $p$ is virtually vanilla MLTT．Therefore，unlike simple types，there are two layers in DeLAM permitting computation．At last，we have layer $m$（for meta－programs）where we have the power to do meta－programming．At this layer，we have access to not only universe－polymorphic functions， but also contextual types and recursive principles for code．All meta－functions must reside at this layer．All layers are related by a strict order of $v<c<p<m$ ．

The reason to introduce layer $p$ becomes obvious by considering which layer the type of a given MLTT term should live in. For instance, given a judgment $L \mid \Psi ; \Gamma \vdash_{c} t: T @ l$, we know $t$ lives at layer $c$ as code, but what about $T$ ? Since $T$ is a type and we want $T$ to compute, it cannot live at layer $c$, but also not $m$ as it must be a wellformed pure MLTT type. Indeed, $T$ ought to live at layer $p$, i.e. $L \mid \Psi ; \Gamma \vdash_{p} T @ l$. What about $L \mid \Psi ; \Gamma \vdash_{p} t: T @ l$ ? In this case, $t$ lives at layer $p$. Since $T$ must still be a well-formed pure MLTT type and compute, we must have $L \mid \Psi ; \Gamma \vdash_{p} T @ l$. The type of a term living at layer $m$ simply also lives at layer $m$. The relation of layers of terms and types is summarized by the following function:

$$
\begin{aligned}
\operatorname{typeof}(v) & :=p \\
\operatorname{typeof}(c) & :=p \\
\operatorname{typeof}(p) & :=p \\
\operatorname{typeof}(m) & :=m
\end{aligned}
$$

The judgment $i$ computable quantifies computable layers:

$$
\overline{p \text { computable }}
$$

$m$ computable
Then we extend our system with five types, following Sec. 2:

- $U^{\delta}$ is a global variable for types. Due to separation of types and terms, we need a way to refer to code of types on the type level.
- $(g:$ Ctx $) \Rightarrow^{l} T$ is a meta-function type for introducing a contextual variable $g$ to the global context. We also have this in Sec. 2.
- $\left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T$ is similarly a meta-function type for introducing a type at layer $p$ to the global context. This type is introduced to provide an index for the contextual type for terms to be discussed in the second next item.
- $\square\left(\Gamma \vdash_{c} @ l\right)$ is a contextual type for types in MLTT. It represents a static code of types.
- Finally, $\square\left(\Gamma \vdash_{c} T @ l\right)$ is a contextual type for terms in MLTT. This $T$ may refer to the index type at layer $p$ introduced by meta-functions above.
Since there are two kinds of contextual types now, there four kinds of bindings in a global context:
- contextual variables $g$ : Ctx representing a local context;
- global variables $U:\left(\Gamma \vdash_{i} @ l\right)$ representing a type in MLTT (note that $i \in\{c, p\}$ );
- global variables $u:\left(\Gamma \vdash_{i} T @ l\right)$ representing a term in MLTT (note that $i \in\{v, c\}$ and there is no way to introduce a term at layer $p$ to global context).
Now, let us move on to discuss the extended terms.
- First, we also introduce global variables and local substitutions. Their syntax is identical to one in Sec. 2.
- Then we have the introduction and elimination forms for meta-functions of contextual variables, $\Lambda^{l}$ g.t and $t \$ \Gamma$.
- Similarly, we have the introduction and elimination forms for meta-functions of types, $\Lambda_{p}^{l, l^{\prime}} U . t$ and $t \$_{p} T$.
- Then we have the introduction forms of two kinds of contextual types.
- Following Hu and Pientka [2024a, Sec. 4], we have two elimination forms for each kind of contextual types, letbox and the recursive principles. Same as before, letbox is responsible for code composition and evaluation. Intentional analyses are done through the recursive principles. In DeLaM, letbox is a bit more complex as it requires a specified motive. We alter the syntax a little bit to make letbox more like an operation: letbox ${ }_{\mathrm{Typ}}^{l^{\prime}} l \Gamma\left(x_{T} . M\right)\left(u . t^{\prime}\right) t$ and letbox $\mathrm{Trm}^{l^{\prime}} l \Gamma T\left(x_{t} . M\right)\left(u . t^{\prime}\right) t$.
- Finally, we extend the recursive principles for code. As indicated before, code of types and terms in MLTT are mutually defined, so the recursive principles must also be mutual. The two recursive principles elim $\mathrm{T}_{\mathrm{Typ}}^{l_{1} l_{2}} \vec{M} \vec{b} l \Gamma t$ and $\operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma T t$ require two motives, one for types and one for terms, and contain all branches for code of types and terms. Their difference is what exactly eventually being eliminated, as indicated by their subscript. The branches $\vec{b}$ are a list of branches $\vec{b}_{\text {Typ }}$ and $\vec{b}_{\text {Trm }}$, where $\vec{b}_{\text {Typ }}$ and $\vec{b}_{\text {Trm }}$ contain all branches for types and terms, respectively.

In the branches, there are four kinds of variables.

- There is a globally introduced contextual variable $g$ which represents the local context where the code lives.
- There could be some universe variables that are used to tell the universe levels of some types.
- There could be some global variables $u$ and $U$ represent the sub-structures of a given case. The subscripts correspond tightly to the syntax given in Sec. 4.2.
- For each sub-structure, there is one corresponding recursive variable $x$. Again, the subscripts correspond tightly to the sub-structure.


### 4.6 More Typing and Equivalence Judgments

In this section, we specify the remainder of the rules. We begin with the well-formedness rule for the global contexts. Recall that layer $v$ does not apply for most rules below, unless the otherwise is specifically stated.

The judgments for local substitutions follow very closely Sec. 2. In these rules, $i$ might take $v$. This permission has a particular effect on the step case, which forces all terms in a local substitution must be variables.

$$
\begin{aligned}
& L\left|\Psi \vdash_{\text {typeof }(i)} \Gamma \quad L\right| \Psi \vdash_{\text {typeof }(i)} \Gamma \quad g: \text { Ctx } \in \Psi \\
& \Gamma \text { ends with } \quad|\Gamma|=k \\
& \overline{L\left|\Psi ; \Gamma \vdash_{i}{ }^{k}: \cdot \quad L\right| \Psi ; \Gamma \vdash_{i}{ }^{k} \approx{ }^{k}: \cdot} \\
& \frac{L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma \quad g: \mathrm{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=k}{L\left|\Psi ; \Gamma \vdash_{i} \mathrm{wk}_{g}^{k}: g \quad L\right| \Psi ; \Gamma \vdash_{i} \mathrm{wk}_{g}^{k} \approx \mathrm{wk}_{g}^{k}: g} \\
& \frac{L \vdash l: \text { Level } \quad L\left|\Psi ; \Gamma \vdash_{i} \delta: \Delta \quad L\right| \Psi ; \Delta \vdash_{\text {typeof }(i)} T @ l \quad L \mid \Psi ; \Gamma \vdash_{i} t: T[\delta] @ l}{L \mid \Psi ; \Gamma \vdash_{i} \delta, t / x: \Delta, x: T @ l} \\
& L \vdash l: \text { Level } \quad L\left|\Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta \quad L\right| \Psi ; \Delta \vdash_{\text {typeof }(i)} T @ l \\
& \frac{L\left|\Psi ; \Gamma \vdash_{i} t: T[\delta] @ l \quad L\right| \Psi ; \Gamma \vdash_{i} t^{\prime}: T[\delta] @ l \quad L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T[\delta] @ l}{L \mid \Psi ; \Gamma \vdash_{i} \delta, t / x \approx \delta^{\prime}, t^{\prime} / x: \Delta, x: T @ l}
\end{aligned}
$$

In the step case for equivalence above, we ask for two redundant premises of the well-typedness of $t$ and $t^{\prime}$ to provide an early presupposition for equivalence of local substitutions. We will need this early presupposition in Lemma 5.15. It breaks the dependency loop so that reaching the full presupposition lemma becomes viable.

Now let us consider the extended types and their equivalence.


The additional equivalence rules are just their congruence rules:

$$
\begin{gathered}
L \mid \Psi \vdash_{\text {typeof }(i) \Gamma \quad U:\left(\Delta \vdash_{i^{\prime}} @ l\right) \in \Psi \quad i^{\prime} \in\{c, p\} \quad i^{\prime} \leq i \quad L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta}^{L \mid \Psi ; \Gamma \vdash_{i} U^{\delta} \approx U^{\delta^{\prime}} @ l} \\
\frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} T \approx T^{\prime} @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m}(g: \mathrm{Ctx}) \Rightarrow^{l} T \approx(g: \mathrm{Ctx}) \Rightarrow^{l^{\prime}} T^{\prime} @ l} \\
\frac{L\left|\Psi \vdash_{p} \Delta \approx \Delta^{\prime} \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T \approx T^{\prime} @ l^{\prime} \quad L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T \approx\left(U:\left(\Delta^{\prime} \vdash_{p} @ l_{3}\right)\right) \Rightarrow^{l_{4}} T^{\prime} @ \operatorname{succ} l_{1} \sqcup l_{2}} \\
\frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \approx \Delta^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} @ l\right) \approx \square\left(\Delta^{\prime} \vdash_{c} @ l^{\prime}\right) @ \operatorname{succ} l} \\
\frac{L\left|\Psi \vdash_{p} \Delta \approx \Delta^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l}{L \vdash \vdash_{m} \Gamma \quad L \approx l^{\prime}: \text { Level }} \\
L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T @ l\right) \approx \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l^{\prime}\right) @ l
\end{gathered}
$$

Next, we list the extended typing judgments:

$$
\begin{aligned}
& \underline{L\left|\Psi \vdash_{\text {typeof }(i)} \Gamma \quad u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi \quad i^{\prime} \in\{v, c\} \quad i \in\{v, c, p, m\} \quad i^{\prime} \leq i \quad L\right| \Psi ; \Gamma \vdash_{i} \delta: \Delta} \\
& L \mid \Psi ; \Gamma \vdash_{i} u^{\delta}: T[\delta] @ l \\
& \begin{array}{cc}
L \mid \Psi \vdash_{m} \Gamma \\
L \mid \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} t: T @ l & L \vdash l: \text { Level } \\
L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} g . t:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l
\end{array} \quad \begin{array}{c}
L \mid \Psi ; \mathrm{Ctx} ; \Gamma \vdash_{m} T @ l \\
L \mid \Psi ; \Gamma \vdash_{m} t \$ \Delta: T[\Delta / g] @ l
\end{array} \\
& \frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} t: T @ l^{\prime} \quad L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda_{p}^{l, l^{\prime}} U . t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T @ \operatorname{succ} l \sqcup l^{\prime}} \\
& L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T^{\prime} @ l^{\prime} \\
& \frac{L\left|\Psi ; \Gamma \vdash_{m} t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow l^{l^{\prime}} T^{\prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T @ l}{L \mid \Psi ; \Gamma \vdash_{m} t \$_{p} T: T^{\prime}[T / U] @ l^{\prime}} \\
& \frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi ; \Delta \vdash_{c} T @ l}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{box} T: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l} \quad \frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi ; \Delta \vdash_{c} t: T @ l}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{box} t: \square\left(\Delta \vdash_{c} T @ l\right) @ l} \\
& L \vdash l^{\prime} \text { : Level } \quad L \vdash l: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Gamma \vdash_{m} t: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l \\
& \frac{L\left|\Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l \vdash_{m} M @ l^{\prime} \quad L\right| \Psi, U:\left(\Delta \vdash_{c} @ l\right) ; \Gamma \vdash_{m} t^{\prime}: M\left[\operatorname{box} U / x_{T}\right] @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{letbox}_{\mathrm{Typ}}^{l^{\prime}} l \Delta\left(x_{T} \cdot M\right)\left(U . t^{\prime}\right) t: M\left[t / x_{T}\right] @ l^{\prime}} \\
& L \vdash l^{\prime} \text { : Level } \quad L \vdash l: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{p} T @ l \quad L \mid \Psi ; \Gamma \vdash_{m} t: \square\left(\Delta \vdash_{c} T @ l\right) @ l \\
& L\left|\Psi ; \Gamma, x_{t}: \square\left(\Delta \vdash_{c} T @ l\right) @ l \vdash_{m} M @ l^{\prime} \quad L\right| \Psi, u:\left(\Delta \vdash_{c} T @ l\right) ; \Gamma \vdash_{m} t^{\prime}: M\left[\operatorname{box} u / x_{t}\right] @ l^{\prime} \\
& L \mid \Psi ; \Gamma \vdash_{m} \text { letbox }{ }_{\text {Trm }}^{l^{\prime}} l \Delta T\left(x_{t} \cdot M\right)\left(u . t^{\prime}\right) t: M\left[t / x_{t}\right] @ l^{\prime}
\end{aligned}
$$

For the typing rule of $t \$ \Delta$, we require $\Delta$ to be a context at layer $p$. This is because a contextual variable represents a local context for an MLTT term. A local context for an MLTT term necessarily lives at layer $p$, so we can only substitute a context living at layer $p$ with a contextual variable.

Now we shall mentally prepare ourselves to write down the typing rules for the two recursive principles. They are conceptually easy but simply verbose to write down. We will only write down the rules for this time for completeness and in later discussions, we simply omit the premises. Our goal is to provide the following conclusions:

$$
\begin{gathered}
L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta t: M\left[l^{\prime} / l, \Delta / g, t / x_{T}\right] @ l_{1} \\
L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T t: M^{\prime}\left[l^{\prime} / l, \Delta / g, T / U_{T}, t / x_{t}\right] @ l_{2}
\end{gathered}
$$

We group the premises into different parts. First we give the premises related to the motives:

$$
\begin{gathered}
L \vdash l_{1}: \text { Level } L \vdash l_{2}: \text { Level } L, \ell \mid \Psi, g: \operatorname{Ctx} ; \Gamma, x_{T}: \square\left(g \vdash_{c} @ \ell\right) \vdash_{m} M @ l_{1} \\
L, \ell \mid \Psi, g: \operatorname{Ctx}, U_{T}:\left(g \vdash_{p} @ \ell\right) ; \Gamma, x_{t}: \square\left(g \vdash_{c} U_{T}^{\text {id }} @ \ell\right) \vdash_{m} M^{\prime} @ l_{2}
\end{gathered}
$$

where $\vec{M}=\left(\ell, g, x_{T} \cdot M\right)\left(\ell, g, U_{T}, x_{t} \cdot M^{\prime}\right)$. In the premises above, we give the well-formedness of two motives for code of types and terms, respectively. Let us call this group $G_{M}$. We move on to considering the branches. We first consider the branches for code of types. It is relatively easy as there are only four cases:
$\bullet$
$L \mid \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} t_{\text {Nat }}: M\left[\right.$ zero $/ \ell, g / g$, box Nat $\left./ x_{T}\right] @ l_{1}$

- We explain this premise more carefully. Consider some code of type box $\Pi^{l, l^{\prime}}(x: S) . T$, then we have the matching premise

$$
L, \ell, \ell^{\prime} \mid \Psi^{\prime} ; \Gamma^{\prime} \vdash_{m} t_{\Pi}: M\left[\ell \sqcup \ell^{\prime} / \ell, g / g, \operatorname{box} \Pi^{\ell, \ell^{\prime}}\left(x: U_{S}^{\mathrm{id}}\right) \cdot U_{T}^{\mathrm{id}} / x_{T}\right] @ l_{1}
$$

where

$$
\begin{aligned}
\Psi^{\prime}:= & \Psi \\
& , g: \operatorname{Ctx} \\
& , U_{S}:\left(g \vdash_{c} @ \ell\right) \\
& , U_{T}:\left(g, x: U_{S}^{\mathrm{id}} @ \ell \vdash_{c} @ \ell^{\prime}\right)
\end{aligned}
$$

$$
\left., U_{S}:\left(g \vdash_{c} @ \ell\right) \quad \text { (the global variable for the input type, which captures } S\right)
$$

(the global variable for the output type, which captures $T$; note that it lives in an extended local context) and
$\Gamma^{\prime}:=\Gamma$

$$
, x_{S}: M\left[\ell / \ell, g / g \text {, box } U_{S}^{\text {id }} / x_{T}\right] @ l_{1} \quad \text { (the recursive call for } S \text { of type } M \text { that is properly substituted) }
$$

$$
, x_{T}: M\left[\ell^{\prime} / \ell,\left(g, x: U_{S}^{\text {id }} @ \ell\right) / g, \operatorname{box} U_{T}^{\text {id }} / x_{T}\right] @ l_{1}
$$

(the recursive call for $T$; see how the local context is extended)

- Further,

$$
L, \ell \mid \Psi, g: \operatorname{Ctx} ; \Gamma \vdash_{m} t_{\mathrm{Ty}}: M\left[\operatorname{succ} \ell / \ell, g / g \text {, box } \mathrm{Ty}_{\ell} / x_{T}\right] @ l_{1}
$$

- 

$$
L, \ell \mid \Psi, g: \operatorname{Ctx}, u_{t}:\left(g \vdash_{c} \mathrm{Ty}_{\ell} @ \operatorname{succ} \ell\right) ; \Gamma^{\prime} \vdash_{m} t_{\mathrm{E} 1}: M\left[\ell / \ell, g / g, \operatorname{box}\left(\mathrm{El}^{\ell} u_{t}^{\mathrm{id}}\right) / x_{T}\right] @ l_{1}
$$

where

$$
\Gamma^{\prime}:=\Gamma, x_{t}: M^{\prime}\left[\operatorname{succ} \ell / \ell, g / g, \mathrm{Ty}_{\ell} / U_{T}, \text { box } u_{t}^{\text {id }} / x_{t}\right] @ l_{2}
$$

Let us call this group $G_{\text {Typ }}$.
Lastly, let us consider the nine cases for terms.

$$
L, \ell \mid \Psi, g: \operatorname{Ctx}, U_{T}:\left(g \vdash_{p} @ \ell\right), u_{x}:\left(g \vdash_{v} U_{T}^{\mathrm{id}} @ \ell\right) ; \Gamma \vdash_{m} t_{x}: M^{\prime}\left[\ell / \ell, g / g, U_{T}^{\mathrm{id}} / U_{T}, \text { box } u_{x} / x_{t}\right] @ l_{2}
$$

In this case, the type of the variable is captured by $U_{T}$. It has to live at layer $p$ because it is not a sub-structure of the variable, i.e. it is obtained externally, from the indexing arguments of the recursive principles.

$$
L \mid \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} t_{\text {Nat }}^{\prime}: M^{\prime}\left[\text { succ zero } / \ell, g / g, \mathrm{Ty}_{\text {zero }} / U_{T} \text {, box Nat } / x_{t}\right] @ l_{2}
$$

- 

$$
L, \ell, \ell^{\prime} \mid \Psi^{\prime} ; \Gamma^{\prime} \vdash_{m} t_{\Pi}^{\prime}: M^{\prime}\left[\operatorname{succ}\left(\ell \sqcup \ell^{\prime}\right) / \ell, g / g, \mathrm{Ty}_{\ell \sqcup \ell^{\prime}} / U_{T}, \operatorname{box} \Pi^{\ell, \ell^{\prime}}\left(x: u_{s}^{\mathrm{id}}\right) . u_{t}^{\mathrm{id}} / x_{t}\right] @ l_{2}
$$

where

$$
\Psi^{\prime}:=\Psi, g: \mathrm{Ctx}, u_{s}:\left(g \vdash_{c} \mathrm{Ty}_{\ell} @ \operatorname{succ} \ell\right), u_{t}:\left(g, x: \mathrm{El}^{\ell} u_{s}^{\mathrm{id}} \vdash_{c} \mathrm{Ty}_{\ell^{\prime}} @ \operatorname{succ} \ell^{\prime}\right)
$$

and

$$
\begin{aligned}
\Gamma^{\prime}:= & \Gamma \\
& , x_{s}: M^{\prime}\left[\operatorname{succ} \ell / \ell, g / g, \mathrm{Ty}_{\ell} / U_{T}, \operatorname{box} u_{s}^{\mathrm{id}} / x_{t}\right] @ l_{2} \\
& , x_{t}: M^{\prime}\left[\operatorname{succ} \ell^{\prime} / \ell,\left(g, x: \mathrm{El}^{\ell} u_{s}^{\mathrm{id}}\right) / g, \mathrm{Ty}_{\ell^{\prime}} / U_{T}, \text { box } u_{t}^{\mathrm{id}} / x_{t}\right] @ l_{2}
\end{aligned}
$$

Notice that this premise for the encoding of $\Pi$ is almost identical to the premise in $G_{\text {Typ }}$ above, with necessary adjustment to return the proper motive $M^{\prime}$ instead.
$L, \ell \mid \Psi, g: \operatorname{Ctx} ; \Gamma \vdash_{m} t_{\text {Ty }}^{\prime}: M^{\prime}\left[\operatorname{succ} \operatorname{succ} \ell / \ell, g / g, T y \operatorname{succ} \ell / U_{T}\right.$, box $\left.\mathrm{Ty}_{\ell} / x_{T}\right] @ l_{2}$
-

$$
L \mid \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} t_{\text {zero }}: M^{\prime}\left[\text { zero } / \ell, g / g, \text { Nat } / U_{T}, \text { box zero } / x_{t}\right] @ l_{2}
$$

- 

$$
L \mid \Psi, g: \text { Ctx, } u_{t}:\left(g \vdash_{c} \text { Nat @ zero }\right) ; \Gamma^{\prime} \vdash_{m} t_{\text {succ }}: M^{\prime}\left[\text { zero } / \ell, g / g, \text { Nat } / U_{T}, \text { box }\left(\operatorname{succ} u_{t}^{\text {id }}\right) / x_{t}\right] @ l_{2}
$$ where $\Gamma^{\prime}:=\Gamma, x_{t}: M^{\prime}\left[\right.$ zero $/ \ell, g / g$, Nat $/ U_{T}$, box $\left.u_{t}^{\text {id }} / x_{t}\right] @ l_{2}$.

- We carefully explain this premise for the code of the elimination of natural numbers. Recall that the syntax is $\lim _{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t$. We use corresponding global variables to capture the sub-structures.

$$
L, \ell \mid \Psi^{\prime} ; \Gamma^{\prime} \vdash_{m} t_{\mathrm{elim}_{\text {Nat }}}: M^{\prime}\left[\ell / \ell, g / g, U_{M}^{\mathrm{id}_{g}, u_{t}^{\mathrm{id}} / x} / U_{T}, \operatorname{elim}_{\mathrm{Nat}}^{\ell}\left(x \cdot U_{M}^{\mathrm{id}_{g, x}}\right) u_{s}^{\mathrm{id}_{g}}\left(x, y \cdot u_{s^{\prime}}^{\mathrm{id}_{g, x, y}}\right) u_{t}^{\mathrm{id}_{g}} / x_{t}\right] @ l_{2}
$$

where

$$
\begin{aligned}
\Psi^{\prime}:= & \Psi \\
& , g: \text { Ctx } \\
& , U_{M}:\left(g, x: \text { Nat @ zero } \vdash_{c} @ \ell\right)
\end{aligned}
$$

(the global variable for the motive; it lives at layer $c$ as it is a sub-structure) , $u_{s}:\left(g \vdash_{c} U_{M}^{\mathrm{id}_{g}, \text { zero } / x} @ \ell\right)$
(the code for the base case; its type refers to the code of the motive with $x$ for zero)

$$
u_{s^{\prime}}:\left(g, x: \text { Nat @ zero, } y: U_{M}^{\mathrm{id} \mathrm{~d}_{g, x}} @ \ell \vdash_{c} U_{M}^{\mathrm{id}_{g}, \text { succ } x / x} @ \ell\right)
$$

(the code for the step case; the local context is extended with the predecessor and the recursive call)

$$
\left., u_{t}:\left(g \vdash_{c} \text { Nat @ zero }\right) \quad \text { (the code for the scrutinee }\right)
$$

and

$$
\begin{aligned}
\Gamma^{\prime}:= & \Gamma \\
& , x_{M}: M\left[\ell / \ell,(g, x: \text { Nat @ zero }) / g, \text { box } U_{M}^{\mathrm{id}_{g, x}} / x_{T}\right] @ l_{1}
\end{aligned}
$$

(since the motive is a sub-structure, a recursive call is available)

$$
, x_{s}: M^{\prime}\left[\ell / \ell, g / g, U_{M}^{\mathrm{id}_{g}, \text { zero } / x} / U_{T}, \operatorname{box} u_{s}^{\mathrm{id}_{g}} / x_{t}\right] @ l_{2}
$$

(the recursive call for the base case; recall that $U_{T}$ is the type of $x_{t}$,)
(which in this case is also the type of $u_{s}$ )
, $x_{s^{\prime}}: M^{\prime}\left[\ell / \ell,\left(g, x:\right.\right.$ Nat @ zero, $\left.y: U_{M}^{\mathrm{id}_{g, x}} @ \ell\right) / g, U_{M}^{\mathrm{id} g_{g}, \text { succ } x / x} / U_{T}$, box $\left.u_{s^{\prime}}^{\mathrm{id}_{g, x, y}} / x_{t}\right] @ l_{2}$
(the recursive call for the step case; similar logic applies but more longer)

$$
, x_{t}: M^{\prime}\left[\ell / \ell, g / g, \text { Nat } / U_{T}, \text { box } u_{t}^{\text {id }} / x_{t}\right] @ l_{2} \quad \text { (the recursive call for the scrutinee) }
$$

- 

$$
L, \ell, \ell^{\prime} \mid \Psi^{\prime} ; \Gamma^{\prime} \vdash_{m} t_{\lambda}: M^{\prime}\left[\ell \sqcup \ell^{\prime} / \ell, g / g, \Pi^{\ell, \ell^{\prime}}\left(x: U_{S}^{\mathrm{id} g}\right) \cdot U_{T}^{\mathrm{id} \mathrm{~d}_{g, x}} / U_{T}, \text { box } \lambda^{\ell, \ell^{\prime}}\left(x: U_{S}^{\mathrm{id} g}\right) \cdot u_{t}^{\mathrm{id} \mathrm{~d}_{g, x}} / x_{t}\right] @ l_{2}
$$

where

$$
\Psi^{\prime}:=\Psi
$$

$$
\begin{aligned}
& , g: \mathrm{Ctx} \\
& , U_{S}:\left(g \vdash_{c} @ \ell\right) \\
& , U_{T}:\left(g, x: U_{S}^{\mathrm{id}_{g}} @ \ell \vdash_{p} @ \ell^{\prime}\right) \\
& , u_{t}:\left(g, x: U_{S}^{\mathrm{id} g} @ \ell \vdash_{c} U_{T}^{\mathrm{id} \mathrm{~d}_{g, x}} @ \ell^{\prime}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma^{\prime}:= & \Gamma \\
& , x_{S}: M\left[\ell / \ell, g / g, \operatorname{box} U_{S}^{\mathrm{id}_{g}} / x_{T}\right] @ l_{1} \\
& , x_{t}: M^{\prime}\left[\ell^{\prime} / \ell,\left(g, x: U_{S}^{\mathrm{id}_{g}} @ \ell\right) / g, U_{T}^{\mathrm{id}_{g, x}} / U_{T}, \operatorname{box} u_{t}^{\mathrm{id}_{g, x}} / x_{t}\right] @ l_{2}
\end{aligned}
$$

Note that here $U_{T}$ is at layer $p$. This is because the return type of not a sub-structure in a function abstraction $\lambda^{l, l^{\prime}}(x: S) . t$, and therefore it must be captured externally from the indexing arguments of the recursive principle. Since it is not a sub-structure, there also is not a recursive call for it. It is possible to include the return type as a sub-structure, e.g. $\lambda^{l, l^{\prime}}(x: S) .(t: T)$ but we decided to show this alternative to demonstrate various design spaces.

- Finally,
$L, \ell, \ell^{\prime} \mid \Psi^{\prime} ; \Gamma^{\prime} \vdash_{m} t_{\mathrm{app}}: M^{\prime}\left[\ell^{\prime} / \ell, g / g, U_{T}^{\mathrm{id} g, u_{t}^{\mathrm{id} g} / x} / U_{T}, \operatorname{box}\left(\left(u_{t}^{\mathrm{id} g}: \Pi^{\ell, \ell^{\prime}}\left(x: U_{S}^{\mathrm{id} \mathrm{d}_{g}}\right) \cdot U_{T}^{\mathrm{id} \mathrm{d}_{g, x}}\right) u_{s}^{\mathrm{id} g}\right) / x_{t}\right] @ l_{2}$
where

$$
\begin{aligned}
\Psi^{\prime}:= & \Psi \\
& , g: \mathrm{Ctx} \\
& , U_{S}:\left(g \vdash_{c} @ \ell\right) \\
& , U_{T}:\left(g, x: U_{S}^{\mathrm{id}} \mathrm{id}_{g} @ \ell \vdash_{c} @ \ell^{\prime}\right) \\
& , u_{t}:\left(g \vdash_{c} \Pi^{\ell, \ell^{\prime}}\left(x: U_{S}^{\mathrm{id}_{g}}\right) \cdot U_{T}^{\mathrm{id}_{g, x}} @ \ell \sqcup \ell^{\prime}\right) \\
& , u_{s}:\left(g \vdash_{c} U_{S}^{\mathrm{id}_{g}} @ \ell\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma^{\prime}:= & \Gamma \\
& , x_{S}: M\left[\ell / \ell, g / g, \operatorname{box} U_{S}^{\mathrm{id}_{g}} / x_{T}\right] @ l_{1} \\
& , x_{T}: M\left[\ell / \ell,\left(g, x: U_{S}^{\mathrm{id}_{g}} @ \ell\right) / g, \text { box } U_{T}^{\mathrm{id} g_{g, x}} / x_{T}\right] @ l_{1} \\
& , x_{t}: M^{\prime}\left[\ell \sqcup \ell^{\prime} / \ell, g / g, \Pi^{\ell, \ell^{\prime}}\left(x: U_{S}^{\mathrm{id}_{g}}\right) \cdot U_{T}^{\mathrm{id}_{g, x}} / U_{T}, \operatorname{box} u_{t}^{\mathrm{id}_{g}} / x_{t}\right] @ l_{2} \\
& , x_{s}: M^{\prime}\left[\ell / \ell, g / g, U_{S}^{\mathrm{id}_{g}} / U_{T}, \operatorname{box} u_{s}^{\mathrm{id}_{g}} / x_{t}\right] @ l_{2}
\end{aligned}
$$

This premise shows why we must use a more verbose syntax for application, i.e. $\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) s$. In the global context, we must introduce the global variables for the input and output types. However, a vanilla function application $t s$ has no such information at all. Since the current syntax has both input and output types as sub-structures, we can also allow their recursive calls.

All premises above conclude the group for terms, which we name $G_{T r m}$. We collectively use $G_{A}$ for all three groups above, i.e. $G_{A}:=G_{M} G_{\mathrm{Typ}} G_{\mathrm{Trm}}$. Then we have the typing rule for the recursive principles as follows:

$$
\begin{gathered}
\frac{G_{A} \quad L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Gamma \vdash_{m} t: \square\left(\Delta \vdash_{c} @ l^{\prime}\right) @ \operatorname{succ} l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta t: M\left[l^{\prime} / \ell, \Delta / g, t / x_{T}\right] @ l_{1}} \\
\frac{L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{p} T @ l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{m} t: \square\left(\Delta \vdash_{c} T @ l^{\prime}\right) @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Trm}}^{l_{1} l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T t: M^{\prime}\left[l^{\prime} / \ell, \Delta / g, T / U_{T}, t / x_{t}\right] @ l_{2}}
\end{gathered}
$$

### 4.7 More Congruence Rules for Typing

The congruence rules for the additional typing rules are naturally derived from the typing rules above.

$$
\begin{gathered}
\frac{L: \left.\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi \quad \begin{array}{c}
\Psi \vdash_{\text {typeof }(i)} \Gamma \\
i^{\prime} \in\{v, c\} \\
i \in\{v, c, p, m
\end{array} \quad i^{\prime} \leq i \quad L \right\rvert\, \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta}{L \mid \Psi ; \Gamma \vdash_{i} u^{\delta} \approx u^{\delta^{\prime}}: T[\delta] @ l} \\
\frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} t \approx t^{\prime}: T @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} g \cdot t \approx \Lambda^{l^{\prime}} g \cdot t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l} \\
\frac{L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l \quad L\right| \Psi \vdash_{p} \Delta \approx \Delta^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} t \$ \Delta \approx t^{\prime} \$ \Delta^{\prime}: T[\Delta / g] @ l} \\
\frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l_{1}\right) ; \Gamma \vdash_{m} t \approx t^{\prime}: T @ l_{2} \quad L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda_{p}^{l_{1} l_{2}} U \cdot t \approx \Lambda_{p}^{l_{3} l_{4}} U \cdot t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T @ \operatorname{succ} l_{1} \sqcup l_{2}} \\
\frac{L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow l^{\prime} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{m} t \$_{p} T \approx t^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}[T / U] @ l^{\prime}}
\end{gathered}
$$

The following rules are related to meta-programming and intensional analysis.


We omit the congruence rules for the recursive principles for code as they are conceptually simple but too long. We simply let equivalence to propagate inwards to all the sub-terms of the recursive principles.

### 4.8 Computation Rules

Finally, we list all the computation rules. In the rules below, we let $i$ computable. We first list the $\beta$ rules for natural numbers:

$$
\begin{gathered}
L \vdash l: \text { Level } \begin{array}{c}
L \mid \Psi ; \Gamma, x: \text { Nat @ zero } \vdash_{i} M @ l \\
L \mid \Psi ; \Gamma \vdash_{i} s: M[z e r o / x] @ l
\end{array} \frac{L \mid \Psi ; \Gamma, x: \text { Nat @ zero, } y: M @ l \vdash_{i} s^{\prime}: M[\operatorname{succ} x / x] @ l}{L \mid \Psi ; \Gamma \vdash_{i} s \approx \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right) \text { zero }: M[\text { zero } / x] @ l} \\
L \vdash l: \text { Level } L \mid \Psi ; \Gamma, x: \text { Nat @ zero } \vdash_{i} M @ l \quad L \mid \Psi ; \Gamma \vdash_{i} s: M[\text { zero } / x] @ l \\
L \mid \Psi ; \Gamma, x: \text { Nat @ zero, } y: M @ l \vdash_{i} s^{\prime}: M[\operatorname{succ} x / x] @ l \quad L \mid \Psi ; \Gamma \vdash_{i} t: \text { Nat @ zero } \\
L \mid \Psi ; \Gamma \vdash_{i} s^{\prime}\left[t / x, \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t / y\right] \approx \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right)(\operatorname{succ} t): M[\operatorname{succ} t / x] @ l
\end{gathered}
$$

Then we have the $\beta$ and $\eta$ rules for dependent functions:

$$
\begin{aligned}
& L \vdash l \text { : Level } \quad L \vdash l^{\prime} \text { : Level } \\
& L\left|\Psi ; \Gamma \vdash_{i} S @ l \quad L\right| \Psi ; \Gamma, x: S @ l \vdash_{i} T @ l^{\prime} \quad L\left|\Psi ; \Gamma, x: S @ l \vdash_{i} t: T @ l^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{i} s: S @ l \\
& L \mid \Psi ; \Gamma \vdash_{i} t[s / x] \approx\left(\lambda^{l, l^{\prime}}(x: S) . t: \Pi^{l, l^{\prime}}(x: S) . T\right) s: T[s / x] @ l^{\prime} \\
& L \vdash l \text { : Level } \\
& L \vdash l^{\prime} \text { : Level } \quad L\left|\Psi ; \Gamma \vdash_{i} S @ l \quad L\right| \Psi ; \Gamma, x: S @ l \vdash_{i} T @ l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{i} t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
& L \mid \Psi ; \Gamma \vdash_{i} \lambda^{l, l^{\prime}}(x: S) .\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) x \approx t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}
\end{aligned}
$$

In the $\eta$ rule, on the right hand side, all $t, S$ and $T$ should be properly locally weakened.
Finally we have $\beta$ and $\eta$ rules for universe-polymorphic functions:

$$
\begin{gathered}
L\left|\Psi \vdash_{m} \Gamma \quad L, \vec{\ell}\right| \Psi ; \Gamma \vdash_{m} t: T @ l \quad L, \vec{\ell}+l: \text { Level } \quad|\vec{\ell}|=|\vec{l}|>0 \quad \forall l^{\prime} \in \vec{l} \cdot L \vdash l^{\prime}: \text { Level } \\
L \mid \Psi ; \Gamma \vdash_{m} t[\vec{l} / \vec{\ell}] \approx\left(\Lambda^{l} \vec{\ell} \cdot t\right) \$ \vec{l}: T[\vec{l} \mid \vec{\ell}] @ l[\vec{l} / \vec{\ell}] \\
\frac{L \mid \Psi ; \Gamma \vdash_{m} t: \vec{\ell} \Rightarrow^{l} T @ \omega}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} \vec{\ell} \cdot(t \$ \vec{\ell}) \approx t: \vec{\ell} \Rightarrow^{l} T @ \omega}
\end{gathered}
$$

Similarly, in the $\eta$ rule, the universe variables appearing in $t$ must also be properly weakened. This concludes all the rules for the MLTT portion of DeLaM.

Then we move on to considering the computation rules for the extended types. Let us finish considering all meta-function types.

$$
\begin{gathered}
\frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, g: C t x ; \Gamma \vdash_{m} t: T @ l \quad L \vdash l: \text { Level } L \mid \Psi \vdash_{p} \Delta}{L \mid \Psi ; \Gamma \vdash_{m} t[\Delta / g] \approx\left(\Lambda^{l} g \cdot t\right) \$ \Delta: T[\Delta / g] @ l} \\
\frac{L \mid \Psi ; \Gamma \vdash_{m} t:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} g \cdot(t \$ g) \approx t:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l} \\
\hline L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} t: T^{\prime} @ l^{\prime} \quad L \vdash l: \text { Level } L \vdash l^{\prime}: \text { Level } \quad L \mid \Psi ; \Delta \vdash_{p} T @ l \\
L \mid \Psi ; \Gamma \vdash_{m} t[T / U] \approx\left(\Lambda_{p}^{l, l^{\prime}} U . t\right) \$_{p} T: T^{\prime}[T / U] @ l^{\prime} \\
\frac{L \mid \Psi ; \Gamma \vdash_{m} t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda_{p}^{l, l^{\prime}} U .\left(t \$_{p} U^{\text {id }}\right) \approx t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}}
\end{gathered}
$$

Now we consider the contextual types. They only have $\beta$ rules. Let us consider letbox first.

$$
\begin{gathered}
L \mid \Psi \vdash_{m} \Gamma \quad L \vdash l^{\prime}: \text { Level } \quad L \vdash l: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{c} T @ l \\
L\left|\Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l \vdash_{m} M @ l^{\prime} \quad L\right| \Psi, U:\left(\Delta \vdash_{c} @ l\right) ; \Gamma \vdash_{m} t^{\prime}: M\left[\operatorname{box} U / x_{T}\right] @ l^{\prime} \\
L \mid \Psi ; \Gamma \vdash_{m} t^{\prime}[T / U] \approx l_{\text {etbox }}^{l_{\mathrm{Typ}}^{\prime}} l \Delta\left(x_{T} \cdot M\right)\left(U \cdot t^{\prime}\right)(\operatorname{box} T): M\left[\operatorname{box} T / x_{T}\right] @ l^{\prime} \\
L \mid \Psi \vdash_{m} \Gamma \quad L \vdash l^{\prime}: \text { Level } \quad L \vdash l: \text { Level } L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{p} T @ l \quad L \mid \Psi ; \Delta \vdash_{c} t: T @ l \\
L\left|\Psi ; \Gamma, x_{t}: \square\left(\Delta \vdash_{c} T @ l\right) @ l \vdash_{m} M @ l^{\prime} \quad L\right| \Psi, u:\left(\Delta \vdash_{c} T @ l\right) ; \Gamma \vdash_{m} t^{\prime}: M\left[\operatorname{box} u / x_{t}\right] @ l^{\prime} \\
\hline L \mid \Psi ; \Gamma \vdash_{m} t^{\prime}[t / u] \approx l e t b o x_{T r m}^{l^{\prime}} l \Delta T\left(x_{t} \cdot M\right)\left(u \cdot t^{\prime}\right)(\operatorname{box} t): M\left[\operatorname{box} t / x_{t}\right] @ l^{\prime}
\end{gathered}
$$

We can also give the $\beta$ rules for the recursive principles for code. There are too many to list them all, and moreover they follow the same pattern, so we just list a selected few of them. We begin with something easy:

$$
\frac{G_{A} \quad L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta}{L \mid \Psi ; \Gamma \vdash_{m} t_{\text {Nat }}[\Delta / g] \approx \operatorname{elim}_{\text {Typ }}^{l_{1}, l_{2}} \vec{M} \vec{b} \text { zero } \Delta(\text { box Nat }): M\left[\text { zero } / \ell, \Delta / g, \text { box Nat } / x_{T}\right] @ l_{1}}
$$

In this case, we provide Nat to the recursive principle for code of types. It hits the base case described by $t_{\mathrm{Nat}}$, and thus the whole program is reduced to $t_{\mathrm{Nat}}$ with $g$ for $\Delta$. Note that the universe level for Nat must be zero as specified by the typing judgment at layer $c$. The recursive principle for code of terms behaves very similarly when encountering the code of Nat. Instead, it picks the right branch $t_{\text {Nat }}^{\prime}$ and returns the right motive instead:

$$
\frac{G_{A} \quad L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \quad l=\text { succ zero }}{L \mid \Psi ; \Gamma \vdash_{m} t_{\text {Nat }}^{\prime}[\Delta / g] \approx \operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta \mathrm{Ty}_{\text {zero }}(\text { box Nat }): M^{\prime}\left[l / \ell, \Delta / g, \mathrm{Ty}_{\text {zero }} / U_{T}, \text { box Nat } / x_{t}\right] @ l_{2}}
$$

In order to have the code to be box Nat as a term, this code must have type $\mathrm{Ty}_{\text {zero }}$, which lives at universe level succ zero. Hence the indices are forced by the typing rules at layer $c$.

Then we specify the variable case:

$$
\frac{G_{A} \quad L \mid \Psi \vdash_{m} \Gamma \quad L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{p} T @ l^{\prime} \quad x: T @ l^{\prime} \in \Delta}{L \mid \Psi ; \Gamma \vdash_{m} t_{x}\left[l^{\prime} / \ell, \Delta / g, T / U_{T}, x / u_{x}\right] \approx \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T(\operatorname{box} x): M^{\prime}\left[l^{\prime} / \ell, \Delta / g, T / U_{T}, \operatorname{box} x / x_{t}\right] @ l_{2}}
$$

The subtlety here is that $u_{x}$ can only receive a variable as it is typed at layer $v$, but it is fine as $x$ is precisely just a variable.

Then let us consider a more complex case of $\Pi$ types.

$$
\begin{aligned}
& G_{A} \quad L \mid \Psi \vdash_{m} \Gamma \\
& L \vdash l \text { : Level } \quad L \vdash l^{\prime} \text { : Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{c} S @ l \quad L \mid \Psi ; \Delta, x: S @ l \vdash_{c} T @ l^{\prime} \\
& t=\operatorname{box} \Pi^{l, l^{\prime}}(x: S) \cdot T \quad s_{S}=\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta(\operatorname{box} S) \quad s_{T}=\operatorname{elim}_{\mathrm{Typ}}^{l_{1} l_{2}} \vec{M} \vec{b} l^{\prime}(\Delta, x: S @ l)(\operatorname{box} T)
\end{aligned}
$$


Notice how $s_{S}$ and $s_{T}$ recurse down the sub-structures, i.e. $S$ and $T$ with the proper universe levels and local contexts. We end our discussion by given the $\beta$ rules for code of function abstractions and applications, as they appear to be rather complex, but their essence is fundamentally simple.

$$
\begin{gathered}
G_{A} L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \quad L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level } \quad L \mid \Psi ; \Delta \vdash_{c} S @ l \\
L \mid \Psi ; \Delta, x: S @ l \vdash_{c} t: T @ l^{\prime} \quad l_{\Pi}=l \sqcup l^{\prime} \quad T_{\Pi}=\Pi^{l, l^{\prime}}(x: S) \cdot T \quad t^{\prime}=\operatorname{box} \lambda^{l^{\prime}}(x: S) \cdot t \\
s_{S}=\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta(\operatorname{box} S) \quad s_{t}=\operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{l^{\prime}}(\Delta, x: S @ l) T(\operatorname{box} t) \quad \delta=s_{S} / x_{S}, s_{t} / x_{t} \\
\hline L \mid \Psi ; \Gamma \vdash_{m} t_{\lambda}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}, \delta\right] \approx \operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l_{\Pi} \Delta T_{\Pi} t^{\prime}: M^{\prime}\left[l_{\Pi} / \ell, \Delta / g, T_{\Pi} / U_{T}, t^{\prime} / x_{t}\right] @ l_{2}
\end{gathered}
$$

Similarly, the recursive principle for code of terms picks the right branch $\left(t_{\lambda}\right)$ with variables properly substituted. Since $S$ is also a sub-structure, the recursive call $s_{S}$ invokes the recursive principle for code of types instead, hence making the recursive principles mutually defined.

Last, we give the case for function applications.

$$
\begin{aligned}
& G_{A} \quad L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \quad L \vdash l \text { : Level } \\
& L \vdash l^{\prime} \text { : Level } \quad L\left|\Psi ; \Delta \vdash_{c} S @ l \quad L\right| \Psi ; \Delta, x: S \text { @ } l \vdash_{c} T @ l^{\prime} \quad L \mid \Psi ; \Delta \vdash_{c} t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
& L \mid \Psi ; \Delta \vdash_{c} s: S @ l \quad T_{\text {app }}=T[s / x] \quad t^{\prime}=\operatorname{box}\left(\left(t: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) s\right) \quad s_{S}=\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta \text { (box } S \text { ) } \\
& s_{T}=\operatorname{elim}_{\operatorname{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime}(\Delta, x: S @ l)(\operatorname{box} T) \quad s_{t}=\operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b}\left(l \sqcup l^{\prime}\right) \Delta\left(\Pi^{l, l^{\prime}}(x: S) \cdot T\right)(\operatorname{box} t) \\
& s_{s}=\operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta S(\text { box } s) \quad \sigma=\Delta / g, S / U_{S}, T / U_{T}, t / u_{t}, s / u_{s} \quad \delta=s_{S} / x_{S}, s_{T} / x_{T}, s_{t} / x_{t}, s_{s} / x_{s} \\
& L \mid \Psi ; \Gamma \vdash_{m} t_{\mathrm{app}}\left[l / \ell, l^{\prime} / \ell^{\prime}, \sigma, \delta\right] \approx \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T_{\mathrm{app}} t^{\prime}: M^{\prime}\left[l^{\prime} / \ell, \Delta / g, T_{\mathrm{app}} / U_{T}, t^{\prime} / x_{t}\right] @ l_{2}
\end{aligned}
$$

Similar to above, we can do recursion on all sub-structures, including $S$ and $T$, which are handled by the recursive principles for code of types. It is not only convenient to put $S$ and $T$ in the syntax of a function application, but also necessary. If we look at $s_{t}$ and $s_{s}$, the recursive calls on the function and the argument, we see that we must supply their types, i.e. $\Pi^{l, l^{\prime}}(x: S) . T$ and $S$, respectively. This information, unfortunately, cannot be recovered, if we employed the more common syntax of $t s$. In practice, the $\Pi$ type can be filled in by a type inference algorithm when we do not care, so it does not truly make the type theory more difficult, but rather enables the recursion on code of function applications.

At this point, we conclude all rules for DeLaM. Next, we shall carefully define all syntactic operations and examine the syntactic properties of DELAM. Then we work out the semantics by following Sec. 2 and Abel et al. [2017], from which we conclude the convertibility problem of DeLaM is decidable.

### 4.9 A Note on Layer $v$ Rules

To summarize, only the following rules can be indexed by layer $v$ :

- the typing rule for local variables and its congruence;
- the local substitution rules and their equivalence rules;
- the typing rule for global variables and its congruence;
- all conversion rules for terms and their equivalence.

In particular, we are not even obliged to include symmetry and transitivity, because they can be derived from existing rules.

## 5 SYNTACTIC OPERATIONS AND PROPERTIES OF DELAM

In the previous section, we have introduced all judgments of DELAM, but we have left out some details. For one, we have not defined the substitution operations yet, though they are very intuitive. For the sake of completeness, we will give their definitions. Then we examine the syntactic properties of DELAM before entering the semantic zone.

### 5.1 Substitution Operations

In Sec. 4.3, we have given the definition of substitutions for universe levels and how to apply one to a universe level. Applying a substitution for universe levels to types and terms simply propagate the substitution downwards.

$$
\begin{aligned}
\operatorname{Nat}[\phi] & :=\mathrm{Nat} \\
\Pi^{l, l^{\prime}}(x: S) \cdot T[\phi] & :=\Pi^{[\phi \phi], l^{\prime}[\phi]}(x: S[\phi]) \cdot(T[\phi])
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{Ty}_{l}[\phi]:=\mathrm{Ty}_{l[\phi]} \\
& \vec{\ell} \Rightarrow^{l} T[\phi]:=\vec{\ell} \Rightarrow{ }^{l[\phi, \overrightarrow{,} / \vec{\ell}]}(T[\phi, \vec{\ell} / \vec{\ell}]) \\
& \mathrm{El}^{l} t[\phi]:=\mathrm{El}^{l[\phi]}(t[\phi]) \\
& U^{\delta}[\phi]:=U^{\delta[\phi]} \\
& (\mathrm{g}: \mathrm{Ctx}) \Rightarrow^{l} T[\phi]:=(\mathrm{g}: \mathrm{Ctx}) \Rightarrow^{l[\phi]}(T[\phi]) \\
& \left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{\prime} T[\phi]:=\left(U:\left(\Gamma[\phi] \vdash_{p} @ l[\phi]\right)\right) \Rightarrow^{l^{\prime}[\phi]}(T[\phi]) \\
& \square\left(\Gamma \vdash_{c} @ l\right)[\phi]:=\square\left(\Gamma[\phi] \vdash_{c} @ l[\phi]\right) \\
& \square\left(\Gamma \vdash_{c} T @ l\right)[\phi]:=\square\left(\Gamma[\phi] \vdash_{c} T[\phi] @ l[\phi]\right) \\
& \text { • }[\phi]:=\text {. } \\
& g[\phi]:=g \\
& \Gamma, x: T @ l[\phi]:=\Gamma[\phi], x: T[\phi] \text { @ } l[\phi] \\
& { }_{g_{k}^{?}}^{k}[\phi]:={ }_{g}^{k} \\
& \mathrm{wk}_{g}^{k}[\phi]:=\mathrm{wk}_{g}^{k} \\
& \delta, t / x[\phi]:=\delta[\phi], t[\phi] / x \\
& x[\phi]:=x \\
& \mathrm{Nat}[\phi]:=\mathrm{Nat} \\
& \Pi^{l, l^{\prime}}(x: s) . t[\phi]:=\Pi^{l[\phi], l^{\prime}}[\phi](x: s[\phi]) .(t[\phi]) \\
& \mathrm{Ty}_{\mathrm{l}}[\phi]:=\mathrm{Ty}_{l[\phi]} \\
& \text { zero }[\phi]:=\text { zero } \\
& \operatorname{succ} t[\phi]:=\operatorname{succ}(t[\phi]) \\
& \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t[\phi]:=\operatorname{elim}_{\text {Nat }}^{l[\phi]}(x . M[\phi])(s[\phi])\left(x, y . s^{\prime}[\phi]\right)(t[\phi]) \\
& \lambda^{l, l^{\prime}}(x: S) . t[\phi]:=\lambda^{\left[\phi \phi,, l^{\prime} \mid \phi\right]}(x: S[\phi]) .(t[\phi]) \\
& \left(t: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) s[\phi]:=\left(t[\phi]: \Pi^{l[\phi], l^{\prime}[\phi]}(x: S[\phi]) \cdot T[\phi]\right)(s[\phi]) \\
& \Lambda^{l} \vec{l} . t[\phi]:=\Lambda^{l[\phi, \vec{l} / \vec{\ell}]} \vec{l} \cdot t[\phi, \vec{\ell} / \vec{\ell}] \\
& t \$ \vec{l}[\phi]:=(t[\phi]) \$(\vec{l}[\phi]) \\
& u^{\delta}[\phi]:=u^{\delta[\phi]} \\
& \Lambda^{l} g . t[\phi]:=\Lambda^{l[\phi]} g .(t[\phi]) \\
& t \$ \Gamma[\phi]:=t[\phi] \$(\Gamma[\phi]) \\
& \Lambda_{p}^{l, l^{\prime}} U . t[\phi]:=\Lambda_{p}^{l\left[\phi, l^{\prime}[\phi]\right.} U .(t[\phi]) \\
& t \$_{p} T[\phi]:=t[\phi] \$_{p}(T[\phi]) \\
& \text { box } T[\phi]:=\operatorname{box}(T[\phi]) \\
& \text { box } t[\phi]:=\operatorname{box}(t[\phi]) \\
& \text { letbox }{ }_{\text {Typ }}^{\prime \prime} l \Gamma\left(x_{T} \cdot M\right)\left(U . t^{\prime}\right) t[\phi]:=\operatorname{letbox}_{\text {Typ }}^{\left.l^{\prime \prime} \mid \phi\right]}(l[\phi])(\Gamma[\phi])\left(x_{T} \cdot M[\phi]\right)\left(U . t^{\prime}[\phi]\right)(t[\phi]) \\
& \text { letbox }{ }_{\text {Trm }}^{l^{\prime}} l \Gamma T\left(x_{t} \cdot M\right)\left(u . t^{\prime}\right) t[\phi]:=\operatorname{letbox}_{T \mathrm{Tr}}^{\prime^{\prime}[\phi]}(l[\phi])(\Gamma[\phi])(T[\phi])\left(x_{t} \cdot M[\phi]\right)\left(u . t^{\prime}[\phi]\right)(t[\phi]) \\
& \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma t[\phi]:=\operatorname{elim}_{\mathrm{Typ}}^{l_{1}[\phi], l_{2}[\phi]}(\vec{M}[\phi])(\vec{b}[\phi])(l[\phi])(\Gamma[\phi])(t[\phi]) \\
& \operatorname{elim}_{\operatorname{Tr} m}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma T t[\phi]:=\operatorname{elim}_{\operatorname{Trm}}^{l_{1}[\mid], l_{2}[\phi]}(\vec{M}[\phi])(\vec{b}[\phi])(l[\phi])(\Gamma[\phi])(T[\phi])(t[\phi]) \\
& \vec{M}[\phi]:=\left(\ell, g, x_{T} \cdot M[\phi, \ell / \ell]\right)\left(\ell, g, U_{T}, x_{t} \cdot M^{\prime}[\phi, \ell / \ell]\right) \\
& \left(g . t_{\mathrm{Nat}}\right)[\phi]:=\left(\mathrm{g} . \mathrm{t}_{\mathrm{Nat}}[\phi]\right) \\
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, x_{S}, x_{T} . t_{\Pi}\right)[\phi]:=\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, x_{S}, x_{T} . t_{\Pi}\left[\phi, \ell / \ell, \ell^{\prime} / \ell^{\prime}\right]\right) \\
& \left(\ell, g . t_{\mathrm{T}}\right)[\phi]:=\left(\ell, \text { g.t } \mathrm{t}_{\mathrm{y}}[\phi, \ell / \ell]\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{El}}\right)[\phi]:=\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{E}}[\phi, \ell / \ell]\right) \\
& \left(\ell, g, U_{T}, u_{x} \cdot t_{x}\right)[\phi]:=\left(\ell, g, U_{T}, u_{x} \cdot t_{x}[\phi, \ell / \ell]\right) \\
& \left(g . t_{\text {Nat }}^{\prime}\right)[\phi]:=\left(g \cdot t_{\text {Nat }}^{\prime}[\phi]\right) \\
& \left(\ell, \ell^{\prime}, g, u_{s}, u_{t}, x_{s}, x_{t} \cdot t_{\Pi}^{\prime}\right)[\phi]:=\left(\ell, \ell^{\prime}, g, u_{s}, u_{t}, x_{s}, x_{t} \cdot t_{\Pi}^{\prime}\left[\phi, \ell / \ell, \ell^{\prime} / \ell^{\prime}\right]\right) \\
& \left(\ell, g . t_{\text {Ty }}^{\prime}\right)[\phi]:=\left(\ell, g \cdot t_{\text {Ty }}^{\prime}[\phi, \ell / \ell]\right) \\
& \left(\text { g. } t_{\text {zero }}\right)[\phi]:=\left(\text { g. } t_{\text {zero }}[\phi]\right) \\
& \left(g, u_{t}, x_{t} . t_{\text {succ }}\right)[\phi]:=\left(g, u_{t}, x_{t} . t_{\text {succ }}[\phi]\right) \\
& \left(\ell, g, U_{M}, u_{s}, u_{s^{\prime}}, u_{t}, x_{M}, x_{s}, x_{s^{\prime}}, x_{t} \cdot t_{\mathrm{elim}_{\text {Nat }}}\right)[\phi]:=\left(\ell, g, U_{M}, u_{s}, u_{s^{\prime}}, u_{t}, x_{M}, x_{s}, x_{s^{\prime}}, x_{t} \cdot t_{\mathrm{elim}}^{\text {Nat }} \text { }[\phi, \ell / \ell]\right) \\
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, x_{S}, x_{t} . t_{\lambda}\right)[\phi]:=\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, x_{S}, x_{t} . t_{\lambda}\left[\phi, \ell / \ell, \ell^{\prime} / \ell^{\prime}\right]\right) \\
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{S}, x_{S}, x_{T}, x_{t}, x_{s} . t_{\text {app }}\right)[\phi]:=\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{s}, x_{S}, x_{T}, x_{t}, x_{s} \cdot t_{\text {app }}\left[\phi, \ell / \ell, \ell^{\prime} / \ell^{\prime}\right]\right)
\end{aligned}
$$

The composition operation and the identity substitution are defined intuitively as:

$$
\begin{aligned}
\cdot \circ \phi & :=\cdot \\
\left(\phi^{\prime}, l / \ell\right) \circ \phi & :=\left(\phi^{\prime} \circ \phi\right), l[\phi] / \ell \\
\mathrm{id.} & := \\
\mathrm{id}_{L, \ell} & :=\mathrm{id}_{L}, \ell / \ell
\end{aligned}
$$

The presentation of the identity substitution is simpler as we do not consider weakenings for universe contexts. We sometimes omit the subscript when it can be inferred from the textual context. We will also need to apply a universe substitution to global context, which does not need to be mutually defined:

$$
\begin{aligned}
\cdot[\phi] & :=\cdot \\
(\Psi, g: \mathrm{Ctx})[\phi] & :=\Psi[\phi], g: \mathrm{Ctx} \\
\left(\Psi, U:\left(\Gamma \vdash_{i} @ l\right)\right)[\phi] & :=\Psi[\phi], U:\left(\Gamma[\phi] \vdash_{i} @ l[\phi]\right) \\
\left(\Psi, u:\left(\Gamma \vdash_{i} T @ l\right)\right)[\phi] & :=\Psi[\phi], u:\left(\Gamma[\phi] \vdash_{i} T[\phi] @ l[\phi]\right)
\end{aligned}
$$

Then we give the the application of a local substitution. Following Sec. 2, we need two auxiliary definitions to query a local substitution in order to define its composition. We repeat their definitions as follows:

$$
\begin{aligned}
\widehat{\widehat{k}_{g}} & :=k \\
\widehat{\mathrm{wk}_{g}^{k}} & :=k \\
\widehat{\delta, t / x} & :=\widehat{\delta} \\
\breve{\breve{k}^{\prime}} & :=g ? \\
\widehat{\mathrm{wk}_{g}^{k}} & :=g \\
\widehat{\delta, t / x} & :=\check{\delta}
\end{aligned}
$$

Then we give the application of local substitutions:

$$
\begin{aligned}
\operatorname{Nat}[\delta] & :=\mathrm{Nat} \\
\Pi^{l, l^{\prime}}(x: S) \cdot T[\delta] & :=\Pi^{l, l^{\prime}}(x: S[\delta]) \cdot(T[\delta, x / x]) \\
\mathrm{Ty}_{l}[\delta] & :=\mathrm{Ty}_{l} \\
\vec{\ell} \Rightarrow^{l} T[\delta] & :=\vec{\ell} \Rightarrow^{l}(T[\delta])
\end{aligned}
$$

- Jason Z. S. Hu and Brigitte Pientka

$$
\begin{aligned}
& \mathrm{El}^{l} t[\delta]:=\mathrm{El}^{l}(t[\delta]) \\
& U^{\delta^{\prime}}[\delta]:=U^{\delta^{\prime} \circ \delta} \\
& (g: \mathrm{Ctx}) \Rightarrow^{l} T[\delta]:=(g: \mathrm{Ctx}) \Rightarrow^{l}(T[\delta]) \\
& \left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T[\delta]:=\left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}}(T[\delta]) \\
& \square\left(\Gamma \vdash_{c} @ l\right)[\delta]:=\square\left(\Gamma \vdash_{c} @ l\right) \\
& \square\left(\Gamma \vdash_{c} T @ l\right)[\delta]:=\square\left(\Gamma \vdash_{c} T @ l\right) \\
& x[\delta]:=\delta(x) \quad \text { (lookup of } x \text { in } \delta \text { ) } \\
& \text { Nat }[\delta]:=\text { Nat } \\
& \Pi^{l, l^{\prime}}(x: s) . t[\delta]:=\Pi^{l, l^{\prime}}(x: s[\delta]) .(t[\delta, x / x]) \\
& \mathrm{Ty}_{l}[\delta]:=\mathrm{Ty}_{l} \\
& \text { zero }[\delta]:=\text { zero } \\
& \operatorname{succ} t[\delta]:=\operatorname{succ}(t[\delta]) \\
& \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right) t[\delta]:=\operatorname{elim}_{\text {Nat }}^{l}(x . M[\delta, x / x])(s[\delta])\left(x, y . s^{\prime}[\delta, x / x, y / y]\right)(t[\delta]) \\
& \lambda^{l, l^{\prime}}(x: S) . t[\delta]:=\lambda^{l, l^{\prime}}(x: S[\delta]) .(t[\delta, x / x]) \\
& \left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) s[\delta]:=\left(t[\delta]: \Pi^{l, l^{\prime}}(x: S[\delta]) \cdot T[\delta, x / x]\right)(s[\delta]) \\
& \Lambda^{l} \vec{\ell} . t[\delta]:=\Lambda^{l} \vec{\ell} . t[\delta] \\
& t \$ \vec{l}[\delta]:=t[\delta] \$ \vec{l} \\
& u^{\delta^{\prime}}[\delta]:=u^{\delta^{\prime} \circ \delta} \\
& \Lambda^{l} g . t[\delta]:=\Lambda^{l} g .(t[\delta]) \\
& t \$ \Gamma[\delta]:=t[\delta] \$ \Gamma \\
& \Lambda_{p}^{l, l^{\prime}} U . t[\delta]:=\Lambda_{p}^{l, l^{\prime}} U .(t[\delta]) \\
& t \$_{p} T[\delta]:=t[\delta] \$_{p} T \\
& \text { box } T[\delta]:=\text { box } T \\
& \text { box } t[\delta]:=\text { box } t \\
& \text { letbox }{\underset{\text { Typ }}{\prime}}_{l^{\prime}} l \Gamma\left(x_{T} . M\right)\left(U . t^{\prime}\right) t[\delta]:=\text { letbox }{ }_{\text {Typ }}^{l^{\prime}} l \Gamma\left(x_{T} . M\left[\delta, x_{T} / x_{T}\right]\right)\left(U . t^{\prime}[\delta]\right)(t[\delta]) \\
& \text { letbox } \mathrm{T}_{\mathrm{rm}}^{l^{\prime}} l \Gamma T\left(x_{t} \cdot M\right)\left(u . t^{\prime}\right) t[\delta]:=\text { letbox }_{\text {Trm }}^{l^{\prime}} l \Gamma T\left(x_{t} \cdot M\left[\delta, x_{t} / x_{t}\right]\right)\left(u . t^{\prime}[\delta]\right)(t[\delta]) \\
& \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma t[\delta]:=\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}}(\vec{M}[\delta])(\vec{b}[\delta]) l \Gamma(t[\delta]) \\
& \operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma T t[\delta]:=\operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}}(\vec{M}[\delta])(\vec{b}[\delta]) l \Gamma T(t[\delta]) \\
& \vec{M}[\delta]:=\left(\ell, g, x_{T} \cdot M\left[\delta, x_{T} / x_{T}\right]\right)\left(\ell, g, U_{T}, x_{t} \cdot M^{\prime}\left[\delta, x_{t} / x_{t}\right]\right) \\
& \left(g . t_{\mathrm{Nat}}\right)[\delta]:=\left(g . \mathrm{t}_{\mathrm{Nat}}[\delta]\right) \\
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, x_{S}, x_{T} . t_{\Pi}\right)[\delta]:=\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, x_{S}, x_{T} . t_{\Pi}\left[\delta, x_{S} / x_{S}, x_{T} / x_{T}\right]\right) \\
& \left(\ell, g . t_{\mathrm{Ty}}\right)[\delta]:=\left(\ell, g . t_{\mathrm{Ty}}[\delta]\right) \\
& \left(\ell, g, u_{t}, x_{t} . t_{\mathrm{EI}}\right)[\delta]:=\left(\ell, g, u_{t}, x_{t} . t_{\mathrm{EI}}\left[\delta, x_{t} / x_{t}\right]\right) \\
& \left(\ell, g, U_{T}, u_{x} \cdot t_{x}\right)[\delta]:=\left(\ell, g, U_{T}, u_{x} \cdot t_{x}[\delta]\right) \\
& \left(g . t_{\text {Nat }}^{\prime}\right)[\delta]:=\left(g . t_{\text {Nat }}^{\prime}[\delta]\right) \\
& \left(\ell, \ell^{\prime}, g, u_{s}, u_{t}, x_{s}, x_{t} . t_{\Pi}^{\prime}\right)[\delta]:=\left(\ell, \ell^{\prime}, g, u_{s}, u_{t}, x_{s}, x_{t} . t_{\Pi}^{\prime}\left[\delta, x_{s} / x_{s}, x_{t} / x_{t}\right]\right) \\
& \left(\ell, g \cdot t_{\text {Ty }}^{\prime}\right)[\delta]:=\left(\ell, g \cdot t_{\text {Ty }}^{\prime}[\delta]\right) \\
& \left(g . t_{\text {zero }}\right)[\delta]:=\left(g . t_{\text {zero }}[\delta]\right) \\
& \left(g, u_{t}, x_{t} \cdot t_{\text {succ }}\right)[\delta]:=\left(g, u_{t}, x_{t} \cdot t_{\text {succ }}\left[\delta, x_{t} / x_{t}\right]\right) \\
& \left(\ell, g, U_{M}, u_{s}, u_{s^{\prime}}, u_{t}, x_{M}, x_{s}, x_{s^{\prime}}, x_{t} . t_{\text {elim }}^{\text {Nat }} \text { }\right)[\delta]:= \\
& \left(\ell, g, U_{M}, u_{s}, u_{s^{\prime}}, u_{t}, x_{M}, x_{s}, x_{s^{\prime}}, x_{t} . t_{\text {elim }}^{\text {Nat }} \text { }\left[\delta, x_{M} / x_{M}, x_{s} / x_{s}, x_{s^{\prime}} / x_{s^{\prime}}, x_{t} / x_{t}\right]\right) \\
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, x_{S}, x_{t} . t_{\lambda}\right)[\delta]:=\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, x_{S}, x_{t} . t_{\lambda}\left[\delta, x_{S} / x_{S}, x_{t} / x_{t}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{s}, x_{S}, x_{T}, x_{t}, x_{s} \cdot t_{\mathrm{app}}\right)[\delta]:= \\
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{s}, x_{S}, x_{T}, x_{t}, x_{s} \cdot t_{\mathrm{app}}\left[\delta, x_{S} / x_{S}, x_{T} / x_{T}, x_{t} / x_{t}, x_{s} / x_{s}\right]\right)
\end{aligned}
$$

where composition is defined in the same way as Sec. 2:

$$
\begin{aligned}
\mathrm{wk}_{g}^{k} \circ \delta & :=\mathrm{wk}_{g}^{\widehat{\delta}} \\
\cdot k \circ \delta & :=\cdot \widehat{\delta} \\
{ }_{g}^{k} \circ \delta & :=\widehat{\delta}_{g}^{\widehat{\delta}} \\
\left(\delta^{\prime}, t / x\right) \circ \delta & :=\left(\delta^{\prime} \circ \delta\right), t[\delta] / x
\end{aligned}
$$

The identity local substitution is characterized as a generalization of local weakening wk.

$$
\begin{aligned}
\mathrm{wk}^{k} & :=\cdot k \\
\mathrm{wk}_{g}^{k} & :=\mathrm{wk}_{g}^{k} \\
\mathrm{wk}_{\Gamma, x: T}^{k} @ l & :=\mathrm{wk}_{\Gamma}^{1+k}, x / x
\end{aligned}
$$

Identity is just $\mathrm{id}_{\Gamma}:=\mathrm{wk}_{\Gamma}^{0}$.
Then we give the global substitutions.

$$
\sigma:=\cdot|\sigma, \Gamma / g| \sigma, T / U \mid \sigma, t / u
$$

(Global substitutions)
Then we define the typing rules as:

$$
\begin{aligned}
& \begin{array}{ccc}
L \vdash \Psi \\
L \mid \Psi \vdash \cdot: & \frac{L|\Psi \vdash \sigma: \Phi \quad L| \Psi \vdash_{p} \Gamma}{L \mid \Psi \vdash \sigma, \Gamma / g: \Phi, g: C t x}
\end{array} \frac{L \vdash l: \text { Level } \left.\begin{array}{c}
L \mid \Psi \vdash \sigma: \Phi \\
i \in\{c, p\}
\end{array} \quad L \right\rvert\, \Phi \vdash_{p} \Gamma}{L \mid \Psi ; \Gamma[\sigma] \vdash_{i} T @ l} \\
& \frac{L|\Psi \vdash \sigma: \Phi \quad L| \Phi ; \Gamma \vdash_{p} T @ l \quad L \vdash l: \text { Level } \quad i \in\{v, c\} \quad L \mid \Psi ; \Gamma[\sigma] \vdash_{i} t: T[\sigma] @ l}{L \mid \Psi \vdash \sigma, t / u: \Phi, u:\left(\Gamma \vdash_{i} T @ l\right)}
\end{aligned}
$$

Then we consider the cases for application:

$$
\begin{aligned}
\mathrm{Nat}[\sigma] & :=\mathrm{Nat} \\
\Pi^{l, l^{\prime}}(x: S) . T[\sigma] & :=\Pi^{l, l^{\prime}}(x: S[\sigma]) .(T[\sigma]) \\
\mathrm{Ty}_{l}[\sigma] & :=\mathrm{Ty}_{l} \\
\vec{\ell} \Rightarrow^{l} T[\sigma] & :=\vec{\ell} \Rightarrow^{l}(T[\sigma]) \\
\mathrm{E} l^{l} t[\sigma] & :=\mathrm{El}^{l}(t[\sigma]) \\
U^{\delta}[\sigma] & :=\sigma(U)[\delta[\sigma]] \\
(g: \mathrm{Ctx}) \Rightarrow^{l} T[\sigma] & :=(g: \mathrm{Ctx}) l^{l}(T[\sigma, g / g]) \\
\left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T[\sigma] & :=\left(U:\left(\Gamma[\sigma] \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}}\left(T\left[\sigma, U^{\mathrm{id}} / U\right]\right) \\
\square\left(\Gamma \vdash_{c} @ l\right)[\sigma] & :=\square\left(\Gamma[\sigma] \vdash_{c} @ l\right) \\
\square\left(\Gamma \vdash_{c} T @ l\right)[\sigma] & :=\square\left(\Gamma[\sigma] \vdash_{c} T[\sigma] @ l\right) \\
\cdot[\sigma] & := \\
g[\sigma] & :=\sigma(g) \\
\Gamma, x: T @ l[\sigma] & :=\Gamma[\sigma], x: T[\sigma] @ l \\
\mathrm{wk}{ }_{g}^{k}[\sigma] & :=\mathrm{wk} \mathrm{k}_{\sigma(g)}^{k}
\end{aligned}
$$

> (a local weakening extending the local context by length $m$ ) ${ }^{k}[\sigma]:={ }^{k}$

$$
\begin{aligned}
& \left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{S}, x_{S}, x_{T}, x_{t}, x_{s} . t_{\mathrm{app}}\right)[\sigma]:= \\
& \quad\left(\ell, \ell^{\prime}, g, U_{S}, U_{T}, u_{t}, u_{S}, x_{S}, x_{T}, x_{t}, x_{S} \cdot t_{\mathrm{app}}\left[\sigma, g / g, U_{S}^{\mathrm{id}} / U_{S}, U_{T}^{\mathrm{id}} / U_{T}, u_{t}^{\mathrm{id}} / u_{t}, u_{S}^{\mathrm{id}} / u_{S}\right]\right)
\end{aligned}
$$

Following Sec. 2, we give the identity global substitution as a special case of global weakenings, and composition.

$$
\begin{aligned}
& \mathrm{wk}_{.}^{k}:= \\
& \mathrm{wk}_{\Psi, g: \mathrm{Ctx}}^{k}:=\mathrm{wk}_{\Psi}^{1+k}, g / g \\
& \mathrm{wk}_{\Psi, U:\left(\Gamma \vdash_{i} @ l\right)}^{k}:=\mathrm{wk}_{\Psi}^{1+k}, U^{\mathrm{id}_{\Gamma}} / U \\
& \mathrm{wk}_{\Psi, u:\left(\Gamma \vdash_{i} T @ l\right)}^{k}:=\mathrm{wk}_{\Psi}^{1+k}, u^{\mathrm{id}_{\Gamma}} / u
\end{aligned}
$$

As a special case, we have

$$
\mathrm{id}_{\Psi}:=\mathrm{wk}_{\Psi}^{0}
$$

Moreover, we have composition

$$
\begin{aligned}
\cdot \circ \sigma^{\prime} & := \\
(\sigma, \Gamma / g) \circ \sigma^{\prime} & :=\left(\sigma \circ \sigma^{\prime}\right), \Gamma\left[\sigma^{\prime}\right] / g \\
(\sigma, T / U) \circ \sigma^{\prime} & :=\left(\sigma \circ \sigma^{\prime}\right), T\left[\sigma^{\prime}\right] / U \\
(\sigma, t / u) \circ \sigma^{\prime} & :=\left(\sigma \circ \sigma^{\prime}\right), t\left[\sigma^{\prime}\right] / u
\end{aligned}
$$

### 5.2 Properties of Substitutions

In the next step, we examine the algebraic properties of all substitutions. In the lemmas below, we always assume well-formedness or well-typedness of the subjects in the lemmas, unless the lemmas are about typing. For conciseness, we do not spell out the conditions as they are routine.

Lemma 5.1.

- $L \vdash i d_{L}: L$
- If $L \vdash \phi: L^{\prime}$ and $L^{\prime} \vdash \phi^{\prime}: L^{\prime \prime}$, then $L \vdash \phi^{\prime} \circ \phi: L^{\prime \prime}$.

Proof. Analyze the definition of identity and composition.
Lemma 5.2 (Algebra of Universe Substitutions).

- $l[\phi]\left[\phi^{\prime}\right]=l\left[\phi \circ \phi^{\prime}\right](T, \Gamma, \delta, t, \Psi$ resp. $)$
- l[id] $=l(T, \Gamma, \delta, t$, $\Psi$ resp.)
- id $\circ \phi=\phi$ and $\phi \circ i d=\phi$
- $\left(\phi_{1} \circ \phi_{2}\right) \circ \phi_{3}=\phi_{1} \circ\left(\phi_{2} \circ \phi_{3}\right)$

Proof. The proofs are routine; the first two statements are proved by induction on $l$ first, and then mutual induction on all applications of universe substitutions. The last two we analyze the definition of composition.

Similar lemmas hold for local and global substitutions.
Lemma 5.3 .

- $L \mid \Psi ; \Gamma, \Delta \vdash_{i} w k_{\Gamma}^{|\Delta|}: \Gamma$
- $L \mid \Psi ; \Gamma \vdash_{i} i d_{\Gamma}: \Gamma$

Note that local substitutions permit $i=v$, so we have $L \mid \Psi ; \Gamma \vdash_{v} \mathrm{id}_{\Gamma}: \Gamma$. This intuitively makes sense, as all terms in $\mathrm{id}_{\Gamma}$ are just local variables.

Lemma 5.4 (Algebra of Local Substitutions).

- $T[\delta]\left[\delta^{\prime}\right]=T\left[\delta \circ \delta^{\prime}\right]$ (t resp.)
- $T[i d]=T$ ( $t$ resp.)
- id $\circ \delta=\delta$ and $\delta \circ i d=\delta$
- $\left(\delta_{1} \circ \delta_{2}\right) \circ \delta_{3}=\delta_{1} \circ\left(\delta_{2} \circ \delta_{3}\right)$

Proof. The first statement is mutually proved with associativity and by mutual induction. The second statement is mutually proved with right identity and also by mutual induction. When proving right identity, we realize that all extended local substitutions under binders are identities.

Then we reason about global substitutions.
Lemma 5.5.

- $L \mid \Psi, \Phi \vdash w k_{\Psi}^{|\Phi|}: \Psi$
- $L \mid \Psi \vdash i d_{\Psi}: \Psi$

Lemma 5.6 (Algebra of Global Substitutions).

- $T[\sigma]\left[\sigma^{\prime}\right]=T\left[\sigma \circ \sigma^{\prime}\right](\Gamma, \delta, t$ resp.)
- $T[i d]=T(\Gamma, \delta, t$ resp. $)$
- id $\circ \sigma=\sigma$ and $\sigma \circ i d=\sigma$
- $\left(\sigma_{1} \circ \sigma_{2}\right) \circ \sigma_{3}=\sigma_{1} \circ\left(\sigma_{2} \circ \sigma_{3}\right)$

Proof. The first two statements require mutual inductions on the applications of global substitutions. Right identity is a natural consequence of the second statement. Left identity is proved by simply looking at the definition of the identity global substitution. Associativity is routine.

Finally, we conclude how all these kinds of substitutions interact.
Lemma 5.7 (Acting on Weakenings).

- $w k_{\Gamma}^{k}[\phi]=w k_{\Gamma[\phi]}^{k}$
- $w k_{\Psi}^{k}[\phi]=w k_{\Psi[\phi]}^{k}$
- $w k_{\Gamma}^{k}[\sigma]=w k_{\Gamma[\sigma]}^{k}$

Proof. The first two statements are pretty straightforward as the lengths of the contexts are not altered. The last one requires a bit more thought. We proceed by induction on $\Gamma$.
Case

$$
\begin{aligned}
\mathrm{wk}^{k}[\sigma] & =\cdot^{k}[\sigma] \\
& =\cdot^{k}
\end{aligned}
$$

Case

$$
\mathrm{wk}_{g}^{k}[\sigma]=\mathrm{wk}_{\sigma(g)}^{k}
$$

which already matches the definition of $w k_{g[\sigma]}^{k}$.
Case

$$
\begin{align*}
\mathrm{wk}_{\Gamma, x: T ~ @ ~}^{k}[\sigma] & =\left(\mathrm{wk}_{\Gamma}^{1+m}, x / x\right)[\sigma] \\
& =\mathrm{wk}_{\Gamma}^{1+m}[\sigma], x / x \\
& =\mathrm{wk}_{\Gamma[\sigma]}^{1+m}, x / x \tag{byIH}
\end{align*}
$$

$$
=\mathrm{wk}{ }_{\Gamma[\sigma], x: T[\sigma] @ l}^{k}
$$

Corollary 5.8 (Acting on Identities).

- $i d_{\Gamma}[\phi]=i d_{\Gamma[\phi]}$
- $i d_{\Psi}[\phi]=i d_{\Psi[\phi]}$
- $i d_{\Gamma}[\sigma]=i d_{\Gamma[\sigma]}$

Lemma 5.9 (Interactions between Different Substitutions).

- $T[\delta][\phi]=T[\phi][\delta[\phi]]$ ( $t$, resp.)
- $\left(\delta \circ \delta^{\prime}\right)[\phi]=(\delta[\phi]) \circ\left(\delta^{\prime}[\phi]\right)$
- $T[\sigma][\phi]=T[\phi][\sigma[\phi]](\Gamma, \delta, t$, resp. $)$
- $T[\delta][\sigma]=T[\sigma][\delta[\sigma]](t, r e s p$.
- $\left(\delta \circ \delta^{\prime}\right)[\sigma]=(\delta[\sigma]) \circ\left(\delta^{\prime}[\sigma]\right)$

Proof. The first two statements are mutually proved. The last two statements are also mutually proved. Most of them can be done by simply following the IHs. We give a few examples.
Case

$$
\begin{aligned}
\left(\lambda^{l, l^{\prime}}(x: S) \cdot t\right)[\delta][\phi] & =\lambda^{l, l^{\prime}}(x: S[\delta][\phi]) \cdot(t[\delta, x / x][\phi]) \\
& =\lambda^{l, l^{\prime}}(x: S[\phi][\delta[\phi]]) \cdot(t[\phi][(\delta, x / x)[\phi]]) \\
& =\lambda^{l, l^{\prime}}(x: S[\phi][\delta[\phi]]) \cdot(t[\phi][(\delta[\phi], x / x)]) \\
& =\left(\lambda^{l, l^{\prime}}(x: S) \cdot t\right)[\phi][\delta[\phi]]
\end{aligned}
$$

Case

$$
\begin{aligned}
\left(\ell, g, x_{T} \cdot M\right)[\delta][\phi]= & \left(\ell, g, x_{T} \cdot M\left[\delta, x_{T} / x_{T}\right][\phi, \ell / \ell]\right) \\
= & \left(\ell, g, x_{T} \cdot M[\phi, \ell / \ell]\left[\left(\delta, x_{T} / x_{T}\right)[\phi, \ell / \ell]\right]\right) \\
= & \left(\ell, g, x_{T} \cdot M[\phi, \ell / \ell]\left[\left(\delta[\phi], x_{T} / x_{T}\right)\right]\right) \\
& (\delta[\phi, \ell / \ell]=\delta[\phi] \text { due to weakening of universe variables }) \\
= & \left(\ell, g, x_{T} \cdot M\right)[\phi][\delta[\phi]]
\end{aligned}
$$

Case

$$
\begin{aligned}
\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{El}}\right)[\sigma][\phi] & =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{El}}\left[\sigma, g / g, u_{t}^{\text {id }} / u_{t}\right][\phi, \ell / \ell]\right) \\
& =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{El}}[\phi, \ell / \ell]\left[\left(\sigma, g / g, u_{t}^{\text {id }} / u_{t}\right)[\phi, \ell / \ell]\right]\right) \\
& =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{El}}[\phi, \ell / \ell]\left[\left(\sigma[\phi], g / g, u_{t}^{\text {id }} / u_{t}\right)\right]\right) \quad(\sigma \text { is universe weakened; Lemma 5.7) } \\
& =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{El}}\right)[\phi][\sigma[\phi]]
\end{aligned}
$$

Case

$$
\begin{align*}
\mathrm{wk}_{g}^{k}[\sigma][\phi] & =\mathrm{wk}_{\sigma(g)}^{k}[\phi] \\
& =\mathrm{wk}_{\sigma(g)[\phi]}^{k}  \tag{byLemma5.7}\\
& =\mathrm{wk}_{g[\sigma[\phi]]}^{k} \\
& =\mathrm{wk}_{g}^{k}[\phi][\sigma[\phi]]
\end{align*}
$$

Case Then we consider ${ }_{g}^{k}$, and we case analyze $\Gamma:=\sigma(g)$ :
Subcase If $\Gamma$ ends with $\cdot$.

$$
._{g}^{k}[\sigma][\phi]=.|\Gamma|+m[\phi]=.|\Gamma|+m
$$

Subcase If $\Gamma$ ends with $g^{\prime}$.

$$
\dot{g}_{g}^{k}[\sigma][\phi]=\cdot_{g^{\prime}}^{|\Gamma|+m}[\phi]=._{g^{\prime}}^{|\Gamma|+m}
$$

Case

$$
\begin{array}{rlr}
\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{E} 1}\right)[\delta][\sigma] & =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{E} 1}\left[\delta, x_{t} / x_{t}\right]\left[\sigma, u_{t}^{\mathrm{id}} / u_{t}\right]\right) \\
& =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{E} 1}\left[\sigma, u_{t}^{\mathrm{id}} / u_{t}\right]\left[\left(\delta, x_{t} / x_{t}\right)\left[\sigma, u_{t}^{\mathrm{id}} / u_{t}\right]\right]\right)  \tag{byIH}\\
& =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{E} 1}\left[\sigma, u_{t}^{\mathrm{id}} / u_{t}\right]\left[\delta[\sigma], x_{t} / x_{t}\right]\right) & \quad(\delta \text { is globally weakened }) \\
& =\left(\ell, g, u_{t}, x_{t} \cdot t_{\mathrm{E} 1}\right)[\sigma][\delta[\sigma]] &
\end{array}
$$

Case Now we consider $\mathrm{wk}_{g}^{k} \circ \delta$. This composition basically cancels out all terms from $\delta$ and leave a weakening behind. In this case, we know that $\delta$ must end with $w k_{g}^{k^{\prime}}$ for some $g$. Moreover, in order to compose, we have that $|\delta|=m$. Therefore,

$$
\begin{aligned}
\left(\mathrm{wk}_{g}^{k} \circ \delta\right)[\sigma] & =\mathrm{wk} \mathrm{k}_{g}^{\widehat{\delta}}[\sigma] \\
& =\mathrm{w} \mathrm{k}_{\sigma(g)}^{k^{\prime}}
\end{aligned}
$$

Moreover,

$$
\left.\begin{array}{rl}
\left(\mathrm{wk}_{g}^{k}[\sigma]\right) \circ(\delta[\sigma]) & =\mathrm{wk}_{\sigma(g)}^{|\delta|} \circ(\delta[\sigma]) \\
& =\mathrm{wk}_{\sigma(g)}^{k^{\prime}} \quad\left(\mathrm{wk}_{\sigma(g)}^{|\delta|} \text { projects away all leading terms kept by } \delta \text { so only } \mathrm{wk}\right. \\
g
\end{array}[\sigma] \text { is left }\right)
$$

Therefore two expressions are equal. Similar reasoning holds for the case of ${ }_{g}^{k} \circ \delta$.
Case Then we consider ${ }^{k}$ and case analyze $\check{\delta}$.
Subcase If $\check{\delta}$ is not a contextual variable, then $\delta$ must end with $\cdot k^{\prime}$ and also $|\delta|=m$ due to well-typedness.

$$
\begin{aligned}
\left(\cdot^{k} \circ \delta\right)[\sigma] & =\cdot^{\widehat{\delta}}[\sigma] \\
& =\cdot^{\prime} \\
& =\cdot^{\prime}[\sigma] \\
& =\left(\cdot^{k}[\sigma]\right) \circ(\delta[\sigma])
\end{aligned}
$$

Subcase If $\check{\delta}$ is contextual variable $g$, then $\delta$ must end with ${ }_{g}^{k^{\prime}}$ and also $|\delta|=m$ due to well-typedness.

$$
\begin{aligned}
\left(\cdot^{k} \circ \delta\right)[\sigma] & ={ }_{g}^{\widehat{\delta}}[\sigma] \\
& ={ }_{g}^{k^{\prime}}[\sigma]
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\cdot^{k}[\sigma] \circ(\delta[\sigma]) & =\cdot^{k} \circ(\delta[\sigma]) \\
& =\cdot \frac{\widehat{\delta[\sigma]}}{\delta[\sigma]} \\
& =\frac{|\sigma(g)|+k^{\prime}}{\delta[\sigma]}
\end{aligned}
$$

Then we consider whether $\sigma(g)$ ends with another contextual variable or not.
Subsubcase If $\sigma(g)$ ends with $\cdot$, then

$$
\begin{aligned}
\left(\cdot^{k} \circ \delta\right)[\sigma] & =\cdot_{g}^{k^{\prime}}[\sigma] \\
& =.|\sigma(g)|+k^{\prime}
\end{aligned}
$$

and also $\delta[\sigma]$ must also end with no global variable.
Subsubcase If $\sigma(g)$ ends with some $g^{\prime}$, then

$$
\begin{aligned}
\left(\cdot^{k} \circ \delta\right)[\sigma] & ={ }_{g}^{k^{\prime}}[\sigma] \\
& =\dot{g}_{g^{\prime}}^{|\sigma(g)|+k^{\prime}}
\end{aligned}
$$

Then $\delta[\sigma]$ must also return $g^{\prime}$.

Lemma 5.10 (Universe Substitutions).

- If $L^{\prime} \vdash \Psi$ and $L \vdash \phi: L^{\prime}$, then $L \vdash \Psi[\phi]$.
- If $L^{\prime} \mid \Psi \vdash_{i} \Gamma$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] \vdash_{i} \Gamma[\phi]$.
- If $L^{\prime} \mid \Psi \vdash_{i} \Gamma \approx \Delta$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] \vdash_{i} \Gamma[\phi] \approx \Delta[\phi]$.
- If $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} T @ l$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} T[\phi] @ l[\phi]$.
- If $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} T[\phi] \approx T^{\prime}[\phi] @ l[\phi]$.
- If $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} t: T$ @ $l$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} t[\phi]: T[\phi] @ l[\phi]$.
- If $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ land $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} t[\phi] \approx t^{\prime}[\phi]: T[\phi] @ l[\phi]$.
- If $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} \delta[\phi]: \Gamma^{\prime}[\phi]$.
- If $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Gamma^{\prime}$ and $L \vdash \phi: L^{\prime}$, then $L \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} \delta[\phi] \approx \delta^{\prime}[\phi]: \Gamma^{\prime}[\phi]$.

Proof. Do a mutual induction. Most rules do not alter the universe context at all so they are discharged naturally. When encountering the only changing cases, i.e. universe-polymorphic functions and branches of the recursive principles for code, we extend the unvierse substitutions with sufficient new universe variables before applying the IHs. When computation rules are encountered, we apply the composition lemma above before applying the IHs.

We consider one complex computational rule in detail to illustrate the proof of the lemma:

$$
\begin{aligned}
& G_{A} \quad L \mid \Psi \vdash_{m} \Gamma \\
& L^{\prime} \vdash l \text { : Level } \quad L^{\prime} \vdash l^{\prime} \text { : Level } \quad L^{\prime}\left|\Psi \vdash_{p} \Delta \quad L^{\prime}\right| \Psi ; \Delta \vdash_{c} S @ l \quad L^{\prime} \mid \Psi ; \Delta, x: S @ l \vdash_{c} T @ l^{\prime} \\
& t=\operatorname{box} \Pi^{l, l^{\prime}}(x: S) . T \quad s_{S}=\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta(\operatorname{box} S) \quad s_{T}=\operatorname{elim}_{\mathrm{Typ}}^{l_{1} l_{2}} \vec{M} \vec{b} l^{\prime}(\Delta, x: S @ l)(\operatorname{box} T)
\end{aligned}
$$

$\overline{L^{\prime}} \mid \Psi ; \Gamma \vdash_{m} t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, s_{S} / x_{S}, s_{T} / x_{T}\right] \approx \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b}\left(l \sqcup l^{\prime}\right) \Delta t: M\left[l \sqcup l^{\prime} / \ell, \Delta / g, t / x_{T}\right] @ l_{1}$
On the right hand side, $\phi$ simply propagates. By IH, we can show that all premises with the universe-substituted sub-terms are well-formed. For example, we have

$$
L \mid \Psi[\phi] ; \Delta[\phi] \vdash_{c} S[\phi] @ l[\phi]
$$

and

$$
L \mid \Psi[\phi] ; \Delta[\phi], x: S[\phi] @ l[\phi] \vdash_{c} T[\phi] @ l^{\prime}[\phi]
$$

Now we consider the left hand side.

$$
\begin{align*}
& t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, s_{S} / x_{S}, s_{T} / x_{T}\right][\phi] \\
= & t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}\right][\phi]\left[\Delta[\phi] / g, S[\phi] / U_{S}, T[\phi] / U_{T}, s_{S}[\phi] / x_{S}, s_{T}[\phi] / x_{T}\right] \tag{byLemma5.9}
\end{align*}
$$

$$
=t_{\Pi}\left[\phi, \ell / \ell, \ell^{\prime} / \ell^{\prime}\right]\left[l[\phi] / \ell, l^{\prime}[\phi] / \ell^{\prime}\right]\left[\Delta[\phi] / g, S[\phi] / U_{S}, T[\phi] / U_{T}, s_{S}[\phi] / x_{S}, s_{T}[\phi] / x_{T}\right]
$$

(by naturality of substitutions)
Note that $t_{\Pi}\left[\phi, \ell / \ell, \ell^{\prime} / \ell^{\prime}\right]$ is precisely how $\phi$ should be propagated in $t_{\Pi}$.
Now we consider the type. A similar reasoning applies:

$$
\begin{align*}
M\left[l \sqcup l^{\prime} / \ell, \Delta / g, t / x_{T}\right][\phi] & =M\left[l \sqcup l^{\prime} / \ell\right][\phi]\left[\Delta[\phi] / g, t[\phi] / x_{T}\right]  \tag{byLemma5.9}\\
& =M[\phi, \ell / \ell]\left[l[\phi] \sqcup l^{\prime}[\phi] / \ell\right]\left[\Delta[\phi] / g, t[\phi] / x_{T}\right]
\end{align*}
$$

(by naturality of substitutions)
This equality verifies the resulting type is correct. The same rule ensures the equivalence lives in the universe level $l_{1}[\phi]$.

Then we consider the properties of local substitutions.
Lemma 5.11 (Partial Presupposition).

- If $L \mid \Psi \vdash_{i} \Gamma$, then $L \vdash \Psi$.
- If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$, then $L \mid \Psi \vdash_{i} \Gamma$ and $L \mid \Psi \vdash_{i} \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $L \mid \Psi \vdash_{i} \Gamma$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta^{\prime}: \Delta$.
- If $L \mid \Psi \vdash \sigma: \Phi$, then $L \vdash \Psi$ and $L \vdash \Phi$.

Proof. Induction on their respective premises. Note that in the second statement, the definition of $L \mid \Psi \vdash_{i}$ $\Gamma \approx \Delta$ is adjusted so that a simple induction would suffice. The third statement requires the extra premises added to the step case of the equivalence judgment.

In fact the lemma above has given full presupposition for local contexts and their equivalence. Therefore, in the forthcoming full presupposition lemma, we do not have to state these cases.

Lemma 5.12 (Symmetry and Transitivity of Local Substitutions).

- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta^{\prime} \approx \delta: \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime \prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime \prime}: \Delta$.

Proof. By induction.
Lemma 5.13 (Reflexivity).

- If $L \mid \Psi ; \Gamma \vdash_{i} T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t: T$ @ $l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta: \Delta$.
- If $L \mid \Psi \vdash_{i} \Gamma$, then $L \mid \Psi \vdash_{i} \Gamma \approx \Gamma$.

Proof. The first two statements are proved by symmetry and then transitivity. The third (fourth) statement is a natural consequence of the second (first, resp.) statement.

Lemma 5.14 (Local Substitutions).

- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} T[\delta] @ l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} T[\delta] \approx T^{\prime}[\delta]$ @ $l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} t: T$ @ $l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t[\delta]: T[\delta]$ @ $l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} t \approx t^{\prime}: T$ @ $l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t[\delta] \approx t^{\prime}[\delta]: T[\delta]$ @ $l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} \delta^{\prime}: \Gamma^{\prime \prime}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta^{\prime} \circ \delta: \Gamma^{\prime \prime}$.
$\bullet$ If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} \delta^{\prime} \approx \delta^{\prime \prime}: \Gamma^{\prime \prime}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta^{\prime} \circ \delta \approx \delta^{\prime \prime} \circ \delta: \Gamma^{\prime \prime}$.

Proof. Do a mutual induction. Many cases go through naturally if their premises do not alter the local context. In the base cases, we use partial presupposition above to obtain $L \mid \Psi \vdash_{i} \Gamma$ and $L \vdash \Psi$. The global variable cases depend on the composition of local substitutions. The local variable case depends on the reflexivity lemma.

We consider a few cases:

## Case

$$
\frac{L\left|\Psi ; \Gamma^{\prime} \vdash_{m} t \approx t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t \$_{p} T \approx t^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}[T / U] @ l^{\prime}}
$$

$$
\begin{equation*}
L \mid \Psi ; \Gamma \vdash_{m} t[\delta] \approx t^{\prime}[\delta]:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}}\left(T^{\prime \prime}[\delta]\right) @ \operatorname{succ} l \sqcup l^{\prime} \tag{byIH}
\end{equation*}
$$

$L \mid \Psi ; \Gamma \vdash_{m} t[\delta] \$_{p} T \approx t^{\prime}[\delta] \$_{p} T^{\prime}: T^{\prime \prime}[\delta][T / U] @ l^{\prime} \quad$ (by the same congruence rule)
Notice that

$$
T^{\prime \prime}[\delta][T / U]=T^{\prime \prime}[T / U][\delta[T / U]]=T^{\prime \prime}[T / U][\delta]
$$

where the second equation holds because $\delta$ is not typed in a context with $U$.
Case

$$
\begin{array}{cccc}
L \mid \Psi \vdash_{p} \Delta & L \vdash l: \text { Level } & L \vdash l^{\prime}: \text { Level } & L \mid \Psi \vdash_{m} \Gamma^{\prime} \\
l_{\Pi}=l \sqcup l^{\prime} & T_{\Pi}=\Pi^{l, l^{\prime}}(x: S) . T \quad t_{c} S @ l & L \mid \Psi ; \Delta, x: S @ l \vdash_{c} t: T @ l^{\prime} \\
& t_{t}^{\prime}=\operatorname{box} \lambda^{l, l^{\prime}}(x: S) \cdot t \quad s_{S}=\operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime}(\Delta, x: S @ l) T(\operatorname{lox} t)[\delta] \quad \delta^{\prime}=s_{S} / x_{S}, s_{t} / x_{t} \\
\hline
\end{array}
$$

$L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t_{\lambda}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}, \delta^{\prime}\right] \approx \operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l_{\Pi} \Delta T_{\Pi} t^{\prime}: M^{\prime}\left[l_{\Pi} / \ell, \Delta / g, T_{\Pi} / U_{T}, t^{\prime} / x_{t}\right] @ l_{2}$
In this case, we first apply IHs so that $\delta$ is propagated into all premises in $G_{A}$ and we must reason about the left hand side and the result type.

$$
\begin{array}{rlr} 
& t_{\lambda}\left[\delta, x_{S} / x_{S}, x_{t} / x_{t}\right]\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}, \delta^{\prime}\right] \\
= & t_{\lambda}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}\right]\left[\delta\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}\right], x_{S} / x_{S}, x_{t} / x_{t}\right]\left[\delta^{\prime}\right] & \text { (by Lemma 5.9) } \\
= & t_{\lambda}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}\right]\left[\delta, x_{S} / x_{S}, x_{t} / x_{t}\right]\left[\delta^{\prime}\right] & \text { ( } \delta \text { has no those variables) } \\
= & t_{\lambda}\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta / g, S / U_{S}, T / U_{T}, t / u_{t}\right]\left[s_{S} / x_{S}, s_{t} / x_{t}\right][\delta] & \text { (naturality of local substitutions) }
\end{array}
$$

On the return type, we have

$$
M^{\prime}\left[\delta, x_{t} / x_{t}\right]\left[l_{\Pi} / \ell, \Delta / g, T_{\Pi} / U_{T}, t^{\prime} / x_{t}\right]
$$

$$
=M^{\prime}\left[l_{\Pi} / \ell, \Delta / g, T_{\Pi} / U_{T}\right]\left[\delta, x_{t} / x_{t}\right]\left[t^{\prime} / x_{t}\right] \quad \text { (by Lemma } 5.9 \text { similarly) }
$$

$$
=M^{\prime}\left[l_{\Pi} / \ell, \Delta / g, T_{\Pi} / U_{T}\right]\left[t^{\prime} / x_{t}\right][\delta] \quad \text { (naturality of local substitutions) }
$$

Both equations conclude this case.

We also need a similar lemma about equivalent local substitutions.

## Lemma 5.15 (Equivalent Local Substitutions).

- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} T$ @ $l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \approx \delta_{2}: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} T\left[\delta_{1}\right] \approx T\left[\delta_{2}\right] @ l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \approx \delta_{2}: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} T\left[\delta_{1}\right] \approx T^{\prime}\left[\delta_{2}\right]$ @ $l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} t: T$ @ $l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \approx \delta_{2}: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t\left[\delta_{1}\right] \approx t\left[\delta_{2}\right]: T\left[\delta_{1}\right] @ l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} t \approx t^{\prime}: T$ @ $l$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \approx \delta_{2}: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t\left[\delta_{1}\right] \approx t^{\prime}\left[\delta_{2}\right]: T\left[\delta_{1}\right] @ l$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} \delta: \Gamma^{\prime \prime}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \approx \delta_{2}: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \circ \delta_{1} \approx \delta \circ \delta_{2}: \Gamma^{\prime \prime}$.
- If $L \mid \Psi ; \Gamma^{\prime} \vdash_{i} \delta \approx \delta^{\prime}: \Gamma^{\prime \prime}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \approx \delta_{2}: \Gamma^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \circ \delta_{1} \approx \delta^{\prime} \circ \delta_{2}: \Gamma^{\prime \prime}$.

Proof. We proceed by mutual induction. When we encounter well-formedness of types and well-typedness of terms, we conclude by IHs and respective congruence rules. What is more difficult are the asymmetric equivalence rules. We must apply IHs a bit more carefully to obtain the conclusions. We elaborate on a few cases.
Case

$$
\frac{L\left|\Psi ; \Gamma^{\prime} \vdash_{i} t \approx t^{\prime}: T @ l \quad L\right| \Psi ; \Gamma^{\prime} \vdash_{i} t^{\prime} \approx t^{\prime \prime}: T @ l}{L \mid \Psi ; \Gamma^{\prime} \vdash_{i} t \approx t^{\prime \prime}: T @ l}
$$

$L \mid \Psi ; \Gamma \vdash_{i} \delta_{1}: \Gamma^{\prime}$
(by presupposition)
$L \mid \Psi ; \Gamma \vdash_{i} t\left[\delta_{1}\right] \approx t^{\prime}\left[\delta_{1}\right]: T\left[\delta_{1}\right] @ l \quad$ (by local substitution lemma)
$L \mid \Psi ; \Gamma \vdash_{i} t^{\prime}\left[\delta_{1}\right] \approx t^{\prime \prime}\left[\delta_{2}\right]: T\left[\delta_{1}\right] @ l \quad$ (by IH)
$L \mid \Psi ; \Gamma \vdash_{i} t\left[\delta_{1}\right] \approx t^{\prime \prime}\left[\delta_{2}\right]: T\left[\delta_{1}\right] @ l$
(by transitivity)
Case

$$
\frac{L\left|\Psi ; \Gamma^{\prime} \vdash_{i} t: T^{\prime} @ l \quad L\right| \Psi ; \Gamma^{\prime} \vdash_{\text {typeof }(i)} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma^{\prime} \vdash_{i} t: T @ l}
$$

| $L \mid \Psi ; \Gamma \vdash_{i} \delta_{1}: \Gamma^{\prime}$ | (by presupposition) |
| :--- | ---: |
| $L \mid \Psi ; \Gamma \vdash_{\text {typeof }(i)} T\left[\delta_{1}\right] \approx T^{\prime}\left[\delta_{1}\right] @ l$ | (by local substitution lemma) |
| $L \mid \Psi ; \Gamma \vdash_{i} t\left[\delta_{1}\right] \approx t\left[\delta_{2}\right]: T^{\prime}\left[\delta_{1}\right] @ l$ | (by IH) |

Case

$\frac{L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma^{\prime} \vdash_{m} t: T^{\prime} @ l^{\prime}$| $L \mid \Psi \vdash_{m} \Gamma^{\prime}$ |
| :---: |
| $L \vdash l: \text { Level }$ |$\quad L \vdash l^{\prime}: \text { Level } \quad L \mid \Psi ; \Delta \vdash_{p} T @ l}{L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t[T / U] \approx\left(\Lambda_{p}^{l, l^{\prime}} U . t\right) \$_{p} T: T^{\prime}[T / U] @ l^{\prime}}$

$L \mid \Psi ; \Gamma \vdash_{m} \delta_{1}: \Gamma^{\prime}$
$L \mid \Psi ; \Gamma \vdash_{m} t[T / U]\left[\delta_{1}\right] \approx\left(\Lambda_{p}^{l, l^{\prime}} U . t\left[\delta_{1}\right]\right) \$_{p} T: T^{\prime}[T / U]\left[\delta_{1}\right] @ l^{\prime} \quad$ (by local substitution lemma)
$L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} t\left[\delta_{1}\right] \approx t\left[\delta_{2}\right]: T^{\prime}\left[\delta_{1}\right] @ l^{\prime} \quad$ (by IH$)$
$L \mid \Psi ; \Gamma \vdash_{m}\left(\Lambda_{p}^{l, l^{\prime}} U . t\left[\delta_{1}\right]\right) \$_{p} T \approx\left(\Lambda_{p}^{l, l^{\prime}} U . t\left[\delta_{2}\right]\right) \$_{p} T: T^{\prime}[T / U]\left[\delta_{1}\right] @ l^{\prime}$
(by congruence; note that $T^{\prime}[T / U]\left[\delta_{1}\right]=T^{\prime}\left[\delta_{1}\right][T / U]$ as $U$ is not in $\delta_{1}$ )

$$
L \mid \Psi ; \Gamma \vdash_{m} t[T / U]\left[\delta_{1}\right] \approx\left(\Lambda_{p}^{l, l^{\prime}} U . t\left[\delta_{2}\right]\right) \$_{p} T: T^{\prime}[T / U]\left[\delta_{1}\right] @ l^{\prime} \quad \text { (by transitivity) }
$$

Case

$$
\begin{gathered}
L \mid \Psi \vdash_{m} \Gamma^{\prime} \quad L \vdash l^{\prime}: \text { Level } \quad L \vdash l: \text { Level } L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{c} T @ l \\
L\left|\Psi ; \Gamma^{\prime}, x_{T}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l \vdash_{m} M @ l^{\prime} \quad L\right| \Psi, U:\left(\Delta \vdash_{c} @ l\right) ; \Gamma^{\prime} \vdash_{m} t^{\prime}: M\left[\operatorname{box} U / x_{T}\right] @ l^{\prime} \\
L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t^{\prime}[T / U] \approx \operatorname{letbox}_{\mathrm{Typ}}^{l^{\prime}} l \Delta\left(x_{T} \cdot M\right)\left(U \cdot t^{\prime}\right)(\operatorname{box} T): M\left[\operatorname{box} T / x_{T}\right] @ l^{\prime}
\end{gathered}
$$

$L \mid \Psi ; \Gamma \vdash_{m} \delta_{1}: \Gamma^{\prime} \quad$ (by presupposition)
$L \mid \Psi \vdash_{m} \Gamma$
$L \mid \Psi ; \Gamma \vdash_{m} t^{\prime}[T / U]\left[\delta_{1}\right] \approx \operatorname{letbox}_{\text {Typ }}^{l^{\prime}} l \Delta\left(x_{T} \cdot M\left[\delta_{1}, x_{T} / x_{T}\right]\right)\left(U . t^{\prime}\left[\delta_{1}\right]\right)(\operatorname{box} T): M\left[\operatorname{box} T / x_{T}\right]\left[\delta_{1}\right] @ l^{\prime}$ (by local substitution lemma)

$$
\begin{aligned}
& L \mid \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l \vdash_{m} M\left[\delta_{1}, x_{T} / x_{T}\right] \approx M\left[\delta_{2}, x_{T} / x_{T}\right] @ l^{\prime} \\
& L \mid \Psi, U:\left(\Delta \vdash_{c} @ l\right) ; \Gamma \vdash_{m} t^{\prime}\left[\delta_{1}\right] \approx t^{\prime}\left[\delta_{2}\right]: M\left[\delta_{1}, x_{T} / x_{T}\right]\left[\operatorname{box} U / x_{T}\right] @ l^{\prime} \\
& \quad l^{\prime} \text { etbox }{ }_{\text {Typ }}^{l^{\prime}} l \Delta\left(x_{T} \cdot M\left[\delta_{1}, x_{T} / x_{T}\right]\right)\left(U \cdot t^{\prime}\left[\delta_{1}\right]\right)(\text { box } T) \\
& \approx \text { letbox } \mathrm{T}_{\mathrm{Typ}}^{l^{\prime}} l \Delta\left(x_{T} \cdot M\left[\delta_{2}, x_{T} / x_{T}\right]\right)\left(U \cdot t^{\prime}\left[\delta_{2}\right]\right)(\text { box } T): M\left[\delta_{1}, x_{T} / x_{T}\right]\left[\text { box } T / x_{T}\right] \\
& L \mid \Psi ; \Gamma \vdash_{m} t^{\prime}[T / U]\left[\delta_{1}\right] \approx l e t b o x_{\mathrm{Typ}}^{l^{\prime}} l \Delta\left(x_{T} \cdot M\left[\delta_{2}, x_{T} / x_{T}\right]\right)\left(U \cdot t^{\prime}\left[\delta_{2}\right]\right)(\text { box } T): M\left[\delta_{1}, x_{T} / x_{T}\right]\left[\text { box } T / x_{T}\right] @ l^{\prime}
\end{aligned}
$$

(by transitivity)
Case Finally we consider a $\eta$ rule.

$$
\frac{L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l}{L \mid \Psi ; \Gamma^{\prime} \vdash_{m} \Lambda^{l} g \cdot(t \$ g) \approx t:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l}
$$

$L \mid \Psi ; \Gamma \vdash_{m} \delta_{1}: \Gamma^{\prime}$
(by presupposition)
$L \mid \Psi ; \Gamma \vdash_{m} t\left[\delta_{1}\right]:(g: \mathrm{Ctx}) \Rightarrow^{l}\left(T\left[\delta_{1}\right]\right) @ l \quad$ (by local substitution lemma)
$L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} g \cdot\left(\left(t\left[\delta_{1}\right]\right) \$ g\right) \approx t\left[\delta_{1}\right]:(g: \mathrm{Ctx}) \Rightarrow^{l}\left(T\left[\delta_{1}\right]\right) @ l \quad$ (by the same $\eta$ rule)
$L \mid \Psi ; \Gamma \vdash_{m} t\left[\delta_{1}\right] \approx t\left[\delta_{2}\right]:(g: \mathrm{Ctx}) \Rightarrow^{l}\left(T\left[\delta_{1}\right]\right) @ l \quad$ (by IH)
$L \mid \Psi ; \Gamma \vdash_{m} \Lambda^{l} g \cdot\left(\left(t\left[\delta_{1}\right]\right) \$ g\right) \approx t\left[\delta_{2}\right]:(g: \mathrm{Ctx}) \Rightarrow^{l}\left(T\left[\delta_{1}\right]\right) @ l \quad$ (by transitivity)

Remark. Note that the statement of this lemma is left-biased. For example, when considering terms, the types are substituted by $\delta_{1}$. This bias causes the whole formulation of equivalence judgment between terms to be left biased as well. Otherwise, this lemma cannot be easily justified as above and requires the global substitution lemma, which definitely causes issues as the latter depends on this very lemma in the global variable cases.

A visible effect of this left bias is especially evident in the computation rules. For example, if we define the following $\beta$ rule instead, then the lemma above suddenly becomes unprovable at this stage:

$$
\left.\left.\frac{L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma^{\prime} \vdash_{m} t: T^{\prime} @ l^{\prime}}{} \begin{array}{c}
L \mid \Psi \vdash_{m} \Gamma^{\prime} \\
L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level }
\end{array} \quad L \right\rvert\, \Psi ; \Delta \vdash_{p} T @ l\right]: \Gamma^{\prime}\left(\Lambda_{p}^{l, l^{\prime}} U . t\right) \$_{p} T \approx t[T / U]: T^{\prime}[T / U] @ l^{\prime} \quad
$$

Let us work on the proof to see what happens. We now must prove

$$
L \mid \Psi ; \Gamma \vdash_{m}\left(\Lambda_{p}^{l, l^{\prime}} U .\left(t\left[\delta_{1}\right]\right)\right) \$_{p} T \approx t[T / U]\left[\delta_{2}\right]: T^{\prime}[T / U]\left[\delta_{1}\right] @ l^{\prime}
$$

The only way to introduce $\delta_{2}$ on the right hand side now is to apply the local substitution lemma, which yields

$$
L \mid \Psi ; \Gamma \vdash_{m}\left(\Lambda_{p}^{l, l^{\prime}} U .\left(t\left[\delta_{2}\right]\right)\right) \$_{p} T \approx t[T / U]\left[\delta_{2}\right]: T^{\prime}[T / U]\left[\delta_{2}\right] @ l^{\prime}
$$

Notice that the return type yields a local substitution $\delta_{2}$, instead of $\delta_{1}$ as required by the goal. At this stage, however, we are not able to prove the equivalence between $T^{\prime}[T / U]\left[\delta_{1}\right]$ and $T^{\prime}[T / U]\left[\delta_{2}\right]$ as we are missing the global substitution lemma to justify that $T^{\prime}[T / U]$ remains well-formed.

Similarly, we are not able to prove a lemma if some $\eta$ rules are flipped either. Consider the following "innocent" $\eta$ rule:

$$
\frac{L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l}{L \mid \Psi ; \Gamma^{\prime} \vdash_{m} t \approx \Lambda^{l} g \cdot(t \$ g):(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l}
$$

Since there is only one premise, we have no choice but to eventually use IH to obtain

$$
L \mid \Psi ; \Gamma \vdash_{m} t\left[\delta_{1}\right] \approx t\left[\delta_{2}\right]:(g: \mathrm{Ctx}) \Rightarrow^{l}\left(T\left[\delta_{1}\right]\right) @ l
$$

This leaves us to prove

$$
L \mid \Psi ; \Gamma \vdash_{m} t\left[\delta_{2}\right] \approx \Lambda^{l} g \cdot\left(\left(t\left[\delta_{2}\right]\right) \$ g\right):(g: \mathrm{Ctx}) \Rightarrow^{l}\left(T\left[\delta_{1}\right]\right) @ l
$$

Notice how the equivalence itself talks about $\delta_{2}$ exclusively while the type refers to $\delta_{1}$. This asymmetry forces us to flip the equivalence to obtain a better proof.

Then we move on to the global substitution lemma. We must first establish a number of other lemmas. The lifting lemma is one of the guiding lemmas of the layering principle, where we require that well-formedness can be carried over to higher layers.

Lemma 5.16 (Lifting). If $i \leq i^{\prime}$, and

- $L \mid \Psi \vdash_{i} \Gamma$, then $L \mid \Psi \vdash_{i^{\prime}} \Gamma$;
- $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$, then $L \mid \Psi \vdash_{i^{\prime}} \Gamma \approx \Delta$;
- $L \mid \Psi ; \Gamma \vdash_{i} T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i^{\prime}} T$ @ $l$;
- $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i^{\prime}} T \approx T^{\prime} @ l$;
- $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i^{\prime}} t: T @ l$.
- $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i^{\prime}} t \approx t^{\prime}: T @ l$;
- $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i^{\prime}} \delta: \Delta$;
- $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i^{\prime}} \delta \approx \delta^{\prime}: \Delta$.

Proof. First, we realize that the typeof function is monotonic, i.e. typeof $(i) \leq \operatorname{typeof}\left(i^{\prime}\right)$. We proceed by a mutual induction. Most cases are obvious by IHs. Notice that there are cases where we have premises like $L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma$, so we must apply IH to obtain $L \mid \Psi \vdash_{\text {typeof }\left(i^{\prime}\right)} \Gamma$ with the monotonicity property above. It works similarly for the conversion rule, where we have $L \mid \Psi ; \Gamma \vdash_{\text {typeof }(i)} T \approx T^{\prime} @ l$. In the cases of global variables, the transitivity of $\leq$ eventually complete the proof of this lemma. We elaborate on one case:

$$
\begin{array}{llr}
L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma \quad u:\left(\Delta \vdash_{i^{\prime \prime}} T @ l\right) \in \Psi \quad i^{\prime \prime} \in\{v, c\} \quad i \in\{v, c, p, m\} & i^{\prime \prime} \leq i & L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta \\
\hline & L \mid \Psi ; \Gamma \vdash_{i} u^{\delta}: T[\delta] @ l & \\
& L \mid \Psi \vdash_{\text {typeof( } \left.i^{\prime}\right)} \Gamma & \\
& L \mid \Psi ; \Gamma \vdash_{i^{\prime}} \delta: \Delta \\
& i^{\prime \prime} \leq i \leq i^{\prime} \\
& L \mid \Psi ; \Gamma \vdash_{i^{\prime}} u^{\delta}: T[\delta] @ l & \text { (by IH) } \\
\text { (by IH) } \\
\text { (by the same rule) }
\end{array}
$$

The inverse of lifting sometimes is possible
Lemma 5.17 (Unlifting).

- $L \mid \Psi ; \Gamma \vdash_{p} T @ l$, then $L \mid \Psi ; \Gamma \vdash_{c} T @ l$;
- $L \mid \Psi ; \Gamma \vdash_{p} t: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{c} t: T @ l$.
- $L \mid \Psi ; \Gamma \vdash_{p} \delta: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{c} \delta: \Delta$;

Proof. Induction. Notice that typeof $(p)=\operatorname{typeof}(c)=p$.
The unlifting lemma says that the typing at layer $p$ can be unlifted back to layer $c$.
As another guiding lemma, we have the static code lemma, which states that code at layer $v$ and $c$ has no computational behavior.

Lemma 5.18 (Static Code). If $i \in\{v, c\}$,

- $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $T=T^{\prime}$;
- $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ $l$, then $t=t^{\prime}$;
- $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $\delta=\delta^{\prime}$.

All equalities above are quotient over the equivalence of universe levels.
Proof. Mutual induction. We are not concerned about the equivalence of types due to the conversion rule.
We emphasize again that the equalities hold modulo the equivalence of universe levels. For example, $\mathrm{Ty}_{\ell \cup \ell^{\prime}}$ and $\mathrm{Ty}_{f^{\prime} \sqcup \ell}$ as code are considered equal, though their universe levels are not exactly syntactically identical. This is fine as we know how to decide the equality between two universe levels as shown in Sec. 4.3.

Lemma 5.19 (Global Substitutions).

- If $L \mid \Phi \vdash_{i} \Gamma, i \in\{p, m\}$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi \vdash_{i} \Gamma[\sigma]$.
- If $L \mid \Phi \vdash_{i} \Gamma \approx \Delta, i \in\{p, m\}$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi \vdash_{i} \Gamma[\sigma] \approx \Delta[\sigma]$.
- If $L \mid \Phi ; \Gamma \vdash_{i} T @ l$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} T[\sigma]$ @ $l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} T[\sigma] \approx T^{\prime}[\sigma] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} t: T$ @ $l$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} t[\sigma]: T[\sigma] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ land $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} t[\sigma] \approx t^{\prime}[\sigma]: T[\sigma] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} \delta: \Delta$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} \delta[\sigma]: \Delta[\sigma]$.
- If $L \mid \Phi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Psi \vdash \sigma: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} \delta[\sigma] \approx \delta^{\prime}[\sigma]: \Delta[\sigma]$.

Proof. We proceed by a mutual induction. Notice that in the first two statements, $i \in\{p, m\}$, namely the range of the typeof function. This ensures a lookup $\sigma(g)$ of a contextual variable $g$ to be well-formed at layer $i$, due to Lemma 5.16. Most cases can be discharged by IHs directly. The complex cases are the computation rules and the global variable cases.

We consider a few cases:

## Case

$$
\frac{u: \left.\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Phi \quad \begin{array}{c}
L \left\lvert\, \begin{array}{c}
\Phi \vdash_{\text {typeof }(i)} \Gamma \\
i \in\{v, c, p, m\}
\end{array} \quad i^{\prime} \leq i\right.
\end{array} \quad L \right\rvert\, \Phi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta}{L \mid \Phi ; \Gamma \vdash_{i} u^{\delta} \approx u^{\delta^{\prime}}: T[\delta] @ l}
$$

$L \mid \Psi ; \Delta[\sigma] \vdash_{i^{\prime}} \sigma(u): T[\sigma] @ l$
(by lookup)
$L \mid \Psi ; \Delta[\sigma] \vdash_{i} \sigma(u): T[\sigma] @ l \quad$ (by lifting)
$L \mid \Psi ; \Gamma[\sigma] \vdash_{i} \delta[\sigma] \approx \delta^{\prime}[\sigma]: \Delta[\sigma]$
(by IH)
$L \mid \Psi ; \Gamma[\sigma] \vdash_{i} \sigma(u)[\delta[\sigma]] \approx \sigma(u)\left[\delta^{\prime}[\sigma]\right]: T[\sigma][\delta[\sigma]] @ l \quad$ (by equivalent local substitution lemma)
Notice that

$$
T[\sigma][\delta[\sigma]]=T[\delta][\sigma]
$$

Case

$$
\begin{aligned}
& L \mid \Phi ; \Gamma \vdash_{m} t \approx t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega \\
& \underline{|\vec{\ell}|=|\vec{l}|=|\vec{l}|>0 \quad \forall 0 \leq n<|\vec{l}| \cdot L \vdash \vec{l}(n) \approx \vec{l}^{\prime}(n) \text { : Level }} \\
& L \mid \Phi ; \Gamma \vdash_{m} t \$ \vec{l} \approx t^{\prime} \$ \vec{l}^{\prime}: T[\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}] \\
& L \mid \Psi ; \Gamma[\sigma] \vdash_{m} t[\sigma] \approx t^{\prime}[\sigma]: \vec{\ell} \Rightarrow^{l}(T[\sigma]) @ \omega \\
& \text { (by IH) } \\
& L \mid \Psi ; \Gamma[\sigma] \vdash_{m}(t[\sigma]) \$ \vec{l} \approx\left(t^{\prime}[\sigma]\right) \$ \vec{l}^{\prime}: T[\sigma][\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}]
\end{aligned}
$$

Note that

$$
T[\sigma][\vec{l} / \vec{\ell}]=T[\vec{l} / \vec{\ell}][\sigma[\vec{l} / \vec{l}]]=T[\vec{l} / \vec{\ell}][\sigma]
$$

because all $\vec{\ell}$ do not occur in $\sigma$.
Case


We first proceed by using IHs on the premises, which include the following judgments:

$$
\begin{aligned}
& L \mid \Psi \vdash_{p} \Delta[\sigma] \\
& L \mid \Psi ; \Delta[\sigma] \vdash_{c} S[\sigma] @ l \\
& L \mid \Psi ; \Delta[\sigma], x: S[\sigma] @ l \vdash_{c} T[\sigma] @ l^{\prime}
\end{aligned}
$$

By using the same $\beta$ rule, we must check the resulting left hand side and the result type are equal to the target goal. Let us first consider the left hand side:

$$
\begin{align*}
& t_{\Pi}\left[\sigma, g / g, U_{S}^{\text {id }} / U_{S}, U_{T}^{\text {id }} / U_{T}\right]\left[l / \ell, l^{\prime} / \ell^{\prime}, \Delta[\sigma] / g, S[\sigma] / U_{S}, T[\sigma] / U_{T}, s_{S}[\sigma] / x_{S}, s_{T}[\sigma] / x_{T}\right] \\
& =t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}\right]\left[\sigma, g / g, U_{S}^{\text {id }} / U_{S}, U_{T}^{\text {id }} / U_{T}\right]\left[\Delta[\sigma] / g, S[\sigma] / U_{S}, T[\sigma] / U_{T}, s_{S}[\sigma] / x_{S}, s_{T}[\sigma] / x_{T}\right] \\
& \text { (by Lemma } 5.9 ; \ell \text { and } \ell^{\prime} \text { do not occur in } \sigma \text { ) } \\
& =t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}\right]\left[\sigma, \Delta[\sigma] / g, S[\sigma] / U_{S}, T[\sigma] / U_{T}\right]\left[s_{S}[\sigma] / x_{S}, s_{T}[\sigma] / x_{T}\right] \\
& =t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}\right]\left[\Delta / g, S / U_{S}, T / U_{T}\right][\sigma]\left[s_{s}[\sigma] / x_{S}, s_{T}[\sigma] / x_{T}\right] \\
& \text { (by naturality) } \\
& =t_{\Pi}\left[l / \ell, l^{\prime} / \ell^{\prime}\right]\left[\Delta / g, S / U_{S}, T / U_{T}\right]\left[s_{S} / x_{S}, s_{T} / x_{T}\right][\sigma] \tag{byLemma5.9}
\end{align*}
$$

Then we consider the result type in a similar way:

$$
\begin{aligned}
& M[\sigma, g / g]\left[l \sqcup l^{\prime} / \ell, \Delta[\sigma] / g, t[\sigma] / x_{T}\right] \\
= & M\left[l \sqcup l^{\prime} / \ell\right][\sigma, g / g]\left[\Delta[\sigma] / g, t[\sigma] / x_{T}\right] \\
= & M\left[l \sqcup l^{\prime} \ell \ell\right][\Delta / g][\sigma]\left[t[\sigma] / x_{T}\right] \\
= & M\left[l \sqcup l^{\prime} / \ell\right][\Delta / g]\left[t / x_{T}\right][\sigma]
\end{aligned}
$$

Both equations allow us to conclude the goal.

Next, we consider the effect of equivalent global substitutions on the judgments. We first define the equivalence relation between global substitutions:


We can then consider similar properties of this equivalence relation.
Lemma 5.20 (Presupposition). If $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi \vdash \sigma: \Phi$ and $L \mid \Psi \vdash \sigma^{\prime}: \Phi$.
Proof. Induction.
Lemma 5.21 (Equivalent Global Substitutions).

- If $L \mid \Phi \vdash_{i} \Gamma, i \in\{p, m\}$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi \vdash_{i} \Gamma[\sigma] \approx \Gamma\left[\sigma^{\prime}\right]$.
- If $L \mid \Phi \vdash_{i} \Gamma \approx \Delta, i \in\{p, m\}$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi \vdash_{i} \Gamma[\sigma] \approx \Delta\left[\sigma^{\prime}\right]$.
- If $L \mid \Phi ; \Gamma \vdash_{i} T @ l$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} T[\sigma] \approx T\left[\sigma^{\prime}\right] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} T[\sigma] \approx T^{\prime}\left[\sigma^{\prime}\right] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} t: T$ @ $l$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} t[\sigma] \approx t\left[\sigma^{\prime}\right]: T[\sigma] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} t[\sigma] \approx t^{\prime}\left[\sigma^{\prime}\right]: T[\sigma] @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} \delta: \Delta$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} \delta[\sigma] \approx \delta\left[\sigma^{\prime}\right]: \Delta[\sigma]$.
- If $L \mid \Phi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi ; \Gamma[\sigma] \vdash_{i} \delta[\sigma] \approx \delta^{\prime}\left[\sigma^{\prime}\right]: \Delta[\sigma]$.

Proof. We apply mutual induction. This lemma is much less sensitive to the exact statement of rules compared to Lemma 5.15. Since now we have proved the global substitution lemma, we could use conversion rules whenever necessary.

At this point, we have concluded that all substitutions are coherent with well-formedness and typing judgments. Next, we shall move towards the full presupposition lemma and end our discussion on syntactic properties with it.

### 5.3 Context Equivalence and Presupposition

In order to establish presupposition, we must concern ourselves with the asymmetry in the congruence rules of the equivalence judgments. Presupposition, intuitively, requires us to show that this asymmetry "does not matter". This intuition is formalized by the context equivalence lemma. In fact, we need two such lemmas, as we need to show one for local contexts and one for global contexts. In light of that, let us proceed with the lemma for local contexts first.

## Lemma 5.22 (Local Context Equivalence).

- If $L \mid \Psi ; \Delta \vdash_{i} T @ l$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} T @ l$.
- If $L \mid \Psi ; \Delta \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Delta \vdash_{i} t: T$ @ $l$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} t: T$ @ $l$.
- If $L \mid \Psi ; \Delta \vdash_{i} t \approx t^{\prime}: T @ l$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$.
- If $L \mid \Psi ; \Delta \vdash_{i} \delta: \Gamma^{\prime}$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$.
- If $L \mid \Psi ; \Delta \vdash_{i} \delta \approx \delta^{\prime}: \Gamma^{\prime}$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Gamma^{\prime}$.

Proof. We start by mutual induction. The base case is the local variable cases, where we simply apply the conversion rule to the equivalence given by $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta \approx \Gamma$. We might also use presupposition (Lemma 5.11) to derive $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta$. Otherwise, most cases can be handled by IHs. In cases where local contexts are extended with variables, we shall carefully use IHs to obtain the necessary premises to extend the equivalence of $\Delta$ and $\Gamma$ as well.

We consider a few cases:

## Case

$$
\begin{gathered}
L \vdash l_{1} \approx l_{3}: \text { Level }
\end{gathered} \begin{gathered}
L \vdash l_{2} \approx l_{4}: \text { Level } \\
L \mid \Psi ; \Delta \vdash_{i} S @ l_{1} \\
L\left|\Psi ; \Delta \vdash_{i} S \approx S^{\prime} @ l_{1} \quad L\right| \Psi ; \Delta, x: S @ l_{1} \vdash_{i} t \approx t^{\prime}: T @ l_{2} \\
L \mid \Psi ; \Delta \vdash_{i} \lambda^{l_{1}, l_{2}}(x: S) \cdot t \approx \lambda^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot t^{\prime}: \Pi^{l l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}
\end{gathered}
$$

In this case, the crucial part is to be able to invoke IH on $t \approx t^{\prime}$. We proceed as follows:

$$
L \mid \Psi ; \Gamma \vdash_{i} S @ l_{1}
$$

(by IH)

$$
L \mid \Psi \vdash_{\text {typeof }(i)} \Delta, x: S @ l_{1} \approx \Gamma, x: S @ l_{1}
$$

$$
\begin{equation*}
L \mid \Psi ; \Gamma, x: S @ l_{1} \vdash_{i} t \approx t^{\prime}: T @ l_{2} \tag{byIH}
\end{equation*}
$$

Then IHs will allow us to conclude the rest.
Case

As a corollary, we can prove the following lemma.

## Lemma 5.23 (Symmetry and Transitivity of Local Contexts).

- If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$, then $L \mid \Psi \vdash_{i} \Delta \approx \Gamma$.
- If $L \mid \Psi \vdash_{i} \Gamma_{1} \approx \Gamma_{2}$ and $L \mid \Psi \vdash_{i} \Gamma_{2} \approx \Gamma_{3}$, then $L \mid \Psi \vdash_{i} \Gamma_{1} \approx \Gamma_{3}$.

Proof. Induction. Note that transitivity replies on the local context equivalence lemma.
A similar lemma replaces the codomain local contexts of local substitutions. This variant is much simpler just by conversion rules.

$$
\begin{aligned}
& L \mid \Psi \vdash_{m} \Delta \quad L \vdash l^{\prime} \text { : Level } \quad L \vdash l \text { : Level } \quad L\left|\Psi \vdash_{p} \Delta^{\prime} \quad L\right| \Psi ; \Delta^{\prime} \vdash_{c} T @ l \\
& \frac{L\left|\Psi ; \Delta, x_{T}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l \vdash_{m} M @ l^{\prime} \quad L\right| \Psi, U:\left(\Delta^{\prime} \vdash_{c} @ l\right) ; \Delta \vdash_{m} t^{\prime}: M\left[\operatorname{box} U / x_{T}\right] @ l^{\prime}}{L \mid \Psi ; \Delta \vdash_{m} t^{\prime}[T / U] \approx \operatorname{letbox}_{\mathrm{Typ}}^{\prime^{\prime}} l \Delta^{\prime}\left(x_{T} \cdot M\right)\left(U . t^{\prime}\right)(\operatorname{box} T): M\left[\operatorname{box} T / x_{T}\right] @ l^{\prime}} \\
& L \mid \Psi \vdash_{m} \Gamma \\
& \text { (by presupposition (Lemma 5.11)) } \\
& L \mid \Psi \vdash_{m} \Delta, x_{T}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l \approx \Gamma, x_{T}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l \\
& \text { (note that well-formedness of } \square\left(\Delta^{\prime} \vdash_{c} @ l\right) \text { does not depend on } \Delta \text { or } \Gamma \text { ) } \\
& L \mid \Psi ; \Gamma, x_{T}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l \vdash_{m} M @ l^{\prime} \\
& \text { (by IH) } \\
& L \mid \Psi, U:\left(\Delta^{\prime} \vdash_{c} @ l\right) \vdash_{m} \Delta \approx \Gamma \\
& \text { (by global weakening) } \\
& L \mid \Psi, U:\left(\Delta^{\prime} \vdash_{c} @ l\right) ; \Gamma \vdash_{m} t^{\prime}: M\left[\operatorname{box} U / x_{T}\right] @ l^{\prime} \\
& \text { (by IH) } \\
& L \mid \Psi ; \Gamma \vdash_{m} t^{\prime}[T / U] \approx l^{2} \operatorname{etbox}_{\text {Typ }}^{l^{\prime}} l \Delta^{\prime}\left(x_{T} \cdot M\right)\left(U . t^{\prime}\right)(\operatorname{box} T): M\left[\operatorname{box} T / x_{T}\right] @ l^{\prime}
\end{aligned}
$$

Lemma 5.24 (Local Context Conversion).

- If $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Gamma^{\prime}$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma^{\prime} \approx \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Gamma^{\prime}$ and $L \mid \Psi \vdash_{\text {typeof }(i)} \Gamma^{\prime} \approx \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$.

Proof. By induction. Propagate conversion rules together with the local substitution lemma in the step case.

Then we work on the global context equivalence lemma. To state this lemma, we should first specify what does that mean for two global contexts are equivalent.

$$
\begin{aligned}
& \frac{L \vdash \Psi \approx \Phi}{L \vdash \cdot \approx \cdot} \quad \frac{L+\Psi, g: \mathrm{Ctx} \approx \Phi, g: \mathrm{Ctx}}{} \\
& \frac{L \vdash \Psi \approx \Phi \quad L\left|\Psi \vdash_{p} \Gamma \quad L\right| \Phi \vdash_{p} \Delta \quad L \mid \Psi \vdash_{p} \Gamma \approx \Delta \quad L \vdash l: \text { Level } \quad i \in\{c, p\}}{L \vdash \Psi, U:\left(\Gamma \vdash_{i} @ l\right) \approx \Phi, U:\left(\Delta \vdash_{i} @ l\right)} \\
& L \vdash \Psi \approx \Phi \quad L \mid \Psi \vdash_{p} \Gamma \approx \Delta \\
& \frac{L\left|\Psi ; \Gamma \vdash_{p} T @ l \quad L\right| \Phi ; \Delta \vdash_{p} T^{\prime} @ l \quad L \mid \Psi ; \Gamma \vdash_{p} T \approx T^{\prime} @ l \quad L \vdash l: \text { Level } \quad i \in\{v, c\}}{L \vdash \Psi, u:\left(\Gamma \vdash_{i} T @ l\right) \approx \Phi, u:\left(\Delta \vdash_{i} T^{\prime} @ l\right)}
\end{aligned}
$$

Essentially, the equivalence of global contexts are just point-wise equivalence of types within. We can reconstruct the well-formedness of both components from the premises:

Lemma 5.25 (Presupposition of Equivalence of Global Contexts). If $L \vdash \Psi \approx \Phi$, then $L \vdash \Psi$ and $L \vdash \Phi$.
For the global context equivalence lemma, we would like to take a shortcut by taking advantage of the global substitution lemma.

Lemma 5.26. If $L \mid \Psi \vdash_{\text {typeof( }(i)} \Gamma \approx \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i}$ id $: \Delta$.
Lemma 5.27. If $L \vdash \Psi \approx \Phi$, then $L \mid \Psi \vdash i d: \Phi$.
Proof. We proceed by induction. In each step case, notice that weakening is used implicitly. Use Lemma 5.26 to derive $L \mid \Psi ; \Gamma \vdash_{p}$ id : $\Delta$ whenever necessary.

Lemma 5.28 (Global Context Equivalence).

- If $L \mid \Phi \vdash_{i} \Gamma$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi \vdash_{i} \Gamma$.
- If $L \mid \Phi \vdash_{i} \Gamma \approx \Delta$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$.
- If $L \mid \Phi ; \Gamma \vdash_{i} T @ l$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi ; \Gamma \vdash_{i} T$ @ $l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} t: T @ l$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ land $L \vdash \Phi \approx \Psi$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ $l$.
- If $L \mid \Phi ; \Gamma \vdash_{i} \delta: \Delta$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$.
- If $L \mid \Phi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$.

Proof. We have $L \mid \Psi \vdash$ id : $\Phi$ due to Lemma 5.27. Then by the global substitution lemma, we have our goal by knowing that a global identity substitution id does no action.

Finally, we prove the presupposition lemma, which is the last guiding lemma of the layering principle.
Lemma 5.29 (Presupposition).

- If $L \mid \Psi ; \Gamma \vdash_{i} T @ l$, then $L \mid \Psi \vdash_{\text {typeof( }(i)} \Gamma$ and $L \vdash l:$ Level or $i=m \wedge l=\omega$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $L\left|\Psi \vdash_{\text {typeof(i) }} \Gamma, L\right| \Psi ; \Gamma \vdash_{i} T @ l, L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} @ l$ and $L \vdash l:$ Level or $i=m \wedge l=\omega$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$, then $L\left|\Psi \vdash_{\text {typeof }(i)} \Gamma, L\right| \Psi ; \Gamma \vdash_{\text {typeof }(i)} T @ l$ and $L \vdash l:$ Level or $i=m \wedge l=\omega$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$, then $L\left|\Psi \vdash_{t y p e o f(i)} \Gamma, L\right| \Psi ; \Gamma \vdash_{i} t: T @ l, L \mid \Psi ; \Gamma \vdash_{i} t^{\prime}: T$ @ $l$, $L \mid \Psi ; \Gamma \vdash_{\text {typeof }(i)} T @ l$ and $L \vdash l:$ Level or $i=m \wedge l=\omega$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Gamma^{\prime}$, then $L \mid \Psi \vdash_{\text {typeof }(i)} \Delta$.

Notice that in the statement of the lemma, we sometimes conclude $i=m \wedge l=\omega$. The only occasion when $\omega$ is used is when universe polymorphic functions are involved. In that case, we know for sure that $i=m$. In any other cases, we obtain $L \vdash l$ : Level, which excludes $l=\omega$.

Proof. We proceed by a mutual induction. In certain congruence rules, we must apply Lemmas 5.15 and 5.21 to resolve the asymmetry in the rules. Otherwise, we simply apply the substitution lemmas whenever necessary. Note that our rules are stated with redundant premises to make sure this lemma eventually checks out.

$$
\begin{aligned}
& \frac{L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow l^{\prime} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{m} t \$_{p} T \approx t^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}[T / U] @ l^{\prime}} \\
& L \mid \Psi ; \Gamma \vdash_{m} t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad \text { (by IH) } \\
& L \mid \Psi ; \Gamma \vdash_{m} t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad \text { (by IH) } \\
& L \mid \Psi ; \Delta \vdash_{p} T @ l \\
& L \mid \Psi ; \Delta \vdash_{p} T^{\prime} @ l \\
& \text { (by IH) } \\
& L \mid \Psi ; \Gamma \vdash_{m} t^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}\left[T^{\prime} / U\right] @ l^{\prime} \\
& L \mid \Psi ; \Gamma \vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow l^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad \text { (by IH) } \\
& L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T^{\prime \prime} @ l^{\prime}
\end{aligned}
$$

Notice that

$$
T / U \approx T^{\prime} / U
$$

We then have the goal by Lemma 5.21.

### 5.4 Coverage and Progress of Recursive Principles

Before moving to the semantics, let us pause a second and think about the recursive principles: is it guaranteed to always pick a case from the branches? In this section, we would like to positively answer this question. The ingredient lies in the typing judgment at layer $c$ and how the recursive principle is formulated. Recall that the recursive principle for code of terms is

$$
\frac{G_{A} \quad L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \quad L\right| \Psi ; \Delta \vdash_{p} T @ l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{m} t: \square\left(\Delta \vdash_{c} T @ l^{\prime}\right) @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T t: M^{\prime}\left[l^{\prime} / \ell, \Delta / g, T / U_{T}, t / x_{t}\right] @ l_{2}}
$$

In this rule, we see that the type of $t$ is indexed by $l, \Delta$ and $T$, both of which live at layer $p$. When $t=$ box $t^{\prime}$, then $t^{\prime}$ must be typed at layer $c$. Then coverage is provided by the exhaustiveness of the branches which should enumerate all possible types and terms at layer $c$. This is simple as we simply check the syntax at layer $c$ and can confirm that the branches are indeed exhaustive. Progress, on the other hand, requires both $\Delta$ and $T$ are in
the right form prescribed by the $\beta$ rules. For example, the following rule gives the $\beta$ rule for the Nat case as a term:

$$
\frac{G_{A} \quad L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \quad l=\text { succ zero }}{L \mid \Psi ; \Gamma \vdash_{m} t_{\mathrm{Nat}}^{\prime}[\Delta / g] \approx \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta \mathrm{Ty}_{\text {zero }}(\text { box Nat }): M^{\prime}\left[l / \ell, \Delta / g, \mathrm{Ty}_{\text {zero }} / U_{T}, \text { box Nat } / x_{t}\right] @ l_{2}}
$$

where $T=T y_{\text {zero }}$ is required. In this section, we show that when $t=$ box $t^{\prime}$ where $t^{\prime}$ is of some concrete form prescribed by a $\beta$ rule, then the indices must have the right form.

We first consider the well-formed types at layer $c$ :
Lemma 5.30. If $L \mid \Psi ; \Gamma \vdash_{c}$ Nat @ $l$, then $L \vdash l \approx$ zero : Level.
Proof. By induction. The only applicable rules are the well-formedness rule and the conversion rule.
Similar lemmas can be stated and proved.
Lemma 5.31. If $L \mid \Psi ; \Gamma \vdash_{c} \Pi^{l, l^{\prime}}(x: S) . T @ l^{\prime \prime}$, then $L \mid \Psi ; \Gamma \vdash_{c} S$ @ $l$ and $L \mid \Psi ; \Gamma, x: S @ l \vdash_{c} T$ @ $l^{\prime}$ and they are sub-derivations of the assumption; moreover, $L \vdash l^{\prime \prime} \approx l \sqcup l^{\prime}:$ Level.

Proof. Induction.
That the judgments in the conclusion are sub-derivations ensures the well-foundedness of the recursion. Effectively, the recursive principles recurse on the structures of the typing derivations, so they are the most general principles that can be formulated on top the syntax of MLTT.

Lemma 5.32. If $L \mid \Psi ; \Gamma \vdash_{c} \mathrm{El}^{l} t$ @ $l^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{c} t: \mathrm{Ty}_{l}$ @ succ $l$ as a sub-derivation and $L \vdash l^{\prime} \approx l:$ Level.
Lemma 5.33. If $L \mid \Psi ; \Gamma \vdash_{c} T y_{l} @ l^{\prime}$, then $L \vdash l^{\prime} \approx$ succ $l$ : Level.
Now we have exhausted all possible cases for types, so we move on to terms.
Lemma 5.34. If $L \mid \Psi ; \Gamma \vdash_{c} x: T$ @ $l$, then $x: T^{\prime} @ l^{\prime} \in \Gamma$ and $L \mid \Psi ; \Gamma \vdash_{p} T \approx T^{\prime} @ l$ and $L \vdash l \approx l^{\prime}:$ Level.
The statement of lemmas for terms need to also consider the equivalence of types. The equivalence of types is implicitly handled by when evaluating the recursive principles: since the equivalence is at layer $p$, computation applies, so the equivalence can be acknowledged by the conversion checking algorithm.

Lemma 5.35. If $L \mid \Psi ; \Gamma \vdash_{c}$ Nat: $T$ @ $l$, then $L \mid \Psi ; \Gamma \vdash_{p} T \approx T y_{\text {zero }} @ l$ and $L \vdash l \approx$ succ zero : Level.
Lemma 5.36. If $L \mid \Psi ; \Gamma \vdash_{c} \Pi^{l, l^{\prime}}(x: s) . t: T @ l^{\prime \prime}$, then as sub-derivations $L \mid \Psi ; \Gamma \vdash_{c} s: T y_{l} @$ succ $l$ and $L\left|\Psi ; \Gamma, x: E l^{l} s @ l \vdash_{c} t: \mathrm{Ty}_{l^{\prime}} @ \operatorname{succ} l^{\prime}, L\right| \Psi ; \Gamma \vdash_{p} T \approx \mathrm{Ty}_{l \sqcup l^{\prime}} @ l^{\prime \prime}$ and $L \vdash l^{\prime \prime} \approx \operatorname{succ}\left(l \sqcup l^{\prime}\right):$ Level.

Lemma 5.37. If $L \mid \Psi ; \Gamma \vdash_{c} T \mathrm{y}_{l}: T @ l^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{p} T \approx T \mathrm{y}_{\text {succ } l} @ l^{\prime}$ and $L \vdash l^{\prime} \approx \operatorname{succ}$ (succ $l$ ) : Level.
Lemma 5.38. If $L \mid \Psi ; \Gamma \vdash_{c}$ zero : $T @ l$, then $L \mid \Psi ; \Gamma \vdash_{p} T \approx$ Nat @ zero and $L \vdash l \approx$ zero: Level.
Lemma 5.39. If $L \mid \Psi ; \Gamma \vdash_{c}$ succ $t: T$ @ $l$, then as a sub-derivation $L \mid \Psi ; \Gamma \vdash_{c} t: N a t$ @ zero, $L \mid \Psi ; \Gamma \vdash_{p} T \approx$ Nat @ zero and $L \vdash l \approx$ zero: Level.

Lemma 5.40. If $L \mid \Psi ; \Gamma \vdash_{c}$ elim $_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t: T$ @ $l^{\prime}$, then as sub-derivations

- $L \mid \Psi ; \Gamma, x:$ Nat @ zero $\vdash_{c} M @ l$,
- $L \mid \Psi ; \Gamma \vdash_{c} s: M[z e r o / x] @ l$,
- $L \mid \Psi ; \Gamma, x: N a t @ z e r o, y: M @ l \vdash_{c} s^{\prime}: M[\operatorname{succ} x / x] @ l$
- $L \mid \Psi ; \Gamma \vdash_{c} t$ : Nat @ zero;
moreover $L \mid \Psi ; \Gamma \vdash_{p} T \approx M[t / x] @ l^{\prime}$ and $L \vdash l^{\prime} \approx l:$ Level.

Lemma 5.41. If $L \mid \Psi ; \Gamma \vdash_{c} \lambda^{l, l^{\prime}}(x: S) . t: T^{\prime} @ l^{\prime \prime}$, then as sub-derivations

- L| $\Psi ; \Gamma \vdash_{c} S @ l$,
- $L \mid \Psi ; \Gamma, x: S @ l \vdash_{c} t: T @ l^{\prime}$;
moreover $L \mid \Psi ; \Gamma \vdash_{p} T^{\prime} \approx \Pi^{l, l^{\prime}}(x: S) . T @ l^{\prime \prime}$ and $L \vdash l^{\prime \prime} \approx l \sqcup l^{\prime}:$ Level.
Notice that in this case, we do not have the well-formedness of $T$ as a sub-derivation. This is reflected in the premises for the $t_{\lambda}$ branch that the global variable $U_{T}$ representing $T$ lives at layer $p$. In general, a global assumption can live at layer $c$ and have a recursive call only if it has a sub-derivation in the typing judgment.

Lemma 5.42. If $L \mid \Psi ; \Gamma \vdash_{c}\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) s: T^{\prime} @ l^{\prime \prime}$, then as sub-derivations

- L| $\Psi ; \Gamma \vdash_{c} S @ l$,
- $L \mid \Psi ; \Gamma, x: S @ l \vdash_{c} T @ l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{c} t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{c} s: S @ l$;
moreover $L \mid \Psi ; \Gamma \vdash_{p} T^{\prime} \approx T[s / x] @ l^{\prime \prime}$ and $L \vdash l^{\prime \prime} \approx l^{\prime}:$ Level.
There is no other possible terms at layer $c$. These lemmas give us a syntactic account of coverage and progress of the recursive principles. In the next section, we give a more rigorous semantic account.


## 6 REDUCTION AND CONVERTIBILITY

We have finished syntactic verification for DeLaM. In this section, let us consider its dynamics by providing the reduction rules for types and terms and the convertibility checking algorithm between two terms. The reduction relations to be given compute the weak head normal forms for types and terms, respectively, and are sub-relations for equivalence judgments for types and terms. We first give the syntax for weak head normal forms and neutral forms, and then give the rules for reduction. We will need the reduction relations to write down the Kripke logical relations in the next section as well as in the convertibility checking algorithm.

### 6.1 Weak Head Normal Forms

The following gives the syntax for weak head normal forms and neutral forms for types and terms. As usual, we use capital case for types and lower case for terms.

$$
\begin{aligned}
& W:=\operatorname{Nat}\left|\Pi^{l, l^{\prime}}(x: S) . T\right| \mathrm{Ty}_{l} \mid \vec{l} \Rightarrow^{l} T \quad \text { (Weak head normal form for types) } \\
& \left|(g: \mathrm{Ctx}) \Rightarrow^{l} T\right|\left(U:\left(\Gamma \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T\left|\square\left(\Gamma \vdash_{c} @ l\right)\right| \square\left(\Gamma \vdash_{c} T @ l\right) \\
& V:=U^{\delta} \mid \mathrm{El}^{l} \mu \quad \text { (Neutral form for types) } \\
& w:=\mu|\operatorname{Nat}| \Pi^{l, l^{\prime}}(x: s) . t\left|\mathrm{Ty}_{l}\right| \text { zero }|\operatorname{succ} t| \lambda^{l, l^{\prime}}(x: S) . t \quad \text { (Weak head normal form for terms (Nf)) } \\
& \left|\Lambda^{l} \vec{\ell} . t\right| \Lambda^{l} \text { g.t }\left|\Lambda_{p}^{l, l^{\prime}} U . t\right| \text { box } T \mid \text { box } t \\
& \mu:=x\left|u^{\delta}\right| \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) \mu\left|\left(\mu: \Pi^{l, l^{\prime}}(x: S) . T\right) s\right| \mu \$ \vec{l} \quad \text { (Neutral form for terms (Ne)) } \\
& |\mu \$ \Gamma| \mu \$_{p} T \mid \text { letbox }_{\text {Typ }}^{l^{\prime}} l \Gamma\left(x_{T} . M\right)\left(U . t^{\prime}\right) \mu \mid \text { letbox }_{\text {Trm }}^{l^{\prime}} l \Gamma T\left(x_{t} . M\right)\left(u . t^{\prime}\right) \mu \\
& \left|\operatorname{elim}_{\mathrm{Typ}}^{l_{1} l_{2}} \vec{M} \vec{b} l \Gamma \mu\right| \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma\left(\operatorname{box} U^{\delta}\right)\left|\operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma T \mu\right| \operatorname{elim}_{\operatorname{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Gamma T\left(\operatorname{box} u^{\delta}\right)
\end{aligned}
$$

Notice that for the recursive principles, we block on global variables, following Hu and Pientka [2024a].

### 6.2 Reduction Relations

There are two required reduction relations, one for types and one for terms. The one for types simply compute types from encodings via El. Unlike Abel et al. [2017] who employed typed reductions, we deliberately use untyped reductions and use preservation later to make sure the reductions are well-defined. This deviation from Abel et al. [2017] requires us to establish enough syntactic theorems before hand. It is particularly important to use untyped reductions because of the way in which the logical relations relate terms. We let $\rightsquigarrow$ * to be the reflexive transitive closure of $\rightsquigarrow$.

$$
\mathrm{El}^{\text {zero }} \mathrm{Nat} \rightsquigarrow \mathrm{Nat} \quad \mathrm{El}^{\operatorname{succ} l} \mathrm{Ty}_{l} \rightsquigarrow \mathrm{Ty}_{l} \quad \mathrm{El}^{l \sqcup l^{\prime}} \Pi^{l, l^{\prime}}(x: s) . t \rightsquigarrow \Pi^{l, l^{\prime}}\left(x: \mathrm{El}^{l} s\right) . \mathrm{El}^{l^{\prime}} t \quad \frac{t \rightsquigarrow t^{\prime}}{\mathrm{El}^{l} t \rightsquigarrow \mathrm{El}^{l} t^{\prime}}
$$

The reduction rules for terms are simply the $\beta$ equivalence rules.

$$
\begin{aligned}
& \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right) \text { zero } \rightsquigarrow s \quad \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right)(\operatorname{succ} t) \rightsquigarrow s^{\prime}\left[t / x, \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t / y\right] \\
& \left(\lambda^{l, l^{\prime}}(x: S) . t: \Pi^{l, l^{\prime}}(x: S) . T\right) s \rightsquigarrow t[s / x] \quad\left(\Lambda^{l} \vec{l} . t\right) \$ \vec{l} \rightsquigarrow t[\vec{l} / \vec{\ell}] \quad\left(\Lambda^{l} g . t\right) \$ \Delta \rightsquigarrow t[\Delta / g] \\
& \left(\Lambda_{p}^{l, l^{\prime}} U . t\right) \$_{p} T \rightsquigarrow t[T / U] \quad \text { letbox }{\underset{T y p}{l}}_{l^{\prime}} l \Delta\left(x_{T} . M\right)\left(U . t^{\prime}\right)(\text { box } T) \rightsquigarrow t^{\prime}[T / U] \\
& \text { letbox }{ }_{T r m}^{l^{\prime}} l \Delta T\left(x_{t} \cdot M\right)\left(u . t^{\prime}\right)(\operatorname{box} t) \rightsquigarrow t^{\prime}[t / u]
\end{aligned}
$$

The reduction rules for recursors follow the same principle. We write down only one rule as an example and omit the rest as they are just $\beta$ rules:

$$
\operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} \text { zero } \Delta(\text { box Nat }) \rightsquigarrow t_{\mathrm{Nat}}[\Delta / g]
$$

The congruence rules reduce the terms at the weak head positions to discover further redices. There are at least one congruence rules for all elimination forms.

$$
\frac{t \rightsquigarrow t^{\prime}}{\operatorname{elim}_{\mathrm{Nat}}^{l}(x \cdot M) s\left(x, y \cdot s^{\prime}\right) t \rightsquigarrow \operatorname{elim}_{\mathrm{Nat}}^{l}(x \cdot M) s\left(x, y \cdot s^{\prime}\right) t^{\prime}} \quad \frac{t \rightsquigarrow t^{\prime}}{\left(t: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) s \rightsquigarrow\left(t^{\prime}: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) s}
$$

For elimination forms for meta-programming, we have

$$
\begin{aligned}
& \frac{t \rightsquigarrow t^{\prime}}{t \$ \vec{l} \rightsquigarrow t^{\prime} \$ \vec{l}} \quad \frac{t \rightsquigarrow t^{\prime}}{t \$ \Delta \rightsquigarrow t^{\prime} \$ \Delta} \quad \frac{t \rightsquigarrow t^{\prime}}{t \$_{p} T \rightsquigarrow t^{\prime} \$_{p} T} \\
& \frac{t \rightsquigarrow t^{\prime}}{\text { letbox }_{\text {Typ }}^{l^{\prime}} l \Delta\left(x_{T} . M\right)(U . s) t \rightsquigarrow \operatorname{letbox}_{\text {Typ }}^{l^{\prime}} l \Delta\left(x_{T} . M\right)(U . s) t^{\prime}} \\
& \frac{t \rightsquigarrow t^{\prime}}{\text { letbox }_{\text {Trm }}^{l^{\prime}} l \Delta T\left(x_{t} . M\right)(\text { u.s }) t \rightsquigarrow \text { letbox }_{\text {Trm }}^{l^{\prime}} l \Delta T\left(x_{t} . M\right)(u . s) t^{\prime}}
\end{aligned}
$$

There are also congruence rules for the recursive principles. For the recursive principle for code of terms, we choose to reduce the type to weak head normal form first and then reduce the term itself. This order is arbitrary
and can be flipped. We simply fix a choice here.

$$
\begin{aligned}
& \frac{t \rightsquigarrow t^{\prime}}{\underset{\operatorname{elim}}{\mathrm{T}_{\mathrm{Tp}}, l_{2}} \vec{M} \vec{b} l^{\prime} \Delta t \rightsquigarrow \operatorname{elim}_{\mathrm{Typ}}^{l_{1} l_{2}} \vec{M} \vec{b} \quad l^{\prime} \Delta t^{\prime}} \\
& \frac{T \rightsquigarrow T^{\prime}}{\operatorname{elim}_{T r m}^{l_{1} l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T t \rightsquigarrow \operatorname{elim}_{T r m}^{l_{1} l_{2}} \vec{M} \vec{b} l^{\prime} \Delta T^{\prime} t} \\
& \overline{\operatorname{elim}_{\operatorname{Trm}}^{l_{1} l_{2}} \vec{M} \vec{b} l^{\prime} \Delta W t \rightsquigarrow \operatorname{elim}_{\operatorname{Trm}}^{l_{1} l_{2}} \vec{M} \vec{b} l^{l^{\prime}} \Delta W t^{\prime}}
\end{aligned}
$$

We first verify the fact that reductions are just sub-relations of the equivalence judgments:
Lemma 6.1 (Soundness). Given i computable,

- if $L \mid \Psi ; \Gamma \vdash_{i} T$ @ $l$ and $T \rightsquigarrow T^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$;
- if $L \mid \Psi ; \Gamma \vdash_{i} t: T$ @l and $t \rightsquigarrow t^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ $l$.

Proof. We proceed by mutual induction on the typing judgments and then invert the reduction relations. We use $i$ computable to make sure computation rules are available. We select a few cases for discussion: Case The following is the only possible rule for types:

$$
\frac{L \vdash l: \text { Level } \quad L \mid \Psi ; \Gamma \vdash_{i} t: T y_{l} @ \operatorname{succ} l}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{El}^{l} t @ l}
$$

Inversion of $\mathrm{El}^{l} t \rightsquigarrow T^{\prime}$ gives four possible subcases. We only consider two:
Subcase

$$
\mathrm{El}^{l_{1} \mathrm{l}_{2}} \Pi^{l_{1}, l_{2}}(x: s) \cdot t \rightsquigarrow \Pi^{l_{1}, l_{2}}\left(x: \mathrm{El}^{l_{1}} s\right) \cdot \mathrm{El}^{l_{2}} t
$$

Then we know

$$
L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l_{1}, l_{2}}(x: s) . t: \mathrm{Ty}_{l} @ \operatorname{succ} l
$$

We further do an inner induction on the typing judgment above after generalizing $\mathrm{Ty}_{l}$ to some arbitrary $T$. There are only three cases to consider:
Subsubcase

$$
\begin{gathered}
L \vdash l_{1}: \text { Level } \\
L \vdash l_{2}: \text { Level } \quad L\left|\Psi ; \Gamma \vdash_{i} s: \mathrm{Ty}_{l_{1}} @ \operatorname{succ} l_{1} \quad L\right| \Psi ; \Gamma, x: E l^{l_{1}} s @ l_{1} \vdash_{i} t: \mathrm{Ty}_{l_{2}} @ \operatorname{succ} l_{2} \\
L \mid \Psi ; \Gamma \vdash \cdot \Pi_{i}^{l_{1}, l_{2}}(x: s) . t: \mathrm{Ty}_{l_{l}} @ \operatorname{succ}\left(l_{1} \sqcup l_{2}\right)
\end{gathered}
$$

Then this case we derive the goal immediately from the El rule for $\Pi$ types.
Subsubcase

$$
\frac{L\left|\Psi ; \Gamma \vdash_{i} \Pi^{l_{1,2}}(x: s) . t: T^{\prime} @ l^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{\text {typeof }(i)} T \approx T^{\prime} @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l_{1}, l_{2}}(x: s) \cdot t: T @ l^{\prime}}
$$

In this case, we simply apply the inner IH to obtain the goal.
Subsubcase

$$
\frac{L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l_{1}, l_{2}}(x: s) \cdot t: T @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l_{1}, l_{2}}(x: s) \cdot t: T @ l}
$$

Similarly, we use the inner IH to obtain the goal.
In general, when we know the form of a term, an inner induction must reveal only three cases to consider. This pattern appears a lot when we consider cases for types.

Subcase We have this case

$$
\frac{t \rightsquigarrow t^{\prime}}{\mathrm{El}^{l} t \rightsquigarrow \mathrm{El}^{l} t^{\prime}}
$$

Then by IH, we have

$$
L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l
$$

We obtain the goal by the congruence rule for El .
Case

$$
\frac{L\left|\Psi ; \Gamma \vdash_{i} t: T^{\prime} @ l \quad L\right| \Psi ; \Gamma \vdash_{\text {typeof }(i)} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} t: T @ l}
$$

By IH, we have

$$
L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T^{\prime} @ l
$$

We obtain the goal by the conversion rule.
Case

$$
\frac{L \mid \Psi ; \Gamma \vdash_{i} t: T @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} t: T @ l}
$$

By IH, we have

$$
L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l^{\prime}
$$

We obtain the goal by the conversion rule.
Case For the recursive principle for natural numbers, there are three subcases after inverting the reduction premise. We apply $\beta$ rules or the congruence rule properly.
The same principle applies for the recursive principles for code, but a bit more complex. We will use theorems from Sec. 5.4 in combination of the congruence rules to obtain our goals.

As a corollary,
Lemma 6.2 (Preservation).

- If $L \mid \Psi ; \Gamma \vdash_{i} T$ @ $l$ and $T \rightsquigarrow T^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$ and $t \rightsquigarrow t^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t^{\prime}: T @ l$.

Proof. We analyze $i$. If $i=v$, then there is no applicable reduction rule. If $i=c$, then we use lifting to lift $i$ to $p$, then use the soundness, presupposition and unlifting lemmas to obtain the goals. Otherwise, we use the soundness lemma and the presupposition lemma.

The substitution lemmas require the well-formedness of types and the well-typedness of terms to make use of algebraic laws of substitutions.

Lemma 6.3 (Universe Substitutions). Given i computable,

- if $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} T @ l, T \rightsquigarrow T^{\prime}$ and $L \vdash \phi: L^{\prime}$, then $T[\phi] \rightsquigarrow T^{\prime}[\phi]$;
- if $L^{\prime} \mid \Psi ; \Gamma \vdash_{i} t: T @ l, t \rightsquigarrow t^{\prime}$ and $L \vdash \phi: L^{\prime}$, then $t[\phi] \rightsquigarrow t^{\prime}[\phi]$.

Lemma 6.4 (Local Substitutions). Given $i$ computable,

- if $L \mid \Psi ; \Delta \vdash_{i} T @ l, T \rightsquigarrow T^{\prime}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $T[\delta] \rightsquigarrow T^{\prime}[\delta]$;
- if $L \mid \Psi ; \Delta \vdash_{i} t: T @ l, t \rightsquigarrow t^{\prime}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $t[\delta] \rightsquigarrow t^{\prime}[\delta]$.

Lemma 6.5 (Global Substitutions). Given i computable,

- if $L \mid \Phi ; \Gamma \vdash_{i} T @ l, T \rightsquigarrow T^{\prime}$ and $L \mid \Psi \vdash \sigma: \Phi$, then $T[\sigma] \rightsquigarrow T^{\prime}[\sigma]$;
- if $L \mid \Phi ; \Gamma \vdash_{i} t: T @ l, t \rightsquigarrow t^{\prime}$ and $L \mid \Psi \vdash \sigma: \Phi$, then $t[\sigma] \rightsquigarrow t^{\prime}[\sigma]$.

All lemmas above also work for the reflexive transitive closure versions of reduction.
Lemma 6.6 (Determinacy).

- If $T \rightsquigarrow T^{\prime}$ and $T \rightsquigarrow T^{\prime}$, then $T^{\prime}=T^{\prime \prime}$.
- If $t \rightsquigarrow t^{\prime}$ and $t \rightsquigarrow t^{\prime \prime}$, then $t^{\prime}=t^{\prime \prime}$.

If a multi-step reduction reaches a normal form, then we know this normal form is also uniquely determined:

## Lemma 6.7 (Determinacy).

- If $T \rightsquigarrow^{*} W$ and $T \rightsquigarrow{ }^{*} W^{\prime}$, then $W=W^{\prime}$.
- If $t \rightsquigarrow^{*} w$ and $t \rightsquigarrow^{*} w^{\prime}$, then $w=w^{\prime}$.

Proof. Induction. Use the fact that weak head normal forms do not reduce and determinacy of single-step reduction.

Due to preservation, we often are interested in keeping track of well-formedness and well-typedness of types and terms. Therefore it is convenient to give the following convenient auxiliary judgments:

$$
\begin{aligned}
\frac{L \mid \Psi ; \Gamma \vdash_{i} T @ l}{L \mid \Psi ; \Gamma \vdash_{i} T \rightsquigarrow T^{\prime} @ l} & \frac{L \mid \Psi ; \Gamma \vdash_{i} T @ l}{L \mid \Psi ; \Gamma \vdash_{i} T ~^{\prime}{ }^{*} T^{\prime} @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} t: T @ l}{L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow t^{\prime}: T @ l} t \rightsquigarrow t^{\prime} \\
& \frac{L \mid \Psi ; \Gamma \vdash_{i} t: T @ l}{L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} t^{\prime}: T @ l} t \rightsquigarrow^{*} t^{\prime} \\
&
\end{aligned}
$$

### 6.3 Convertibility Checking

The convertibility checking is standard: we first reduce types or terms to their weak head normal forms using reduction, and then recursively compare the sub-structures. Either we detect a mismatch which causes a failure, or everything checks out and the convertibility is verified.

Following this line, we give the following judgments for convertibility checking. Here we always quantify $i$ computable. The layering index $i$ restricts only types (i.e. those in MLTT or in DeLAM), but not terms. In other words, it is possible for convertibility checking to relate at layer $p$ two terms only well-typed at layer $m$, as long as these two terms have type well-formed at layer $p$ (i.e. MLTT). This is a critical property to establish a relation between the logical relations at both layers.

- $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime} @ l$ denotes that $T$ and $T^{\prime}$ are convertible at universe level $l$.
- $L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l$ denotes that $W$ and $W^{\prime}$ are convertible normal types.
- $L \mid \Psi ; \Gamma \vdash_{i} V \longleftrightarrow V^{\prime} @ l$ denotes that $V$ and $V^{\prime}$ are convertible neutral types.
$\bullet L \mid \Psi \vdash_{i} \Gamma \stackrel{\Longleftrightarrow}{\Longleftrightarrow}$ denotes that $\Gamma$ and $\Delta$ are convertible contexts. This judgment is defined by using $L \mid \Psi ; \Gamma \vdash_{i} T \stackrel{\wedge}{\Longleftrightarrow} T^{\prime}$ @ $l$ pairwise.
- $L \mid \Psi ; \Gamma \vdash_{i} t \Longleftrightarrow t^{\prime}: T$ @ $l$ denotes that $t$ and $t^{\prime}$ of type $T$ are convertible.
- $L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l$ denotes that $w$ and $w^{\prime}$ are convertible normal terms of a normal type $W$.
- $L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\mu}{\longleftrightarrow}$ 信 $T$ @ $l$ denotes that $\mu$ and $\mu^{\prime}$ are convertible neutral terms. $T$ is the result of inference.
- $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l$ denotes that $\mu$ and $\mu^{\prime}$ are convertible neutral terms of a normal type $W$. $W$ is the result of inference.
$\bullet L \mid \Psi ; \Gamma \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta$ denotes that $\delta$ and $\delta^{\prime}$ are convertible local substitutions. This judgment is defined by using $L \mid \Psi ; \Gamma \vdash_{i} t \stackrel{ }{\Longleftrightarrow} t^{\prime}: T @ l$ pairwise.

We give the following convertibility checking rules for types first:

$$
\begin{aligned}
& \frac{L\left|\Psi ; \Gamma \vdash_{i} T \rightsquigarrow^{*} W @ l \quad L\right| \Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow^{*} W^{\prime} @ l \quad L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime} @ l} \\
& \frac{L \mid \Psi \vdash_{i} \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} \text { Nat } \Longleftrightarrow \text { Nat @ zero }} \quad \frac{L \mid \Psi \vdash_{i} \Gamma \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Ty}_{l} \Longleftrightarrow \mathrm{Ty}_{l^{\prime}} \text { @ succ } l} \\
& \frac{L\left|\Psi ; \Gamma \vdash_{i} S \Longleftrightarrow S^{\prime} @ l \quad L\right| \Psi ; \Gamma, x: S @ l \vdash_{i} T \Longleftrightarrow T^{\prime} @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}(x: S) . T \Longleftrightarrow \Pi^{l, l^{\prime}}\left(x: S^{\prime}\right) . T^{\prime} @ l \sqcup l^{\prime}} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} V \Longleftrightarrow V^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} V \Longleftrightarrow V^{\prime} @ l} \\
& \frac{L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l^{\prime} \quad \Gamma \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l} \\
& \frac{L\left|\Psi \vdash_{i} \Gamma \quad U:\left(\Delta \vdash_{i^{\prime}} @ l\right) \in \Psi \quad i^{\prime} \in\{c, p\} \quad i^{\prime} \leq i \quad L\right| \Psi ; \Gamma \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta}{L \mid \Psi ; \Gamma \vdash_{i} U^{\delta} \longleftrightarrow U^{\delta^{\prime}} @ l} \\
& \frac{L \vdash l \approx l^{\prime}: \text { Level } \quad L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{El}^{l} \mu \longleftrightarrow \mathrm{El}^{l^{\prime}} \mu^{\prime} @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} V \longleftrightarrow V^{\prime} @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} V \longleftrightarrow V^{\prime} @ l}
\end{aligned}
$$

For the types only available at layer $m$ :

$$
\begin{aligned}
& \frac{L\left|\Psi \vdash_{m} \Gamma \quad L, \vec{\ell}\right| \Psi ; \Gamma \vdash_{m} T \Longleftrightarrow T^{\prime} @ l \quad L, \vec{\ell}+l \approx l^{\prime} \text { : Level }}{L \mid \Psi ; \Gamma \vdash_{m} \vec{\ell} \Rightarrow^{l} T \Longleftrightarrow \vec{\ell} \Rightarrow^{l^{\prime}} T^{\prime} @ \omega} \\
& L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} T \Longleftrightarrow T^{\prime} @ l \quad L \vdash l \approx l^{\prime} \text { : Level } \\
& L \mid \Psi ; \Gamma \vdash_{m}(g: \mathrm{Ctx}) \Rightarrow^{l} T \Longleftrightarrow(g: \mathrm{Ctx}) \Rightarrow^{l^{\prime}} T^{\prime} @ l \\
& L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l_{1}\right) ; \Gamma \vdash_{m} T \Longleftrightarrow T^{\prime} @ l_{2} \\
& \frac{L \mid \Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \quad L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4} \text { : Level }}{L \mid \Psi ; \Gamma \vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T \Longleftrightarrow\left(U:\left(\Delta^{\prime} \vdash_{p} @ l_{3}\right)\right) \Rightarrow^{l_{4}} T^{\prime} @ \text { succ } l_{1} \sqcup l_{2}} \\
& \frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \stackrel{\Delta^{\prime}}{\Longleftrightarrow}}{L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} @ l\right) \Longleftrightarrow \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \text { succ } l} \\
& \frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \quad L \mid \Psi ; \Delta \vdash_{p} T \Longleftrightarrow T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T @ l\right) \Longleftrightarrow \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l\right) @ l}
\end{aligned}
$$

We propagate the convertibility for types pairwise to obtain the convertibility for local contexts.

$$
\begin{aligned}
& \frac{L \vdash \Psi}{L \mid \Psi \vdash_{i} \cdot \Leftarrow} \quad \frac{L \vdash \Psi \quad g: \mathrm{Ctx} \in \Psi}{\Longleftrightarrow} \\
& \frac{L\left|\Psi \vdash_{i} \Gamma \stackrel{\Longleftrightarrow}{\Longleftrightarrow} \quad L\right| \Psi ; \Gamma \vdash_{i} T \stackrel{\Longleftrightarrow}{\Longleftrightarrow} T^{\prime} @ l \quad L \vdash l \approx l^{\prime} \text { : Level }}{L \mid \Psi \vdash_{i} \Gamma, x: T @ l \Longleftrightarrow \Delta, x: T^{\prime} @ l^{\prime}}
\end{aligned}
$$

The convertibility of terms proceeds similarly. The following are checking rules that are available at both layers:

$$
\begin{aligned}
& \frac{T \rightsquigarrow^{*} W \quad L\left|\Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} w: T @ l \quad L\right| \Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} w^{\prime}: T @ l \quad L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l}{L \mid \Psi ; \Gamma \vdash_{i} t \Longleftrightarrow t^{\prime}: T @ l} \\
& \frac{L \mid \Psi \vdash_{i} \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} \text { Nat } \Longleftrightarrow \text { Nat: Ty }{ }_{\text {zero }} @ \text { succ zero }} \\
& \frac{L \mid \Psi \vdash_{i} \Gamma \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Ty}_{l} \Longleftrightarrow \mathrm{Ty}_{l^{\prime}}: \mathrm{Ty}_{\text {succ } l} @ \text { succ succ } l} \\
& L\left|\Psi ; \Gamma \vdash_{i} s \Longleftrightarrow s^{\prime}: \operatorname{Ty}_{l} @ \operatorname{succ} l \quad L\right| \Psi ; \Gamma, x: \mathrm{El}^{l} s @ l \vdash_{i} t \stackrel{ }{\Longleftrightarrow} t^{\prime}: \mathrm{Ty}_{l^{\prime}} @ \operatorname{succ} l^{\prime} \\
& L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}(x: s) . t \Longleftrightarrow \Pi^{l, l^{\prime}}\left(x: s^{\prime}\right) \cdot t^{\prime}: \mathrm{Ty}_{l \sqcup l^{\prime}} @ \operatorname{succ}\left(l \sqcup l^{\prime}\right) \\
& L \mid \Psi \vdash_{i} \Gamma \\
& \overline{L \mid \Psi ; \Gamma \vdash_{i} \text { zero } \Longleftrightarrow \text { zero : Nat @ zero }} \\
& L \mid \Psi ; \Gamma \vdash_{i} t \Longleftrightarrow t^{\prime} \text { : Nat @ zero } \\
& L \mid \Psi ; \Gamma \vdash_{i} \text { succ } t \Longleftrightarrow \operatorname{succ} t^{\prime}: \text { Nat @ zero } \\
& \frac{L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: \text { Nat @ zero }}{L \mid \Psi ; \Gamma \vdash_{i} \mu \Longleftrightarrow \mu^{\prime}: \text { Nat @ zero }} \\
& L\left|\Psi ; \Gamma \vdash_{i} S @ l \quad L\right| \Psi ; \Gamma \vdash_{i} w: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{i} w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
& L \mid \Psi ; \Gamma, x: S @ l \vdash_{i}\left(w: \Pi^{l, l^{\prime}}(x: S) . T\right) x \Longleftrightarrow\left(w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T\right) x: T @ l^{\prime} \\
& L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
& \frac{L \mid \Psi ; \Gamma \vdash_{i} \mu \Longleftrightarrow \mu^{\prime}: W @ l}{L \mid \Psi ; \Gamma \vdash_{i} \mu \Longleftrightarrow \mu^{\prime}: V @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l}
\end{aligned}
$$

The following rules check terms that are available only at layer $m$ :

$$
\begin{gathered}
L\left|\Psi ; \Gamma \vdash_{m} w: \vec{l} \Rightarrow^{l} T @ \omega \quad L\right| \Psi ; \Gamma \vdash_{m} w^{\prime}: \vec{l} \Rightarrow^{l} T @ \omega \quad L, \vec{l} \mid \Psi ; \Gamma \vdash_{m} t \$ \vec{l} \Longleftrightarrow t^{\prime} \$ \vec{\ell}: T @ l \\
L \mid \Psi ; \Gamma \vdash_{m} w \Longleftrightarrow w^{\prime}: \vec{l} \Rightarrow^{l} T @ \omega \\
L \mid \Psi ; \Gamma \vdash_{m} w:(g: C t x) \Rightarrow^{l} T @ l \\
\frac{L\left|\Psi ; \Gamma \vdash_{m} w^{\prime}:(g: C t x) \Rightarrow^{l} T @ l \quad L\right| \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} w \$ g \Longleftrightarrow w^{\prime} \$ g: T @ l}{L \mid \Psi ; \Gamma \vdash_{m} w \Longleftrightarrow w^{\prime}:(g: \mathrm{Ctx}) \Rightarrow^{l} T @ l}
\end{gathered}
$$

$L \mid \Psi ; \Gamma \vdash_{m} w:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T$ @ succ $l \sqcup l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{m} w^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T @ \operatorname{succ} l \sqcup l^{\prime}$ $L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} w \$_{p} U^{\mathrm{id} d_{\Delta}} \Longleftrightarrow w^{\prime} \$_{p} U^{\mathrm{id} \Delta}: T @ l^{\prime}$
$L \mid \Psi ; \Gamma \vdash_{m} w \Longleftrightarrow w^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T @ \operatorname{succ} l \sqcup l^{\prime}$

$$
\begin{array}{cll}
\frac{L \mid \Psi \vdash_{m} \Gamma}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{box} T \Longleftrightarrow \Psi ; \Delta \vdash_{c} T @ l \quad T=T^{\prime}} & & \frac{L \mid \Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l}{L \mid \Psi ; \Gamma \vdash_{m} \mu \Longleftrightarrow T^{\prime}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l} \\
\frac{L \mid \Psi \vdash_{m} \Gamma}{\left.L \mid \Psi ; \Gamma \vdash_{m} \operatorname{box} t \Longleftrightarrow \vdash_{c} @ l\right) @ \operatorname{succ} l} \\
L \mid \Psi ; \Delta \vdash_{c} t: T @ l \quad t=t^{\prime} \\
\operatorname{box} t^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ l & & \frac{L \mid \Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ l}{L \mid \Psi ; \Gamma \vdash_{m} \mu \Longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ l}
\end{array}
$$

Notice here convertibility of box'ed types and terms are checked simply with syntactic equality. The convertibility of neutral terms proceeds as follows. Similarly, we first give the checking rules that are available at both
layers:

$$
L \vdash l \approx l^{\prime} \text { : Level } \quad L \mid \Psi ; \Gamma, x: \text { Nat @ zero } \vdash_{i} M \Longleftrightarrow M^{\prime} @ l \quad L \mid \Psi ; \Gamma \vdash_{i} s_{1} \Longleftrightarrow s_{3}: M[\text { zero } / x] @ l
$$

$$
L \mid \Psi ; \Gamma, x: \text { Nat @ zero, } y: M @ l \vdash_{i} s_{2} \Longleftrightarrow s_{4}: M[\operatorname{succ} x / x] @ l \quad L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: \text { Nat @ zero }
$$

$$
L \mid \Psi ; \Gamma \vdash_{i} \operatorname{elim}_{\mathrm{Nat}}^{l}(x . M) s_{1}\left(x, y \cdot s_{2}\right) \mu \stackrel{ }{\longleftrightarrow} \operatorname{elim}_{\mathrm{Nat}}^{l^{\prime}}\left(x \cdot M^{\prime}\right) s_{3}\left(x, y \cdot s_{4}\right) \mu^{\prime}: M[\mu / x] @ l
$$

$L \vdash l_{1} \approx l_{3}$ : Level $\quad L \vdash l_{2} \approx l_{4}$ : Level $L\left|\Psi ; \Gamma \vdash_{i} S \Longleftrightarrow S^{\prime} @ l_{1} \quad L\right| \Psi ; \Gamma, x: S @ l_{1} \vdash_{i} T \Longleftrightarrow T^{\prime} @ l_{2}$ $L\left|\Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S^{\prime \prime}\right) \cdot T^{\prime \prime} @ l_{1} \sqcup l_{2} \quad L\right| \Psi ; \Gamma \vdash_{i} s \Longleftrightarrow s^{\prime}: S @ l_{1}$

$$
L \mid \Psi ; \Gamma \vdash_{i}\left(\mu: \Pi^{l_{1}, l_{2}}(x: S) . T\right) s \stackrel{\leftrightarrow}{\longleftrightarrow}\left(\mu^{\prime}: \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot T^{\prime}\right) s^{\prime}: T[s / x] @ l_{2}
$$

When checking applications, we simply ignore the type inferred by checking $\mu$ and $\mu^{\prime}$. This is fine because we already know $\mu$ and $\mu^{\prime}$ are well-typed so the type annotations must be equivalent.

Then we give the rules only available at layer $m$ :

The remaining piece of the convertibility checking for neutral recursive principles. The recursive principles get stuck when the scrutinees are neutral or box'ed global variables. To check the convertibility of neutral recursive principles, we recursively check the convertibility between motives, corresponding branches and the indexing

$$
\begin{aligned}
& L\left|\Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \vec{l} \Rightarrow^{l} T @ \omega \quad\right| \vec{\ell}\left|=|\vec{l}|=\left|\vec{l}^{\prime}\right|>c \quad \forall c \leq n<|\vec{l}| \cdot L \vdash \vec{l}(n) \approx \vec{l}^{\prime}(n):\right. \text { Level } \\
& L \mid \Psi ; \Gamma \vdash_{m} \mu \$ \vec{l} \stackrel{\mu^{\prime}}{\stackrel{ }{\longleftrightarrow} \vec{l}^{\prime}: T[\vec{l} / \vec{l}] @ l[\vec{l} / \vec{l}]} \\
& \frac{L\left|\Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}:(g: C t x) \Rightarrow^{l} T @ l \quad L\right| \Psi \vdash_{p} \Delta \stackrel{\Longleftrightarrow}{\Longleftrightarrow} \Delta^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \mu \$ \Delta \longleftrightarrow \mu^{\prime} \$ \Delta^{\prime}: T[\Delta / g] @ l} \\
& \frac{L\left|\Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T \Longleftrightarrow T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{m} \mu \$_{p} T \longleftrightarrow \mu^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}[T / U] @ l^{\prime}} \\
& L \mid \Psi \vdash_{m} \Gamma \quad L \vdash l_{1} \approx l_{3} \text { : Level } \quad L \vdash l_{2} \approx l_{4} \text { : Level } \quad L \mid \Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \\
& L\left|\Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l_{2}\right) @ \operatorname{succ} l_{2} \quad L\right| \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} @ l_{2}\right) @ \operatorname{succ} l_{2} \vdash_{m} M \Longleftrightarrow M^{\prime} @ l_{1} \\
& L \mid \Psi, U:\left(\Delta \vdash_{c} @ l_{2}\right) ; \Gamma \vdash_{m} t_{1} \Longleftrightarrow t_{2}: M\left[\text { box } U / x_{T}\right] @ l_{1} \\
& L \mid \Psi ; \Gamma \vdash_{m} \text { letbox }_{\text {Typ }}^{l_{1}} l_{2} \Delta\left(x_{T} \cdot M\right)\left(U . t_{1}\right) \mu \stackrel{\longleftrightarrow}{\longleftrightarrow} \text { letbox }_{\text {Typ }}^{l_{3}} l_{4} \Delta^{\prime}\left(x_{T} \cdot M^{\prime}\right)\left(U . t_{2}\right) \mu^{\prime}: M\left[t / x_{T}\right] @ l_{1} \\
& L \mid \Psi \vdash_{m} \Gamma \quad L \vdash l_{1} \approx l_{3} \text { : Level } \quad L \vdash l_{2} \approx l_{4} \text { : Level } L\left|\Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{p} T \stackrel{\Longleftrightarrow}{\Longleftrightarrow} \text { Q } l_{2} \\
& L\left|\Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{2} \quad L\right| \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{1} \vdash_{m} M \Longleftrightarrow M^{\prime} @ l_{1} \\
& L \mid \Psi, u:\left(\Delta \vdash_{c} T @ l_{2}\right) ; \Gamma \vdash_{m} t_{1} \stackrel{ }{\Longleftrightarrow} t_{2}: M\left[\operatorname{box} u / x_{t}\right] @ l_{1} \\
& L \mid \Psi ; \Gamma \vdash_{m} \text { letbox }_{\text {Trm }}^{l_{1}} l_{2} \Delta T\left(x_{t} \cdot M\right)\left(U . t_{1}\right) \mu \stackrel{\longleftrightarrow}{\longleftrightarrow} \text { letbox }_{\text {Trm }}^{l_{3}} l_{4} \Delta^{\prime} T^{\prime}\left(x_{T} . M^{\prime}\right)\left(U . t_{2}\right) \mu^{\prime}: M\left[t / x_{t}\right] @ l_{1}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\longleftrightarrow}{\longleftrightarrow} \mu^{\prime}: T @ l \quad T \rightsquigarrow^{*} W}{L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\mu}{\longleftrightarrow}: T @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\mu^{\prime}}{ }: T @ l} \\
& \frac{L \mid \Psi \vdash_{i} \Gamma \quad x: T @ l \in \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} x \stackrel{\leftrightarrow}{\longleftrightarrow}: T @ l} \\
& L\left|\Psi \vdash_{i} \Gamma \quad u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi \quad i^{\prime} \in\{v, c\} \quad i^{\prime} \leq i \quad L\right| \Psi ; \Gamma \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta \\
& L \mid \Psi ; \Gamma \vdash_{i} u^{\delta} \stackrel{ }{\longleftrightarrow} u^{\delta^{\prime}}: T[\delta] @ l
\end{aligned}
$$

universe levels, local contexts and potentially types. To derive the following two conclusions:

$$
\begin{gathered}
L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Typ}}^{l_{1} l_{2}} \vec{M} \vec{b} l \Delta \mu \stackrel{\longleftrightarrow}{\longleftrightarrow} \operatorname{elim}_{\mathrm{Typ}}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} \mu^{\prime}: M\left[\mu / x_{T}\right] @ l_{1} \\
L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta T \mu \stackrel{\operatorname{elim}_{\mathrm{Typ}}^{l_{3}, l_{4}}}{\overrightarrow{M_{1}}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} T^{\prime} \mu^{\prime}: M^{\prime}\left[\mu / x_{t}\right] @ l_{1}
\end{gathered}
$$

We proceed by checking the convertibility of motives:

$$
\begin{gathered}
L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level } L, \ell \mid \Psi, g: \mathrm{Ctx} ; \Gamma, x_{T}: \square\left(g \vdash_{c} @ \ell\right) \vdash_{m} M \Longleftrightarrow M_{1} @ l_{1} \\
L, \ell \mid \Psi, g: \mathrm{Ctx}, U_{T}:\left(g \vdash_{p} @ \ell\right) ; \Gamma, x_{t}: \square\left(g \vdash_{c} U_{T}^{\text {id }} @ \ell\right) \vdash_{m} M^{\prime} \Longleftrightarrow M_{1}^{\prime} @ l_{2}
\end{gathered}
$$

We do the same for all the branches as well. Following the previous conventions, we group all these checking into $C_{A}$ for convertibility checking for all premises. Then what we have left is to make sure the scrutinees are convertible.

$$
\begin{aligned}
& C_{A} \quad L \vdash l \approx l^{\prime} \text { : Level } \quad L\left|\Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l\right) @ l \\
& L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta \mu \stackrel{\wedge}{\longleftrightarrow} \operatorname{elim}_{\mathrm{Typ}}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} \mu^{\prime}: M\left[\mu / x_{T}\right] @ l_{1} \\
& C_{A} \quad L \vdash l \approx l^{\prime} \text { : Level } \\
& L\left|\Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{p} T \stackrel{\Longleftrightarrow}{\Longleftrightarrow} T^{\prime} @ \operatorname{succ} l \quad L \mid \Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ \operatorname{succ} l \\
& L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Trm}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta T \mu \stackrel{\operatorname{elim}}{\mathrm{Typ}} \operatorname{l}_{3} l_{4} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} T^{\prime} \mu^{\prime}: M^{\prime}\left[\mu / x_{t}\right] @ l_{1}
\end{aligned}
$$

If the scrutinees are box'ed global variables, then the check is always the same, except that the global variables are compared syntactically:

$$
\begin{aligned}
& C_{A} \quad L \vdash l \approx l^{\prime}: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \quad U:\left(\Delta \vdash_{c} @ l\right) \in \Psi \quad L\right| \Psi ; \Gamma \vdash_{c} \delta: \Delta \\
& L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\mathrm{Typ}}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta\left(\operatorname{box} U^{\delta}\right) \stackrel{\wedge}{\longleftrightarrow} \operatorname{elim}_{\mathrm{Typ}}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime}\left(\operatorname{box} U^{\delta}\right): M\left[\operatorname{box} U^{\delta} / x_{T}\right] @ l_{1} \\
& C_{A} \quad L \vdash l \approx l^{\prime} \text { : Level } \quad L \mid \Psi \vdash_{p} \Delta \Longleftrightarrow \Delta^{\prime} \\
& L\left|\Psi ; \Gamma \vdash_{p} T \Longleftrightarrow T^{\prime} @ \operatorname{succ} l \quad u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi \quad i^{\prime} \in\{v, c\} \quad L\right| \Psi ; \Gamma \vdash_{c} \delta: \Delta \\
& L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{\operatorname{Trm}}^{l_{1} l_{2}} \vec{M} \vec{b} l \Delta T\left(\operatorname{box} u^{\delta}\right) \stackrel{\operatorname{elim}_{\mathrm{Typ}}}{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} T^{\prime}\left(\operatorname{box} u^{\delta}\right): M^{\prime}\left[\operatorname{box} u^{\delta} / x_{t}\right] @ l_{1}
\end{aligned}
$$

Now we have finished all the convertibility rules for neutral terms.
We simply let the convertibility for terms to propagate pairwise to derive the convertibility for local substitutions:

$$
\begin{aligned}
& \frac{L \mid \Psi \vdash_{i} \Gamma \quad \Gamma \text { ends with } \cdot|\Gamma|=m}{L \mid \Psi ; \Gamma \vdash_{i}{ }^{m} \stackrel{{ }^{m}}{\Longleftrightarrow} \cdot} \quad \frac{L \mid \Psi \vdash_{i} \Gamma \quad g: \operatorname{Ctx} \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m}{L \mid \Psi ; \Gamma \vdash_{i}{ }_{g}^{m} \stackrel{{ }_{g}^{m}}{\Longleftrightarrow} \cdot} \\
& \begin{array}{l}
L \mid \Psi \vdash_{i} \Gamma \quad g: \operatorname{Ctx} \in \Psi \\
\Gamma \text { ends with } g \quad|\Gamma|=m
\end{array} \\
& L \mid \Psi ; \Gamma \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta \\
& \frac{L\left|\Psi ; \Delta \vdash_{i} T @ l \quad L\right| \Psi ; \Gamma \vdash_{i} t \stackrel{\wedge}{\Longleftrightarrow} t^{\prime}: T[\delta] @ l}{L \mid \Psi ; \Gamma \vdash_{i} \delta, t / x \Longleftrightarrow \delta^{\prime}, t^{\prime} / x: \Delta, x: T @ l}
\end{aligned}
$$

The convertibility algorithm is obtained by reading all the components for convertibility rules as inputs and the neutral judgments consider types as outputs. If there is no corresponding rule, then two terms are not convertible; otherwise, two terms are convertible. We verify some basic properties as follows:

Lemma 6.8 (Soundness). Assuming i computable,
$\bullet$ if $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$;

- if $L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} W \approx W^{\prime} @ l$;
- if $L \mid \Psi ; \Gamma \vdash_{i} V \longleftrightarrow V^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} V \approx V^{\prime} @ l$;
- if $L \mid \Psi \vdash_{i} \Gamma \stackrel{\wedge}{\Longleftrightarrow}$, then $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$;
- if $L \mid \Psi ; \Gamma \vdash_{i} t \Longleftrightarrow t^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$;
- if $L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} w \approx w^{\prime}: W @ l$;
- if $L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\mu}{\longleftrightarrow}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} \mu \approx \mu^{\prime}: T @ l$;
- if $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} \mu \approx \mu^{\prime}: W @ l$;
- if $L \mid \Psi ; \Gamma \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$.

Proof. Mutual induction. Use $\eta$ rules for all kinds of function types. Use congruence rules, presupposition and conversion rules when checking neutral terms.

## Lemma 6.9 .

- If $L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W$ @ l.

Other lemmas like PER require the fundamental theorems so we postpone their proofs until we have the semantic models.

## 7 LOGICAL RELATIONS FOR DELAM

Previously, we have given the judgments of DeLaM, verified its syntactic properties and given its reduction and convertibility algorithms. Starting this section, we establish the logical relation and prove the (weak) normalization and convertibility properties of DeLAM. Following Abel et al. [2017], we proceeds as follows:

- First we give a set of generic equivalence conditions for a parameterized discussion of the logical relation.
- Then we give the definition of the Kripke logical relations of types and terms. The logical relations are parameterized by layers. In this step, we are only concerned about types that are available at all layers, i.e. those in MLTT and unrelated to meta-programming.
- Then we give the definition of the Kripke logical relations of local contexts and local substitutions.
- Then we branch off two orthogonal developments.
- We give the definition of the Kripke logical relations of global contexts and global substitutions.
- We give the definition of the Kripke logical relations of types and terms, again. But in this case, we must also give the definition for types that are related to meta-programming, i.e. contextual types.
In fact, the definitions given by the two sub-steps above must consider each other. Otherwise, we will not able to extend related global substitutions during the proof of the fundamental theorems.
- Next we give the semantic judgments. The semantic judgments require types, terms, etc. to be stable under all universe, global and local substitutions.
- Finally, we establish the fundamental theorems for the semantic judgments. Instantiating the generic equivalence gives us the proof of convertibility.

Due to layering, following Sec. 3, the generic equivalence, logical relation and validity judgments are all layered. In fact, since computation exists at both layers $p$ and $m$, the situation is very complex. Abel et al. [2017] instantiate their generic equivalence twice to obtain the decidability of convertibility checking, and we will also be doing the same. Due to the complication of layering, our fundamental theorems must talk about all layers. The difficulties of the logical relations lie in that how we can support code running and recursions on code at the same time and justify them in the semantics.

### 7.1 Generic Equivalence

Similar to Sec. 3, we first quantify four generic equivalence relations, which will be instantiated to syntactic equivalence and convertibility later, and their laws. This step provides modularity to logical relation argument: we simply instantiate this generic equivalence to obtain different versions of the fundamental theorems. Due to dependent types, we define generic equivalence over $i \in\{p, m\}$ :

- $L \mid \Psi ; \Gamma \vdash_{i} V \sim V^{\prime} @ l$ describes a generic type equivalence between two neutral types at universe level $l$ at layer $i$.
- $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l$ describes a generic type equivalence between two types at universe level $l$ at layer i.
- $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T @ l$ describes a generic type equivalence between two neutral terms of type $T$ at universe level $l$ at layer $i$.
- $L \| \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$ describes a generic type equivalence between two terms of type $T$ at universe level $l$ at layer $i$.
From the four generic equivalence, we induce two equivalence of local contexts and local substitutions by using the generic equivalence pairwise:

$$
\begin{aligned}
& \frac{L \vdash \Psi}{L \mid \Psi \vdash_{i} \cdot \simeq \cdot} \quad \frac{L \vdash \Psi \quad g: C t x \in \Psi}{L \mid \Psi \vdash_{i} g \simeq g} \quad \frac{L\left|\Psi \vdash_{i} \Gamma \simeq \Delta \quad L\right| \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l \quad L \vdash l \approx l^{\prime} \text { : Level }}{L \mid \Psi \vdash_{i} \Gamma, x: T @ l \simeq \Delta, x: T^{\prime} @ l^{\prime}} \\
& \frac{L \mid \Psi \vdash_{i} \Gamma \quad \Gamma \text { ends with } \cdot \quad|\Gamma|=m}{L \mid \Psi ; \Gamma \vdash_{i} \cdot{ }^{m} \simeq{ }^{m}: \cdot} \quad \frac{L \mid \Psi \vdash_{i} \Gamma \quad g: C t x \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=m}{L \mid \Psi ; \Gamma \vdash_{i} \cdot{ }_{g}^{m} \simeq{ }_{g}^{m}: \cdot} \\
& L \mid \Psi \vdash_{i} \Gamma \quad g: \operatorname{Ctx} \in \Psi \\
& L \mid \Psi ; \Gamma \vdash_{i} \delta \simeq \delta^{\prime}: \Delta \\
& \frac{\Gamma \text { ends with } g \quad|\Gamma|=m}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{wk}_{g}^{m} \simeq \mathrm{wk}_{g}^{m}: g} \quad \frac{L\left|\Psi ; \Delta \vdash_{i} T @ l \quad L\right| \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T[\delta] @ l}{L \mid \Psi ; \Gamma \vdash_{i} \delta, t / x \simeq \delta^{\prime}, t^{\prime} / x: \Delta, x: T @ l}
\end{aligned}
$$

The generic equivalence and the logical relations are invariant under all weakenings. Therefore, we should make these notions clear here. Since there are three different contexts, we have three corresponding kinds of weakenings. In particular, $\theta:: L \Longrightarrow L^{\prime}$ is the universe weakening. Following previous conventions, $\gamma::$ $L \mid \Psi \Longrightarrow{ }_{g} \theta$ is a global weakening, and $\tau:: L \mid \Psi ; \Gamma \Longrightarrow_{i} \Delta$ is a local weakening. The subscript $i$ denotes which layer the contexts $\Gamma$ and $\Delta$ live in. We can simultaneously weaken all three contexts at the same time. We simply apply $\theta, \gamma$ and $\tau$ in this order. We let $\psi$ represent this triple:

$$
\psi:=(\theta, \gamma, \tau):: L\left|\Psi ; \Gamma \Longrightarrow_{i} L^{\prime}\right| \Phi ; \Delta
$$

where

$$
\begin{aligned}
& \theta:: L \Longrightarrow L^{\prime} \\
& \gamma:: L^{\prime} \mid \Psi[\theta] \Longrightarrow g \\
& \tau:: L^{\prime} \mid \theta ; \Gamma[\theta][\gamma] \Longrightarrow_{i} \Delta
\end{aligned}
$$

Similarly we let

$$
\alpha:=(\theta, \gamma):: L\left|\Psi \Longrightarrow L^{\prime}\right| \Phi
$$

We can apply weakenings like substitutions to universe levels, types, terms, contexts and substitutions as expected. The action is to shift the variables according to the specified weakenings. This is a standard action, despite having three separate notions, so we take it for granted here. When it is clear from the context, we do not write down the weakening action at all to avoid clutter.

Then we give the laws of the generic equivalence. Since the generic equivalence at layer $m$ subsumes that at layer $p$, we first give the laws that hold for both layers, and then incrementally add those that only hold at layer m.

Law 7.1 (Subsumption).

- If $L \mid \Psi ; \Gamma \vdash_{i} V \sim V^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} V \simeq V^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T$ @ $l$, then $L \mid \Psi ; \Gamma \vdash_{i} \mu \simeq \mu^{\prime}: T$ @ $l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$.

As a lemma, subsumption propagates to contexts and local substitutions:
Lemma 7.1 (Subsumption).

- If $L \mid \Psi \vdash_{i} \Gamma \simeq \Delta$, then $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \simeq \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$.

Due to subsumption, we know that components in generic equivalence are well-formed or well-typed:
Lemma 7.2 (Presupposition).

- If $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} t^{\prime}: T @ l$.

Proof. By subsumption and presupposition.
Law 7.2 (PER). All four relations are PERs.
Law 7.3 (Type Conversion).

- If $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T^{\prime} @ l$.

Law 7.4 (Context Equivalence).

- If $L \mid \Psi ; \Gamma \vdash_{i} V \sim V^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} V \sim V^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} T \simeq T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} \mu \sim \mu^{\prime}: T$ @ $l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} t \simeq t^{\prime}: T @ l$.

Law 7.5 (Weakening).

- If $L \mid \Psi ; \Gamma \vdash_{i} V \sim V^{\prime} @ l$ and $\psi:: L\left|\Psi ; \Gamma \Longrightarrow_{i} L^{\prime}\right| \Phi ; \Delta$, then $L^{\prime} \mid \Phi ; \Delta \vdash_{i} V[\psi] \sim V^{\prime}[\psi]$ @ $l[\psi]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l$ and $\psi:: L\left|\Psi ; \Gamma \Longrightarrow_{i} L^{\prime}\right| \Phi ; \Delta$, then $L^{\prime} \mid \Phi ; \Delta \vdash_{i} T[\psi] \simeq T^{\prime}[\psi] @ l[\psi]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T$ @ $l$ and $\psi:: L\left|\Psi ; \Gamma \Longrightarrow{ }_{i} L^{\prime}\right| \Phi ; \Delta$, then $L^{\prime} \mid \Phi ; \Delta \vdash_{i} \mu[\psi] \sim \mu^{\prime}[\psi]: T[\psi] @ l[\psi]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$ and $\psi:: L\left|\Psi ; \Gamma \Longrightarrow{ }_{i} L^{\prime}\right| \Phi ; \Delta$, then $L^{\prime} \mid \Phi ; \Delta \vdash_{i} t[\psi] \simeq t^{\prime}[\psi]: T[\psi] @ l[\psi]$.

Law 7.6 (Weak Head Closure).

- If $L\left|\Psi ; \Gamma \vdash_{i} T \rightsquigarrow^{*} W @ l, L\right| \Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow^{*} W^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} W \simeq W^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \simeq$ $T^{\prime} @ l$.
- If $L\left|\Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} w: T @ l, L\right| \Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} w^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} w \simeq w^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$.

Law 7.7 (Type Constructors). If $L \mid \Psi \vdash_{i} \Gamma$,

- if $L \vdash l \approx l^{\prime}$ : Level, then $L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Ty}_{l} \simeq \mathrm{Ty}_{l}^{\prime} @ \operatorname{succ} l$ and $L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Ty}_{l} \simeq \mathrm{Ty}_{l}^{\prime}: \mathrm{Ty}_{\text {succ } l} @ \operatorname{succ}(\operatorname{succ} l)$;
- then $L \mid \Psi ; \Gamma \vdash_{i} N a t \simeq N a t @ z e r o$ and $L \mid \Psi ; \Gamma \vdash_{i} N a t \simeq N a t: T y_{z e r o} @$ succ zero;
- if $L \quad \Psi ; \Gamma \quad \vdash_{i} S \simeq S^{\prime} @ l$ and $L \mid \Psi ; \Gamma, x: S @ l+\vdash_{i} T \simeq T^{\prime} @ l^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}(x: S) . T \simeq \Pi^{l, l^{\prime}}\left(x: S^{\prime}\right) . T^{\prime} @ l \sqcup l^{\prime}$;
- if $L \mid \Psi ; \Gamma \vdash_{i} s \simeq s^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l$ and $L \mid \Psi ; \Gamma, x: E l^{l} s @ l \vdash_{i} t \simeq t^{\prime}: T y_{l^{\prime}} @ \operatorname{succ} l^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}(x: s) . t \simeq \Pi^{l, l^{\prime}}\left(x: s^{\prime}\right) . t^{\prime}: \mathrm{Ty}_{l \sqcup l^{\prime}} @ \operatorname{succ}\left(l \sqcup l^{\prime}\right)$.
Law 7.8 (Neutral Types).
- If $L \mid \Psi \vdash_{i} \Gamma, U:\left(\Delta \vdash_{i^{\prime}} @ l\right) \in \Psi, i^{\prime} \in\{c, p\}, i^{\prime} \leq i$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta \simeq \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} U^{\delta} \sim U^{\delta^{\prime}} @ l$.
- If $L \vdash l \approx l^{\prime}:$ Level and $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l$, then $L \mid \Psi ; \Gamma \vdash_{i} \mathrm{El}^{l} \mu \sim \mathrm{El}^{l^{\prime}} \mu^{\prime} @ l$.

Law 7.9 (Congruence).

- If $L \mid \Psi \vdash_{i} \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i}$ zero $\simeq$ zero : Nat @ zero.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}:$ Nat @ zero, then $L \mid \Psi ; \Gamma \vdash_{i}$ succ $t \simeq \operatorname{succ} t^{\prime}:$ Nat @ zero.
- If $L\left|\Psi ; \Gamma \vdash_{i} S @ l, L\right| \Psi ; \Gamma \vdash_{i} t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}, L \mid \Psi ; \Gamma \vdash_{i} t^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}$ and $L \mid \Psi ; \Gamma, x$ : $S @ l \vdash_{i}\left(t: \Pi^{l, l^{\prime}}(x: S) . T\right) x \simeq\left(t^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T\right) x: T$ @ $l^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}$.

Law 7.10 (Congruence for Neutrals).

- If $L \mid \Psi \vdash_{i} \Gamma$ and $x: T @ l \in \Gamma$, then $L \mid \Psi ; \Gamma \vdash_{i} x \sim x: T @ l$.
- If $L \mid \Psi \vdash_{i} \Gamma, u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi, i^{\prime} \in\{v, c\}, i^{\prime} \leq i$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta \simeq \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} u^{\delta} \sim u^{\delta^{\prime}}: T[\delta] @ l$.
- If $L \vdash l \approx l^{\prime}:$ Level, $L \mid \Psi ; \Gamma, x:$ Nat @ zero $\vdash_{i} M \simeq M^{\prime} @ l, L \mid \Psi ; \Gamma \vdash_{i} s_{1} \simeq s_{3}: M[z e r o / x] @ l$, $L \mid \Psi ; \Gamma, x$ : Nat @ zero, $y: M @ l \vdash_{i} s_{2} \simeq s_{4}: M[\operatorname{succ} x / x] @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}:$ Nat @ zero, then $L \mid \Psi ; \Gamma \vdash_{i} e^{2} \lim _{\mathrm{Nat}}^{l}(x . M) s_{1}\left(x, y . s_{2}\right) \mu \sim \operatorname{elim}_{\mathrm{Nat}}^{l^{\prime}}\left(x . M^{\prime}\right) s_{3}\left(x, y . s_{4}\right) \mu^{\prime}: M[\mu / x] @ l$.
- If $L \vdash l_{1} \approx l_{3}$ : Level, $L \vdash l_{2} \approx l_{4}$ : Level, $L\left|\Psi ; \Gamma \vdash_{i} S \simeq S^{\prime} @ l_{1}, L\right| \Psi ; \Gamma, x: S @ l_{1} \vdash_{i} T \simeq T^{\prime} @ l_{2}$, $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S^{\prime \prime}\right) \cdot T^{\prime \prime} @ l_{1} \sqcup l_{2}$ and $L \mid \Psi ; \Gamma \vdash_{i} s \simeq s^{\prime}: S$ @ $l_{1}$, then $L \mid \Psi ; \Gamma \vdash_{i}\left(\mu: \Pi^{l_{1}, l_{2}}(x: S) . T\right) s \sim\left(\mu^{\prime}: \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) . T^{\prime}\right) s^{\prime}: T[s / x] @ l_{2}$.

We derive that
Lemma 7.3 (Reflexivity of Local Identity Substitutions). If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} i d_{\Gamma} \simeq i d_{\Gamma}: \Delta$.
Lemma 7.4 (Congruence of Global Variables).

- If $L \Psi \vdash_{i} \Gamma \approx \Delta, u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi, i^{\prime} \in\{v, c\}$ and $i^{\prime} \leq i$, then $L \mid \Psi ; \Gamma \vdash_{i} u^{i d_{\Gamma}} \sim u^{i d_{\Gamma}}: T @ l$.
- If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta, U:\left(\Delta \vdash_{i^{\prime}} @ l\right) \in \Psi, i^{\prime} \in\{c, p\}$ and $i^{\prime} \leq i$, then $L \mid \Psi ; \Gamma \vdash_{i} U^{i d_{\Gamma}} \sim U^{i d d_{\Gamma}} @ l$.

At this point, we give the laws that should hold for both layers. Next, we consider laws that only hold at layer $i=m$. We first give the laws for type constructors.

Law 7.11 (Type Constructors).

- If $L\left|\Psi \quad \vdash_{m} \Gamma, L, \vec{l}\right| \Psi ; \Gamma \quad \vdash_{m} T \simeq T^{\prime} @ l$ and $L, \vec{l}+l \approx l^{\prime}:$ Level, then $L \mid \Psi ; \Gamma \vdash_{m} \vec{\ell} \Rightarrow^{l} T \simeq \vec{\ell} \Rightarrow l^{\prime} T^{\prime} @ \omega$.
 $L \mid \Psi ; \Gamma \vdash_{m}(g: C t x) \Rightarrow^{l} T \simeq(g: C t x) \Rightarrow^{l^{\prime}} T^{\prime} @ l$.
- If $L\left|\Psi \vdash_{m} \Gamma, L\right| \Psi, U:\left(\Delta \vdash_{p} @ l_{1}\right) ; \Gamma \vdash_{m} T \simeq T^{\prime} @ l_{2}, L \mid \Psi \vdash_{p} \Delta \simeq \Delta^{\prime}, L \vdash l_{1} \approx l_{3}$ : Level and $L \vdash l_{2} \approx l_{4}:$ Level, then $L \mid \Psi ; \Gamma \vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T \simeq\left(U:\left(\Delta^{\prime} \vdash_{p} @ l_{3}\right)\right) \Rightarrow^{l_{4}} T^{\prime} @ \operatorname{succ} l_{1} \sqcup l_{2}$.
- If $L \mid \Psi \vdash_{m} \Gamma$ and $L \mid \Psi \vdash_{p} \Delta \simeq \Delta^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} @ l\right) \simeq \square\left(\Delta^{\prime} \vdash_{c} @ l\right)$ @ succ $l$.
 $L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T @ l\right) \simeq \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l\right) @ l$.

Law 7.12 (Congruence).

- If $L\left|\Psi ; \Gamma \vdash_{m} t: \vec{\ell} \Rightarrow^{l} T @ \omega, L\right| \Psi ; \Gamma \vdash_{m} t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega$ and $L, \vec{\ell} \mid \Psi ; \Gamma \vdash_{m} t \$ \vec{\ell} \simeq t^{\prime} \$ \vec{\ell}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{m} t \Longleftrightarrow t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega$.
- If $L\left|\Psi ; \Gamma \vdash_{m} t:(g: C t x) \Rightarrow^{l} T @ l, L\right| \Psi ; \Gamma \vdash_{m} t^{\prime}:(g: C t x) \Rightarrow^{l} T$ @ land $L \mid \Psi, g: C t x ; \Gamma \vdash_{m} t \$ g \simeq t^{\prime} \$ g: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{m} t \simeq t^{\prime}:(g: C t x) \Rightarrow^{l} T @ l$.
- If $L\left|\Psi ; \Gamma \vdash_{m} t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow l^{\prime} T @ \operatorname{succ} l \sqcup l^{\prime}, L\right| \Psi ; \Gamma \vdash_{m} t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow l^{\prime} T$ @ succ $l \sqcup l^{\prime}$, and $L \quad \mid \quad \Psi, U:\left(\Delta \vdash_{p} \quad @ l\right) ; \Gamma \quad \vdash_{m} t \$_{p} U^{i d_{\Delta}} \simeq t^{\prime} \$_{p} U^{i d_{\Delta}}: T$ @ $l^{\prime}$, then $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T @ \operatorname{succ} l \sqcup l^{\prime}$.
- If $L \mid \Psi \vdash_{m} \Gamma$ and $L \mid \Psi ; \Delta \vdash_{c} T @ l$, then $L \mid \Psi ; \Gamma \vdash_{m}$ box $T \simeq \operatorname{box} T: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l$.
- If $L \mid \Psi \vdash_{m} \Gamma$ and $L \mid \Psi ; \Delta \vdash_{c} t: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{m} \operatorname{box} t \simeq \operatorname{box} t: \square\left(\Delta \vdash_{c} T @ l\right) @ l$.

Law 7.13 (Congruence for Neutrals).

- If $L \mid \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: \vec{l} \Rightarrow^{l} T$ @ $\omega,|\vec{\ell}|=|\vec{l}|=\left|\vec{l}^{\prime}\right|>c$ and $\forall c \leq n<|\vec{l}| \cdot L \vdash \vec{l}(n) \approx \vec{l}^{\prime}(n):$ Level, then $L \mid \Psi ; \Gamma \vdash_{m} \mu \$ \vec{l} \sim \mu^{\prime} \$ \vec{l}^{\prime}: T[\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}]$.
 $L \mid \Psi ; \Gamma \vdash_{m} \mu \$ \Delta \sim \mu^{\prime} \$ \Delta^{\prime}: T[\Delta / g] @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime}$ and $L \mid \Psi ; \Delta \vdash_{p} T \simeq T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{m} \mu \$_{p} T \sim \mu^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}[T / U] @ l^{\prime}$.
 $L\left|\Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l_{2}\right) @ \operatorname{succ} l_{2}, L\right| \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} @ l_{2}\right) @ \operatorname{succ} l_{2} \vdash_{m} M \simeq M^{\prime} @ l_{1}$ and $L \quad \mid \Psi, U \quad:\left(\Delta \quad \vdash_{c} \quad @ \quad l_{2}\right) ; \Gamma \quad \vdash_{m} \quad t_{1} \simeq t_{2}: M\left[\operatorname{box} U^{i d} / x_{T}\right]$ @ $l_{1}$, then $L \mid \Psi ; \Gamma \vdash_{m}$ letbox $_{T y p}^{l_{1}} l_{2} \Delta\left(x_{T} \cdot M\right)\left(U . t_{1}\right) \mu \sim \operatorname{letbox}_{T y p}^{l_{3}} l_{4} \Delta^{\prime}\left(x_{T} . M^{\prime}\right)\left(U . t_{2}\right) \mu^{\prime}: M\left[t / x_{T}\right] @ l_{1}$.
- If $L \mid \Psi \vdash_{m} \Gamma, L \vdash l_{1} \approx l_{3}:$ Level, $L \vdash l_{2} \approx l_{4}:$ Level, $L\left|\Psi \vdash_{p} \Delta \simeq \Delta^{\prime}, L\right| \Psi ; \Gamma \vdash_{p} T \simeq T^{\prime} @ l_{2}$, $L\left|\Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{2}, L\right| \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{1} \vdash_{m} M \simeq M^{\prime} @ l_{1}$ and $L \quad \mid \quad \Psi, u \quad:\left(\Delta \vdash_{c} T @ l_{2}\right) ; \Gamma \quad \vdash_{m} \quad t_{1} \simeq t_{2}: M\left[b o x u^{i d} / x_{t}\right] @ l_{1}$, then $L \mid \Psi ; \Gamma \vdash_{m}$ letbox $_{T r m}^{l_{1}} l_{2} \Delta T\left(x_{t} \cdot M\right)\left(U . t_{1}\right) \mu \sim \operatorname{letbox}_{T r m}^{l_{3}} l_{4} \Delta^{\prime} T^{\prime}\left(x_{T} \cdot M^{\prime}\right)\left(U . t_{2}\right) \mu^{\prime}: M\left[t / x_{t}\right] @ l_{1}$.

Then we consider the law for neutral forms for recursive principles for code. The law follows a similar line to the equivalence judgments and the convertibility checking: the evaluation is blocked when the scrutinee is neutral or is a box'ed global variable.

Law 7.14 (Neutral Recursion on Code).

- If all motives and branches are related by corresponding generic equivalence, moreover, $L \vdash l \approx l^{\prime}:$ Level, $L \quad \Psi \quad \vdash_{p} \Delta \simeq \Delta^{\prime}$ and $L \mid \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l\right) @ l$, then $L \mid \Psi ; \Gamma \vdash_{m}$ elim $_{T y p}^{l_{1} l_{2}} \vec{M} \vec{b} l \Delta \mu \sim \operatorname{elim}_{T y p}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} \mu^{\prime}: M\left[\mu / x_{T}\right] @ l_{1}$.
- If all motives and branches are related by corresponding generic equivalence, moreover, $L \vdash l \approx l^{\prime}$ : Level, $L \quad \mid \Psi \vdash_{p} \Delta \simeq \Delta^{\prime}, U:\left(\Delta \vdash_{c} @ l\right) \in \Psi$ and $L \mid \Psi ; \Gamma \quad \vdash_{c} \delta \quad: \quad \Delta$, then $L \mid \Psi ; \Gamma \vdash_{m} \operatorname{elim}_{T y p}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta\left(\operatorname{box} U^{\delta}\right) \sim \operatorname{elim}_{T y p}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime}\left(\operatorname{box} U^{\delta}\right): M\left[\operatorname{box} U^{\delta} / x_{T}\right] @ l_{1}$.
- If all motives and branches are related by corresponding generic equivalence, moreover, $L \vdash l \approx l^{\prime}$ : Level, $L\left|\Psi \vdash_{p} \Delta \simeq \Delta^{\prime}, L\right| \Psi ; \Gamma \vdash_{p} T \Longleftrightarrow T^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{m} \mu \longleftrightarrow \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ l$, then $L \mid \Psi ; \Gamma \vdash_{m}$ elim $_{T r m}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta T \mu \sim \operatorname{elim}_{T y p}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} T^{\prime} \mu^{\prime}: M^{\prime}\left[\mu / x_{t}\right] @ l_{1}$.
- If all motives and branches are related by corresponding generic equivalence, moreover, $L \vdash l \approx l^{\prime}$ : Level, $L\left|\Psi \vdash_{p} \Delta \simeq \Delta^{\prime}, L\right| \Psi ; \Gamma \vdash_{p} T \Longleftrightarrow T^{\prime} @ l, u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi, i^{\prime} \in\{v, c\}$ and $L \mid \Psi ; \Gamma \vdash_{c} \delta: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{m}$ elim $_{T r m}^{l_{1}, l_{2}} \vec{M} \vec{b} l \Delta T\left(\operatorname{box} u^{\delta}\right) \sim \operatorname{elim}_{T y p}^{l_{3}, l_{4}} \overrightarrow{M_{1}} \overrightarrow{b_{1}} l^{\prime} \Delta^{\prime} T^{\prime}\left(\operatorname{box} u^{\delta}\right): M^{\prime}\left[\operatorname{box} u^{\delta} / x_{t}\right] @ l_{1}$.

We conclude all the laws here fore the generic equivalence.
This generic equivalence will be instantiated twice times: the syntactic equivalence judgments, and the convertibility checking judgments. DeLaM is way more complex than Abel et al. [2017]'s work because DeLaM has computations at two layers, $p$ and $m$. Therefore, we must derive the necessary properties from the fundamental theorems to be proved shortly at each layer.

### 7.2 Kripke Logical Relations for MLTT

The Kripke logical relations are parameterized by the generic equivalence. It is additionally indexed by another layering index $j \in\{p, m\}$, which quantifies the types described by the relations. When $j=p$, we consider types from MLTT. When $j=m$, we consider all possible types. The reason for this distinction is to handle the lifting property from layer $c$ (which has the terms as $p$ ) or $p$ to $m$, where terms from MLTT are brought to DeLaM. On the semantic side, we need to make sure that terms from MLTT can interact with "native" terms in DeLaM coherent. We further restrict $j=p$ when $i=p$.

The Kripke logical relations are defined by
(1) recursion on $i$,
(2) recursion on $j$, which effectively means the logical relations are 2-layered; also note when $i=p, j=p$ is determined;
(3) a transfinite well-founded recursion on the universe levels, and
(4) induction-recursion on related types and terms.

In particular, the recursion on $j$ is necessary, as the relations when $j=m$ depend on the validity judgments of $j=p$. When we do a recursion on universe levels, we must mind the well-foundedness of universe levels. As we have discussed in Sec. 4.3, we are sure that all universe levels must find a finite number of steps to descend to zero. The only problem is $\omega$, which is not finite. Thus we must include one large cardinal to handle this level, hence the transfinite recursion. Luckily, we do not have to think about it most of the time as we cannot really use $\omega$ to do anything special at all. Note that our relations do not exactly follow Abel et al. [2017] tightly, where two relations are defined for types and terms respectively. In our case, we provide simpler inductive-recursive definitions, where only one relation is defined for types and for terms respectively. This style is more akin to the PER models in untyped domains by Abel [2013]; Hu et al. [2023], except that our logical relations are Kripke. We follow a proof schema that combines that of Abel et al. [2017] and that of Abel [2013]; Hu et al. [2023]. We define the following judgments:

- $\mathcal{D}:: L \mid \Psi ; \Gamma \| \models_{i}^{j} T \approx T^{\prime} @ l$ denotes that two types $T$ and $T^{\prime}$ are related. This relation is defined inductively. We use $\mathcal{D}$ to mark give a name to the derivation as we will do recursion on it.
- $L \mid \Psi ; \Gamma \| \models_{i}^{j} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ denotes that two terms $t$ and $T^{\prime}$ related by $\mathcal{D}$. This relation is defined by a recursion on $\mathcal{D}$.
- $\mathcal{E}:: L|\Psi| \vDash_{i}^{j} \Gamma \approx \Delta$ denotes that two contexts are related. It is a generalization of $\mathcal{D}$.
- $L \mid \Psi ; \Gamma \| \vDash_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{E}$ denotes that two local substitutions $\delta$ and $\delta^{\prime}$ are related. It is a generalization of $L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$ by doing recursion on $\mathcal{E}$.
For convenience, we define the following:

$$
\begin{aligned}
L \mid \Psi ; \Gamma \| \vDash_{i}^{j} T @ l & :=L \mid \Psi ; \Gamma \| \vDash_{i}^{j} T \approx T @ l \\
L \mid \Psi ; \Gamma \| \Vdash_{i}^{j} t \approx t^{\prime}: T @ l & :=\text { for some } T^{\prime}, \mathcal{D}:: L \mid \Psi ; \Gamma \| \vDash_{i}^{j} T \approx T^{\prime} @ l \text { and } L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t \approx t^{\prime}: \mathrm{El}(\mathcal{D}) \\
L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t: T @ l & :=L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t \approx t: T @ l \\
L \mid \Psi \|{ }_{i}^{j} \Gamma & :=L \mid \Psi \|{ }_{i}^{j} \Gamma \approx \Gamma
\end{aligned}
$$

[^0]$L \mid \Psi ; \Gamma \| \vDash_{i}^{j} \delta \approx \delta^{\prime}: \Delta:=$ for some $\Delta^{\prime}, \mathcal{E}:: L \mid \Psi \models_{i}^{j} \Delta \approx \Delta^{\prime}$ and $L \mid \Psi ; \Gamma \| \vDash_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{E}$
$L\left|\Psi ; \Gamma\left\|\vDash_{i}^{j} \delta: \Delta:=L \mid \Psi ; \Gamma\right\| \vDash_{i}^{j} \delta \approx \delta: \Delta\right.$
Now we proceed to define the relations. We begin with the natural numbers.
$$
\mathcal{D}:: \frac{L \mid \Psi ; \Gamma \vdash_{i} T \rightsquigarrow^{*} \text { Nat @ zero } \quad L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow{ }^{*} \text { Nat @ zero }}{L \mid \Psi ; \Gamma \| \Vdash_{i}^{j} T \approx T^{\prime} @ \text { zero }}
$$

Then $L \mid \Psi ; \Gamma \| \models_{i}^{j} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ is defined by $L \mid \Psi ; \Gamma \| \vDash_{i} t \approx t^{\prime}:$ Nat, which we define as follows:

| $L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} w$ : Nat @ zero |  |  |
| :---: | :---: | :---: |
| $L \mid \Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} w^{\prime}$ : Nat @ zero | $L \mid \Psi ; \Gamma \vdash_{i} w \simeq w^{\prime}$ : Nat @ zero | $L \mid \Psi ; \Gamma \\| \vDash_{i} w \simeq w^{\prime}:$ Nat |
| $L \mid \Psi ; \Gamma \\| \vdash_{i} t \approx t^{\prime}$ : Nat |  |  |
|  | $L \mid \Psi ; \Gamma \\| F_{i} t \approx t^{\prime}: \mathrm{Nat}$ | $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}:$ Nat @ zero |
| $\Psi ; \Gamma \\|=_{i}$ zero $\simeq$ zero : Nat | ; Г $\\| \vDash_{i} \operatorname{succ} t \simeq \operatorname{succ} t^{\prime}: \mathrm{Nat}$ | $L \mid \Psi ; \Gamma \\| \vDash_{i} \mu \simeq \mu^{\prime}$ : Nat |

Then we consider universes.

$$
\mathcal{D}:: \frac{L \mid \Psi ; \Gamma \vdash_{i} T \rightsquigarrow^{*} \mathrm{Ty}_{l_{1}} @ \operatorname{succ} l_{1} \quad}{L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow{ }^{*} \mathrm{Ty}_{l_{2}} @ \operatorname{succ} l_{2}} \begin{aligned}
& L \vdash l_{1} \approx l: \text { Level } \\
& L \mid \Psi ; \Gamma \| \vdash_{i}^{j} T \approx T^{\prime} @ \operatorname{succ} l
\end{aligned}
$$

Then $L \mid \Psi ; \Gamma \| \Vdash_{i}^{j} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} w: \operatorname{Ty}_{l} @ \operatorname{succ} l$,
- L | $\Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} w^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l$,
- $L \mid \Psi ; \Gamma \vdash_{i} w \simeq w^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l$, which means that $t$ and $t^{\prime}$ are equivalent types at level $l$,
$\bullet L \mid \Psi ; \Gamma\| \|_{i}^{j} \mathrm{El} l^{l} w \approx \mathrm{El}^{l} w^{\prime} @ l$, which means that the corresponding types of $w$ and $w^{\prime}$ are related.
The last condition requires the well-founded recursion on the universe levels in order to refer back to the relation for types. Notice that the universe level decreases by one so this definition is valid.

Then we define the relation for $\Pi$ types.

$$
\begin{gathered}
\mathcal{D}:: \\
L\left|\Psi ; \Gamma \vdash_{i} T \rightsquigarrow *^{*} \Pi^{l, l^{\prime}}\left(x: S_{1}\right) . T_{1} @ l \sqcup l^{\prime} L\right| \Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow^{*} \Pi^{l, l^{\prime}}\left(x: S_{2}\right) \cdot T_{2} @ l \sqcup l^{\prime} \\
L\left|\Psi ; \Gamma \vdash_{i} S_{1} @ l \quad L\right| \Psi ; \Gamma \vdash_{i} S_{2} @ l \quad L \mid \Psi ; \Gamma, x: S_{1} @ l \vdash_{i} T_{1} @ l^{\prime} \\
L\left|\Psi ; \Gamma, x: S_{2} @ l \vdash_{i} T_{2} @ l^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}\left(x: S_{1}\right) \cdot T_{1} \simeq \Pi^{l, l^{\prime}}\left(x: S_{2}\right) \cdot T_{2} @ l \sqcup l^{\prime} \\
\mathcal{E}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{j} S_{1} \approx S_{2} @ l\right) \\
\mathcal{F}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \text { and } L^{\prime}\left|\Phi ; \Delta\left\|\Vdash_{i}^{j} s \approx s^{\prime}: \operatorname{El}(\mathcal{E}(\psi)) \cdot L^{\prime} \mid \Phi ; \Delta\right\| \vDash_{i}^{j} T_{1}[s / x] \approx T_{2}\left[s^{\prime} \mid x\right] @ l^{\prime}\right)\right. \\
L \mid \Psi ; \Gamma \| \vDash_{i}^{j} T \approx T^{\prime} @ l \sqcup l^{\prime}
\end{gathered}
$$

This case is particularly complex. Let us digest the premises one by one. First, we require that both types $T$ and $T^{\prime}$ to reduce to some $\Pi$ types. The typing judgments require the components of the $\Pi$ types are well-formed. Further, the $\Pi$ types themselves are equivalent. Then $\mathcal{E}$ requires that $S_{1}$ and $S_{2}$ are related under any weakening. This makes the relation Kripke. $\mathcal{F}$ is similar but require $T_{1}$ and $T_{2}$ remain related given any two related terms $s$ and $s^{\prime}$. Furthermore, in reality, we should put down sufficient premises for equivalent universes to allow syntactically different universes to appear in both $T$ and $T^{\prime}$, but due to the size of this rule, we only use $l$ and $l^{\prime}$ here. We apply the same principle in other rules in the rest of this technical report.

Then $L \mid \Psi ; \Gamma \| \models_{i}^{j} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} w: \Pi^{l, l^{\prime}}\left(x: S_{1}\right) . T_{1} @ l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} w^{\prime}: \Pi^{l, l^{\prime}}\left(x: S_{2}\right) \cdot T_{2} @ l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{i} w \simeq w^{\prime}: \Pi^{l, l^{\prime}}\left(x: S_{1}\right) \cdot T_{1} @ l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{i} w \simeq w^{\prime}: \Pi^{l, l^{\prime}}\left(x: S_{2}\right) \cdot T_{2} @ l \sqcup l^{\prime}$, which is duplicated to make symmetry a simpler property,
- $\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma$ and $\mathcal{A}:: L^{\prime} \mid \Phi ; \Delta\| \|_{i}^{j} s \approx s^{\prime}: \operatorname{El}(\mathcal{E}(\psi)) \cdot$, moreover, we add equivalence assumptions to remove the effect of type annotations, $L^{\prime}\left|\Phi ; \Delta \vdash_{i} S_{1} \simeq S_{1}^{\prime} @ l, L^{\prime}\right| \Phi ; \Delta \vdash_{i} S_{2} \simeq S_{2}^{\prime} @ l$, $L^{\prime}\left|\Phi ; \Delta, x: S_{1} @ l \vdash_{i} T_{1} \simeq T_{1}^{\prime} @ l^{\prime}, L^{\prime}\right| \Phi ; \Delta, x: S_{2} @ l \vdash_{i} T_{2} \simeq T_{2}^{\prime} @ l^{\prime}$, then $L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{j}\left(w: \Pi^{l, l^{\prime}}\left(x: S_{1}^{\prime}\right) \cdot T_{1}^{\prime}\right) s \approx\left(w^{\prime}: \Pi^{l, l^{\prime}}\left(x: S_{2}^{\prime}\right) \cdot T_{2}^{\prime}\right) s^{\prime}: \operatorname{El}(\mathcal{F}(\psi, \mathcal{A}))$.
The next case is neutral types.

$$
\left.\mathcal{D}:: \frac{L \mid \Psi ; \Gamma \vdash_{i} T \rightsquigarrow *^{*} V @ l}{} \quad L\left|\Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow{ }^{*} V^{\prime} @ l \quad L\right| \Psi ; \Gamma \vdash_{i} V \sim V^{\prime} @ l\right]\left(\Psi ; \Gamma \| \Vdash_{i}^{j} T \approx T^{\prime} @ l\right.
$$

Then $L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} \mu: V @ l$,
- $L \mid \Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} \mu^{\prime}: V^{\prime} @ l$,
- $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: V @ l$,
- $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: V^{\prime} @ l$.

Again, we duplicate $\mu \sim \mu^{\prime}$ to make symmetry easy.
The last case possible for both layers is equivalence of universe levels. This case is introduced for bureaucratic purposes and often ignored.

$$
\mathcal{D}:: \frac{\mathcal{E}:: L \mid \Psi ; \Gamma \| \vDash_{i}^{j} T \approx T^{\prime} @ l^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \| \vDash_{i}^{j} T \approx T^{\prime} @ l}
$$

Then $L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ is defined by $L \mid \Psi ; \Gamma \| \vDash_{i}^{j} t \approx t^{\prime}: \operatorname{El}(\mathcal{E})$.
Now we have given all logical relations for types and terms that are available at both layers. In order to give the logical relations for layer $m$, we need to first give the logical relations for local contexts and local substitutions, as the semantics of contextual types depend on them. The complexity of our logical relations primarily comes from the fact that we must be able to handle computation at two layers ( $p$ and $m$ ) and different lifting behaviors.

We proceed by defining related local contexts inductively and then the corresponding recursive case for local substitutions.

$$
\mathcal{E}:: \frac{L \vdash \Psi}{L \mid \Psi \vDash_{i}^{j} \cdot \approx .}
$$

Then $L \mid \Psi ; \Gamma \| \models_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{E}$ is defined by $L \mid \Psi \vdash_{i} \Gamma$ and then checking $\Gamma$ :

- if $\Gamma$ ends with $\cdot$, then $\delta=\delta^{\prime}=.|\Gamma|$;
- if $\Gamma$ ends with $g$, then $g: \operatorname{Ctx} \in \Psi$ and $\delta=\delta^{\prime}={ }_{g}^{|\Gamma|}$;

$$
\mathcal{E}:: \frac{L \vdash \Psi \quad g: \mathrm{Ctx} \in \Psi}{L \mid \Psi \vDash_{i}^{j} g \approx g}
$$

Then $L \mid \Psi ; \Gamma \| \models_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{E}$ is defined as $L \mid \Psi \vdash_{i} \Gamma$ and

- $\Gamma$ also ends with $g$,
- $\delta=\delta^{\prime}=\mathrm{wk}_{g}^{|\Gamma|}$.

$$
\mathcal{E}:: \frac{\mathcal{F}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime}\left|\Phi \vDash_{i}^{j} \Delta \approx \Delta^{\prime} \quad \mathcal{D}:: L\right| \Psi ; \mathcal{F} \vDash_{i}^{j} T \approx T^{\prime} @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi \vDash_{i}^{j} \Delta, x: T @ l \approx \Delta^{\prime}, x: T^{\prime} @ l^{\prime}}
$$

Then $L \mid \Psi ; \Gamma \| \models_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{E}$ is defined as

- $\delta=\delta_{1}, t / x$,
- $\delta^{\prime}=\delta_{1}^{\prime}, t^{\prime} / x$,
- $C:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi ; \Gamma \| \models_{i}^{j} \delta_{1} \approx \delta_{1}^{\prime}: \mathcal{F}(\alpha)$,
- $\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi ; \Gamma \| \vDash_{i}^{j} t \approx t^{\prime}: \operatorname{El}(\mathcal{D}(\alpha, \mathcal{C}(\alpha)))$.
where we let $\mathcal{D}:: L|\Psi ; \mathcal{F}| \vDash_{i}^{j} T \approx T^{\prime} @ l$ to be

$$
\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime}\left|\Phi ; \Gamma\left\|\vDash_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{F}(\alpha) \cdot L^{\prime} \mid \Phi ; \Gamma\right\| \Vdash_{i}^{j} T[\delta] \approx T^{\prime}\left[\delta^{\prime}\right] @ l\right.
$$

The judgment $L \mid \Psi ; \mathcal{F} \vDash \vDash_{i}^{j} T \approx T^{\prime} @ l$ requires the stability under local substitutions of the relation between $T$ and $T^{\prime}$. We apply the same principle to related terms and derive the judgment $L \mid \Psi ; \mathcal{F} \|{ }_{i}^{j} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, which is given by

$$
\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot \mathcal{C}:: L^{\prime}\left|\Phi ; \Gamma\left\|\vDash_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{F}(\alpha) \cdot L^{\prime} \mid \Phi ; \Gamma\right\| \Vdash_{i}^{j} t[\delta] \approx t^{\prime}\left[\delta^{\prime}\right]: \operatorname{El}(\mathcal{D}(\alpha, C))\right.
$$

### 7.3 Properties for Logical Relations When $j=p$

In this section, we pause our progress of defining the logical relations when $j=m$. Following previous lines of work, when we give the Kripke logical relations to contextual types, we will have to refer to the validity judgments of types and terms in the logical relations. Therefore, in this section, we first work out a list of properties of the logical relations when $j=p$, and then in the next section, we define semantic judgments for global contexts and global substitutions, and then validity judgments. Once we have the validity judgments when $j=p$, we can then finish writing down the logical relations for contextual types. Without further ado, let us start proving some lemmas. The list of lemmas mainly follows Abel [2013]; Hu et al. [2023] though we also take Abel et al. [2017] into consideration.

Lemma 7.5 (Weakening).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} T \approx T^{\prime} @ l$ and $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma$, then $\mathcal{E}:: L^{\prime} \mid \Phi ; \Delta \| \Vdash_{i}^{p} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \| \models_{i}^{p} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ and $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma$, then $L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{p} t^{\prime} \approx t: \operatorname{El}(\mathcal{E})$.

Proof. Induction on $\mathcal{D}$. Note that typing judgments are invariant under weakenings. Also use the weakening law for generic equivalence.

Lemma 7.6 (Escape).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T^{\prime} @ l$.

Proof. Induction on $\mathcal{D}$. Use the subsumption law of the generic equivalence.
Lemma 7.7 (Reflexivity of Neutral). If $\mathcal{D}:: L\left|\Psi ; \Gamma \| \vDash_{i}^{p} T \approx T^{\prime} @ l, L\right| \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \mu \approx \mu^{\prime}: \mathrm{El}(\mathcal{D})$.

Proof. Induction on $\mathcal{D}$. Use the conversion and subsumption laws of generic equivalence and then the soundness of multi-step reductions and the escape lemma to obtain the equivalence between $\mu$ and $\mu^{\prime}$ as normal forms. In the function case, we use the congruence for neutral law of generic equivalence to show that the results of applying two neutral function are related.

## Lemma 7.8 (Weak Head Expansion).

- If $\mathcal{D}:: L\left|\Psi ; \Gamma \| \Vdash_{i}^{p} T \approx T^{\prime} @ l, L\right| \Psi ; \Gamma \vdash_{i} T_{1} \rightsquigarrow^{*} T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} T_{1}^{\prime} \rightsquigarrow^{*} T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \|=_{i}^{p} T_{1} \approx T_{1}^{\prime} @ l$.
 $L \mid \Psi ; \Gamma \| F_{i}^{p} t_{1} \approx t_{1}^{\prime}: \operatorname{El}(\mathcal{D})$.

Proof. Induction on $\mathcal{D}$. Use transitivity of reductions.
Lemma 7.9 (Symmetry).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \| \approx_{i}^{p} T \approx T^{\prime} @ l$, then $\mathcal{E}:: L \mid \Psi ; \Gamma \| \vDash_{i}^{p} T^{\prime} \approx T$ @ $l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t^{\prime} \approx t: \mathrm{El}(\mathcal{E})$.

Proof. We do induction on $\mathcal{D}$. Since our definition is designed with symmetry in mind, symmetry is rather immediate.

Lemma 7.10 (Right Irrelevance). If $\mathcal{D}:: L\left|\Psi ; \Gamma\left\|\vDash_{i}^{p} T \approx T^{\prime} @ l, \mathcal{E}:: L \mid \Psi ; \Gamma\right\| \Vdash_{i}^{p} T \approx T^{\prime \prime} @ l\right.$ and $L \mid \Psi ; \Gamma\| \|_{i}^{p} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma\| \|_{i}^{p} t \approx t^{\prime}: \mathrm{El}(\mathcal{E})$.

Proof. We do induction on $\mathcal{D}$ and then invert $\mathcal{E}$. We consider the function case. In this case, we have premises

$$
\begin{aligned}
& \mathcal{D}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \Vdash_{i}^{p} S_{1} \approx S_{2} @ l_{1}\right) \\
& \mathcal{D}_{2}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot \forall L^{\prime}\left|\Phi ; \Delta \Vdash_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}(\psi)\right) \cdot L^{\prime}\right| \Phi ; \Delta \| \Vdash_{i}^{p} T_{1}[s / x] \approx T_{2}\left[s^{\prime} / x\right] @ l_{2}\right) \\
& \mathcal{E}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \Vdash_{i}^{p} S_{1} \approx S_{2}^{\prime} @ l_{1}\right) \\
& \mathcal{E}_{2}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot \forall L^{\prime}\left|\Phi ; \Delta\left\|_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{E}_{1}(\psi)\right) \cdot L^{\prime} \mid \Phi ; \Delta\right\| \Vdash_{i}^{p} T_{1}[s / x] \approx T_{2}^{\prime}\left[s^{\prime} \mid x\right] @ l_{2}\right)\right.
\end{aligned}
$$

from $\mathcal{D}$ and $\mathcal{E}$. Here $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are the premises of $\mathcal{D}$, and likewise for $\mathcal{E}$. By determinacy, we know that

$$
T \rightsquigarrow^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{1}\right) \cdot T_{1}
$$

must be unique. Moreover, we also know

$$
\begin{gathered}
T^{\prime} \rightsquigarrow \leadsto^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{2}\right) \cdot T_{2} \\
T^{\prime \prime} \rightsquigarrow \Vdash^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{2}^{\prime}\right) \cdot T_{2}^{\prime}
\end{gathered}
$$

The most difficult part is to show that given

- $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma$,
- $\mathcal{A}:: L^{\prime} \mid \Phi ; \Delta\| \|_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{E}_{1}(\psi)\right)$
- $L^{\prime} \mid \Phi ; \Delta \vdash_{i} S_{1} \simeq S_{3} @ l$,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{i} S_{2} \simeq S_{4} @ l$,
- $L^{\prime} \mid \Phi ; \Delta, x: S_{1} @ l_{1} \vdash_{i} T_{1} \simeq T_{3} @ l_{2}$, and
- $L^{\prime} \mid \Phi ; \Delta, x: S_{2}^{\prime} @ l_{1} \vdash_{i} T_{2}^{\prime} \simeq T_{4} @ l_{2}$,
then

$$
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{i}^{p}\left(w: \Pi^{l_{1}, l_{2}}\left(x: S_{3}\right) \cdot T_{3}\right) s \approx\left(w^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot T_{4}\right) s^{\prime}: \operatorname{El}\left(\mathcal{E}_{2}(\psi, \mathcal{A})\right)
$$

From $L \mid \Psi ; \Gamma\| \|_{i}^{p} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$, together with

- $\mathcal{B}:: L^{\prime} \mid \Phi ; \Delta \| F_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}(\psi)\right)$ by IH ,
- $L^{\prime} \mid \Phi ; \Delta, x: S_{2} @ l_{1} \vdash_{i} T_{2} \simeq T_{4} @ l_{2}$, where we know $L \mid \Psi ; \Gamma \vdash_{i} S_{2} \approx S_{2}^{\prime} @ l_{1}$ by escape, the subsumption law of generic equivalence, transitivity and local context equivalence of syntactic and generic equivalence.
we have

$$
L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{p}\left(w: \Pi^{l_{1}, l_{2}}\left(x: S_{3}\right) \cdot T_{3}\right) s \approx\left(w^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot T_{4}\right) s^{\prime}: \operatorname{El}\left(\mathcal{D}_{2}(\psi, \mathcal{B})\right)
$$

By another IH, we have the goal.
Lemma 7.11 (Left Irrelevance). If $\mathcal{D}:: L\left|\Psi ; \Gamma\left\|\vDash_{i}^{p} T^{\prime} \approx T @ l, \mathcal{E}:: L \mid \Psi ; \Gamma\right\| \vDash_{i}^{p} T^{\prime \prime} \approx T\right.$ @ $l$ and $L \mid \Psi ; \Gamma \| \models_{i}^{p} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t \approx t^{\prime}: \mathrm{El}(\mathcal{E})$.

Proof. Immediate by symmetry and right irrelevance.
The left and right irrelevance lemmas are called the irrelevance lemma. It says that the exact relation between types is not important as long as their normal forms are related.

Lemma 7.12 (Reflexivity and Transitivity).

- If $\mathcal{D}_{1}:: L \mid \Psi ; \Gamma \| \vDash_{i}^{p} T_{1} \approx T_{2} @ l$ and $\mathcal{D}_{2}:: L \mid \Psi ; \Gamma \| \vDash_{i}^{p} T_{2} \approx T_{3} @ l$, then $\mathcal{D}_{3}:: L \mid \Psi ; \Gamma \| \vDash_{i}^{p} T_{1} \approx T_{3} @ l$.
- If $\mathcal{E}:: L\left|\Psi ; \Gamma\left\|\vDash_{i}^{p} T_{1} \approx T_{1} @ l, L \mid \Psi ; \Gamma\right\| \vDash_{i}^{p} t_{1} \approx t_{2}: \operatorname{El}\left(\mathcal{D}_{1}\right)\right.$ and $\left.L\right| \Psi ; \Gamma \| \vDash_{i}^{p} t_{2} \approx t_{3}: \operatorname{El}\left(\mathcal{D}_{2}\right)$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t_{1} \approx t_{3}: \operatorname{El}\left(\mathcal{D}_{3}\right)$.
- $\mathcal{F}:: L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} T_{1} \approx T_{1} @ l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t_{1} \approx t_{2}: \operatorname{El}\left(\mathcal{D}_{1}\right)$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t_{1} \approx t_{1}: \operatorname{El}(\mathcal{F})$.

Proof. We do induction on $\mathcal{D}_{1}$ and then invert $\mathcal{D}_{2}$. The function case is the most complex one. We have the following premises:

$$
\begin{aligned}
& \mathcal{A}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{p} S_{1} \approx S_{2} @ l_{1}\right) \\
& \mathcal{A}_{2}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot \forall L^{\prime}\left|\Phi ; \Delta\left\|\models_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{A}_{1}(\psi)\right) \cdot L^{\prime} \mid \Phi ; \Delta\right\| \models_{i}^{p} S_{1}^{\prime}[s / x] \approx S_{2}^{\prime}\left[s^{\prime} / x\right] @ l_{2}\right)\right. \\
& \mathcal{B}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{p} S_{2} \approx S_{3} @ l_{1}\right) \\
& \mathcal{B}_{2}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot \forall L^{\prime}\left|\Phi ; \Delta\left\|\models_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{E}_{1}(\psi)\right) \cdot L^{\prime} \mid \Phi ; \Delta\right\| \vDash_{i}^{p} S_{2}^{\prime}[s / x] \approx S_{3}^{\prime}\left[s^{\prime} / x\right] @ l_{2}\right)\right.
\end{aligned}
$$

from $\mathcal{D}$ and $\mathcal{E}$. Here $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are the premises of $\mathcal{D}_{1}$, and $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ are from $\mathcal{D}_{2}$. By determinacy, we know that

$$
\begin{aligned}
& T_{1} \rightsquigarrow^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{1}\right) \cdot S_{1}^{\prime} \\
& T_{2} \rightsquigarrow^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{2}\right) \cdot S_{2}^{\prime} \\
& T_{3} \rightsquigarrow^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{3}\right) \cdot S_{3}^{\prime}
\end{aligned}
$$

Now, we should construct the transitivity for types and terms at the same time to understand how this proof is going to check out. First we let

$$
\mathcal{C}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta\| \|_{i}^{p} S_{1} \approx S_{3} @ l_{1}\right)
$$

be the result of IH on $\mathcal{A}_{1}$ and $\mathcal{B}_{1}$. Then our goal is to show that
The most difficult part is to show that given

- $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma$,
- $\mathcal{F}:: L^{\prime} \mid \Phi ; \Delta \| \Vdash_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(C_{1}(\psi)\right)$
- $L^{\prime} \mid \Phi ; \Delta \vdash_{i} S_{1} \simeq S_{4} @ l$,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{i} S_{3} \simeq S_{5} @ l$,
- $L^{\prime} \mid \Phi ; \Delta, x: S_{1} @ l_{1} \vdash_{i} S_{1}^{\prime} \simeq S_{4}^{\prime} @ l_{2}$, and
- $L^{\prime} \mid \Phi ; \Delta, x: S_{3} @ l_{1} \vdash_{i} S_{3}^{\prime} \simeq S_{5}^{\prime} @ l_{2}$,
then
- 

$$
C_{2}:: L^{\prime} \mid \Phi ; \Delta \| \vDash_{i}^{p} S_{1}^{\prime}[s / x] \approx S_{3}^{\prime}\left[s^{\prime} / x\right] @ l_{2}
$$

- 

$$
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{i}^{p}\left(w: \Pi^{1}\left(l_{1}: l_{2}\right) \cdot x\right) S_{4} S_{4}^{\prime} s \approx\left(w_{3}: \Pi^{l_{1}, l_{2}}\left(x: S_{5}\right) \cdot S_{5}^{\prime}\right) s^{\prime}: \operatorname{El}\left(C_{2}\right)
$$

where $t_{k} \rightsquigarrow^{*} w_{k}$ for $k \in\{1,2,3\}$.
Our plan is the following:
(1) We first relate $S_{1}^{\prime}[s / x]$ and $S_{2}^{\prime}[s / x]$, resp. $\left(w_{1}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot S_{4}^{\prime}\right) s$ and $\left(w_{2}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot S_{4}^{\prime}\right) s$, which requires

$$
L^{\prime} \mid \Phi ; \Delta \| \models_{i}^{p} s \approx s: \operatorname{El}\left(\mathcal{A}_{1}(\psi)\right)
$$

which is derived from reflexivity and irrelevance;
(2) and then relate $S_{2}^{\prime}[s / x]$ and $S_{3}^{\prime}[s / x]$, resp. $\left(w_{2}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot S_{4}^{\prime}\right) s$ and $\left(w_{2}: \Pi^{l_{1}, l_{2}}\left(x: S_{5}\right) \cdot S_{5}^{\prime}\right) s^{\prime}$, which requires

$$
L^{\prime} \mid \Phi ; \Delta \| \models_{i}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{B}_{1}(\psi)\right)
$$

which is immediate from irrelevance.
The final missing piece is reflexivity. For types, it is just a result from transitivity and symmetry. For terms, it is reflexivity and transitivity of related types and then irrelevance.

This concludes the function case.
Next, we work on the properties for related local contexts and local substitutions. We first consider the built property of weakening. Weakening can be seen as a case of the Yoneda lemma, where we only depend on composition of weakenings.

Lemma 7.13 (Weakening).

- If $\mathcal{D}:: L \mid \Psi \not \vDash_{i}^{p} \Delta \approx \Delta^{\prime}$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $\mathcal{E}:: L^{\prime} \mid \Phi \Vdash_{i}^{p} \Delta \approx \Delta^{\prime}$.
- If $L\left|\Psi ; \Gamma \| \vDash_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{D}, \alpha:: L^{\prime}\right| \Phi \Longrightarrow L \mid \Psi$ and $\tau:: L^{\prime} \mid \Phi ; \Gamma^{\prime} \Longrightarrow_{i} \Gamma$, then $L^{\prime} \mid \Phi ; \Gamma^{\prime}\| \|_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{E}$.

Proof. Induction on $\mathcal{D}$. In the two base cases, we know $L^{\prime} \vdash \Phi$ by weakening. In the step case, we simply store the given weakenings in a composition.

Notice that the second statement is a little bit strengthened. It is possible due to local weakening of related types and terms.
We should first prove the reflexivity between identity local substitutions before proving the the escape lemma.
Lemma 7.14 (Reflexive Local Weakenings). If $L \mid \Psi \vdash_{i} \Delta, \Gamma \approx \Delta^{\prime}, \Gamma$ and $\mathcal{D}:: L \mid \Psi \vDash_{i}^{p} \Delta \approx \Delta^{\prime}$, then $L \mid \Psi ; \Delta, \Gamma \| \models_{i}^{p} w k_{\Delta}^{|\Gamma|} \approx w k_{\Delta}^{|\Gamma|}: \mathcal{D}$.

Proof. We do induction on $\mathcal{D}$ and consider the step case

$$
\begin{aligned}
& \mathcal{D}:: \frac{\mathcal{E}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime}\left|\Phi \Vdash_{i}^{p} \Delta \approx \Delta^{\prime} \quad \mathcal{F}:: L\right| \Psi ; \mathcal{E} \Vdash_{i}^{p} T \approx T^{\prime} @ l \quad L+l \approx l^{\prime}: \text { Level }}{L \mid \Psi \Vdash_{i}^{p} \Delta, x: T @ l \approx \Delta^{\prime}, x: T^{\prime} @ l^{\prime}} \\
& \mathcal{B}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi ; \Delta, x: T @ l, \Gamma \nVdash_{i}^{p} \mathrm{wk}_{\Delta}^{1+|\Gamma|} \approx \mathrm{wk}_{\Delta}^{1+|\Gamma|}: \mathcal{E}(\alpha)
\end{aligned}
$$

$$
\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi
$$ (by assumption) $L^{\prime} \mid \Phi ; \Delta, x: T @ l, \Gamma \| \models_{i}^{p} T\left[\mathrm{wk}_{\Delta}^{1+|\Gamma|}\right] \approx T^{\prime}\left[\mathrm{wk}_{\Delta}^{1+|\Gamma|}\right] @ l \quad \quad($ as $\mathcal{F}(\alpha, \mathcal{B}(\alpha)))$ $\mathcal{A}:: L^{\prime} \mid \Phi ; \Delta, x: T @ l, \Gamma\| \|_{i}^{p} T \approx T^{\prime} @ l$ $L^{\prime} \mid \Phi \vdash_{i} \Gamma, x: T @ l, \Delta \quad$ (by weakening and presupposition)

$$
\begin{aligned}
& L^{\prime} \mid \Phi ; \Delta, x: T @ l, \Gamma \vdash_{i} x \sim x: T @ l \quad \text { (by congruence for neutrals of generic equivalence) } \\
& L^{\prime} \mid \Phi ; \Delta, x: T @ l, \Gamma \| \vDash_{i}^{p} x \approx x: \mathrm{El}(\mathcal{A}) \\
& L \mid \Psi ; \Gamma, x: T @ l, \Delta \| \vDash_{i}^{p} \mathrm{wk}_{\Delta}^{1+|\Gamma|}, x / x \approx \mathrm{wk}_{\Delta}^{1+|\Gamma|}, x / x: \mathcal{D}
\end{aligned}
$$

Thus we conclude the goal.
Corollary 7.15 (Reflexive Local Identity Substitutions). If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$ and $\mathcal{D}:: L \mid \Psi \Vdash_{i}^{p} \Gamma \approx \Delta$, then
$L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} i d \approx i d: \mathcal{D}$.
Proof. This is a specialization of the previous lemma.
Knowing all local identity substitutions are reflexively related, we can then prove the escape lemma.
Lemma 7.16 (Escape).

- If $\mathcal{D}:: L \mid \Psi \Vdash_{i}^{p} \Delta_{1} \approx \Delta_{2}$, then $L \mid \Psi \vdash_{i} \Delta_{1} \simeq \Delta_{2}$.
- If $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{D}$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \simeq \delta^{\prime}: \Delta_{1}$ and $L \mid \Psi ; \Gamma \vdash_{i} \delta \simeq \delta^{\prime}: \Delta_{2}$.

Proof. Induction on $\mathcal{D}$. We consider the step case.

$$
\begin{aligned}
& \mathcal{D}:: \frac{\mathcal{E}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime}\left|\Phi \vDash_{i}^{p} \Delta_{1} \approx \Delta_{2} \quad \mathcal{F}:: L\right| \Psi ; \mathcal{E} \Vdash_{i}^{p} T \approx T^{\prime} @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi \Vdash_{i}^{p} \Delta_{1}, x: T @ l \approx \Delta_{2}, x: T^{\prime} @ l^{\prime}} \\
& L \mid \Psi \vdash_{i} \Delta_{1} \simeq \Delta_{2} \\
& \text { (by IH) } \\
& L \mid \Psi ; \Delta_{1} \| \vDash_{i}^{p} \mathrm{id} \approx \mathrm{id}: \mathcal{E}(\mathrm{id}) \\
& \text { (by Corollary 7.15) } \\
& L \mid \Psi ; \Delta_{1} \| \vDash_{i}^{p} T \approx T^{\prime} @ l \\
& \text { (as } \mathcal{F}(\mathrm{id}, \mathcal{E}(\mathrm{id}))) \\
& L \mid \Psi ; \Delta_{1} \vdash_{i} T \simeq T^{\prime} @ l \\
& \text { (by escape) }
\end{aligned}
$$

Hence we conclude the first statement.
In the second statement, we have $\delta=\delta_{1}, t / x$ and $\delta^{\prime}=\delta_{1}^{\prime}, t^{\prime} / x$, then

$$
\begin{aligned}
& L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta_{1} \approx \delta_{1}^{\prime}: \mathcal{E}(\mathrm{id}) \\
& L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \simeq \delta_{1}^{\prime}: \Delta_{1} \text { and } L \mid \Psi ; \Gamma \vdash_{i} \delta_{1} \simeq \delta_{1}^{\prime}: \Delta_{2} \\
& L \mid \Psi ; \Gamma \| \vdash_{i}^{p} t \approx t^{\prime}: \operatorname{El}(\mathcal{F}(\mathrm{id}, \mathcal{E}(\mathrm{id}))) \\
& L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T\left[\delta_{1}\right] @ l
\end{aligned} \quad \text { (by assumption) }
$$

Therefore we conclude the second statement as well.
Lemma 7.17 (Symmetry).

- If $\mathcal{D}:: L|\Psi| \models_{i}^{p} \Delta_{1} \approx \Delta_{2}$, then $\mathcal{E}:: L \mid \Psi \Vdash_{i}^{p} \Delta_{2} \approx \Delta_{1}$.
- If $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{D}$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta^{\prime} \approx \delta: \mathcal{E}$.

Proof. Induction on $\mathcal{D}$. Use the symmetry of related types to obtain the goal.
Lemma 7.18 (Right Irrelevance). If $\mathcal{D}:: L\left|\Psi \vDash_{i}^{p} \Delta_{1} \approx \Delta_{2}, \mathcal{E}:: L\right| \Psi \Vdash_{i}^{p} \Delta_{1} \approx \Delta_{3}$ and $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta \approx \delta^{\prime}$ : $\mathcal{D}$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{E}$.

Proof. Do induction on $\mathcal{D}$ and then invert $\mathcal{E}$. Use the right irrelevance of related terms.

Lemma 7.19 (Left Irrelevance). If $\mathcal{D}:: L\left|\Psi \vDash_{i}^{p} \Delta_{1} \approx \Delta_{2}, \mathcal{E}:: L\right| \Psi \models_{i}^{p} \Delta_{3} \approx \Delta_{1}$ and $L \mid \Psi ; \Gamma \| \models_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{D}$, then $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta \approx \delta^{\prime}: \mathcal{E}$.

Proof. A direct consequence of right irrelevance and symmetry.
Lemma 7.20 (Reflexivity and Transitivity).

- If $\mathcal{D}_{1}:: L \mid \Psi \vDash_{i}^{p} \Delta_{1} \approx \Delta_{2}$ and $\mathcal{D}_{2}:: L|\Psi| \models_{i}^{p} \Delta_{2} \approx \Delta_{3}$, then $\mathcal{D}_{3}:: L \mid \Psi \vDash_{i}^{p} \Delta_{1} \approx \Delta_{3}$.
- If $L \mid \Psi ; \Gamma \| \models_{i}^{p} \delta_{1} \approx \delta_{2}: \mathcal{D}_{1}$ and $L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} \delta_{2} \approx \delta_{3}: \mathcal{D}_{2}$, then $L \mid \Psi ; \Gamma \| \models_{i}^{p} \delta_{1} \approx \delta_{3}: \mathcal{D}_{3}$.
- $\mathcal{E}:: L|\Psi| \vDash_{i}^{p} \Delta_{1} \approx \Delta_{1}$.
- If $L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} \delta_{1} \approx \delta_{2}: \mathcal{D}_{1}$, then $L \mid \Psi ; \Gamma \| \models_{i}^{p} \delta_{1} \approx \delta_{1}: \mathcal{E}$.

Proof. Do induction on $\mathcal{D}_{1}$ and then invert $\mathcal{D}_{2}$. The proof proceeds very similarly to the relations for types and terms. We only consider the step case. In this case, we have the following premises:

- $\mathcal{F}_{1}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi \vDash_{i}^{p} \Delta_{1}^{\prime} \approx \Delta_{2}^{\prime}$,
- $\mathcal{F}_{2}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi \vDash_{i}^{p} \Delta_{2}^{\prime} \approx \Delta_{3}^{\prime}$,
- $\mathcal{A}_{1}:: L \mid \Psi ; \mathcal{F}_{1} \vDash_{i}^{p} T_{1} \approx T_{2} @ l$,
- $\mathcal{A}_{2}:: L \mid \Psi ; \mathcal{F}_{2} \vDash_{i}^{p} T_{2} \approx T_{3} @ l$,
- If $L \mid \Psi ; \mathcal{F}_{1} \vDash_{i}^{p} T \approx T^{\prime} @ l$ and $L \mid \Psi ; \mathcal{F}_{2} \vDash_{i}^{p} T^{\prime} \approx T^{\prime \prime} @ l$, then $L \mid \Psi ; \mathcal{F}_{3} \vDash_{i}^{p} T \approx T^{\prime \prime} @ l$.
- $\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi ; \Gamma \| \models_{i}^{p} t_{1} \approx t_{2}: \operatorname{El}\left(\mathcal{A}_{1}\left(\alpha, \mathcal{F}_{1}(\alpha)\right)\right)$,
- $\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi ; \Gamma \| F_{i}^{p} t_{2} \approx t_{3}: \operatorname{El}\left(\mathcal{A}_{2}\left(\alpha, \mathcal{F}_{2}(\alpha)\right)\right)$,

By IH, it is easy to show

$$
\mathcal{F}_{3}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi \vDash_{i}^{p} \Delta_{1}^{\prime} \approx \Delta_{3}^{\prime}
$$

The difficult goals are

- $\mathcal{A}_{3}:: L \mid \Psi ; \mathcal{F}_{3} \vDash_{i}^{p} T_{1} \approx T_{3} @ l$, and
- $\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi ; \Gamma \| \models_{i}^{p} t_{1} \approx t_{3}: \operatorname{El}\left(\mathcal{A}_{3}\left(\alpha, \mathcal{F}_{3}(\alpha)\right)\right)$.

We first assume $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$. To prove the first statement, we further assume $L^{\prime} \mid \Phi ; \Gamma \| \models_{i}^{j} \delta \approx \delta^{\prime}: \mathcal{F}_{3}(\alpha)$. Then we do a similar reasoning to the function case for transitivity of related types.
(1) We first relate $T_{1}[\delta]$ and $T_{2}[\delta]$ by using reflexivity.
(2) Then we related $T_{2}[\delta]$ and $T_{3}\left[\delta^{\prime}\right]$.
(3) Then we apply transitivity of related types.

To prove the second statement, we can simply use irrelevance so that transitivity can eventually apply on $\mathcal{A}_{3}\left(\alpha, \mathcal{F}_{3}(\alpha)\right)$.

Lemma 7.21 (Transitivity). Given

- $\mathcal{F}_{1}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime}|\Phi| \models_{i}^{p} \Delta_{1} \approx \Delta_{2}$,
- $\mathcal{F}_{2}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime} \mid \Phi \Vdash_{i}^{p} \Delta_{2} \approx \Delta_{3}$, and
- $\mathcal{F}_{3}:: \forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \cdot L^{\prime}|\Phi| \vDash_{i}^{p} \Delta_{1} \approx \Delta_{3}$,
then
- if $\mathcal{D}_{1}: e: L \mid \Psi ; \mathcal{F}_{1} \vDash_{i}^{p} T \approx T^{\prime} @ l$ and $\mathcal{D}_{2}:: L \mid \Psi ; \mathcal{F}_{2} \vDash_{i}^{p} T^{\prime} \approx T^{\prime \prime} @ l$, then $\mathcal{D}_{3}:: L \mid \Psi ; \mathcal{F}_{3} \vDash_{i}^{p} T \approx T^{\prime} @ l ;$
- if $L \mid \Psi ; \mathcal{F}_{1} \vDash_{i}^{p} t \approx t^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}\right)$ and $L \mid \Psi ; \mathcal{F}_{2} \vDash_{i}^{p} t^{\prime} \approx t^{\prime \prime}: \operatorname{El}\left(\mathcal{D}_{3}\right)$, then $L \mid \Psi ; \mathcal{F}_{3} \vDash_{i}^{p} t \approx t^{\prime \prime}: \operatorname{El}\left(\mathcal{D}_{3}\right)$.

Proof. The first statement is what we have proved in transitivity previously. The second statement uses the transitivity of related terms.

### 7.4 Semantic Well-formedness of Global Contexts and Related Global Substitutions

Following previous lines of work, when we give the Kripke logical relations to contextual types, we will have to refer to related types and terms in the logical relations under some invariants. Therefore, as early as it might seem, we must consider the semantics for global contexts and global substitutions to see what it needs to learn how exactly the semantics of contextual types can be defined. We then describe the semantics of global contexts and global substitutions, similar to that of local contexts and local substitutions, as inductive-recursive definitions. Types and terms in a global substitution consist of two components:
(1) the use of logical relations showing related terms having related computation,
(2) together with the maintenance of syntactic structures, if the type or the term comes from layer $v$ or $c$.

To handle the first component, we define several auxiliary definitions. These definitions essentially state that logical relations should remain stable under weakenings and local substitutions. We give the definitions as follows. We define $L \mid \Psi ; \Gamma \|_{\geq p}^{p} T \approx T^{\prime} @ l$ to be given

- $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$,
- $k \geq p$, and
- $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma$,
it holds that

$$
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{k}^{p} T[\delta] \approx T^{\prime}\left[\delta^{\prime}\right] @ l
$$

Similarly, we define $L|\Psi ; \Gamma|_{\geq p}^{p} t \approx t^{\prime}: T @ l$ to be given

- $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$,
- $k \geq p$, and
- $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma$,
we have

$$
L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} t[\delta] \approx t^{\prime}\left[\delta^{\prime}\right]: T[\delta] @ l
$$

We also need a similar relation for local contexts and local substitutions. $L \mid \Psi \|_{\geq p}^{p} \Delta \approx \Delta^{\prime}$ is defined as

$$
\forall \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi \text { and } k \geq\left. p \cdot L^{\prime}|\Phi|\right|_{k} ^{j} \Delta \approx \Delta
$$

We define $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Delta$ as

- $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$,
- $k \geq p$, and
- $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{k}^{p} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma$,
we have

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{k}^{p} \delta \circ \delta_{1} \approx \delta^{\prime} \circ \delta_{1}: \Delta
$$

We define their non-relational version:

$$
\begin{aligned}
& L \mid \Psi ; \Gamma \|_{\geq p}^{p} T @ l:=L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} T \approx T @ l \\
& L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t: T @ l:=L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t: T @ l \\
& L \mid \Psi \Vdash_{\geq p}^{p} \Delta:=L \mid \Psi \Vdash_{\geq p}^{p} \Delta \approx \Delta \\
& L\left|\Psi ; \Gamma \Vdash_{\geq p}^{p} \delta: \Delta:=L\right| \Psi ; \Gamma \vDash_{\geq p}^{p} \delta \approx \delta: \Delta
\end{aligned}
$$

These would have been the semantic judgments for the fundamental theorems for MLTT. However, they are not enough if we want to establish the fundamental theorems for the whole DeLaM. That is because the semantic
judgments for DELAM require stability under all sorts of substitutions, including universe, global and local substitutions. Thus, in this section, we will define when a global context and global substitutions are semantically well-formed in order to state the semantic judgments.

These definitions are closed under weakenings and local substitutions by design:
Lemma 7.22 (Weakenings).

- If $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} T \approx T^{\prime} @ l$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi ; \Gamma \vDash_{\geq p}^{p} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t^{\prime}: T$ @ $l$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi ; \Gamma \Vdash_{\geq p}^{p} t \approx t^{\prime}: T$ @ $l$.
- If $L \mid \Psi \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi \|_{\geq p}^{p} \Delta \approx \Delta^{\prime}$.
- If $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Delta$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi ; \Gamma \Vdash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Delta$.

Lemma 7.23 (Local Substitutions).

- If $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} T \approx T^{\prime} @ l$ and $L \mid \Psi ; \Delta \| \vDash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Gamma$, then $L|\Psi ; \Delta| \vDash_{\geq p}^{p} T[\delta] \approx T^{\prime}\left[\delta^{\prime}\right] @ l$.
- If $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t^{\prime}: T @ l$ and $L \mid \Psi ; \Delta \vDash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Gamma$, then $L \mid \Psi ; \Delta \Vdash_{\geq p}^{p} t[\delta] \approx t^{\prime}\left[\delta^{\prime}\right]: T[\delta] @ l$.
- If $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Psi ; \Delta^{\prime} \Vdash_{\geq p}^{p} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma$, then $L \mid \Phi ; \Delta^{\prime} \Vdash_{\geq p}^{p} \delta \circ \delta_{1} \approx \delta^{\prime} \circ \delta_{1}^{\prime}: \Delta$.

Lemma 7.24 (Local Weakenings). If $L \mid \Psi \vdash_{p} \Delta, \Gamma \approx \Delta^{\prime}, \Gamma$ and $L \mid \Psi \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$, then $L|\Psi ; \Delta, \Gamma|{ }_{\geq i}^{j} w k_{\Delta}^{|\Gamma|} \approx w k_{\Delta}^{|\Gamma|}: \Delta$.

Proof. Notice that a local weakening only shorten a well-formed local substitution.
Lemma 7.25 (Semantic Conversion). If $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t^{\prime}: T$ @ $l$ and $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} T \approx T^{\prime} @ l$, $L|\Psi ; \Gamma| \vDash_{\geq p}^{p} t \approx t^{\prime}: T^{\prime} @ l$.

Lemma 7.26 (PER). $L\left|\Psi ; \Gamma \vDash_{\geq p}^{p} T \approx T^{\prime} @ l, L\right| \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t^{\prime}: T @ l, L \mid \Psi \Vdash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$ and $L \mid \Psi ; \Gamma \not \vDash_{\geq i}^{j} \delta \approx \delta^{\prime}: \Delta$ are PERs.

Next, we handle the second component to record the syntactic information of types and terms. Following Hu and Pientka [2024a,b], we use an inductive definition to keep track of the syntactic structure, in which semantic information is also maintained. In fact, we need three mutually inductive judgments very similar to typing judgments.

- $L \mid \Psi ; \Gamma \| \models_{c}^{p} T @ l$ stores the syntactic information and the semantic information of $T$ and all its substructures.
- $L \mid \Psi ; \Gamma \| F_{i}^{p} t: T @ l$ stores the syntactic information and the semantic information of $t$ and all its sub-structures.
- $L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} \delta: \Delta$ stores the syntactic information and the semantic information of all terms in $\delta$.

We restrict the parameter layer $i \in\{v, c\}$. These definitions will make use of $L\left|\Psi ; \Gamma \Vdash_{\geq p}^{p} T @ l, L\right| \Psi ; \Gamma \mid \models_{\geq p}^{p} t$ : $T @ l$ and $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta: \Delta$.

Their definitions are designed to imply semantic information:
Lemma 7.27.

- If $L \mid \Psi ; \Gamma \| \Vdash_{c}^{p} T$ @ $l$, then $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} T$ @ $l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} t: T$ @ $l$, then $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} t: T$ @ $l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta: \Delta$, then $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} \delta: \Delta$.

Further we let

$$
\begin{gathered}
L\left|\Psi ; \Gamma\left\|\models_{c}^{p} T \approx T^{\prime} @ l:=L \mid \Psi ; \Gamma\right\| \Vdash_{c}^{p} T @ l \text { and } T=T^{\prime}\right. \\
L\left|\Psi ; \Gamma\left\|\vDash_{i}^{p} t \approx t^{\prime}: T @ l:=L \mid \Psi ; \Gamma\right\| \models_{i}^{p} t: T @ l \text { and } t=t^{\prime}\right. \\
L\left|\Psi ; \Gamma\left\|\models_{i}^{p} \delta \approx \delta^{\prime}: \Delta:=L \mid \Psi ; \Gamma\right\| \Vdash_{i}^{p} \delta: \Delta \text { and } \delta=\delta^{\prime}\right.
\end{gathered}
$$

We first consider the judgment for types:

$$
\begin{aligned}
& \frac{L \vdash l \approx \text { zero : Level } \quad L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} \text { Nat @ } l}{L \mid \Psi ; \Gamma \| \models_{c}^{p} \text { Nat @ } l} \quad \frac{L \vdash l \approx \operatorname{succ} l^{\prime}: \text { Level } \quad L \mid \Psi ; \Gamma \|_{\geq p}^{p} \mathrm{Ty}_{l^{\prime}} @ l}{L \mid \Psi ; \Gamma \| \models_{c}^{p} \mathrm{Ty}_{l^{\prime}} @ l} \\
& L \vdash l \approx l_{1} \sqcup l_{2} \text { : Level } L \vdash l_{1} \text { : Level } \\
& L \vdash l_{2} \text { : Level } \quad L\left|\Psi ; \Gamma\left\|\vDash_{c}^{p} S @ l_{1} \quad L\left|\Psi ; \Gamma, x: S @ l_{1} \| \vDash_{c}^{p} T @ l_{2} \quad L\right| \Psi ; \Gamma \vDash_{\geq p}^{p} \Pi^{l_{1}, l_{2}}(x: S) . T @ l\right.\right. \\
& L \mid \Psi ; \Gamma \| \models_{c}^{p} \Pi^{l_{1}, l_{2}}(x: S) . T @ l \\
& L \vdash l \approx l^{\prime}: \text { Level } \quad U:\left(\Delta \vdash_{c} @ l^{\prime}\right) \in \Psi \quad L\left|\Psi ; \Gamma \| \vDash_{c}^{p} \delta: \Delta \quad L\right| \Psi ; \Gamma \mid \vDash_{\geq p}^{p} U^{\delta} @ l \\
& L \mid \Psi ; \Gamma \| \Vdash_{c}^{p} U^{\delta} @ l \\
& L \vdash l^{\prime} \approx l \text { : Level } L\left|\Psi ; \Gamma \| \models_{c}^{p} t: \mathrm{Ty}_{l^{\prime}} @ l \quad L\right| \Psi ; \Gamma \Vdash_{\geq p}^{p} \mathrm{El}^{l^{\prime}} t @ l \\
& L \mid \Psi ; \Gamma \| \vDash_{c}^{p} \mathrm{El}^{l^{\prime}} t @ l
\end{aligned}
$$

The judgment not only keeps track of the syntactic structure of types but also the semantic information for both layers $p$ and $m$. The semantic information for both layers is critical to enable code running, when we refer to the code of types at layer $p$ and $m$, resp..

The judgment for local substitutions is simple, which simply generalizes that for terms:

$$
\begin{aligned}
& \frac{L \mid \Psi ; \Gamma \models_{\geq p}^{p} \cdot{ }^{k}:}{L \mid \Psi ; \Gamma \| \models_{i}^{p} \cdot{ }^{k}: \cdot} \quad \frac{L \mid \Psi ; \Gamma \models_{\geq p}^{p} \cdot k}{L \mid \Psi ; \Gamma \| \models_{i}^{p}:{ }_{g}^{k}: \cdot} \quad \frac{L \mid \Psi ; \Gamma \models_{\geq p}^{p} \mathrm{wk}_{g}^{k}: g}{L \mid \Psi ; \Gamma \| \models_{i}^{p} \mathrm{wk}_{g}^{k}: g} \\
& \frac{L\left|\Psi ; \Gamma\left\|\vDash_{i}^{p} \delta: \Delta \quad L\left|\Psi ; \Gamma \| \vDash_{i}^{p} t: T[\delta] @ l \quad L\right| \Psi ; \Gamma \vDash_{\geq p}^{p} \delta, t / x: \Delta, x: T @ l\right.\right.}{L \mid \Psi ; \Gamma \| \vDash_{i}^{p} \delta, t / x: \Delta, x: T @ l}
\end{aligned}
$$

We also keep track of both syntactic and semantic information of terms. We give the definition as follows (the parameter $i$ is restricted to be in $\{v, c\}$ when used only for the rules below):


These rules strictly decrease on the structures of types, terms and local substitutions. The purpose of these rules are to remember all computational and syntactic information of all sub-structures. Notice that the annotated
types do not necessarily match up with the precise types which terms might have. For example, zero is quantified to have seemingly some arbitrary type $T$. This, however, is not an accurate understanding. In fact, $T$ is quantified by $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p}$ zero : $T @ l$, so if we provide a proof of $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} T \approx T^{\prime} @ l$, then we obtain $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p}$ zero : $T^{\prime} @ l$ by irrelevance. More specifically,

Lemma 7.28 (Conversion). If $L \mid \Psi ; \Gamma \| \Vdash_{c}^{p} t: T$ @ $l$ and $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \| \vDash_{c}^{p} t: T^{\prime} @ l$.
The rules are designed in this way so that there is a semantic structure which the recursors can recurse on. Moreover, these rules are closed under weakenings and local substitutions, which is particularly important for use of global variables.

Lemma 7.29 (Weakenings).

- If $L \mid \Psi ; \Gamma \| \models_{c}^{p} T$ @ l and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi ; \Gamma \| \Vdash_{c}^{p} T$ @ $l$.
- If $i \in\{v, c\}, L \mid \Psi ; \Gamma \| \Vdash_{i}^{p} t: T$ @ $l$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi ; \Gamma \| \vDash_{i}^{p} t: T$ @ $l$.
- If $i \in\{v, c\}, L \mid \Psi ; \Gamma \| \models_{i}^{p} \delta: \Delta$ and $\alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$, then $L^{\prime} \mid \Phi ; \Gamma \| \models_{i}^{p} \delta: \Delta$.

Lemma 7.30 (Local Substitutions).

- If $L \mid \Psi ; \Gamma \| \models_{c}^{p} T @ l$ and $L \mid \Psi ; \Delta \| \Vdash_{c}^{p} \delta: \Gamma$, then $L \mid \Psi ; \Delta \| \models_{c}^{p} T[\delta] @ l$.
- If $i \in\{v, c\}, L \mid \Psi ; \Gamma \| \models_{i}^{p} t: T$ @ $l$ and $L \mid \Psi ; \Delta \| \vDash_{i}^{p} \delta: \Gamma$, then $L \mid \Psi ; \Delta \| \vDash_{i}^{p} t[\delta]: T[\delta] @ l$.
- If $i \in\{v, c\}, L \mid \Psi ; \Gamma \| \models_{i}^{p} \delta: \Delta$ and $L \mid \Psi ; \Delta^{\prime} \| \vDash_{i}^{p} \delta^{\prime}: \Gamma$, then $L \mid \Phi ; \Delta^{\prime} \| \vDash_{i}^{p} \delta \circ \delta^{\prime}: \Delta$.

Proof. We proceed by mutual induction. The syntax is already closed under local substitutions. The semantics is also closed under local substitutions by Lemma 7.23.

We can lift from layer $v$ to layer $c$.
Lemma 7.31 (Lifting).

- If $L \mid \Psi ; \Gamma \| \vDash_{v}^{p} t: T$ @ $l$, then $L \mid \Psi ; \Gamma \| \vDash_{c}^{p} t: T$ @ $l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{v}^{p} \delta: \Delta$, then $L \mid \Psi ; \Gamma \| \Vdash_{c}^{p} \delta: \Delta$.

Lemma 7.32 (Local Weakenings). If $L \mid \Psi \vdash_{p} \Delta, \Gamma \approx \Delta^{\prime}, \Gamma$ and $L \mid \Psi \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$, then $L \mid \Psi ; \Delta, \Gamma \| \models_{c}^{p} w k_{\Delta}^{|\Gamma|}: \Delta$.

In the semantic rules for global contexts and global substitutions, we use the auxiliary definitions above. Similar to the counterparts for local contexts and local substitutions, the rules are defined in an induction-recursion. The key idea is to make sure the relation is invariant under universe weakenings. We also impose invariance under global substitutions for related local contexts, types and terms. We are going to define the following definitions:

- $\mathcal{D}:: L \vDash \Psi \approx \Phi$ denotes the semantic related global contexts.
- $L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$ denotes the semantic relation between the global substitutions $\sigma$ and $\sigma^{\prime}$. The two definitions above are defined inductive-recursively.
- $\mathcal{F}:: L \mid \mathcal{E} \models_{p}^{p} \Gamma$ given $\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L \| \neq \Phi$ denotes $\Gamma$ is semantically well-formedness and is stable under universe weakening and related global substitutions.
- $L \mid \Psi ; \mathcal{F} \vDash_{p}^{p} T @ l$ denotes that $T$ is semantically well-formedness and is stable under universe weakening and related global substitutions.
We now move on to the actual definitions.

$$
\mathcal{D}:: \overline{L \vDash \cdot \approx}
$$

$L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$ is defined as $L \vdash \Psi$ and $\sigma=\sigma^{\prime}=\cdot$.

$$
\mathcal{D}:: \frac{\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi \approx \Phi^{\prime}}{L \vDash \Phi, g: \operatorname{Ctx} \approx \Phi^{\prime}, g: \operatorname{Ctx}}
$$

$L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$ is defined as

- $\sigma=\sigma_{1}, \Gamma / g$ and $\sigma^{\prime}=\sigma_{1}^{\prime}, \Gamma^{\prime} / g$,
- $\forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \mid \Psi \Vdash \sigma_{1} \approx \sigma_{1}^{\prime}: \mathcal{E}(\theta)$, and
- $L \mid \Psi \|_{\geq p}^{p} \Gamma \approx \Gamma^{\prime}$.

$$
\mathcal{D}:: \frac{\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi \approx \Phi^{\prime} \quad L \mid \mathcal{E} \vDash_{p}^{p} \Gamma \approx \Gamma^{\prime} \quad i \in\{c, p\}}{L \vDash \Phi, U:\left(\Gamma \vdash_{i} @ l\right) \approx \Phi^{\prime}, U:\left(\Gamma^{\prime} \vdash_{i} @ l\right)}
$$

$L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$ is defined as

- $\sigma=\sigma_{1}, T / U$ and $\sigma^{\prime}=\sigma_{1}^{\prime}, T^{\prime} / U$,
- $\forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \mid \Psi \|=\sigma_{1} \approx \sigma_{1}^{\prime}: \mathcal{E}(\theta)$,
- We analyze $i$.
- If $i=c$, then $T=T^{\prime}, L \mid \Psi ; \Gamma\| \|_{c}^{p} T @ l$ and $L \mid \Psi ; \Gamma^{\prime} \| \models_{c}^{p} T @ l$.
- Otherwise, $i=p$, then $L \mid \Psi ; \Gamma \|_{\geq p}^{p} T \approx T^{\prime} @ l$ and $L\left|\Psi ; \Gamma^{\prime}\right|_{\geq p}^{p} T \approx T^{\prime} @ l$.
where $L \mid \mathcal{E} \vDash_{p}^{p} \Gamma \approx \Delta$ is defined as

$$
\begin{gathered}
\forall \theta:: L^{\prime} \Longrightarrow L \text { and } L^{\prime} \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \mathcal{E}(\theta) \text { and } k \in\{p, m\} \cdot L^{\prime} \mid \Psi \Vdash_{k}^{p} \Gamma[\sigma] \approx \Delta\left[\sigma^{\prime}\right] \\
\mathcal{D}:: \frac{\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi \approx \Phi^{\prime} \quad \mathcal{F}:: L\left|\mathcal{E} \vDash_{p}^{p} \Gamma \approx \Gamma^{\prime} \quad \mathcal{A}:: L\right| \Psi ; \mathcal{F} \vDash_{p}^{p} T \approx T^{\prime} @ l \quad i \in\{v, c\}}{L \vDash \Phi, u:\left(\Gamma \vdash_{i} T @ l\right) \approx \Phi^{\prime}, u:\left(\Gamma^{\prime} \vdash_{i} T^{\prime} @ l\right)}
\end{gathered}
$$

$L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$ is defined as

- $\sigma=\sigma_{1}, t / u$ and $\sigma^{\prime}=\sigma_{1}^{\prime}, t^{\prime} / u$,
- $\forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \mid \Psi \vDash \sigma_{1} \approx \sigma_{1}^{\prime}: \mathcal{E}(\theta)$,
- Since we know $i \in\{v, c\}$, we have $t=t^{\prime}, L \mid \Psi ; \Gamma\left[\sigma_{1}\right] \quad \| F_{i}^{p} t: T\left[\sigma_{1}\right] @ l$ and $L \mid \Psi ; \Gamma^{\prime}\left[\sigma_{1}^{\prime}\right]\| \|_{i}^{p} t: T^{\prime}\left[\sigma_{1}^{\prime}\right] @ l$. In an extension where $i=p$ is possible, then we use $L \mid \Psi ; \Gamma\left[\sigma_{1}\right] \Vdash_{\geq p}^{p} t \approx t^{\prime}: T\left[\sigma_{1}\right] @ l$ and $L \mid \Psi ; \Gamma^{\prime}\left[\sigma_{1}^{\prime}\right] \Vdash_{\geq p}^{p} t \approx t^{\prime}: T^{\prime}\left[\sigma_{1}^{\prime}\right] @ l$.
where $L \mid \Psi ; \mathcal{F} \vDash_{p}^{p} T \approx T^{\prime} @ l$ is defined as given
- $\theta:: L^{\prime} \Longrightarrow L$,
- $\mathcal{B}:: L^{\prime} \mid \Psi \|=\sigma \approx \sigma^{\prime}: \mathcal{E}(\theta)$,
- $k \in\{p, m\}$, and
- $L^{\prime} \mid \Psi ; \Delta\| \|_{k}^{p} \delta \approx \delta^{\prime}: \mathcal{F}(\theta, \mathcal{B}, k)$,
we have

$$
L^{\prime} \mid \Psi ; \Delta \| \Vdash_{k}^{p} T[\sigma][\delta] \approx T^{\prime}\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ l
$$

Now we consider the properties of the new definitions.
Lemma 7.33 (Weakening).

- If $\mathcal{D}:: L \vDash \Phi \approx \Phi^{\prime}$ and $\theta:: L^{\prime} \Longrightarrow L$, then $\mathcal{D}^{\prime}:: L^{\prime} \vDash \Phi \approx \Phi^{\prime}$.
- If $L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$ and $\theta:: L^{\prime} \Longrightarrow L$, then $L^{\prime} \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}^{\prime}$.

Proof. Immediate by induction on $\mathcal{D}$. We simply use the composition of universe weakenings.

Lemma 7.34 (Reflexive Global Weakening). If $L \quad \vdash \Phi, \Psi$ and $\mathcal{D}:: L \vDash \Phi \approx \Phi^{\prime}$, then $L \mid \Phi, \Psi \| w k_{\Phi}^{|\Psi|} \approx w k_{\Phi}^{|\Psi|}: \mathcal{D}$.

Proof. We do induction on $\mathcal{D}$.
Case

$$
\mathcal{D}:: \frac{\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi_{1} \approx \Phi_{1}^{\prime}}{L \vDash \Phi_{1}, g: \operatorname{Ctx} \approx \Phi_{1}^{\prime}, g: \mathrm{Ctx}}
$$

We have given $\theta:: L^{\prime} \Longrightarrow L$,

$$
\begin{aligned}
& L \mid \Phi_{1}, g: \mathrm{Ctx}, \Psi \| \mathrm{wk}_{\Phi_{1}}^{1+|\Psi|} \approx \mathrm{wk}_{\Phi_{1}}^{1+|\Psi|}: \mathcal{E}(\theta) \\
& \alpha:: L^{\prime \prime}\left|\Psi^{\prime} \Longrightarrow L^{\prime}\right| \Phi_{1}, g: \mathrm{Ctx}, \Psi \\
& k \in\{p, m\} \\
& L^{\prime \prime}\left|\Psi^{\prime}\right| \vDash_{k}^{p} g \approx g
\end{aligned} \quad \text { (by IH) }
$$

In the last line, we know $g$ exits in $\Psi^{\prime}$ because $\alpha$ is a weakening. Hence we have this case.
Case

$$
\mathcal{D}:: \frac{\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi_{1} \approx \Phi_{1}^{\prime} \quad L \mid \mathcal{E} \vDash_{p}^{p} \Gamma \approx \Gamma^{\prime} \quad i \in\{c, p\}}{L \vDash \Phi_{1}, U:\left(\Gamma \vdash_{i} @ l\right) \approx \Phi_{1}^{\prime}, U:\left(\Gamma^{\prime} \vdash_{i} @ l\right)}
$$

Subcase If $i=p$, then given $\theta:: L^{\prime} \Longrightarrow L$ and $k \in\{p, m\}$, then we further assume $\alpha:: L^{\prime \prime}\left|\Psi^{\prime} \Longrightarrow L^{\prime}\right| \Phi_{1}, U:\left(\Gamma \vdash_{i} @ l\right), \Psi$ and $L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma$, we should prove

$$
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \Vdash_{k}^{p} U^{\delta} \approx U^{\delta^{\prime}} @ l
$$

There is a symmetric proof for $\Gamma^{\prime}$ which we omit here. We proceed as follows:

$$
\begin{array}{ll}
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \vdash_{k} \delta \simeq \delta^{\prime}: \Gamma & \text { (by escape, Lemma 7.27) } \\
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \vdash_{k} U^{\delta} \sim U^{\delta^{\prime}} @ l & \text { (by the neutral types law) } \\
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \Vdash_{k}^{p} U^{\delta} \approx U^{\delta^{\prime}} @ l & \text { (by the neutral type case) }
\end{array}
$$

Subcase If $i=c$, then we know $L \mid \Psi ; \Gamma \| \vDash_{c}^{p}$ id : $\Gamma$. Combining the previous case, we have the goal.
Case

$$
\begin{gathered}
\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi_{1} \approx \Phi_{1}^{\prime} \\
\mathcal{D}:: \frac{\mathcal{F}:: L\left|\mathcal{E} \vDash_{p}^{p} \Gamma \approx \Gamma^{\prime} \quad \mathcal{A}:: L\right| \Psi ; \mathcal{F} \vDash_{p}^{p} T \approx T^{\prime} @ l \quad i \in\{v, c\}}{L \vDash \Phi_{1}, u:\left(\Gamma \vdash_{i} T @ l\right) \approx \Phi_{1}^{\prime}, u:\left(\Gamma^{\prime} \vdash_{i} T @ @^{\prime}\right) l}
\end{gathered}
$$

We first have

$$
\mathcal{B}:: L \mid \Phi_{1}, u:\left(\Gamma \vdash_{i} T @ l\right), \Psi \Vdash \mathrm{wk}_{\Phi_{1}}^{1+|\Psi|} \approx \mathrm{wk}_{\Phi_{1}}^{1+|\Psi|}: \mathcal{E}(\theta)
$$

by IH.
Then without loss of generality, we should prove

$$
L \mid \Phi_{1}, u:\left(\Gamma \vdash_{i} T @ l\right), \Psi ; \Gamma \| \vDash_{c}^{p} u^{\mathrm{id}}: T @ l
$$

which in turn requires

$$
L\left|\Phi_{1}, u:\left(\Gamma \vdash_{i} T @ l\right), \Psi ; \Gamma\right| \models_{\geq p}^{p} u^{\text {id }}: T @ l
$$

Given $\theta:: L^{\prime} \Longrightarrow L$ and $k \geq i, \alpha:: L^{\prime \prime}\left|\Psi^{\prime} \Longrightarrow L^{\prime}\right| \Phi_{1}, g: C t x, \Psi$ and $C:: L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma$, we should prove

$$
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \Vdash_{k}^{p} u^{\delta} \approx u^{\delta^{\prime}}: T[\delta] @ l
$$

We proceed as follows:

$$
\begin{aligned}
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \vdash_{k} \delta \simeq \delta^{\prime}: \Gamma \\
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \models_{k}^{p} T\left[\mathrm{wk}_{\Phi_{1}}^{1+|\Psi|}\right][\delta] \approx T\left[\mathrm{wk}_{\Phi_{1}}^{1+|\Psi|}\right]\left[\delta^{\prime}\right] @ l \\
C^{\prime}:: L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \models_{k}^{p} T[\delta] \approx T\left[\delta^{\prime}\right] @ l
\end{aligned}
$$

$$
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \vdash_{k} u^{\delta} \sim u^{\delta^{\prime}}: T[\delta] @ l \quad \text { (by the congruence for neutrals law) }
$$

$$
L^{\prime \prime} \mid \Psi^{\prime} ; \Delta \| \vDash_{k}^{p} u^{\delta} \approx u^{\delta^{\prime}}: \mathrm{El}\left(C^{\prime}\right) \quad \text { (by reflexivity of neutral) }
$$

Corollary 7.35 (Reflexive Global Identity). If $L \vdash \Phi$ and $\mathcal{D}:: L \vDash \Phi$, then $L \mid \Phi \vDash i d_{\Phi} \approx i d_{\Phi}: \mathcal{D}$.
Lemma 7.36 (Escape).

- If $\mathcal{D}:: L \vDash \Phi \approx \Phi^{\prime}$, then $L \vdash \Phi$ and $L \vdash \Phi^{\prime}$.
- If $L \mid \Psi \|=\sigma \approx \sigma^{\prime}: \mathcal{D}$, then $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi$ and $L \mid \Psi \vdash \sigma \approx \sigma^{\prime}: \Phi^{\prime}$.

Proof. We do induction on $\mathcal{D}$. We consider one case:
$\mathcal{D}::$
$\frac{\mathcal{E}:: \forall \theta:: L^{\prime} \Longrightarrow L \cdot L^{\prime} \vDash \Phi_{1} \approx \Phi_{1}^{\prime} \quad \mathcal{F}:: L\left|\mathcal{E} \vDash_{p}^{p} \Gamma \approx \Gamma^{\prime} \quad \mathcal{A}:: L\right| \Psi ; \mathcal{F} \vDash_{p}^{p} T \approx T^{\prime} @ l \quad i \in\{v, c\}}{L \vDash \Phi_{1}, u:\left(\Gamma \vdash_{i} T @ l\right) \approx \Phi_{1}^{\prime}, u:\left(\Gamma^{\prime} \vdash_{i} T^{\prime} @ l\right)}$
We proceed as follows:

$$
\begin{array}{lr}
L \vdash \Phi_{1} \text { and } L \vdash \Phi_{1}^{\prime} \\
L \mid \Psi \Vdash_{p}^{p} \Gamma \approx \Gamma^{\prime} \\
L \mid \Psi \vdash_{p} \Gamma \approx \Gamma^{\prime} & \text { (by IH) } \\
L \mid \Psi ; \Gamma \| \vdash_{p}^{p} T @ l & \text { (by } \mathcal{F}) \\
L \mid \Psi ; \Gamma \vdash_{p} T @ l \text { and } L \mid \Psi ; \Gamma \vdash_{p} T^{\prime} @ l \\
L \vdash \Phi_{1}, u:\left(\Gamma \vdash_{i} T @ l\right) \text { and } L \vdash \Phi_{1}^{\prime}, u:\left(\Gamma^{\prime} \vdash_{i} T^{\prime} @ l\right) & \text { (applying escape to the previous line) } \\
\text { (by } \mathcal{A} \text {, using reflexive global and local identities) } \\
\text { (by escape) }
\end{array}
$$

Then we consider the global substitutions. We know $\sigma=\sigma_{1}, t / u$ and $\sigma^{\prime}=\sigma_{1}^{\prime}, t^{\prime} / u$ and $t=t^{\prime}$. Then by IH, we have

$$
L \mid \Psi \vdash \sigma_{1} \approx \sigma_{1}^{\prime}: \Phi_{1} \text { and } L \mid \Psi \vdash \sigma_{1} \approx \sigma_{1}^{\prime}: \Phi_{1}^{\prime}
$$

Since we know $i \in\{v, c\}$, by passing in a local identity substitution, we have

$$
L \mid \Psi ; \Gamma\left[\sigma_{1}\right] \| \models_{p}^{p} t \approx t^{\prime}: T\left[\sigma_{1}\right] @ l
$$

We then have $L \mid \Psi ; \Gamma\left[\sigma_{1}\right] \vdash_{i} t \approx t^{\prime}: T\left[\sigma_{1}\right] @ l$ by escape. We do it similarly for $L \mid \Psi ; \Gamma^{\prime}\left[\sigma_{1}\right] \vdash_{p} t \approx t^{\prime}:$ $T^{\prime}\left[\sigma_{1}\right] @ l$. By presupposition, unlifting and analysis using $i$, we can conclude the desired goal.

Lemma 7.37 (Symmetry).

- If $\mathcal{D}:: L \vDash \Phi \approx \Phi^{\prime}$, then $\mathcal{E}:: L \vDash \Phi^{\prime} \approx \Phi$.
- If $L \mid \Psi \|=\sigma \approx \sigma^{\prime}: \mathcal{D}$, then $L \mid \Psi \| \sigma^{\prime} \approx \sigma: \mathcal{E}$.
- If $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \|_{\geq p}^{p} T^{\prime} \approx T @ l$.
- If $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t^{\prime}: T$ @ $l$, then $L \mid \Psi ; \Gamma \vDash_{\geq p}^{p} t \approx t^{\prime}: T$ @ $l$.
- If $L|\Psi| \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$, then $L \mid \Psi \|_{\geq p}^{p} \Delta^{\prime} \approx \Delta$.
- If $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta^{\prime} \approx \delta: \Delta$.

Proof. We only consider the first statement and do Induction on $\mathcal{D}$. The other three statements are proved as part of the first statement. Immediate as we only need to be concerned about global substitutions. Apply symmetry for previous definitions.

Lemma 7.38 (Right Irrelevance). If $\mathcal{D}:: L \vDash \Phi \approx \Phi^{\prime}, \mathcal{E}:: L \vDash \Phi \approx \Phi^{\prime \prime}$ and $L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$, then $L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{E}$.

Proof. We do induction on $\mathcal{D}$ and then invert $\mathcal{E}$. This lemma is proved similar to previous irrelevance lemma. IHs are sufficient to discharge proof obligations.

Lemma 7.39 (Left Irrelevance). If $\mathcal{D}:: L \vDash \Phi^{\prime} \approx \Phi, \mathcal{E}:: L \vDash \Phi^{\prime \prime} \approx \Phi$ and $L \mid \Psi \| \sigma \approx \sigma^{\prime}: \mathcal{D}$, then $L \mid \Psi \|=\sigma \approx \sigma^{\prime}: \mathcal{E}$.

Proof. Apply symmetry and right irrelevance.

## Lemma 7.40 (Transitivity).

- If $\mathcal{D}_{1}:: L \vDash \Phi_{1} \approx \Phi_{2}$ and $\mathcal{D}_{2}:: L \vDash \Phi_{2} \approx \Phi_{3}$, then $\mathcal{D}_{3}:: L \vDash \Phi_{1} \approx \Phi_{3}$.
- If $\mathcal{E}:: L \vDash \Phi_{1} \approx \Phi_{1}, L \mid \Psi \vDash \sigma_{1} \approx \sigma_{2}: \mathcal{D}_{1}$ and $L \mid \Psi \| \sigma_{2} \approx \sigma_{3}: \mathcal{D}_{2}$, then $L \mid \Psi \vDash \sigma_{1} \approx \sigma_{3}: \mathcal{D}_{3}$.
- If $i \geq c, L \mid \Psi ; \Gamma \vDash_{\geq i}^{p} T \approx T^{\prime} @ l$ and $L|\Psi ; \Gamma|_{\geq i}^{p} T^{\prime} \approx T^{\prime \prime} @ l$, then $L|\Psi ; \Gamma|_{\geq i}^{p} T \approx T^{\prime \prime} @ l$.
- If $L \mid \Psi ; \Gamma \vDash_{\geq i}^{p} t \approx t^{\prime}: T$ @ $l$ and $L \mid \Psi ; \Gamma \vDash_{>i}^{p} t^{\prime} \approx t^{\prime \prime}: T$ @ $l$, then $L|\Psi ; \Gamma| \vDash_{\geq i}^{p} t \approx t^{\prime \prime}: T @ l$.
- If $i \geq p, L \mid \Psi \|_{\geq i}^{p} \Delta \approx \Delta^{\prime}$ and $L \mid \Psi \|_{\geq i}^{p} \Delta^{\prime} \approx \Delta^{\prime \prime}$, then $L \mid \Psi \|_{\geq i}^{p} \Delta \approx \Delta^{\prime \prime}$.
- If $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta^{\prime} \approx \delta^{\prime \prime}: \Delta$, then $L \mid \Psi ; \Gamma \Vdash_{\geq p}^{p} \delta \approx \delta^{\prime \prime}: \Delta$.

Proof. We only focus on the first two statements and do induction on $\mathcal{D}_{1}$ and then invert $\mathcal{D}_{2}$. The other three statements are proved during the process of proving the first two statements. The lemma is immediate after use of reflexivity and transitivity of previous definitions.

### 7.5 Logical Relations When $j=m$

We are ready for defining the rest of the logical relations. The only cases left are those when $j=m$. In this case, we also know that $i=m$. We first begin with the contextual types for types.
$\mathcal{D}:: \frac{L\left|\Psi ; \Gamma \vdash_{m} T \rightsquigarrow^{*} \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l \quad L\right| \Psi ; \left.\Gamma \vdash_{m} T^{\prime} \rightsquigarrow_{*}^{*} \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l \begin{array}{c}L \mid \Psi \vdash_{m} \Gamma \\ L \mid \Psi \vdash_{p} \Delta \\ L \mid \Psi \vdash_{p} \Delta^{\prime} \\ L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} @ l\right) \simeq \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l \\ L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ \operatorname{succ} l\end{array} \right\rvert\, \Psi \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime}}{L}$
Then $L \mid \Psi ; \Gamma\| \|_{m}^{m} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{m} t \rightsquigarrow^{*} w: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l$,
- $L \mid \Psi ; \Gamma \vdash_{m} t^{\prime} \rightsquigarrow^{*} w^{\prime}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l$,
- then we have two auxiliary definitions $L \mid \Psi ; \Gamma \| \vDash w \simeq w^{\prime}: \square\left(\Delta \vdash_{c} @ l\right)$, and
- $L \mid \Psi ; \Gamma \| \vDash w \simeq w^{\prime}: \square\left(\Delta^{\prime} \vdash_{c} @ l\right)$.

$$
\frac{L \mid \Psi ; \Delta \| \vDash_{c}^{p} T_{1} @ l}{L \mid \Psi ; \Gamma \| \vDash \operatorname{box} T_{1} \simeq \operatorname{box} T_{1}: \square\left(\Delta \vdash_{c} @ l\right)} \quad \frac{L \mid \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l}{L \mid \Psi ; \Gamma \| \vDash \mu \simeq \mu^{\prime}: \square\left(\Delta \vdash_{c} @ l\right)}
$$

Notice how the relation relies on previous relations when $j=p$. In particular, the local contexts $\Delta$ and $\Delta^{\prime}$ are related when $k$ takes different values. This is because we need to make sure that they can be the domain of local substitutions at any layer $\geq c$. When we relate $t$ and $t^{\prime}$, we do the routine: we first reduce them to weak head normal forms, and say that the normal forms are generically equivalent under two types to make symmetry easy. Then the real deal is the auxiliary relation $L \mid \Psi ; \Gamma \| \vDash w \simeq w^{\prime}: \square\left(\Delta \vdash_{c} @ l\right)$. This relation treats a contextual type as a sum type. There are two possibilities: a normal form of a contextual type is either neutral, then there is nothing else to do, or is the code of a type. Due to the static code lemma, we know that these types must be syntactically equal. In this case, if the type is $T_{1}$, we require $L \mid \Psi ; \Delta \|_{\geq c}^{p} T_{1} @ l$. This predicate is what we just defined: it asks $T_{1}$ to be stable under local substitutions at layer $\geq c$. In this way, we can be sure that $T_{1}$ semantically works fine when its syntactic information and/or its semantic information is needed. The same principle is applied to contextual types for terms.

$$
\begin{aligned}
& L\left|\Psi ; \Gamma \vdash_{m} T \rightsquigarrow^{*} \square\left(\Delta \vdash_{c} T_{1} @ l\right) @ l \quad L\right| \Psi ; \Gamma \vdash_{m} T^{\prime} \rightsquigarrow^{*} \square\left(\Delta^{\prime} \vdash_{c} T_{1}^{\prime} @ l\right) @ l \quad L \mid \Psi \vdash_{m} \Gamma \\
& L\left|\Psi ; \Delta \vdash_{p} T_{1} @ l \quad L\right| \Psi ; \Delta^{\prime} \vdash_{p} T_{1}^{\prime} @ l \quad L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T_{1} @ l\right) \simeq \square\left(\Delta^{\prime} \vdash_{c} T_{1}^{\prime} @ l\right) @ l \\
& \mathcal{D}:: \frac{L\left|\Psi \Vdash_{\geq p}^{p} \Delta \approx \Delta^{\prime} \quad L\right| \Psi ; \Delta \|_{\geq p}^{p} T_{1} \approx T_{1}^{\prime} @ l \quad L \mid \Psi ; \Delta^{\prime} \Vdash_{\geq p}^{p} T_{1}^{\prime} \approx T_{1} @ l}{L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l}
\end{aligned}
$$

Then $L \mid \Psi ; \Gamma \|=_{m}^{m} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{m} t \rightsquigarrow^{*} w: \square\left(\Delta \vdash_{c} T @ l\right) @ l$,
- $L \mid \Psi ; \Gamma \vdash_{m} t^{\prime} \rightsquigarrow^{*} w^{\prime}: \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l\right) @ l$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ l$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}: \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l\right) @ l$,
- then we have two auxiliary definitions $L \mid \Psi ; \Gamma \| \vDash w \simeq w^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right)$, and
- $L \mid \Psi ; \Gamma \| \vDash w \simeq w^{\prime}: \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l\right)$.

$$
\frac{L \mid \Psi ; \Delta \| \vDash_{c}^{p} t_{1}: T @ l}{L \mid \Psi ; \Gamma \| \vDash \text { box } t_{1} \simeq \operatorname{box} t_{1}: \square\left(\Delta \vdash_{c} T @ l\right)} \quad \frac{L \mid \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right) @ l}{L \mid \Psi ; \Gamma \| \vDash \mu \simeq \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l\right)}
$$

Then we take a look at the two kinds of meta-functions. Let us first consider the meta-functions for local contexts.

$$
\begin{gathered}
L \mid \Psi ; \Gamma \vdash_{m} T \rightsquigarrow^{*}(g: \mathrm{Ctx}) \Rightarrow^{l} T_{1} @ l \\
L\left|\Psi ; \Gamma \vdash_{m} T^{\prime} \rightsquigarrow^{*}(g: \mathrm{Ctx}) \Rightarrow^{l} T_{1}^{\prime} @ l \quad L\right| \Psi \vdash_{m} \Gamma L L \mid \Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} T_{1} @ l \\
L\left|\Psi, g: \mathrm{Ctx} ; \Gamma \vdash_{m} T_{1}^{\prime} @ l \mathrm{~L}\right| \Psi ; \Gamma \vdash_{m}(g: \mathrm{Ctx}) \Rightarrow^{l} T_{1} \simeq(\mathrm{~g}: \mathrm{Ctx}) \Rightarrow^{l} T_{1}^{\prime} @ l \\
\mathcal{D}:: \frac{\mathcal{E}:: \forall \psi:: L^{\prime}\left|\Phi ; \Delta^{\prime \prime} \Longrightarrow_{i} L\right| \Psi ; \Gamma \text { and } L^{\prime}\left|\Phi\left\|_{\geq p}^{p} \Delta \approx \Delta^{\prime} \cdot L^{\prime} \mid \Phi ; \Delta^{\prime \prime}\right\| \vDash_{m}^{m} T_{1}[\Delta / g] \approx T_{1}^{\prime}\left[\Delta^{\prime} / g\right] @ l\right.}{L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l}
\end{gathered}
$$

Then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{m} t \rightsquigarrow^{*} w:(g: \mathrm{Ctx}) \Rightarrow^{l} T_{1} @ l$,
- $L \mid \Psi ; \Gamma \vdash_{m} t^{\prime} \rightsquigarrow^{*} w^{\prime}:(g: \mathrm{Ctx}) \Rightarrow^{l} T_{1}^{\prime} @ l$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}:(g: \mathrm{Ctx}) \Rightarrow^{l} T_{1} @ l$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}:(g: C t x) \Rightarrow^{l} T_{1}^{\prime} @ l$, and
- given
$-\psi:: L^{\prime}\left|\Phi ; \Delta^{\prime \prime} \Longrightarrow_{i} L\right| \Psi ; \Gamma$, and
- $\mathcal{A}:: L^{\prime} \mid \Phi \|_{\geq p}^{p} \Delta \approx \Delta^{\prime}$,
then

$$
L^{\prime} \mid \Phi ; \Delta^{\prime \prime} \|=_{m}^{m} w \$ \Delta \approx w^{\prime} \$ \Delta^{\prime}: \operatorname{El}(\mathcal{E}(\psi, \mathcal{A}))
$$

We apply a similar principle to the meta-functions for types.

$$
\mathcal{D}::
$$

$$
\begin{gathered}
L \mid \Psi ; \Gamma \vdash_{m} T \rightsquigarrow^{*}\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1} @ \operatorname{succ} l \sqcup l^{\prime} \\
L \mid \Psi ; \Gamma \vdash_{m} T^{\prime} \rightsquigarrow^{*}\left(U:\left(\Delta^{\prime} \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1}^{\prime} @ \operatorname{succ} l \sqcup l^{\prime} \\
L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T_{1} @ l \quad L \mid \Psi, U:\left(\Delta^{\prime} \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T_{1} @ l \\
L\left|\Psi ; \Gamma \vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow_{l^{\prime}} T_{1} \simeq\left(U:\left(\Delta^{\prime} \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1}^{\prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L^{\prime}\right| \Phi \| \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime} \\
\mathcal{E}_{1}:: \forall \psi:: L^{\prime}\left|\Phi ; \Delta^{\prime \prime} \Longrightarrow_{i} L\right| \Psi ; \Gamma \text { and } L^{\prime}|\Phi ; \Delta| \vDash_{\geq p}^{p} T_{2} \approx T_{2}^{\prime} @ l \cdot L^{\prime} \mid \Phi ; \Delta^{\prime \prime} \| \vDash_{m}^{m} T_{1}\left[T_{2} / U\right] \approx T_{1}^{\prime}\left[T_{2}^{\prime} / U\right] @ l^{\prime} \\
\mathcal{E}_{2}:: \forall \psi:: L^{\prime}\left|\Phi ; \Delta^{\prime \prime} \Longrightarrow_{i} L\right| \Psi ; \Gamma \text { and } L^{\prime}\left|\Phi ; \Delta^{\prime}\right| \vDash_{\geq p}^{p} T_{2} \approx T_{2}^{\prime} @ l \cdot L^{\prime} \mid \Phi ; \Delta^{\prime \prime} \| \Vdash_{m}^{m} T_{1}^{\prime}\left[T_{2} / U\right] \approx T_{1}\left[T_{2}^{\prime} / U\right] @ l^{\prime} \\
\hline L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}
\end{gathered}
$$

Notice the symmetry between $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$. Then $L \mid \Psi ; \Gamma\| \|_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{m} t \rightsquigarrow^{*} w:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1} @ \operatorname{succ} l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{m} t^{\prime} \rightsquigarrow^{*} w^{\prime}:\left(U:\left(\Delta^{\prime} \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1}^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1} @ \operatorname{succ} l \sqcup l^{\prime}$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}:\left(U:\left(\Delta^{\prime} \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T_{1}^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}$, and
- given
$-\psi:: L^{\prime}\left|\Phi ; \Delta^{\prime \prime} \Longrightarrow_{i} L\right| \Psi ; \Gamma$, and
- $\mathcal{A}:: L^{\prime}|\Phi ; \Delta| \vDash_{\geq p}^{p} T_{2} \approx T_{2}^{\prime} @ l$,
then

$$
L^{\prime} \mid \Phi ; \Delta^{\prime \prime} \| \vDash_{m}^{m} w \$_{p} T_{2} \approx w^{\prime} \$_{p} T_{2}^{\prime}: \operatorname{El}\left(\mathcal{E}_{1}(\psi, \mathcal{A})\right)
$$

- and last symmetrically given
$-\psi:: L^{\prime}\left|\Phi ; \Delta^{\prime \prime} \Longrightarrow_{i} L\right| \Psi ; \Gamma$, and
- $\mathcal{A}:: L^{\prime} \mid \Phi ; \Delta^{\prime} \vDash_{\geq p}^{p} T_{2} \approx T_{2}^{\prime} @ l$,
then

$$
L^{\prime} \mid \Phi ; \Delta^{\prime \prime} \|=_{m}^{m} w^{\prime} \$_{p} T_{2} \approx w \$_{p} T_{2}^{\prime}: \operatorname{El}\left(\mathcal{E}_{2}(\psi, \mathcal{A})\right)
$$

The last type is the universe-polymorphic function types. They are special because they parameterize over universe levels. Therefore, a universe-polymorphic function cannot live in any finite universe as it can be instantiated to any small universe. Our semantics must incorporate this fact and introduce the $\omega$ level, which further requires a transfinite recursion on the universe levels.
$\mathcal{D}::$
$L\left|\Psi ; \Gamma \vdash_{m} T \rightsquigarrow^{*} \vec{\ell} \Rightarrow^{l} T_{1} @ \omega \quad L\right| \Psi ; \Gamma \vdash_{m} T^{\prime} \rightsquigarrow^{*} \vec{l} \Rightarrow^{l} T_{1}^{\prime} @ \omega \quad L \mid \Psi \vdash_{m} \Gamma$
$L, \vec{\ell}\left|\Psi ; \Gamma \vdash_{m} T_{1} @ l \quad L, \vec{\ell}\right| \Psi ; \Gamma \vdash_{m} T_{1}^{\prime} @ l \quad L\left|\Psi ; \Gamma \vdash_{m} \vec{\ell} \Rightarrow{ }^{l} T_{1} \simeq \vec{\ell} \Rightarrow T_{1}^{l} @ \omega \quad\right| \vec{\ell} \mid>0$ $\mathcal{E}:: \forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma$ and $|\vec{\ell}|=|\vec{l}|=\left|\vec{l}^{\prime}\right|$ and
$\left(\forall 0 \leq n<|\vec{l}| \cdot L^{\prime} \vdash \vec{l}(n) \approx \vec{l}^{\prime}(n):\right.$ Level $) \cdot L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} T_{1}[\vec{l} / \vec{\ell}] \approx T_{1}^{\prime}[\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}]$

$$
L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ \omega
$$

Then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$ is defined by

- $L \mid \Psi ; \Gamma \vdash_{m} t \rightsquigarrow^{*} w: \vec{\ell} \Rightarrow^{l} T_{1} @ \omega$,
- $L \mid \Psi ; \Gamma \vdash_{m} t^{\prime} \rightsquigarrow^{*} w^{\prime}: \vec{\ell} \Rightarrow^{l} T_{1}^{\prime} @ \omega$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}: \vec{\ell} \Rightarrow^{l} T_{1} @ \omega$,
- $L \mid \Psi ; \Gamma \vdash_{m} w \simeq w^{\prime}: \vec{\ell} \Rightarrow^{l} T_{1}^{\prime} @ \omega$, and
- given
$-\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow{ }_{i} L\right| \Psi ; \Gamma$,
$-\mathcal{A}::|\vec{l}|=|\vec{l}|=|\vec{l}|$, and
$-\mathcal{B}:: \forall 0 \leq n<\vec{l} \mid \cdot L+\vec{l}(n) \approx \vec{l}^{\prime}(n):$ Level, then

$$
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{m}^{m} w \$ \vec{l} \approx w \$ \vec{l}^{\prime}: \operatorname{El}(\mathcal{E}(\psi, \mathcal{A}, \mathcal{B}))
$$

Notice how the premise $\mathcal{E}$ changes its universe levels based on different universes, where exactly transfinite recursion becomes necessary.
Now we have finished defining all definitions, including the Kripke logical relations for types and terms. The logical relations for local contexts and local substitutions have been given generally in Sec. 7.2. Then we will examine the properties of the logical relations when $i=j=m$. Then we are ready for giving the definitions for the semantic judgments as well as establish the fundamental theorems. By instantiating the fundamental theorems with syntactic equivalence, we are able to obtain a few consequence lemmas, which will be subsequently used in our second instantiation. In the second instantiation of the fundamental theorems, we use the convertibility checking judgments, from which we derive our final desired the decidability theorem for convertibility checking.

### 7.6 Properties for Logical Relations When $i=j=m$

Now we move on to consider the properties of the logical relations at the final layers, i.e. when $i=j=m$. In this case, we are considering the relations of all possible types among all possible terms. We first begin with the regular properties, when we will state an important property, layering restriction, which states how the logical relations are transferred between $i \in\{p, m\}$ when $j=m$. Let us first begin with the simple ones. These properties follow Sec. 7.3 quite closely as for the types that exist at both layers $p$ and $m$ the lemma proceeds in a similar way. The only difference comes in for types only available at layer $m$.

## Lemma 7.41 (Weakening).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \|=_{m}^{m} T \approx T^{\prime} @ l$ and $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{m} L\right| \Psi ; \Gamma$, then $\mathcal{E}:: L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$ and $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{m} L\right| \Psi ; \Gamma$, then $L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} t^{\prime} \approx t: \operatorname{El}(\mathcal{E})$.

Proof. Induction on $\mathcal{D}$. For the cases that overlap with $j=p$, they can be ported directly to this lemma. Therefore, the only cases we need to consider are the new rules. The meta-functions are rather routine as weakenings are built in their definitions. Let us consider contextual types.

$$
\begin{gathered}
L \mid \Psi ; \Gamma \vdash_{m} T \rightsquigarrow^{*} \square\left(\Delta \vdash_{c} T_{1} @ l\right) @ l \\
\mathcal{D}:: \frac{L\left|\Psi ; \Gamma \vdash_{m} T^{\prime} \rightsquigarrow_{*}^{*} \square\left(\Delta^{\prime} \vdash_{c} T_{1}^{\prime} @ l\right) @ l \quad L\right| \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T_{1} @ l\right) \simeq \square\left(\Delta^{\prime} \vdash_{c} T_{1}^{\prime} @ l\right) @ l}{L\left|\Psi \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime} \quad L\right| \Psi ; \Delta \vDash_{\geq p}^{p} T @ l \quad L \mid \Psi ; \Delta^{\prime} \vDash_{\geq p}^{p} T^{\prime} @ l} \\
L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l
\end{gathered}
$$

This case is almost immediate. For judgments like $L \mid \Psi \Vdash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$ and $L \mid \Psi ; \Delta \vDash_{\geq p}^{p} T$ @ $l$, we know we can extract from $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{m} L\right| \Psi ; \Gamma \alpha:: L^{\prime}|\Phi \Longrightarrow L| \Psi$. In particular, the local context in $L \mid \Psi ; \Delta \| \vDash_{\geq p}^{p} T @ l$ is not impacted by the weakening. For $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, we shall also apply the weakening lemma to $\alpha$ to weaken $L \mid \Psi ; \Delta \| \Vdash_{c}^{p} t_{1}: T @ l$.

Lemma 7.42 (Escape).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{m} T \simeq T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \vdash_{m} t \simeq t^{\prime}: T$ @ land $L \mid \Psi ; \Gamma \vdash_{m} t \simeq t^{\prime}: T^{\prime} @ l$.

Proof. Case analyze $\mathcal{D}$. The lemma holds by construction.
Lemma 7.43 (Reflexivity of Neutral). If $\mathcal{D}:: L\left|\Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l, L\right| \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{m} \mu \sim \mu^{\prime}: T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} \mu \approx \mu^{\prime}: \operatorname{El}(\mathcal{D})$.

Proof. Induction on $\mathcal{D}$. We proceed similarly to the counterpart when $j=p$.
Lemma 7.44 (Weak Head Expansion).

- If $\mathcal{D}:: L\left|\Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l, L\right| \Psi ; \Gamma \vdash_{m} T_{1} \rightsquigarrow^{*} T @ l$ and $L \mid \Psi ; \Gamma \vdash_{m} T_{1}^{\prime} \rightsquigarrow^{*} T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T_{1} \approx T_{1}^{\prime} @ l$.
- If $L\left|\Psi ; \Gamma\| \|_{m}^{m} t \approx t^{\prime}: \operatorname{El}(\mathcal{D}), L\right| \Psi ; \Gamma \vdash_{m} t_{1} \rightsquigarrow^{*} t: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{m} t_{1}^{\prime} \rightsquigarrow^{*} t^{\prime}: T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \|=_{m}^{m} t_{1} \approx t_{1}^{\prime}: \operatorname{El}(\mathcal{D})$.

Proof. Induction on $\mathcal{D}$. Use transitivity of multi-step reductions.
Lemma 7.45 (Symmetry).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \|=_{m}^{m} T \approx T^{\prime} @ l$, then $\mathcal{E}:: L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T^{\prime} \approx T$ @ $l$.
- If $L \mid \Psi ; \Gamma \|=_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t^{\prime} \approx t: \mathrm{El}(\mathcal{E})$.

Proof. Induction on $\mathcal{D}$. Symmetry holds by design. The verbose case is the meta-functions for types. Effectively, $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$ are duplicated to make sure that symmetry can be easily proved.

Lemma 7.46 (Right Irrelevance). If $\mathcal{D}:: L\left|\Psi ; \Gamma\left\|\vDash_{m}^{m} T \approx T^{\prime} @ l, \mathcal{E}:: L \mid \Psi ; \Gamma\right\| \|_{m}^{m} T \approx T^{\prime \prime} @ l\right.$ and $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{E})$.

Proof. Induction on $\mathcal{D}$. Again, the overlapping cases for $j \in\{p, m\}$ can still be ported immediately, with only layers changed. For contextual types, it is obvious as the logical relations do not depend on the premises at all. The remaining cases are meta-functions and universe-polymorphic functions. These cases are simpler than that of dependent functions because they only take simple IH to go through.

Lemma 7.47 (Left Irrelevance). If $\mathcal{D}:: L\left|\Psi ; \Gamma\left\|\vDash_{m}^{m} T^{\prime} \approx T @ l, \mathcal{E}:: L \mid \Psi ; \Gamma\right\| \vDash_{m}^{m} T^{\prime \prime} \approx T\right.$ @ $l$ and $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \operatorname{El}(\mathcal{D})$, then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \operatorname{El}(\mathcal{E})$.

Proof. Immediate by symmetry and right irrelevance.
Lemma 7.48 (Reflexivity and Transitivity).

- If $\mathcal{D}_{1}:: L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T_{1} \approx T_{2} @ l$ and $\mathcal{D}_{2}:: L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T_{2} \approx T_{3} @ l$, then $\mathcal{D}_{3}:: L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T_{1} \approx T_{3} @ l$.
- If $\mathcal{E}:: L\left|\Psi ; \Gamma\left\|\vDash_{m}^{m} T_{1} \approx T_{1} @ l, L \mid \Psi ; \Gamma\right\| \vDash_{m}^{m} t_{1} \approx t_{2}: \mathrm{El}\left(\mathcal{D}_{1}\right)\right.$ and $\left.L\right| \Psi ; \Gamma \| \vDash_{m}^{m} t_{2} \approx t_{3}: \mathrm{El}\left(\mathcal{D}_{2}\right)$, then $L \mid \Psi ; \Gamma\| \|_{m}^{m} t_{1} \approx t_{3}: \operatorname{El}\left(\mathcal{D}_{3}\right)$.
- $\mathcal{F}:: L \mid \Psi ; \Gamma\| \|_{m}^{m} T_{1} \approx T_{1} @ l$.
- If $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t_{1} \approx t_{2}: \operatorname{El}\left(\mathcal{D}_{1}\right)$, then $L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t_{1} \approx t_{1}: \operatorname{El}(\mathcal{F})$.

Proof. We do induction on $\mathcal{D}_{1}$ and then invert $\mathcal{D}_{2}$. The cases available when $j=p$ still can be ported to this lemma. In fact, all remaining cases are simpler than the case for dependent functions. This is because for dependent functions, we must consider reflexivity of related input arguments, where for all new cases for metaprogramming, we only need transitivity from layer $p$, which has been an established fact at this point.

At last, we must prove one very important lemma which connects the logical relations when $j$ takes different values. We need this lemma to explain the fact that a variable at a lower layer suddenly can be substituted by a term that is not well-typed at its original layer. This phenomenon typically occurs when we run code from MLTT at layer $m$. In this case, a variable originally only expected to be substituted by a term from MLTT must
also be able to handle a term only available at layer $m$, which might contain, for example, a recursion principle for code. A similar lemma occurs in Sec. 2 and 3, and a dependently typed version must also be proved here.

Lemma 7.49 (Layering Restriction).

- If $\mathcal{D}:: L \mid \Psi ; \Gamma \| \models_{m}^{p} T \approx T^{\prime} @ l$, then $\mathcal{E}:: L \mid \Psi ; \Gamma \| \vDash_{m}^{m} T \approx T^{\prime} @ l$.
- The following two relations are equivalent:

$$
L \mid \Psi ; \Gamma \| \models_{m}^{p} t \approx t^{\prime}: \mathrm{El}(\mathcal{D}) \text { and } L \mid \Psi ; \Gamma \| \vDash_{m}^{m} t \approx t^{\prime}: \mathrm{El}(\mathcal{E})
$$

The idea of this lemma is that, if we know a type is coming from MLTT only, then it is possible to regard its term as a term at both layers $p$ and $m$. The direction going from $p$ to $m$ should be intuitive; it resembles the lifting lemma on the syntactic side. The backward direction, however, might appear counter-intuitive. Yet, if we consider the example we discussed above, then we should consider this lemma describing a process of bringing a term from $m$ back to $p$, performing the substitution, and then finally lifting the result back to $m$.

Proof. We do induction on $\mathcal{D}$. Since we know $T$ and $T^{\prime}$ are related when $j=p$, we do not need to consider the cases for meta-programming. The most complex case is the function case. In this case, we have premises

```
\(\mathcal{D}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{m} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{p} S_{1} \approx S_{2} @ l_{1}\right)\)
\(\mathcal{D}_{2}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow{ }_{m} L\right| \Psi ; \Gamma \cdot \forall L^{\prime}\left|\Phi ; \Delta\left\|\Vdash_{m}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}(\psi)\right) \cdot L^{\prime} \mid \Phi ; \Delta\right\| \models_{m}^{p} T_{1}[s / x] \approx T_{2}\left[s^{\prime} / x\right] @ l_{2}\right)\right.\)
```

By determinacy, we know that

$$
\begin{gathered}
T \rightsquigarrow \rightsquigarrow^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{1}\right) \cdot T_{1} \\
T^{\prime} \rightsquigarrow^{*} \Pi^{l_{1}, l_{2}}\left(x: S_{2}\right) \cdot T_{2}
\end{gathered}
$$

must be unique. First we show

$$
\mathcal{E}_{1}::\left(\forall \psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{i} L\right| \Psi ; \Gamma \cdot L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} S_{1} \approx S_{2}^{\prime} @ l_{1}\right)
$$

by a simple IH .
Next we must show that given

- $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow{ }_{m} L\right| \Psi ; \Gamma$,
- $\mathcal{A}:: L^{\prime} \mid \Phi ; \Delta \|=_{m}^{m} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{E}_{1}(\psi)\right)$
then

$$
\left.\mathcal{E}_{2}:: L^{\prime} \mid \Phi ; \Delta \|=_{m}^{m} T_{1}[s / x] \approx T_{2}^{\prime}\left[s^{\prime} / x\right] @ l_{2}\right)
$$

holds. Notice that the goal is almost applicable for $\mathcal{D}_{2}$ except that $\mathcal{A}$ does not satisfy the required premise $L^{\prime} \mid \Phi ; \Delta \| \Vdash_{m}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}(\psi)\right)$. But this is fine as we apply IH to use the iff in the second statement to obtain the required premise.

For the second statement, we must establish an iff relation. That boils down to a symmetric proof. Then in this case, we are given

- $\psi:: L^{\prime}\left|\Phi ; \Delta \Longrightarrow_{m} L\right| \Psi ; \Gamma$,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{i} S_{1} \simeq S_{3} @ l$,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{i} S_{2} \simeq S_{4} @ l$,
- $L^{\prime} \mid \Phi ; \Delta, x: S_{1} @ l_{1} \vdash_{i} T_{1} \simeq T_{3} @ l_{2}$, and
- $L^{\prime} \mid \Phi ; \Delta, x: S_{2} @ l_{1} \vdash_{i} T_{2} \simeq T_{4} @ l_{2}$,
and then we have to show the following equivalence:

$$
\mathcal{B}:: L^{\prime} \mid \Phi ; \Delta \| \models_{m}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}(\psi)\right)
$$

implying

$$
L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{p}\left(w: \Pi^{l_{1}, l_{2}}\left(x: S_{3}\right) \cdot T_{3}\right) s \approx\left(w^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot T_{4}\right) s^{\prime}: \operatorname{El}\left(\mathcal{D}_{2}(\psi, \mathcal{B})\right)
$$

is equivalent to

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{E}_{1}(\psi)\right)
$$

implying

$$
L^{\prime} \mid \Phi ; \Delta \|=_{m}^{m}\left(w: \Pi^{l_{1}, l_{2}}\left(x: S_{3}\right) \cdot T_{3}\right) s \approx\left(w^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot T_{4}\right) s^{\prime}: \operatorname{El}\left(\mathcal{E}_{2}\right)
$$

let us consider the inverse direction. In this case, we assume $L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} s \approx s^{\prime}: \mathrm{El}\left(\mathcal{E}_{1}(\psi)\right)$. By IH, we have $L^{\prime} \mid \Phi ; \Delta \| \models_{m}^{p} s \approx s^{\prime}: \operatorname{El}\left(\mathcal{D}_{1}(\psi)\right)$. From this, we further obtain

$$
L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{p}\left(w: \Pi^{l_{1}, l_{2}}\left(x: S_{3}\right) \cdot T_{3}\right) s \approx\left(w^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S_{4}\right) \cdot T_{4}\right) s^{\prime}: \operatorname{El}\left(\mathcal{D}_{2}(\psi, \mathcal{B})\right)
$$

Another IH is sufficient to establish the goal.
This lemma, when combined with premises like $L \mid \Psi ; \Delta \Vdash_{\geq p}^{p} T @ l$, is the bridge to reveal the true complication of supporting lifting in DeLaM.

We also need a counterpart for local contexts and local substitutions.
Lemma 7.50 (Layering Restriction).

- If $\mathcal{D}:: L \mid \Psi \vDash_{m}^{p} \Delta \approx \Delta^{\prime}$, then $\mathcal{E}:: L \mid \Psi \vDash_{m}^{m} \Delta \approx \Delta^{\prime}$.
- The following two relations are equivalent:

$$
L \mid \Psi ; \Gamma \| \vDash_{m}^{p} \delta \approx \delta^{\prime}: \mathcal{D} \text { and } L \mid \Psi ; \Gamma \| \Vdash_{m}^{m} \delta \approx \delta^{\prime}: \mathcal{E}
$$

Proof. Induction on $\mathcal{D}$. The step case is very similar to function case above.

### 7.7 Semantic Judgments and Fundamental Theorems

After establishing all logical relations and their properties, we are ready for giving the definitions for semantic judgments and then moving on to proving the fundamental theorems. The semantic judgments intuitively should say that logical relations are stable under all substitutions. More concretely, we have

$$
L \Vdash \Psi:=\forall L^{\prime} \vdash \phi \approx \phi^{\prime}: L \cdot L^{\prime} \vDash \Psi[\phi] \approx \Psi\left[\phi^{\prime}\right]
$$

$L \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi$ is defined as $L \Vdash \Psi, L \Vdash \Phi$ and given

- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$, and
- $L^{\prime} \mid \Psi^{\prime} \| \sigma_{1} \approx \sigma_{1}^{\prime}: \Psi[\phi]$,
then

$$
L^{\prime} \mid \Psi^{\prime} \| \vDash \sigma[\phi] \circ \sigma_{1} \approx \sigma_{1}[\phi] \circ \sigma_{1}^{\prime}: \Phi[\phi]
$$

We also define

$$
L|\Psi \Vdash \sigma: \Phi:=L| \Psi \Vdash \sigma \approx \sigma: \Phi
$$

$L \mid \Psi \Vdash_{i} \Gamma \approx \Delta$ where $i \in\{p, m\}$ is defined as $L \Vdash \Psi$ and given

- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, and
- $k \geq i$,
then

$$
L^{\prime} \mid \Phi \vDash_{k}^{\operatorname{typeof}(i)} \Gamma[\phi][\sigma] \approx \Delta\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]
$$

We also define

$$
L\left|\Psi \Vdash_{i} \Gamma:=L\right| \Psi \Vdash_{i} \Gamma \approx \Gamma
$$

$L \mid \Psi ; \Gamma \Vdash_{i} T \approx T^{\prime} @ l$ is defined as $L \Vdash \Psi$ and $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$ and given

- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$,
- $k \geq i$,
- $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{\text {typeof }(i)} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$,
then

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{\text {typeof }(i)} T[\phi][\sigma][\delta] \approx T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ l[\phi]
$$

We also define

$$
L\left|\Psi ; \Gamma \Vdash_{i} T @ l:=L\right| \Psi ; \Gamma \Vdash_{i} T \approx T @ l
$$

$L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: T @ l$ is defined as $L \Vdash \Psi$ and $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$ and $L \mid \Psi ; \Gamma \Vdash_{\text {typeof }(i)} T @ l$ given

- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$,
- $k \geq i$,
- $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{\text {typeof }(i)} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$,
then

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{\text {typeof }(i)} t[\phi][\sigma][\delta] \approx t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: T[\phi][\sigma][\delta] @ l[\phi]
$$

We also define

$$
L\left|\Psi ; \Gamma \Vdash_{i} t: T @ l:=L\right| \Psi ; \Gamma \Vdash_{i} t \approx t: T @ l
$$

$L \mid \Psi ; \Gamma \Vdash_{i} \delta \approx \delta^{\prime}: \Delta$ is defined as $L \Vdash \Psi$ and $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$ and $L \mid \Psi \Vdash_{\text {typeof }(i)} \Delta$ given

- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$,
- $k \geq i$,
- $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{k}^{\text {typeof }(i)} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma[\phi][\sigma]$,
then

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{k}^{\operatorname{typeof}(i)} \delta[\phi][\sigma] \circ \delta_{1} \approx \delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}: \Delta[\phi][\sigma]
$$

We also define

$$
L\left|\Psi ; \Gamma \Vdash_{i} \delta: \Delta:=L\right| \Psi ; \Gamma \Vdash_{i} \delta \approx \delta: \Delta
$$

Notice that for definitions above, when $k \in\{v, c\}$ is possible, we can use the local substitution lemma and ignore the local substitutions completely. This will be a frequent pattern in the proofs of the semantic rules.

Summarizing the semantic judgments, we shall arriving at our statement of the fundamental theorems:

## Theorem 7.51 (Fundamental).

- If $L \vdash \Psi$, then $L \Vdash \Psi$.
- If $L \mid \Psi \vdash_{i} \Gamma$ and $i \in\{p, m\}$, then $L \mid \Psi \vdash_{i} \Gamma$.
- If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$ and $i \in\{p, m\}$, then $L \mid \Psi \vdash_{i} \Gamma \approx \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T$ @ $l$, then $L \mid \Psi ; \Gamma \vdash_{i} T @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ $l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta: \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$, then $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$.

The following lemma is obvious.
Lemma 7.52 (PER). All semantic judgments are PER.
, Vol. 1, No. 1, Article . Publication date: April 2024.

## Lemma 7.53 (Reduction Expansion).

- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l, i \in\{p, m\}$ and $L \mid \Psi ; \Gamma \vdash_{i} T_{1} \rightsquigarrow T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \approx T_{1} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l, i \in\{p, m\}$ and $L \mid \Psi ; \Gamma \vdash_{i} t_{1} \rightsquigarrow t^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \approx t_{1}: T @ l$.

Proof. It is easy to easy due to the weak head expansion lemma for the logical relations and the stability of reduction under all substitutions.

The escape lemma recovers Kripke logical relations by passing in corresponding identity substitutions:
Lemma 7.54 (Escape).

- If $L \Vdash \Psi$, then $L \vDash \Psi \approx \Psi$.
- If $L \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi$, then $L \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi$.
- If $L \mid \Psi \vdash_{i} \Gamma \approx \Delta, i \in\{p, m\}$ and $k \geq i$, then $L \mid \Psi \Vdash_{k}^{\text {typeof }(i)} \Gamma \approx \Delta$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $k \geq i$, then $L \mid \Psi ; \Gamma \| \Vdash_{k}^{\text {typeof }(i)} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: T$ @ $l$ and $k \geq i$, then $L \mid \Psi ; \Gamma \| \Vdash_{k}^{\text {typeof( }(i)} t \approx t^{\prime}: T$ @ $l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $k \geq i$, then $L \mid \Psi ; \Gamma \| \Vdash_{k}^{\text {typeof }(i)} \delta \approx \delta^{\prime}: \Delta$.

By chaining escape lemmas, we can obtain that semantic equivalence judgments generic and syntactic equivalences.

The semantic judgments are stable under substitutions.
Lemma 7.55 (Universe Substitutions).

- If $L \Vdash \Psi$ and $L^{\prime} \vdash \phi: L$, then $L^{\prime} \Vdash \Psi[\phi]$.
- If $L \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi$ and $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$, then $L^{\prime} \mid \Psi[\phi] \Vdash \sigma[\phi] \approx \sigma^{\prime}\left[\phi^{\prime}\right]: \Phi[\phi]$.
- If $L \mid \Psi \Vdash_{i} \Gamma \approx \Delta, i \in\{p, m\}$ and $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$, then $L^{\prime} \mid \Psi[\phi] \Vdash_{i} \Gamma[\phi] \approx \Delta\left[\phi^{\prime}\right]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$, then $L^{\prime} \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} T[\phi] \approx T^{\prime}\left[\phi^{\prime}\right] @ l[\phi]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ $l$ and $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$, then $L^{\prime} \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} t[\phi] \approx t^{\prime}\left[\phi^{\prime}\right]: T[\phi] @ l[\phi]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$, then $L^{\prime} \mid \Psi[\phi] ; \Gamma[\phi] \vdash_{i} \delta[\phi] \approx \delta^{\prime}\left[\phi^{\prime}\right]: \Delta[\phi]$.

Proof. Use composition of universe substitutions.
Lemma 7.56 (Global Substitutions).

- If $L \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi$ and $L \mid \Psi^{\prime} \Vdash \sigma_{1} \approx \sigma_{1}^{\prime}: \Psi$, then $L \mid \Psi^{\prime} \Vdash \sigma \circ \sigma_{1} \approx \sigma^{\prime} \circ \sigma_{1}^{\prime}: \Phi$.
- If $L \mid \Psi \Vdash_{i} \Gamma \approx \Delta, i \in\{p, m\}$ and $L \mid \Phi \Vdash \sigma \approx \sigma^{\prime}: \Psi$, then $L \mid \Phi \Vdash_{i} \Gamma[\sigma] \approx \Delta\left[\sigma^{\prime}\right]$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Phi \Vdash \sigma \approx \sigma^{\prime}: \Psi$, then $L \mid \Phi ; \Gamma[\sigma] \vdash_{i} T[\sigma] \approx T^{\prime}\left[\sigma^{\prime}\right] @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$ and $L \mid \Phi \Vdash \sigma \approx \sigma^{\prime}: \Psi$, then $L \mid \Phi ; \Gamma[\sigma] \vdash_{i} t[\sigma] \approx t^{\prime}\left[\sigma^{\prime}\right]: T[\sigma] @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Phi \Vdash \sigma \approx \sigma^{\prime}: \Psi$, then $L \mid \Phi ; \Gamma[\sigma] \Vdash_{i} \delta[\sigma] \approx \delta^{\prime}\left[\sigma^{\prime}\right]: \Delta[\sigma]$.

Proof. The principle is also to use the composition of global substitutions. We also make use of the commutativity of substitutions, e.g.

$$
t[\phi][\sigma[\phi]]=t[\sigma][\phi]
$$

This equation allows us to swap $\sigma$ forwards, which will be frequently used in this proof.
Lemma 7.57 (Local Substitutions).

- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$ and $L \mid \Psi ; \Delta \Vdash_{i} \delta \approx \delta^{\prime}: \Gamma$, then $L \mid \Psi ; \Delta \vdash_{i} T[\delta] \approx T^{\prime}\left[\delta^{\prime}\right] @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T$ @ $l$ and $L \mid \Psi ; \Delta \vdash_{i} \delta \approx \delta^{\prime}: \Gamma$, then $L \mid \Psi ; \Delta \Vdash_{i} t[\delta] \approx t^{\prime}\left[\delta^{\prime}\right]: T[\delta] @ l$.
- If $L \mid \Psi ; \Gamma \Vdash_{i} \delta \approx \delta^{\prime}: \Delta$ and $L \mid \Psi ; \Gamma^{\prime} \Vdash_{i} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma$, then $L \mid \Psi ; \Gamma^{\prime} \Vdash_{i} \delta \circ \delta_{1} \approx \delta^{\prime} \circ \delta_{1}^{\prime}: \Delta$.

Proof. We apply the similar technique here. We make use equations similar to below:

$$
\begin{aligned}
& t[\phi][\delta[\phi]]=t[\delta][\phi] \\
& t[\sigma][\delta[\sigma]]=t[\delta][\sigma]
\end{aligned}
$$

These equations will swap $\delta$ forwards.
The theorem proceeds by doing induction on the derivations. The proof though is rather verbose due to how the semantic judgments are defined. One pattern that is worth mentioning is that the proof should work "backwards" from the layers. That is, we should work out the proofs from layer $m$, and then $p$ and then $c$ and finally $v$. This pattern makes sense if we consider what information layers contain. The semantics of a term at layer $m$ only contains its computational contents at layer $m$. However, for a term at layer $p$, due to lifting, its semantics must explain how this term computes at both layers $p$ and $m$. For a term at layer $c$, in addition to its collective information as a term at layer $p$, it should also has all information about its sub-structures. At last, if a term is at layer $v$, then we know it must be well-formed at layer $c$ but also it represents a variable. Thus, due to lifting, information contained at each layer strictly increases as the layer decreases. To build up information at a smaller layer, we should prove the fundamental theorems from a higher layer. In the next section, we will start proving the fundamental theorems and make sure that all syntactically well-formed types and terms at all layers are semantically well-formed.

### 7.8 Proving Fundamental Theorems

To demonstrate the idea described at the end of the previous subsection, let us first consider the simplest case. We often proceed by first proving the semantic rule for types and then go on and prove the rules for terms.

Lemma 7.58.

$$
\frac{L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma}{L \mid \Psi ; \Gamma \Vdash_{i} \text { Nat } \approx \text { Nat @ zero }}
$$

Proof. From $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$, we also know $L \Vdash \Psi$. Now assume $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \geq c$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \text { Nat } \approx \text { Nat @ zero }
$$

This holds by definition.
Case $i=p$ Then assuming some $k \geq p$ and $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{p} \text { Nat } \approx \text { Nat @ zero }
$$

Again, this also holds by definition as we see that $k \in\{p, m\}$.
Case $i=c$ This is the last case. Assuming some $k \geq c$ and $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{p} \text { Nat } \approx \text { Nat @ zero }
$$

In the previous case of $i=p$, we have given the proof for $k \in\{p, m\}$, so essentially we only have one case $k=c$ left. In this case, we apply the local substitution lemma so we do not have to introduce any local substitutions at all. Looking up the rules in Sec. 7.4, we need to show

$$
L^{\prime} \mid \Phi ; \Gamma[\phi][\sigma] \Vdash_{\geq p}^{p} \text { Nat @ zero }
$$

This is again have been given by the case of $i=p$, modulo converting weakenings to substitutions. Hence we conclude the proof.

Lemma 7.59.

$$
\frac{L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{i} \mathrm{Ty}_{l} \approx \mathrm{Ty}_{l^{\prime}} @ \operatorname{succ} l}
$$

Proof. Similar to the previous lemma.
Lemma 7.60.

$$
\frac{L \mid \Psi \Vdash_{\text {typeof(i) }} \Gamma}{L \mid \Psi ; \Gamma \Vdash_{i} \text { zero } \approx \text { zero }: \text { Nat @ zero }}
$$

Proof. From the previous lemma, we obtain $L \Vdash \Psi, L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$ and $L \mid \Psi ; \Gamma \Vdash_{i}$ Nat @ zero. Simulating the previous lemma, assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \geq c$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta \|=_{m}^{m} \text { zero } \approx \text { zero : Nat @ zero }
$$

This is the same as showing

$$
\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta \| \models_{m}^{m} \text { Nat } \approx \text { Nat @ zero and } L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \text { zero } \approx \text { zero }: \operatorname{El}(\mathcal{D})
$$

This is immediate by the congruence law of the generic equivalence and the definition of the logical relations.
Case $i=p$ Then assuming some $k \geq p$ and $L^{\prime} \mid \Phi ; \Delta \| \Vdash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{p} \text { Nat } \approx \text { Nat @ zero and } L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{p} \text { zero } \approx \text { zero }: \operatorname{El}(\mathcal{D})
$$

Following a similar proof to the case above, we also establish this case knowing $k \in\{p, m\}$.
Case $i=c$ Assuming some $k \geq c$ and $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta \| \models_{k}^{p} \text { zero } \approx \text { zero : Nat @ zero }
$$

The only additional analysis is when $k=c$ as other values for $k$ have been considered in the previous case. In this case, we apply local substitution lemma so we only need to prove

$$
L^{\prime} \mid \Phi ; \Gamma[\phi][\sigma] \vDash_{\geq p}^{p} \text { zero : Nat @ zero }
$$

This clearly has been proven in the previous case.

Lemma 7.61.

$$
\frac{L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: \text { Nat @ zero }}{L \mid \Psi ; \Gamma \vdash_{i} \text { succ } t \approx \operatorname{succ} t^{\prime}: \text { Nat @ zero }}
$$

Proof. From $L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}$ : Nat @ zero, we also know $L \Vdash \Psi, L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$ and $L \mid \Psi ; \Gamma \vdash_{i}$ Nat $\approx$ Nat @ zero. Now assume $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \geq c$.

Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \text { Nat } \approx \text { Nat @ zero and } L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \operatorname{succ} t[\phi][\sigma][\delta] \approx \operatorname{succ} t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \operatorname{El}(\mathcal{D})
$$

From $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}:$ Nat @ zero, we obtain such $\mathcal{D}$ and also

$$
L^{\prime} \mid \Phi ; \Delta \|=_{m}^{m} t[\phi][\sigma][\delta] \approx t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \mathrm{El}(\mathcal{D})
$$

From here we obtain $L \mid \Psi ; \Gamma \| \vDash_{i}$ succ $t \simeq \operatorname{succ} t^{\prime}:$ Nat which leads to our desired goal.
Case $i=p$ Then assuming some $k \geq p$ and $L^{\prime} \mid \Phi ; \Delta \| \vDash_{k}^{p} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show
$\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta \| \Vdash_{k}^{p}$ Nat $\approx$ Nat @ zero and $L^{\prime} \mid \Phi ; \Delta \| \Vdash_{k}^{p} \operatorname{succ} t[\phi][\sigma][\delta] \approx \operatorname{succ} t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \operatorname{El}(\mathcal{D})$
We follow the previous case.
Case $i=c$ In this case, we only consider the most interesting case of $k=c$. In this case, we apply local substitution lemma, so we know $L^{\prime} \mid \Phi ; \Gamma[\phi][\sigma] \vDash_{\geq p}^{p} t[\phi][\sigma]$ : Nat @ zero and we must prove

$$
L^{\prime} \mid \Phi ; \Gamma[\phi][\sigma] \Vdash_{\geq p}^{p} \operatorname{succ} t[\phi][\sigma]: \text { Nat @ zero }
$$

This clearly has been given by the previous case. As we have seen in the last few proofs with $k=c$, it is a common pattern that we use the local substitution lemma to get rid of the local substitution lemma. Then what is left for the proof obligation is given by $i=p$ modulo converting weakenings to substitutions. Essentially, the semantics of layer $c$ simply remembers the derivation given by the semantic rules. For this reason, we will keep cases of $i=c$ short.

The semantic rules are pretty sensitive to the orders in which they are proved. To handle $\Pi$ types, it is more convenient if we have the rules for contexts ready.

Lemma 7.62.

$$
\frac{L \Vdash \Psi}{L \mid \Psi \Vdash_{i} \cdot \approx}
$$

Proof. Immediate.
Lemma 7.63.

$$
\frac{L \Vdash \Psi \quad g: C t x \in \Psi}{L \mid \Psi \Vdash_{i} g \approx g}
$$

Proof. Now assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \vDash \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \in\{p, m\}$. We only consider $i=m$ here as the proof for $i=p$ is very similar. We know $g: \operatorname{Ctx} \in \Psi[\phi]$ as well. Then we have the following after lookup

$$
L^{\prime} \mid \Phi \Vdash_{\geq p}^{p} \sigma(g) \approx \sigma^{\prime}(g)
$$

We are very close to our goal

$$
L^{\prime} \mid \Phi \|_{m}^{m} \sigma(g) \approx \sigma^{\prime}(g)
$$

First we obtain $L^{\prime} \mid \Phi \vDash_{m}^{p} \sigma(g) \approx \sigma^{\prime}(g)$. Then by layering restriction, we have the goal by lifting $p$ to $m$.
Lemma 7.64.

$$
\frac{L\left|\Psi \Vdash_{i} \Gamma \approx \Delta \quad L\right| \Psi ; \Gamma \Vdash_{i} T \approx T^{\prime} @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi \vdash_{i} \Gamma, x: T @ l \approx \Delta, x: T^{\prime} @ l^{\prime}}
$$

[^1]Proof. Now assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \in\{p, m\}$. We only consider $i=m$ here as the proof for $i=p$ is very similar. We should prove

$$
L^{\prime}|\Phi| \vDash_{m}^{m} \Gamma[\phi][\sigma], x: T[\phi][\sigma] @ l[\phi] \approx \Delta\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right], x: T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] @ l^{\prime}\left[\phi^{\prime}\right]
$$

We first obtain

$$
L^{\prime} \mid \Phi \Vdash_{m}^{m} \Gamma[\phi][\sigma] \approx \Delta\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]
$$

To obtain the goal, we must show $T[\phi][\sigma] \approx T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]$ is stable under local substitutions. This is immediate by the semantic judgment, after converting weakenings into universe and global substitutions.

Lemma 7.65.

$$
\frac{L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma \quad \Gamma \text { ends with } \cdot}{L \mid \Psi ; \Gamma \Vdash_{i} \cdot{k^{\prime}}^{k^{k^{\prime}}}: \cdot}
$$

Proof. From $L \mid \Psi \Vdash_{\text {typeof( }(i)} \Gamma$, we also know $L \Vdash \Psi$. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$. We should consider all possible $i$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
\mathcal{D}:: L^{\prime}|\Phi| \models_{m}^{m} \cdot \approx \cdot \text { and } L^{\prime} \mid \Phi ; \Delta \| \models_{m}^{m} \cdot k^{\prime} \circ \delta \approx \cdot k^{k^{\prime}} \circ \delta^{\prime}: \mathcal{D}
$$

$\mathcal{D}$ is immediate. Now we should consider the composition. We know

$$
\cdot k^{\prime} \circ \delta=\cdot k^{\prime} \circ \delta^{\prime}=\frac{\widehat{\delta}}{\check{\delta}}
$$

We then have the goal by definition.
Case $i=p$ Similar.
Case $i=c$ By the local substitution lemma and the rule in Sec. 7.4, we conclude this case by repeating the previous case.
Case $i=v$ Similar.

Lemma 7.66.

Proof. Similar to the previous lemma. We will need to do a case analysis on the result of lookup of $g$, but otherwise the result is straightforward.

Lemma 7.67.

$$
\frac{L \mid \Psi \Vdash_{t y p e o f(i)} \Gamma \quad g: C t x \in \Psi \quad \Gamma \text { ends with } g \quad|\Gamma|=k^{\prime}}{L \mid \Psi ; \Gamma \Vdash_{i} w k_{g}^{k^{\prime}} \approx w k_{g}^{k^{\prime}}: g}
$$

Proof. From $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$, we also know $L \Vdash \Psi$. The semantic well-formedness for $g$ is established by a previous lemma.

Now we assume $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \vDash \vDash \approx \sigma^{\prime}: \Psi[\phi]$. We should consider all possible $i$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
\mathcal{D}:: L^{\prime} \mid \Phi \Vdash_{m}^{m} \sigma(g) \approx \sigma^{\prime}(g) \text { and } L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} w \mathrm{k}_{g}^{k^{\prime}}[\sigma] \circ \delta \approx \mathrm{wk}_{g}^{k^{\prime}}\left[\sigma^{\prime}\right] \circ \delta^{\prime}: \mathcal{D}
$$

$\mathcal{D}$ is obtained by looking up $g$ in $\sigma$, from which we get

$$
L^{\prime} \mid \Phi \Vdash_{m}^{p} \sigma(g) \approx \sigma^{\prime}(g)
$$

We have $\mathcal{D}$ by layering restriction.
For composition, we have

$$
\begin{aligned}
& \mathrm{wk} \\
& g \\
& \mathrm{wk}_{g}^{k^{\prime}}[\sigma] \circ \delta=\mathrm{wk}_{\sigma(g)}^{k^{\prime}} \circ \delta \circ \delta^{\prime}
\end{aligned}=\mathrm{wk}_{\sigma^{\prime}(g)}^{k^{\prime}} \circ \delta^{\prime}
$$

Effectively, this is the same as popping off $k^{\prime}$ terms from $\delta$ and $\delta^{\prime}$ simultaneously. We have the goal by irrelevance.
Case $i=p$ Similar.
Case $i \in\{v, c\}$ Similarly, we use the previous case.

Lemma 7.68.

$$
\frac{L \vdash l: \text { Level } \quad L\left|\Psi ; \Gamma \Vdash_{i} \delta \approx \delta^{\prime}: \Delta \quad L\right| \Psi ; \Delta \Vdash_{\text {typeof }(i)} T @ l \quad L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: T[\delta] @ l}{L \mid \Psi ; \Gamma \Vdash_{i} \delta, t / x \approx \delta^{\prime}, t^{\prime} / x: \Delta, x: T @ l}
$$

Proof. From $L \mid \Psi ; \Gamma \Vdash_{i} \delta \approx \delta^{\prime}: \Delta$ we have $L\left|\Psi \Vdash_{\text {typeof }(i)} \Gamma, L\right| \Psi \Vdash_{\text {typeof }(i)} \Delta$ and $L \Vdash \Psi$. Then from a previous lemma, we further have $L \mid \Psi \Vdash_{\text {typeof }(i)} \Delta, x: T$ @ $l$.

Now we assume $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$. We should consider all possible $i$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma[\phi][\sigma]$, we must show

- $\mathcal{D}:: L^{\prime}|\Phi| \vDash_{m}^{m}(\Delta, x: T @ l)[\phi][\sigma] \approx(\Delta, x: T @ l)\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]$ and
$-L^{\prime} \mid \Phi ; \Delta^{\prime} \|=_{m}^{m}(\delta, t / x)[\phi][\sigma] \circ \delta_{1} \approx\left(\delta^{\prime}, t^{\prime} / x\right)\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}: \mathcal{D}$.
We expand the composition:

$$
\begin{aligned}
(\delta, t / x)[\phi][\sigma] \circ \delta_{1} & =\left(\delta[\phi][\sigma] \circ \delta_{1}\right),\left(t[\phi][\sigma]\left[\delta_{1}\right]\right) / x \\
\left(\delta^{\prime}, t^{\prime} / x\right)\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime} & =\left(\delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}\right),\left(t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta_{1}^{\prime}\right]\right) / x
\end{aligned}
$$

We can conclude the goal by using $L\left|\Psi \Vdash_{\text {typeof }(i)} \Delta, x: T @ l, L\right| \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: T[\delta] @ l$ and irrelevance.
Case $i=p$ Similar.
Case $i=c$ Similar to the previous pattern, we apply the local substitution lemma and use the previous case to discharge the obligations.
Case $i=v$ Similar.

Lemma 7.69.

Proof. Immediate. We take advantage of the fact that a universe weakening is a special universe substitution.

Lemma 7.70.

$$
\frac{L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma \quad x: T @ l \in \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} x \approx x: T @ l}
$$

Proof. From $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$, we also know $L \Vdash \Psi$.
To construct the semantic judgment for type $T$, we first assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and some $k \geq i$. We have

$$
L^{\prime} \mid \Phi \vDash \vDash_{k}^{\operatorname{typeof}(i)} \Gamma[\phi][\sigma] \approx \Gamma\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]
$$

Our goal is to construct

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{\text {typeof }(i)} T[\phi][\sigma][\delta] \approx T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ l[\phi]
$$

with further assuming $L^{\prime} \mid \Phi ; \Delta I \| \vDash_{k}^{\text {typeof }(i)} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$. This is done by doing induction on $x: T @ l \in \Gamma$.
Then we consider the term. Since it is the variable case, $i$ can take all four layers.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} T[\phi][\sigma][\delta] \approx T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ l[\phi] \text { and } L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \delta(x) \approx \delta^{\prime}(x): \mathrm{El}(\mathcal{D})
$$

We proceed by doing induction on $x: T @ l \in \Gamma$. We weaken the universe and global contexts to obtain the goal.
Case $i=p$ This case works similarly at different layers. We omit it here.
Case $i=c$ In this case, we consider $k=c$ and apply the local substitution lemma. Based on the rule in Sec. 7.4 and the previous case, we have the goal.
Case $i=v$ This case makes use of the entire previous case and also in addition must prove the same for $k=v$. But this is virtually identical to the previous case.

Combining the semantic rules for local substitutions, we derive that
Corollary 7.71 (Local Weakening Substitutions). $L \mid \Psi ; \Gamma, \Delta \Vdash_{i} w k_{\Gamma}^{|\Delta|}: \Gamma$
Lemma 7.72.

$$
\frac{u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi \quad i^{\prime} \in\{v, c\} \quad \begin{array}{c}
L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma \\
i \in\{v, c, p, m\}
\end{array} \quad i^{\prime} \leq i}{L \mid \Psi ; \Gamma \vdash_{i} u^{\delta} \approx u^{\delta^{\prime}}: T[\delta] @ l}
$$

Proof. From $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$, we also know $L \Vdash \Psi$.
To construct the semantic judgment for type $T$, we first assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and some $k \geq i$, and then $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{k}^{\text {typeof }(i)} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma[\phi][\sigma]$. We have

$$
L^{\prime} \mid \Phi \Vdash_{k}^{\operatorname{typeof}(i)} \Gamma[\phi][\sigma] \approx \Gamma\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]
$$

Our goal is to construct

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{k}^{\text {typeof }(i)} T[\phi][\sigma]\left[\delta[\phi][\sigma] \circ \delta_{1}\right] \approx T\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}\right] @ l[\phi]
$$

We obtain this by looking up $L^{\prime} \| \neq \Psi[\phi]$ using $u:\left(\Delta \vdash_{i^{\prime}} T @ l\right) \in \Psi$, and use the invariant that $T$ is stable under global and local substitutions.

Then we consider the term.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \delta_{1} \approx \delta_{1}^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \sigma(u)\left[\delta[\phi][\sigma] \circ \delta_{1}\right] \approx \sigma^{\prime}(u)\left[\delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}\right]: T[\phi][\sigma]\left[\delta[\phi][\sigma] \circ \delta_{1}\right] @ l[\phi]
$$

Looking up $L^{\prime} \mid \Phi \| \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we know that

$$
\mathcal{B}:: L^{\prime} \mid \Phi ; \Delta[\phi][\sigma] \Vdash_{\geq i^{\prime}}^{p} \sigma(u) \approx \sigma^{\prime}(u): T[\phi][\sigma] @ l
$$

and

$$
L^{\prime}|\Phi| \models_{p}^{p} \Delta[\phi][\sigma]
$$

We also know the following from $\mathcal{A}$

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \|=_{m}^{m} \delta[\phi][\sigma] \circ \delta_{1} \approx \delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}: \Delta[\phi][\sigma]
$$

Therefore, we can apply layering restriction and obtain

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \models_{m}^{p} \delta[\phi][\sigma] \circ \delta_{1} \approx \delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}: \Delta[\phi][\sigma]
$$

This is because we are sure $\Delta[\phi][\sigma]$ only contains types from MLTT.
We want to lift $\sigma(u)$ and $\sigma^{\prime}(u)$ to $m$, so we instantiate $\mathcal{B}$ with the related local substitutions above and obtain

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{p} \sigma(u)\left[\delta[\phi][\sigma] \circ \delta_{1}\right] \approx \sigma^{\prime}(u)\left[\delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right] \circ \delta_{1}^{\prime}\right]: T[\phi][\sigma]\left[\delta[\phi][\sigma] \circ \delta_{1}\right] @ l[\phi]
$$

The goal is achieved by another layering restriction.
Case $i=p$ Similar to the previous case except that there is no need for layering restriction, as we are evaluating terms right inside of MLTT and therefore no lifting occurs.
Case $i \in\{v, c\}$ In this case, we also follow similar footsteps as the results of looking up global substitutions must be stable under local substitutions.

Lemma 7.73.

$$
\frac{L\left|\Psi \Vdash_{\text {typeof }(i)} \Gamma \quad U:\left(\Delta \vdash_{i^{\prime}} @ l\right) \in \Psi \quad i^{\prime} \in\{c, p\} \quad i^{\prime} \leq i \quad L\right| \Psi ; \Gamma \Vdash_{i} \delta \approx \delta^{\prime}: \Delta}{L \mid \Psi ; \Gamma \Vdash_{i} U^{\delta} \approx U^{\delta^{\prime}} @ l}
$$

Proof. Similar to above, but simpler. Use layering restriction as well when $i=m$.
Lemma 7.74.

$$
\frac{L \Vdash \Psi}{L \mid \Psi \Vdash \cdot \approx \cdot: \cdot} \quad \frac{L\left|\Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi \quad L\right| \Psi \Vdash_{p} \Gamma \approx \Delta}{L \mid \Psi \Vdash \sigma, \Gamma / g \approx \sigma^{\prime}, \Delta / g: \Phi, g: C t x}
$$

Proof. Immediate.
Lemma 7.75.

$$
\begin{array}{ll}
L \mid \Phi ; \Gamma \Vdash_{p} T @ l & L \vdash l: \text { Level } \begin{array}{c}
L \mid \Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi \\
i \in\{v, c\} \quad \mathcal{A}:: L \mid \Psi ; \Gamma[\sigma] \vdash_{i} t \approx t^{\prime}: T[\sigma] @ l \\
L\left|\Psi \Vdash \sigma, t / u \approx \sigma^{\prime}, t^{\prime}\right| u: \Phi, u:\left(\Gamma \vdash_{i} T @ l\right)
\end{array}
\end{array}
$$

Proof. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime}\left|\Psi^{\prime}\right| \vDash \sigma_{1} \approx \sigma_{1}^{\prime}: \Psi[\phi]$, we have to show

$$
L^{\prime} \mid \Psi^{\prime} \| \vDash \sigma[\phi] \circ \sigma_{1}, t[\phi]\left[\sigma_{1}\right] / u \approx \sigma^{\prime}[\phi] \circ \sigma_{1}^{\prime}, t[\phi]\left[\sigma_{1}^{\prime}\right] / u:\left(\Phi, u:\left(\Gamma \vdash_{i} T @ l\right)\right)[\phi]
$$

Our goal is to show, without loss of generality,

$$
L \mid \Psi ; \Gamma\left[\sigma[\phi] \circ \sigma_{1}\right] \Vdash_{\geq i}^{p} t[\phi]\left[\sigma_{1}\right] \approx t^{\prime}[\phi]\left[\sigma_{1}^{\prime}\right]: T\left[\sigma[\phi] \circ \sigma_{1}\right] @ l
$$

This is given by $\mathcal{A}$, modulo converting weakenings to substitutions.
Lemma 7.76.

$$
\frac{L\left|\Psi \Vdash \sigma \approx \sigma^{\prime}: \Phi \quad L\right| \Phi \Vdash_{p} \Gamma \quad L \vdash l: \text { Level } \quad i \in\{c, p\} \quad L \mid \Psi ; \Gamma[\sigma] \Vdash_{i} T \approx T^{\prime} @ l}{L \mid \Psi \Vdash \sigma, T / U \approx \sigma^{\prime}, T^{\prime} / U: \Phi, u:\left(\Gamma \vdash_{i} @ l\right)}
$$

Proof. Similar to the previous case but simpler.
Corollary 7.77 (Global Weakening Substitutions). $L \mid \Psi, \Phi \Vdash w k_{\Psi}^{|\Phi|}: \Psi$
Lemma 7.78 .

$$
\frac{L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level } \quad L\left|\Psi ; \Gamma \Vdash_{i} S \approx S^{\prime} @ l_{1} \quad L\right| \Psi ; \Gamma, x: S @ l_{1} \Vdash_{i} T \approx T^{\prime} @ l_{2}}{L \mid \Psi ; \Gamma \Vdash_{i} \Pi^{l_{1}, l_{2}}(x: S) . T \approx \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot T^{\prime} @ l_{1} \sqcup l_{2}}
$$

Proof. From $L \mid \Psi ; \Gamma \Vdash_{i} S \approx S^{\prime} @ l_{1}$, we can conclude $L \Vdash \Psi$ and $L \mid \Psi \Vdash_{\text {typeof(i) }} \Gamma$. Now assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \geq c$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

$$
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{m}^{m} \Pi^{l_{1}, l_{2}}(x: S) \cdot T[\phi][\sigma][\delta] \approx \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @\left(l_{1} \sqcup l_{2}\right)[\phi]
$$

From escape lemmas, we are able to establish the reduction premises, typing premises and the generic equivalence. We then only focus on the semantic premises. First, from $L \mid \Psi ; \Gamma \vdash_{i} S \approx S^{\prime} @ l_{1}$, we obtain

$$
L^{\prime} \mid \Phi ; \Delta \| \models_{m}^{m} S[\phi][\sigma][\delta] \approx S^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ l_{1}[\phi]
$$

Then we further assume $\psi:: L^{\prime \prime}\left|\Phi^{\prime} ; \Delta^{\prime} \Longrightarrow \Longrightarrow_{m} L^{\prime}\right| \Phi ; \Delta$ and $L^{\prime \prime} \mid \Phi^{\prime} ; \Delta^{\prime} \| \vDash_{m}^{m} s \approx s^{\prime}: S[\phi][\sigma][\delta] @ l_{1}[\phi]$, we should prove

$$
L^{\prime \prime} \mid \Phi^{\prime} ; \Delta^{\prime} \|=_{m}^{m} T[\phi][\sigma][\delta, s / x] \approx T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, s^{\prime} / x\right] @ l_{1}[\phi]
$$

We are almost there, as long as we can provide

$$
L^{\prime \prime} \mid \Phi^{\prime} ; \Delta^{\prime} \| \vDash_{m}^{m} \delta, s / x \approx \delta^{\prime}, s^{\prime} / x:\left(\Gamma, x: S @ l_{1}\right)[\phi][\sigma]
$$

To prove these two local substitutions are related, we are interested in showing $L \mid \Psi \Vdash_{\operatorname{typeof}(i)} \Gamma, x: S @ l_{1} \approx \Gamma, x: S^{\prime} @ l_{3}$, but this is immediate from a previous lemma.
Case $i=p$ This case follows similarly to the previous case. It must range over $k \in\{p, m\}$ so a similar reasoning must be repeated twice.
Case $i=c$ This case is much simpler by using the local substitution lemma to remove the need to assume another local substitution. Then we can simply apply identity local substitutions to all premises and use the previous case to conclude

$$
L^{\prime} \mid \Phi ; \Gamma[\phi][\sigma] \Vdash_{\geq p}^{p} \Pi^{l_{1}, l_{2}}(x: S) . T @\left(l_{1} \sqcup l_{2}\right)[\phi]
$$

If we introduce another local substitution, then we must reason about extending a local variable to an arbitrary local substitution, which is quite verbose and unnecessary.

Lemma 7.79.

$$
\frac{L \vdash l \approx l^{\prime}: \text { Level } \quad L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l}{L \mid \Psi ; \Gamma \Vdash_{i} \mathrm{El} l \approx \mathrm{El}^{l^{\prime}} t^{\prime} @ l}
$$

Proof. We use the fact that $\mathrm{Ty}_{l}$ reduces only to itself and therefore it is only possible to expand $t \approx t^{\prime}$ to the universe case.

Lemma 7.80.

$$
\begin{gathered}
L \vdash l_{1} \approx l_{3}: \text { Level } L \vdash l_{2} \approx l_{4}: \text { Level } \\
L\left|\Psi ; \Gamma \Vdash_{i} s \approx s^{\prime}: \mathrm{Ty}_{l_{1}} @ \operatorname{succ} l_{1} \quad L\right| \Psi ; \Gamma, x: \mathrm{El}^{l_{1}} s @ l_{1} \Vdash_{i} t \approx t^{\prime}: \mathrm{Ty}_{l_{2}} @ \operatorname{succ} l_{2} \\
L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l_{1}, l_{2}}(x: s) \cdot t \approx \Pi^{l_{3} l_{4}}\left(x: s^{\prime}\right) \cdot t^{\prime}: \mathrm{Ty}_{l_{1} \sqcup l_{2}} @ \operatorname{succ}\left(l_{1} \sqcup l_{2}\right)
\end{gathered}
$$

Proof. Very similar to the previous proof. We use the previous lemma and know that

$$
L \mid \Psi ; \Gamma \Vdash_{i} \mathrm{El}^{l_{1}} s \approx \mathrm{El}^{l_{3}} s^{\prime} @ l_{1}
$$

We do the same for $t \approx t^{\prime}$. This gives us

$$
L \mid \Psi ; \Gamma \Vdash_{i} \Pi^{l_{1}, l_{2}}\left(x: \mathrm{El}^{l_{1}} s\right) \cdot \mathrm{El}^{l_{2}} t \approx \Pi^{l_{3}, l_{4}}\left(x: \mathrm{El}^{l_{3}} s^{\prime}\right) \cdot \mathrm{El}^{l_{4}} t^{\prime} @ l_{1} \sqcup l_{2}
$$

When $i \in\{p, m\}$, we use

$$
\begin{aligned}
& \mathrm{El}^{l_{1} \sqcup l_{2}} \Pi^{l_{1}, l_{2}}(x: s) \cdot t \rightsquigarrow \Pi^{l_{1}, l_{2}}\left(x: \mathrm{El}^{l_{1}} s\right) \cdot \mathrm{El}^{l_{2}} t \\
& \mathrm{El}^{l_{3} \sqcup l_{4}} \Pi^{l_{3}, l_{4}}\left(x: s^{\prime}\right) \cdot t^{\prime} \rightsquigarrow \Pi^{l_{3}, l_{4}}\left(x: \mathrm{El}^{l_{3}} s^{\prime}\right) \cdot \mathrm{El}^{l_{4}} t^{\prime}
\end{aligned}
$$

We have the goal using reduction expansion.
Lemma 7.81.

$$
\frac{L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi ; \Gamma \Vdash_{i} s: \mathrm{Ty}_{l} @ \operatorname{succ} l \quad L\right| \Psi ; \Gamma, x: \mathrm{El}^{l} s @ l \Vdash_{i} t: \mathrm{Ty}_{l^{\prime}} @ \operatorname{succ} l^{\prime}}{L \mid \Psi ; \Gamma \Vdash_{i} \Pi^{l, l^{\prime}}\left(x: \mathrm{El}^{l} s\right) . \mathrm{El}^{l^{\prime}} t \approx \mathrm{El}^{l \sqcup l^{\prime}} \Pi^{l, l^{\prime}}(x: s) . t @ l \sqcup l^{\prime}}
$$

Proof. Here $i \in\{p, m\}$. The proof is similar to the previous lemma except that we only use reduction expansion on one side.

Lemma 7.82.

$$
\frac{L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level } \quad L\left|\Psi ; \Gamma \Vdash_{i} S \approx S^{\prime} @ l_{1} \quad L\right| \Psi ; \Gamma, x: S @ l_{1} \Vdash_{i} t \approx t^{\prime}: T @ l_{2}}{L \mid \Psi ; \Gamma \vdash_{i} \lambda^{l_{1}, l_{2}}(x: S) . t \approx \lambda^{l_{3,} l_{4}}\left(x: S^{\prime}\right) \cdot t^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l_{1} \sqcup l_{2}}
$$

Proof. From $L \mid \Psi ; \Gamma \Vdash_{i} S \approx S^{\prime} @ l_{1}$, we can conclude $L \Vdash \Psi$ and $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$. The premise $L \mid \Psi ; \Gamma, x$ : $S @ l_{1} \vdash_{i} t \approx t^{\prime}: T @ l_{2}$ also gives us $L \mid \Psi ; \Gamma, x: S @ l_{1} \Vdash_{i} T @ l_{2}$. Combining symmetry, transitivity and the previous lemma, we have

$$
L \mid \Psi ; \Gamma \Vdash_{i} \Pi^{l_{1}, l_{2}}(x: S) . T @ l_{1} \sqcup l_{2}
$$

Now assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \geq c$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

- $\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \Pi^{l_{1}, l_{2}}(x: S) \cdot T[\phi][\sigma][\delta] \approx \Pi^{l_{3}, l_{4}}(x: S) \cdot T\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @\left(l_{1} \sqcup l_{2}\right)[\phi]$, and
$-L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} \lambda^{l_{1}, l_{2}}(x: S) \cdot t[\phi][\sigma][\delta] \approx \lambda^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \mathrm{El}(\mathcal{D})$.
To obtain the goal we shall proceed in two steps. First, we obtain

$$
L^{\prime} \mid \Phi ; \Delta, x: S[\phi][\sigma][\delta] @ l_{1}[\phi]\| \|_{m}^{m} \delta, x / x \approx \delta^{\prime}, x / x:\left(\Gamma, x: S @ l_{1}\right)[\phi][\sigma]
$$

. Giving it to $L \mid \Psi ; \Gamma, x: S @ l_{1} \Vdash_{i} t \approx t^{\prime}: T @ l_{2}$, we have

$$
L^{\prime} \mid \Phi ; \Delta, x: S[\phi][\sigma][\delta] @ l_{1}[\phi] \| \vDash_{m}^{m} t[\phi][\sigma][\delta, x / x] \approx t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, x / x\right]: T[\phi][\sigma][\delta, x / x] @ l_{2}[\phi]
$$

A further escape gives us
$L^{\prime} \mid \Phi ; \Delta, x: S[\phi][\sigma][\delta] @ l_{1}[\phi] \vdash_{m} t[\phi][\sigma][\delta, x / x] \simeq t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, x / x\right]: T[\phi][\sigma][\delta, x / x] @ l_{2}[\phi]$
From this, we conclude

$$
L^{\prime} \mid \Phi ; \Delta \vdash_{m}\left(\lambda^{l_{1}, l_{2}}(x: S) \cdot t\right)[\phi][\sigma][\delta] \simeq\left(\lambda^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot t^{\prime}\right)\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: T[\phi][\sigma][\delta] @ l_{2}[\phi]
$$

modulo the law of weak head closure.
In the second step, we assume $\psi:: L^{\prime \prime}\left|\quad \Phi^{\prime} ; \Delta^{\prime} \quad \Longrightarrow_{m} \quad L^{\prime}\right| \quad \Phi ; \Delta$ and $L^{\prime \prime} \mid \Phi^{\prime} ; \Delta^{\prime} \| \models_{m}^{m} s \approx s^{\prime}: S[\phi][\sigma][\delta] @ l_{1}[\phi]$, we should prove

$$
L^{\prime \prime} \mid \Phi^{\prime} ; \Delta^{\prime} \|=_{m}^{m} t[\phi][\sigma][\delta, s / x] \approx t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, s^{\prime} / x\right]: T[\phi][\sigma][\delta, s / x] @ l_{1}[\phi]
$$

then we have the goal modulo weak head expansion. We are almost there, as long as we can provide

$$
L^{\prime \prime} \mid \Phi^{\prime} ; \Delta^{\prime} \| \vDash_{m}^{m} \delta, s / x \approx \delta^{\prime}, s^{\prime} / x:\left(\Gamma, x: S @ l_{1}\right)[\phi][\sigma]
$$

But in the previous lemma, we have seen it obvious.
Case $i=p$ Repeat the previous case at different layers twice.
Case $i=c$ This case is very similar to that of $\Pi$.

Lemma 7.83.
$L \vdash l_{1} \approx l_{3}:$ Level $\quad L \vdash l_{2} \approx l_{4}:$ Level $L\left|\Psi ; \Gamma \Vdash_{i} S \approx S^{\prime} @ l_{1} \quad L\right| \Psi ; \Gamma, x: S @ l_{1} \Vdash_{i} T \approx T^{\prime} @ l_{2}$ $\mathcal{A}:: L\left|\Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: \Pi^{l_{1}, l_{2}}(x: S) . T @ l_{1} \sqcup l_{2} \quad L\right| \Psi ; \Gamma \Vdash_{i} s \approx s^{\prime}: S @ l_{1}$
$L \mid \Psi ; \Gamma \Vdash_{i}\left(t: \Pi^{l_{1}, l_{2}}(x: S) \cdot T\right) s \approx\left(t^{\prime}: \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot T^{\prime}\right) s^{\prime}: T[s / x] @ l_{2}$
Proof. From $L \mid \Psi ; \Gamma \Vdash_{i} S \approx S^{\prime} @ l_{1}$, we can conclude $L \Vdash \Psi$ and $L \mid \Psi \Vdash_{\text {typeof }(i)} \Gamma$. To show $L \mid \Psi ; \Gamma \Vdash_{\text {typeof(i) }} T[s / x] @ l_{2}$, we first show $L \mid \Psi ; \Gamma \Vdash_{\text {typeof }(i)} s / x: \Gamma, x: S$ @ $l_{1}$ and get the goal using the local substitution lemma. This is immediate by the semantic rule for local substitutions and Corollary 7.71.

Now assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$ and $L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$, we have to consider all $i \geq c$.
Case $i=m$ Then assuming $L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, we must show

- $\mathcal{D}:: L^{\prime} \mid \Phi ; \Delta \|=_{m}^{m} T[\phi][\sigma][\delta, s[\phi][\sigma][\delta] / x] \approx T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, s^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] / x\right] @ l_{2}[\phi]$, and
- $L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m}\left(t: \Pi^{l_{1}, l_{2}}(x: S) . T\right) s[\phi][\sigma][\delta] \approx\left(t^{\prime}: \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) . T^{\prime}\right) s^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \operatorname{El}(\mathcal{D})$.
$\mathcal{D}$ is easily concluded from $L \mid \Psi ; \Gamma \vdash_{m} T[s / x] @ l_{2}$.
We obtain the goal by instantiating $\mathcal{A}$, from which we get

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{m}^{m} t[\phi][\sigma][\delta] \approx t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \mathrm{El}\left(\Pi^{l_{1}, l_{2}}(x: S) \cdot T[\phi][\sigma][\delta]\right)\left(l_{1} \sqcup l_{2}\right)[\phi]
$$

The semantics of $\Pi^{l_{1}, l_{2}}(x: S) \cdot T[\phi][\sigma][\delta]$ gives us the goal, up to irrelevance.
Case $i=p$ Similar.
Case $i=c$ Follow the previous pattern, we use the local substitution lemma.

Lemma 7.84.

$$
\frac{L \vdash l: \text { Level } \quad L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi ; \Gamma, x: S @ l \Vdash_{i} t: T @ l^{\prime} \quad L\right| \Psi ; \Gamma \Vdash_{i} s: S @ l}{L \mid \Psi ; \Gamma \Vdash_{i} t[s / x] \approx\left(\lambda^{l, l^{\prime}}(x: S) . t: \Pi^{l, l^{\prime}}(x: S) . T\right) s: T[s / x] @ l^{\prime}}
$$

 similar lines as the previous lemma.

From the local substitution lemma, we also have $L \mid \Psi ; \Gamma \vdash_{i} t[s / x]: T @ l^{\prime}$ so we are one reduction step away, which can be concluded by the reduction expansion lemma.

Lemma 7.85.

$$
\frac{L \vdash l^{\prime}: \text { Level } \quad L\left|\Psi ; \Gamma \Vdash_{i} S @ l \quad L\right| \Psi ; \Gamma, x: S @ l \Vdash_{i} T @ l^{\prime} \quad L \mid \Psi ; \Gamma \Vdash_{i} t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} \lambda^{l, l^{\prime}}(x: S) .\left(t: \Pi^{l l^{\prime}}(x: S) . T\right) x \approx t: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}}
$$

Proof. Here $i \in\{p, m\}$. We assume $\phi \approx \phi^{\prime}, \sigma \approx \sigma^{\prime}$ and $\delta \approx \delta^{\prime}$, and finally $s \approx s^{\prime}$. Then see

$$
\begin{aligned}
& \left(\lambda^{l, l^{\prime}}(x: S) .\left(t[\phi][\sigma][\delta]: \Pi^{l, l^{\prime}}(x: S) . T\right) x: \Pi^{l, l^{\prime}}(x: S) . T\right) s \\
\rightsquigarrow & \left(t[\phi][\sigma][\delta]: \Pi^{l, l^{\prime}}(x: S) . T\right) x[s / x]
\end{aligned}
$$

$$
\begin{aligned}
& =\left(t[\phi][\sigma][\delta]: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) s \\
& \approx\left(t\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) s^{\prime}
\end{aligned}
$$

Therefore, we obtain the goal by weak head expansion of logical relations.

## Lemma 7.86. If

- $(k, i) \in\{(p, p),(m, p),(m, m)\}$,
- $L \vdash l \approx l^{\prime}$ : Level,
- $L \mid \Psi ; \Gamma, x:$ Nat @ zero $\Vdash_{i} M \approx M^{\prime} @ l$,
- $L \mid \Psi ; \Gamma \vdash_{i} s_{1} \approx s_{3}: M[z e r o / x] @ l$,
- $\mathcal{B}:: L \mid \Psi ; \Gamma, x:$ Nat @ zero, $y: M @ l \vdash_{i} s_{2} \approx s_{4}: M[\operatorname{succ} x / x] @ l$,
- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$,
- $L^{\prime} \mid \Phi ; \Delta \| \Vdash_{k}^{\text {typeof }(i)} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{k} t \rightsquigarrow^{*} w:$ Nat @ zero,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{k} t^{\prime} \rightsquigarrow^{*} w^{\prime}$ : Nat @ zero,
- $L^{\prime} \mid \Phi ; \Delta \vdash_{k} w \simeq w^{\prime}$ : Nat @ zero,
- $\mathcal{A}:: L^{\prime} \mid \Phi ; \Delta \| \vDash_{k} w \simeq w^{\prime}$ : Nat,
- $t_{1}=\operatorname{elim}_{\text {Nat }}^{l}(x . M[\phi][\sigma][\delta, x / x])\left(s_{1}[\phi][\sigma][\delta]\right)\left(x, y \cdot s_{2}[\phi][\sigma][\delta, x / x, y / y]\right) t$,
- $t_{2}=e \operatorname{elim}_{\text {Nat }}^{l}\left(x . M^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, x / x\right]\right)\left(s_{3}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]\right)\left(x, y \cdot s_{4}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}, x / x, y / y\right]\right) t^{\prime}$
then

$$
L^{\prime} \mid \Phi ; \Delta\| \|_{k}^{\text {typeof }(i)} t_{1} \approx t_{2}: M[\phi][\sigma][\delta, t / x] @ l[\phi]
$$

Proof. We do induction on $\mathcal{A}$.

- If $w=w^{\prime}=$ zero, then we hit the base case. In this case, we use $L \mid \Psi ; \Gamma \Vdash_{i} s_{1} \approx s_{3}: M[$ zero $/ x] @ l$ and reduction expansion to almost obtain the goal. The only missing piece is to prove

$$
M[\phi][\sigma][\delta, t / x] \approx M[\phi][\sigma][\delta, \text { zero } / x]
$$

This holds from symmetry and the fact that $t \approx$ zero, so that we can extend $\delta \approx \delta$.

- If $w=\operatorname{succ} s$ and $w^{\prime}=\operatorname{succ} s^{\prime}$, then we apply IH and also $\mathcal{B}$ to obtain the relation between recursive calls for $s$ and $s^{\prime}$. We perform a similar analysis to handle the types.
- If $w=v$ and $w=v^{\prime}$ for some neutrals, then we relate them using the reflexivity of neutral.

Lemma 7.87.
$L \vdash l \approx l^{\prime}:$ Level $\quad L \mid \Psi ; \Gamma, x: N a t$ @ zero $\Vdash_{i} M \approx M^{\prime} @ l$
$L \mid \Psi ; \Gamma, x:$ Nat @ zero, $y: M @ l \Vdash_{i} s_{2} \approx s_{4}: M[\operatorname{succ} x / x] @ l$
$L \mid \Psi ; \Gamma \Vdash_{i} \operatorname{elim}_{\mathrm{Nat}}^{l}(x . M) s_{1}\left(x, y \cdot s_{2}\right) t \approx \operatorname{elim}_{\mathrm{Nat}}^{l^{\prime}}\left(x . M^{\prime}\right) s_{3}\left(x, y \cdot s_{4}\right) t^{\prime}: M[t / x] @ l: M[$ zero $/ x] @ l$

Proof. Instantiate $\mathcal{A}$ and use the previous lemma for $i \in\{p, m\}$. Notice that we know $t$ and $t^{\prime}$ are related by Nat which must reduce to itself so we can supply all premises required by the previous lemma. When $i=c$, we reuse the proof when $i=p$.

Lemma 7.88.

$$
\begin{aligned}
& L \vdash l: \text { Level } \quad L \mid \Psi ; \Gamma, x \text { : Nat @ zero } \Vdash_{i} M @ l \\
& L\left|\Psi ; \Gamma \Vdash_{i} s: M[z e r o / x] @ l \quad L\right| \Psi ; \Gamma, x: N a t @ z e r o, y: M @ l \vdash_{i} s^{\prime}: M[\operatorname{succ} x / x] @ l \\
& L \mid \Psi ; \Gamma \vdash_{i} s \approx \operatorname{elim}_{\mathrm{Nat}}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right) \text { zero : M[zero/x] @ } l \\
& L \vdash l: \text { Level } \quad L \mid \Psi ; \Gamma, x: \text { Nat @ zero } \vdash_{i} M @ l \quad L \mid \Psi ; \Gamma \vdash_{i} s: M[z e r o / x] @ l \\
& L \mid \Psi ; \Gamma, x \text { : Nat @ zero, } y: M @ l \Vdash_{i} s^{\prime}: M[\operatorname{succ} x / x] @ l \quad L \mid \Psi ; \Gamma \Vdash_{i} t \text { : Nat @ zero } \\
& L \mid \Psi ; \Gamma \Vdash_{i} s^{\prime}\left[t / x, \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y . s^{\prime}\right) t / y\right] \approx \operatorname{elim}_{\text {Nat }}^{l}(x . M) s\left(x, y \cdot s^{\prime}\right)(\operatorname{succ} t): M[\operatorname{succ} t / x] @ l
\end{aligned}
$$

Proof. We use reduction expansion, semantic local substitution lemma and the previous lemma.
Lemma 7.89.

$$
\frac{L\left|\Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: T^{\prime} @ l \quad L\right| \Psi ; \Gamma \Vdash_{t y p e o f(i)} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \Vdash_{i} t \approx t^{\prime}: T @ l}
$$

Proof. Use irrelevance when $i \in\{p, m\}$. When $i \in\{v, c\}$, we do a case analysis on the semantics of related terms. Notice that all rules for terms in Sec. 7.4 contain a type relation. We apply transitivity of related types and irrelevance.

### 7.9 More Semantic Rules

In the previous section, we have considered all possible (non-trivial) rules for all layers. Among these rules, we have looked at the rules for global variables and see how layering restriction enables code running in the semantics. In this section, we will finish the proof by considering rules that are available at layer $m$. This will make our proofs in some sense simpler; there is only one layer to consider. On the other hand, we will look into another important feature, recursors for code, and how it is semantically justified.

Lemma 7.90.

$$
\frac{L\left|\Psi \vdash_{m} \Gamma \quad L\right| \Psi \Vdash_{p} \Delta \approx \Delta^{\prime} \quad L \mid \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T @ l\right) \approx \square\left(\Delta^{\prime} \vdash_{c} T^{\prime} @ l^{\prime}\right) @ l}
$$

Proof. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \| \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and $L^{\prime} \mid \Phi ; \Delta_{1} \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, the focuses are

- $\mathcal{A}:: L^{\prime} \mid \Phi \Vdash_{\geq p}^{p} \Delta[\phi][\sigma] \approx \Delta^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]$,
- $\mathcal{B}:: L^{\prime} \mid \Phi ; \Delta[\phi][\sigma] \vDash_{\geq p}^{p} T[\phi][\sigma] @ l[\phi]$ and
- $C:: L^{\prime} \mid \Psi ; \Delta^{\prime}[\phi][\sigma] \stackrel{\vDash_{\geq p}^{p}}{T^{\prime}}[\phi][\sigma] @ l[\phi]$.
$\mathcal{A}$ is simple as we only need to convert weakenings to substitutions. Then we can use $L \mid \Psi \vdash_{p} \Delta \approx \Delta^{\prime} . \mathcal{B}$ and $C$ are symmetric so we only focus on $C$ which is slightly more complex. We similarly want to convert weakenings to substitutions so that we can apply $L \mid \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l$ to obtain the goal. The local contexts are mismatched, though it is not a problem. The local contexts are used when further assuming related local substitutions, and by $L \mid \Psi \Vdash_{p} \Delta \approx \Delta^{\prime}$, we can use irrelevance to swap the local contexts.

Lemma 7.91.

$$
\frac{L\left|\Psi \Vdash_{m} \Gamma \quad L\right| \Psi \Vdash_{p} \Delta \approx \Delta^{\prime} \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} @ l\right) \approx \square\left(\Delta^{\prime} \vdash_{c} @ l^{\prime}\right) @ \operatorname{succ} l}
$$

Proof. Similar to above but simpler.

Lemma 7.92.

$$
\frac{L\left|\Psi \Vdash_{m} \Gamma \quad L\right| \Psi ; \Delta \Vdash_{c} t: T @ l}{L \mid \Psi ; \Gamma \Vdash_{m} \operatorname{box} t \approx \operatorname{box} t: \square\left(\Delta \vdash_{c} T @ l\right) @ l}
$$

Proof. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \vDash \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, then the goal requires

$$
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{c}^{p} t[\phi][\sigma]: T[\phi][\sigma] @ l[\phi]
$$

Notice that the effect of local substitutions is irrelevant anymore. This goal can be instantiate by $L \mid \Psi ; \Delta \Vdash_{c} t$ : $T @ l$ by passing in the same universe and global substitutions, and the identity local substitution.

Lemma 7.93.

$$
\frac{L\left|\Psi \Vdash_{m} \Gamma \quad L\right| \Psi ; \Delta \vdash_{c} T @ l}{L \mid \Psi ; \Gamma \vdash_{m} \operatorname{box} T \approx \operatorname{box} T: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l}
$$

Proof. Similar to above but simpler.
Lemma 7.94.

$$
\begin{gathered}
L \vdash l_{1} \approx l_{3}: \text { Level } L \vdash l_{2} \approx l_{4}: \text { Level } \quad L\left|\Psi \vdash_{p} \Delta \approx \Delta^{\prime} \quad L\right| \Psi ; \Gamma \vdash_{p} T \approx T^{\prime} @ l_{2} \\
L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{2} \quad L\right| \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{1} \vdash_{m} M \approx M^{\prime} @ l_{1} \\
L \mid \Psi, u:\left(\Delta \vdash_{c} T @ l_{2}\right) ; \Gamma \vdash_{m} t_{1} \approx t_{2}: M\left[\operatorname{box} u^{i d} / x_{t}\right] @ l_{1} \\
\hline
\end{gathered}
$$

$\overline{L \mid \Psi ; \Gamma \Vdash_{m} \text { letbox }_{T r m}^{l_{1}} l_{2} \Delta T\left(x_{t} \cdot M\right)\left(U . t_{1}\right) t \approx \operatorname{letbox}_{T r m}^{l_{3}} l_{4} \Delta^{\prime} T^{\prime}\left(x_{T} \cdot M^{\prime}\right)\left(U . t_{2}\right) t^{\prime}: M\left[t / x_{t}\right] @ l_{1}}$
Proof. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime} \mid \Phi \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, then the goal requires

$$
L^{\prime} \mid \Phi ; \Delta \| \vDash_{m}^{m} s_{1} \approx s_{2}: M[\phi][\sigma]\left[\delta, t[\phi][\sigma][\delta] / x_{t}\right] @ l_{1}[\phi]
$$

where $s_{1}=$ letbox ${ }_{T r m}^{l_{1}} l_{2} \Delta T\left(x_{t} \cdot M\right)\left(U . t_{1}\right) t[\phi][\sigma][\delta]$ and $s_{2}=$ letbox Trm $_{3}^{l_{3}} l_{4} \Delta^{\prime} T^{\prime}\left(x_{T} \cdot M^{\prime}\right)\left(U . t_{2}\right) t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]$. By instantiating $L \mid \Psi ; \Gamma \vdash_{m} t \approx t^{\prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{2}$, we have

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} t[\phi][\sigma][\delta] \approx t^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right]: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]
$$

Unfolding it, there are two possibilities for this relation.
Case We know for some $t^{\prime \prime}$,
$-L^{\prime} \mid \Phi ; \Delta^{\prime} \vdash_{k} t[\phi][\sigma][\delta] \rightsquigarrow^{*}$ box $t^{\prime \prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]$,
$-L^{\prime} \mid \Phi ; \Delta^{\prime} \vdash_{k} t\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] \rightsquigarrow^{*}$ box $t^{\prime \prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]$, and
$-L^{\prime} \mid \Phi ; \Delta[\phi][\sigma] \| \vDash_{c}^{p} t^{\prime \prime}: T[\phi][\sigma] @ l_{2}[\phi]$
We observe that

$$
L^{\prime} \mid \Phi \| \sigma, t^{\prime \prime} / u \approx \sigma^{\prime}, t^{\prime \prime} / u: \Psi[\phi], u:\left(\Delta[\phi][\sigma] \vdash_{c} T[\phi][\sigma] @ l_{2}[\phi]\right)
$$

Giving it to $L \mid \Psi, u:\left(\Delta \vdash_{c} T @ l_{2}\right) ; \Gamma \vdash_{m} t_{1} \approx t_{2}: M\left[\operatorname{box} u^{\text {id }} / x_{t}\right] @ l_{1}$, we obtain

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \|=_{m}^{m} t_{1}[\phi]\left[\sigma, t^{\prime \prime} / u\right][\delta] \approx t_{2}\left[\phi^{\prime}\right]\left[\sigma^{\prime}, t^{\prime \prime} / u\right]\left[\delta^{\prime}\right]: M[\phi][\sigma]\left[\delta, \text { box } t^{\prime \prime} / x_{t}\right] @ l_{1}[\phi]
$$

By reduction expansion, we have

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \|=_{m}^{m} s_{1} \approx s_{2}: M[\phi][\sigma]\left[\delta, \text { box } t^{\prime \prime} / x_{t}\right] @ l_{1}[\phi]
$$

which is almost the goal. To tame the goal, we observe that

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} t[\phi][\sigma][\delta] \approx \operatorname{box} t^{\prime \prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]
$$

which further allows us to conclude

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \|=_{m}^{m} \delta, t[\phi][\sigma][\delta] / x_{t} \approx \delta^{\prime} \text {, box } t^{\prime \prime} / x_{t}:\left(\Gamma, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{1}\right)[\phi][\sigma]
$$

Apply $L \mid \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{1} \vdash_{m} M \approx M^{\prime} @ l_{1}$ gives us the goal using irrelevance and weak head expansion.
Case We know for some $\mu$ and $\mu^{\prime}$,
$-L^{\prime} \mid \Phi ; \Delta^{\prime} \vdash_{k} t[\phi][\sigma][\delta] \rightsquigarrow^{*} \mu: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]$,

- $L^{\prime} \mid \Phi ; \Delta^{\prime} \vdash_{k} t\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] \rightsquigarrow^{*} \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]$, and
$-L^{\prime} \mid \Phi ; \Delta^{\prime} \vdash_{m} \mu \sim \mu^{\prime}: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma][\delta] @ l_{2}[\phi]$
The idea here is to use the law of congruence for neutrals to establish a generic equivalence between neutrals, and then we use reflexivity for neutrals to relate two neutrals using the logical relations.
The process requires us to provide
$L^{\prime} \mid \Phi ; \Delta^{\prime}, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma] @ l_{1}[\phi] \| F_{m}^{m} \delta, x_{T} / x_{T} \approx \delta^{\prime}, x_{T} / x_{T}:\left(\Gamma, x_{T}: \square\left(\Delta \vdash_{c} T @ l_{2}\right) @ l_{1}\right)[\phi][\sigma]$
This is immediate. We also need

$$
L^{\prime} \mid \Phi, u:\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma] \| \sigma, u^{\text {id }} / u \approx \sigma^{\prime}, u^{\mathrm{id}} / u:\left(\Psi, u:\left(\Delta \vdash_{c} T @ l_{2}\right)\right)[\phi]
$$

In this case, we should prove

$$
L^{\prime} \mid \Phi, u:\left(\Delta \vdash_{c} T @ l_{2}\right)[\phi][\sigma] ; \Delta[\phi][\sigma] \Vdash_{\geq p}^{p} u^{\text {id }}: T[\phi][\sigma] @ l_{2}[\phi]
$$

This turns out to have been checked by the lemma of reflexive global weakening in Sec. 7.4. Finally, $L^{\prime} \mid \Phi ; \Delta^{\prime}\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$ is weakened before applying.

Lemma 7.95.

$$
\begin{aligned}
& L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level } \quad L \mid \Psi \vdash_{p} \Delta \approx \Delta^{\prime} \\
& L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}: \square\left(\Delta \vdash_{c} @ l_{2}\right) @ \operatorname{succ} l_{2} \quad L\right| \Psi ; \Gamma, x_{T}: \square\left(\Delta \vdash_{c} @ l_{2}\right) @ \operatorname{succ} l_{2} \vdash_{m} M \approx M^{\prime} @ l_{1} \\
& L \mid \Psi, U:\left(\Delta \vdash_{c} @ l_{2}\right) ; \Gamma \vdash_{m} t_{1} \approx t_{2}: M\left[\operatorname{box} U^{\text {id }} / x_{T}\right] @ l_{1} \\
& L \mid \Psi ; \Gamma \Vdash_{m} \text { letbox }_{T_{y p}}^{l_{1}} l_{2} \Delta\left(x_{T} . M\right)\left(U . t_{1}\right) t \approx \operatorname{letbox}_{T y p}^{l_{3}} l_{4} \Delta^{\prime}\left(x_{T} . M^{\prime}\right)\left(U . t_{2}\right) t^{\prime}: M\left[t / x_{T}\right] @ l_{1}
\end{aligned}
$$

Proof. Similar to above but simpler.
Next, we consider the semantics for the recursive principles. The following lemma needs to be mutually proved.

Lemma 7.96. If

- $S_{A}$,
- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$,
- $L^{\prime} \mid \Phi ; \Delta_{1}\| \|_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$,
- $L^{\prime}+l \approx l^{\prime}$ : Level,
- $L^{\prime} \mid \Phi \|_{\geq p}^{p} \Delta \approx \Delta^{\prime}$,
- $L^{\prime} \mid \Phi ; \Delta_{1} \vdash_{m} t \rightsquigarrow^{*}$ box $T_{1}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l$,
- $L^{\prime} \mid \Phi ; \Delta_{1} \vdash_{m} t^{\prime} \rightsquigarrow^{*}$ box $T_{1}: \square\left(\Delta \vdash_{c} @ l\right) @ \operatorname{succ} l$,
- $\mathcal{A}:: L^{\prime} \mid \Psi ; \Delta \| \equiv_{c}^{p} T_{1} @ l$,
- $t_{1}=\operatorname{elim}_{T y p}^{l_{1}[\phi], l_{2}[\phi]}(\vec{M}[\phi][\sigma][\delta])(\vec{b}[\phi][\sigma][\delta]) l \Delta t$, and
- $t_{2}=e e l i m_{T y p}^{l_{1}[\phi], l_{2}[\phi]}\left(\vec{M}^{\prime}[\phi][\sigma][\delta]\right)\left(\vec{b}^{\prime}[\phi][\sigma][\delta]\right) l^{\prime} \Delta^{\prime} t^{\prime}$,
then

$$
L^{\prime} \mid \Phi ; \Delta_{1} \| \vDash_{m}^{m} t_{1} \approx t_{2}: M\left[l / \ell, \Delta / g, t / x_{T}\right][\phi][\sigma][\delta] @ l_{1}[\phi]
$$

Lemma 7.97. If

- $S_{A}$,
- $L^{\prime} \vdash \phi \approx \phi^{\prime}: L$,
- $L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$,
- $L^{\prime} \mid \Phi ; \Delta_{1} \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$,
- $L^{\prime}+l \approx l^{\prime}:$ Level,
- $L^{\prime}|\Phi| \vDash_{\geq p}^{p} \Delta \approx \Delta^{\prime}$,
- $L^{\prime} \mid \Phi ; \Delta \|_{\geq p}^{p} T \approx T^{\prime} @ l$,
- $L^{\prime} \mid \Phi ; \Delta_{1} \vdash_{m} t \rightsquigarrow^{*}$ box $t: \square\left(\Delta \vdash_{c} T @ l\right) @ l$,
- $L^{\prime} \mid \Phi ; \Delta_{1} \vdash_{m} t^{\prime} \rightsquigarrow *$ box $t_{1}: \square\left(\Delta \vdash_{c} T @ l\right) @ l$,
- $\mathcal{B}:: L^{\prime} \mid \Psi ; \Delta \| \vDash_{c}^{p} t_{1}: T @ l$,
- $t_{1}=e \operatorname{elim}_{T r m}^{l_{1}[\phi], l_{2}[\phi]}(\vec{M}[\phi][\sigma][\delta])(\vec{b}[\phi][\sigma][\delta]) l \Delta T t$, and
- $t_{2}=e \operatorname{elim} T_{T r m}^{l_{1}[\phi], l_{2}[\phi]}\left(\vec{M}^{\prime}[\phi][\sigma][\delta]\right)\left(\vec{b}^{\prime}[\phi][\sigma][\delta]\right) l^{\prime} \Delta^{\prime} T^{\prime} t^{\prime}$,
then

$$
L^{\prime} \mid \Phi ; \Delta_{1}\| \|_{m}^{m} t_{1} \approx t_{2}: M^{\prime}\left[l / \ell, \Delta / g, T / U_{T}, t / x_{t}\right][\phi][\sigma][\delta] @ l_{2}[\phi]
$$

where $S_{A}$ is the set containing all semantic judgments for motives and branches in $L \mid \Psi ; \Gamma$.
Proof. The idea is to do a mutual induction on $\mathcal{A}$ and $\mathcal{B}$. Recall that they are mutually defined structures as shown in Sec. 7.4. Let us pick two cases to discuss:
Case

$$
\begin{gathered}
u:\left(\Delta_{2} \vdash_{i^{\prime}} T_{2} @ l^{\prime \prime}\right) \in \Phi \quad i^{\prime} \in\{v, c\} \quad L^{\prime} \mid \Phi ; \Delta \| \Vdash_{c}^{p} \delta_{2}: \Delta_{2} \\
L \vdash l \approx l^{\prime \prime}: \text { Level } \\
L^{\prime}\left|\Phi ; \Delta\left\|_{\geq p}^{p} T \approx T_{2}\left[\delta_{2}\right] @ l \quad L^{\prime} \mid \Phi ; \Delta\right\|_{\geq p}^{p} u^{\delta_{2}}: T @ l\right. \\
L^{\prime} \mid \Phi ; \Delta \| \Vdash_{c}^{p} u^{\delta_{2}}: T @ l
\end{gathered}
$$

In this case, we must block the evaluation. The idea follows closely to the neutral case for letbox and we should apply the law of neutral recursion on code. In this case, we see that $L^{\prime}\left|\Phi \vdash_{p} \Delta \simeq \Delta^{\prime}, L^{\prime}\right| \Phi ; \Delta \vdash_{p}$ $T \Longleftrightarrow T^{\prime} @$ succ $l$. Then we have to show that the motives and the branches are related by generic equivalence. The idea is to extend all substitutions if necessary. We have shown that all substitutions can be extended by identities in the semantics (and universe substitutions are the same in both syntax and semantics).
Case

$$
\begin{aligned}
& L^{\prime} \vdash l_{3} \text { : Level } \quad L^{\prime} \vdash l_{4} \text { : Level } \\
& L^{\prime}\left|\Phi ; \Delta\left\|\Vdash_{c}^{p} S_{2} @ l_{3} \quad L^{\prime} \mid \Phi ; \Delta, x: S_{2} @ l_{3}\right\| \vDash_{c}^{p} t_{2}: T_{2} @ l_{4} \quad L^{\prime} \vdash l \approx l_{3} \sqcup l_{4}\right. \text { : Level } \\
& L^{\prime}\left|\Phi ; \Delta\left\|_{\geq p}^{p} T \approx \Pi^{l_{3}, l_{4}}\left(x: S_{2}\right) \cdot T_{2} @ l \quad L^{\prime} \mid \Phi ; \Delta\right\|_{\geq p}^{p} \lambda^{l_{3}, l_{4}}\left(x: S_{2}\right) \cdot t_{2}: T @ l\right. \\
& L^{\prime} \mid \Phi ; \Delta \| \models_{c}^{p} \lambda^{l_{3}, l_{4}}\left(x: S_{2}\right) \cdot t_{2}: T @ l
\end{aligned}
$$

In this case, we should go down and recurse on $S_{2}$ and $t_{2}$. We then use the semantic rule for $t_{\lambda}$ to substitute in the results of the recursive calls for $S_{2}$ and $t_{2}$. We also obtain $L^{\prime} \mid \Phi ; \Delta, x: S_{2} @ l_{3} \vDash_{\geq p}^{p} T_{2} @ l$. Therefore we have everything we need to use the semantic rule for $t_{\lambda}$. The only missing piece is that the conclusion requires relation between $T$ and $T^{\prime}$. Meanwhile, $t_{\lambda}$ only gives us $\Pi^{l_{3}, l_{4}}\left(x: S_{2}\right) \cdot T_{2}$. The solution lies in

$$
L^{\prime} \mid \Phi ; \Delta \vDash_{\geq p}^{p} T \approx T^{\prime} @ l
$$

$$
L^{\prime} \mid \Phi ; \Delta \Vdash_{\geq p}^{p} T \approx \Pi^{l_{3}, l_{4}}\left(x: S_{2}\right) \cdot T_{2} @ l
$$

Thus these three types are related. Since $\Pi^{l_{3}, l_{4}}\left(x: S_{2}\right) . T_{2}$ is already in normal form, so we know both $T^{\prime}$ and $T^{\prime}$ must reduce to it. Together with the $\beta$ rule when hitting the $\lambda$ case, we use weak head expansion to obtain the desired goal.

These two lemmas in the semantics give the recursions on code of types and terms and actual do the recursions. For the semantic rules for the recursors, we first use these lemmas to prove the congruence rules. We are almost done wit the congruence rules except that we have to handle the neutral cases. This is virtually identical to the global variable cases, where we use the law of neutral recursion of code to relate neutral terms and use reflexivity for neutrals to establish the logical relations. The rest are the $\beta$ rules. They are even simpler due to access to the congruence rules. Then we use reduction expansion to achieve the goals.

What are left now are the meta-functions including universe-polymorphic functions. Fist the meta-functions for local contexts and types are very similar.

Lemma 7.98.

$$
\begin{gathered}
L \mid \Psi \Vdash_{m} \Gamma \\
L\left|\Psi \Vdash_{p} \Delta \approx \Delta^{\prime} \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T \approx T^{\prime} @ l^{\prime} \quad L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level } \\
L \mid \Psi ; \Gamma \Vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T \approx\left(U:\left(\Delta^{\prime} \vdash_{p} @ l_{3}\right)\right) \Rightarrow^{l_{4}} T^{\prime} @ \operatorname{succ} l_{1} \sqcup l_{2}
\end{gathered}
$$

Proof. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, then we should prove

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m}\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T[\phi][\sigma][\delta] \approx\left(U:\left(\Delta^{\prime} \vdash_{p} @ l_{3}\right)\right) \Rightarrow^{l_{4}} T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ \operatorname{succ} l_{1} \sqcup l_{2}[\phi]
$$

Most premises are simple. The tricky part is to show $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$, which are symmetric. Knowing $L \mid \Psi \vdash_{p} \Delta \approx \Delta^{\prime}$, it is sufficient to only prove one of them. Using $L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T \approx T^{\prime} @ l^{\prime}$, it is easy to see that both $\mathcal{E}_{1}$ and $E_{2}$ hold, by extending $\sigma$ and $\sigma^{\prime}$.

Lemma 7.99.

$$
\frac{L\left|\Psi \Vdash_{m} \Gamma \quad L\right| \Psi, g: C t x ; \Gamma \Vdash_{m} T \approx T^{\prime} @ l \quad L \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \Vdash_{m}(g: C t x) \Rightarrow^{l} T \approx(g: C t x) \Rightarrow^{l^{\prime}} T^{\prime} @ l}
$$

Proof. Similar to above but simpler.
Lemma 7.100.

$$
\frac{L\left|\Psi \Vdash_{m} \Gamma \quad L\right| \Psi, U:\left(\Delta \vdash_{p} @ l_{1}\right) ; \Gamma \vdash_{m} t \approx t^{\prime}: T @ l_{2} \quad L \vdash l_{1} \approx l_{3}: \text { Level } \quad L \vdash l_{2} \approx l_{4}: \text { Level }}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda_{p}^{l_{1} l_{2}} U . t \approx \Lambda_{p}^{l_{3} l_{4}} U . t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l_{1}\right)\right) \Rightarrow^{l_{2}} T @ \text { succ } l_{1} \sqcup l_{2}}
$$

Proof. In this proof, we are asked to prove two symmetric proof which is essentially to show that the results of applying $\Lambda_{p}^{l_{1} l_{2}} U . t$ and $\Lambda_{p}^{l_{3}, l_{4}} U . t^{\prime}$ are related. This is immediate by extending related global substitutions and applying them to $L \mid \Psi, U:\left(\Delta \vdash_{p} @ l_{1}\right) ; \Gamma \vdash_{m} t \approx t^{\prime}: T @ l_{2}$.

Lemma 7.101.

$$
\frac{L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime} @ \operatorname{succ} l \sqcup l^{\prime} \quad L\right| \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{m} t \$_{p} T \approx t^{\prime} \$_{p} T^{\prime}: T^{\prime \prime}[T / U] @ l^{\prime}}
$$

Proof. We obtain the goal quite easily by using the semantics of related terms of type $\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime \prime}$.

Lemma 7.102.

$$
\begin{gathered}
L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \Vdash_{m} t: T^{\prime} @ l^{\prime} \begin{array}{c}
L \mid \Psi \Vdash_{m} \Gamma \\
L \vdash l: \text { Level }
\end{array} \quad L \vdash l^{\prime}: \text { Level } \quad L \mid \Psi ; \Delta \Vdash_{p} T @ l \\
L \mid \Psi ; \Gamma \vdash_{m} t[T / U] \approx\left(\Lambda_{p}^{l, l^{\prime}} U . t\right) \$_{p} T: T^{\prime}[T / U] @ l^{\prime} \\
\frac{L \mid \Psi ; \Gamma \Vdash_{m} t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{m} \Lambda_{p}^{l, l^{\prime}} U .\left(t \$_{p} U^{i d}\right) \approx t:\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime} @ \text { succ } l \sqcup l^{\prime}}
\end{gathered}
$$

Proof. These two rules are also very simple to prove and follow similar lines to those of $\Pi$ types. For the $\beta$ rule, we use reduction expansion. For the $\eta$ rule, we see it by noticing applying related global substitutions.

The corresponding introduction, elimination, $\beta$ and $\eta$ rules for meta-functions for local contexts are proved similarly but just simpler.

Now the only last piece is universe-polymorphic functions.
Lemma 7.103.

$$
\frac{L, \vec{\ell}\left|\Psi ; \Gamma \Vdash_{m} T @ l \quad L, \vec{\ell}\right| \Psi ; \Gamma \Vdash_{m} T \approx T^{\prime} @ l \quad|\vec{\ell}|>0 \quad L, \vec{\ell} \vdash l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \Vdash_{m} \vec{\ell} \Rightarrow^{l} T \approx \vec{\ell} \Rightarrow^{l^{\prime}} T^{\prime} @ \omega}
$$

Proof. Assuming $L^{\prime} \vdash \phi \approx \phi^{\prime}: L, L^{\prime}|\Phi| \vDash \sigma \approx \sigma^{\prime}: \Psi[\phi]$ and $L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \delta \approx \delta^{\prime}: \Gamma[\phi][\sigma]$, then we should prove

$$
L^{\prime} \mid \Phi ; \Delta^{\prime} \| \vDash_{m}^{m} \vec{\ell} \Rightarrow^{l} T[\phi][\sigma][\delta] \approx \vec{\ell} \Rightarrow^{l^{\prime}} T^{\prime}\left[\phi^{\prime}\right]\left[\sigma^{\prime}\right]\left[\delta^{\prime}\right] @ \omega
$$

Looking at the semantics of universe-polymorphic functions, we see that $L, \vec{l} \mid \Psi ; \Gamma \vdash_{m} T$ @ $l$ has already provided the goal. It is important to see that $L^{\prime} \vdash l[\phi, \vec{l} / \vec{\ell}]$ : Level if all universe levels in $\vec{l}$ are well-formed so it is strictly smaller than $\omega$. Therefore, we still have a well-founded semantics.

Lemma 7.104.

$$
\begin{aligned}
& \frac{L, \vec{\ell}\left|\Psi ; \Gamma \Vdash_{m} t \approx t^{\prime}: T @ l \quad\right| \vec{\ell} \mid>0 \quad L, \vec{\ell}+l \approx l^{\prime}: \text { Level }}{L \mid \Psi ; \Gamma \Vdash_{m} \Lambda^{l} \vec{\ell} . t \approx \Lambda^{l^{\prime}} \vec{\ell} \cdot t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega} \\
& L, \vec{\ell} \mid \Psi ; \Gamma \Vdash_{m} T @ l \\
& L\left|\Psi ; \Gamma \vdash_{m} t \approx t^{\prime}: \vec{\ell} \Rightarrow^{l} T @ \omega \quad\right| \vec{\ell}\left|=|\vec{l}|=\left|\vec{l}^{\prime}\right|>0 \quad \forall 0 \leq n<|\vec{l}| \cdot L \vdash \vec{l}(n) \approx \vec{l}^{\prime}(n):\right. \text { Level } \\
& L \mid \Psi ; \Gamma \vdash_{m} t \$ \vec{l} \approx t^{\prime} \$ \vec{l}^{\prime}: T[\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}] \\
& L\left|\Psi \Vdash_{m} \Gamma \quad L, \vec{l}\right| \Psi ; \Gamma \Vdash_{m} t: T @ l \quad L, \vec{l}+l: \text { Level } \quad|\vec{\ell}|=|\vec{l}|>0 \quad \forall l^{\prime} \in \vec{l} \cdot L \vdash l^{\prime}: \text { Level } \\
& L \mid \Psi ; \Gamma \vdash_{m} t[\vec{l} / \vec{\ell}] \approx\left(\Lambda^{l} \vec{\ell} . t\right) \$ \vec{l}: T[\vec{l} / \vec{\ell}] @ l[\vec{l} / \vec{\ell}] \\
& \frac{L \mid \Psi ; \Gamma \Vdash_{m} t: \vec{\ell} \Rightarrow^{l} T @ \omega}{L \mid \Psi ; \Gamma \Vdash_{m} \Lambda^{l} \vec{\ell} \cdot(t \$ \vec{\ell}) \approx t: \vec{\ell} \Rightarrow^{l} T @ \omega}
\end{aligned}
$$

Proof. All these rules are relatively simple. The congruence rule for introduction can be proved by following the definition of related terms of type $\vec{\ell} \Rightarrow^{l} T$.

The congruence rule for elimination can be proved by using the definition of related terms of type $\vec{\ell} \Rightarrow^{l} T$.

The $\beta$ rule can be derived from using reduction expansion.
The $\eta$ rule can be given after applying arbitrary equivalent universe substitutions.
At this point, we have proved all semantic rules and thus the fundamental theorems hold.

## 8 CONSEQUENCES AND DECIDABILITY OF CONVERTIBILITY

In the previous section, we have established the fundamental theorems, and using the escape lemma, we see that all syntactic equivalent types and terms are also semantically related by the Kripke logical relations. In this section, we will instantiate the generic equivalence so that we can obtain desired properties that are difficult to prove syntactically.

### 8.1 First Instantiation: Syntactic Equivalence

First we instantiate the generic equivalence with syntactic equivalences of types and terms. The laws are all easily instantiated by the corresponding equivalence rules. We also see that the derived equivalences between local contexts and local substitutions are equivalent to the corresponding syntactic equivalences. From the fundamental theorems and the escape lemma, we are able to derive the following lemmas.

Lemma 8.1 (Injectivity of Type Constructors).

- If $L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}(x: S) . T \approx \Pi^{l, l^{\prime}}\left(x: S^{\prime}\right) . T^{\prime} @ l \sqcup l^{\prime}$ and $i \in\{p, m\}$, then $L \mid \Psi ; \Gamma \vdash_{i} S \approx S^{\prime} @ l$ and $L \mid \Psi ; \Gamma, x: S$ @ $l \vdash_{i} T \approx T^{\prime} @ l^{\prime}$.
- If $L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}(x: s) . t \approx \Pi^{l, l^{\prime}}\left(x: s^{\prime}\right) \cdot t^{\prime}: \operatorname{Ty}_{l \sqcup l^{\prime}} @ \operatorname{succ}\left(l \sqcup l^{\prime}\right)$ and $i \in\{p, m\}$, then $L \mid \Psi ; \Gamma \vdash_{i} s \approx s^{\prime}: \mathrm{Ty}_{l} @ \operatorname{succ} l$ and $L \mid \Psi ; \Gamma, x: \mathrm{El}^{l} s @ l \vdash_{i} t \approx t^{\prime}: \mathrm{Ty}_{l^{\prime}} @ \operatorname{succ} l^{\prime}$.
- If $L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} @ l\right) \approx \square\left(\Delta^{\prime} \vdash_{c} @ l\right) @ \operatorname{succ} l$, then $L \mid \Psi \vdash_{p} \Delta \approx \Delta^{\prime}$.
- If $L \mid \Psi ; \Gamma \vdash_{m} \square\left(\Delta \vdash_{c} T @ l\right) \approx \square\left(\Delta^{\prime} \vdash_{c} T @ l\right) @ l$, then $L \mid \Psi \vdash_{p} \Delta \approx \Delta^{\prime}$ and $L \mid \Psi ; \Delta \vdash_{p} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{m}(g: C t x) \Rightarrow^{l} T \approx(g: C t x) \Rightarrow^{l} T^{\prime} @ l$, then $L \mid \Psi, g: C t x ; \Gamma \vdash_{m} T \approx T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{m}\left(U:\left(\Delta \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T \approx\left(U:\left(\Delta^{\prime} \vdash_{p} @ l\right)\right) \Rightarrow^{l^{\prime}} T^{\prime} @ \operatorname{succ} l \sqcup l^{\prime}$, then $L \mid \Psi \vdash_{p} \Delta \approx \Delta^{\prime}$ and $L \mid \Psi, U:\left(\Delta \vdash_{p} @ l\right) ; \Gamma \vdash_{m} T \approx T^{\prime} @ l^{\prime}$.
- If $L \mid \Psi ; \Gamma \vdash_{m} \vec{\ell} \Rightarrow^{l} T \approx \vec{\ell} \Rightarrow^{l} T^{\prime} @ \omega$, then $L, \vec{\ell} \mid \Psi ; \Gamma \vdash_{m} T \approx T^{\prime} @ l$.

Proof. By fundamental theorems, we pass in all identity substitutions and then we can extract this fact from the logical relations of types.

We follow Hu et al. [2023] and give us the consistency proof of DeLaM.
Lemma 8.2 (Consistency). There is no term $t$ that satisfies this typing judgment:

$$
\ell \mid \cdot ; \cdot \vdash_{m} t: \Pi^{\text {succ } \ell, \ell}\left(x: \mathrm{Ty}_{\ell}\right) \cdot x @ \operatorname{succ} \ell
$$

That is, there is not a generic way to construct a term of an arbitrary type.
Proof. The lemma effectively asks to reject the following derivation after applying the fundamental theorems:

$$
\ell \mid \cdot ; x: \mathrm{Ty}_{\ell} @ \operatorname{succ} \ell \vdash_{m} t: x @ \ell
$$

Now by the logical relations of the neutral type $x$, we know that $t$ must also be neutral, so we now move on to rejecting

$$
\ell \mid \cdot ; x: \mathrm{Ty}_{\ell} @ \operatorname{succ} \ell \vdash_{m} v: x @ \ell
$$

Then we do induction on $v$. It is impossible to do any operation on $v$ but to refer to $x$ ultimately, but then $x$ has type $\mathrm{Ty}_{\ell}$ which cannot be equivalent to $x$, as they both have reached normal forms and it is not possible to relate any two distinguished normal forms in any case by the logical relations.

### 8.2 Second Instantiation: Convertibility Checking

In this section, we perform our second instantiation by specifying the generic equivalence to be the convertibility checking judgments. This second instantiation is not very immediate for some laws for the generic equivalence, so we must do some verification, using the results from the first instantiation.

- $L \mid \Psi ; \Gamma \vdash_{i} V \sim V^{\prime} @ l$ is $L \mid \Psi ; \Gamma \vdash_{i} V \longleftrightarrow V^{\prime} @ l$.
- $L \mid \Psi ; \Gamma \vdash_{i} T \simeq T^{\prime} @ l$ is $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime} @ l$.
$\bullet L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T$ @ $l$ is $L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\mu^{\prime}}{\longleftrightarrow} T^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$.
- $L \mid \Psi ; \Gamma \vdash_{i} t \simeq t^{\prime}: T @ l$ is $L \mid \Psi ; \Gamma \vdash_{i} t \Longleftrightarrow t^{\prime}: T @ l$.

Most laws are quite straightforward, except the conversion laws and the PER laws. Let us consider the conversion laws first. We must show that the convertibility checking algorithm is invariant under contexts and types. Intuitively, this should be true, but it is not very easy to prove, until we obtain the injectivity lemma after the first instantiation.

Lemma 8.3 (Conversion).

- if $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} T \Longleftrightarrow T^{\prime} @ l$.
- if $L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} W \Longleftrightarrow W^{\prime} @ l$.
- if $L \mid \Psi ; \Gamma \vdash_{i} V \longleftrightarrow V^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} V \longleftrightarrow V^{\prime} @ l$.
- if $L \mid \Psi \vdash_{i} \Gamma \Longleftrightarrow \Delta$ and $L \vdash \Phi \approx \Psi$, then $L \mid \Phi \vdash_{i} \Gamma \Longleftrightarrow \Delta$.
- if $L\left|\Psi ; \Gamma \vdash_{i} t \stackrel{\wedge}{\Longleftrightarrow} t^{\prime}: T @ l, L \vdash \Phi \approx \Psi, L\right| \Phi \vdash_{i} \Delta \approx \Gamma$ and $L \mid \Phi ; \Delta \vdash_{i} T^{\prime} \approx T$ @ $l$, then $L \mid \Phi ; \Delta \vdash_{i} t \stackrel{\Longleftrightarrow}{\Longleftrightarrow}: T^{\prime}$ @ $l$.
- if $L\left|\Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l, L \vdash \Phi \approx \Psi, L\right| \Phi \vdash_{i} \Delta \approx \Gamma$ and $L \mid \Phi ; \Delta \vdash_{i} W^{\prime} \approx W @ l$, then $L \mid \Phi ; \Delta \vdash_{i} w \approx w^{\prime}: W^{\prime} @ l$;
$\bullet$ if $L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\mu}{ } \mu^{\prime}: T @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} \mu \stackrel{\mu}{\mu}: T^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} \approx T @ l$ for some $T^{\prime}$.
- if $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then $L \mid \Phi ; \Delta \vdash_{i} \mu \approx \mu^{\prime}: W^{\prime} @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} W^{\prime} \approx W$ @ $l$ for some $W^{\prime}$.
$\bullet$ if $L\left|\Psi ; \Gamma \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta, L \vdash \Phi \approx \Psi, L\right| \Phi \vdash_{i} \Gamma^{\prime} \approx \Gamma$ and $L \mid \Phi \vdash_{i} \Delta^{\prime} \approx \Delta$, then $L \mid \Phi ; \Gamma^{\prime} \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta^{\prime}$.
Proof. We do induction on all derivations. Most cases are immediate. When we get under binders, we need to extend the context equivalences, in which case we should use the soundness lemma to obtain the wellformedness of the type that needs to be extended to the contexts.

We consider a few cases.

## Case

$$
\left.\left.\frac{L \mid \Psi ; \Gamma \vdash_{i} t \rightsquigarrow^{*} w: T @ l}{L \mid \Psi ; \Gamma \vdash_{i} t^{\prime} \rightsquigarrow^{*} w^{\prime}: T @ l} \quad L \right\rvert\, \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: W @ l\right]\left(\begin{array}{l}
\text { * }
\end{array}\right.
$$

The most important thing to notice is that by $L \mid \Phi ; \Delta \vdash_{i} T^{\prime} \approx T @ l$ and $T \rightsquigarrow * W$, we know that $T^{\prime} \rightsquigarrow^{*} W^{\prime}$ for some $W^{\prime}$. Therefore we can prove this case using IH. Other premises are satisfied by syntactic context equivalence lemmas.
Case

$$
\begin{gathered}
L\left|\Psi ; \Gamma \vdash_{i} S @ l \quad L\right| \Psi ; \Gamma \vdash_{i} w: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \quad L \mid \Psi ; \Gamma \vdash_{i} w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
L \mid \Psi ; \Gamma, x: S @ l \vdash_{i}\left(w: \Pi^{l, l^{\prime}}(x: S) . T\right) x \Longleftrightarrow\left(w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T\right) x: T @ l^{\prime} \\
L \mid \Psi ; \Gamma \vdash_{i} w \Longleftrightarrow w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}
\end{gathered}
$$

In this case, we know $L \mid \Phi ; \Delta \vdash_{i} W^{\prime} \approx \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}$. By the fundamental theorems, we know that $W^{\prime}$ can only reduce to some $\Pi$ type. Since $W^{\prime}$ is already normal, it is only possible for $W^{\prime}$ to be some $\Pi$ type. Say $W^{\prime}=\Pi^{l, l^{\prime}}\left(x: S^{\prime}\right) \cdot T^{\prime}$. Then by injectivity, we know $S \approx S^{\prime}$ and $T \approx T^{\prime}$. We obtain our goal by extending the local contexts.
Case

$$
\frac{L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l}{L \mid \Psi ; \Gamma \vdash_{i} \mu \Longleftrightarrow \mu^{\prime}: V @ l}
$$

In this case, the type equivalence is irrelevant. When hitting neutral types, we simply ignore the type $W$ inferred by $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l$.
Case

$$
\frac{L \mid \Psi ; \Gamma \vdash_{i} \mu \stackrel{\longleftrightarrow}{\longleftrightarrow} \mu^{\prime}: T @ l \quad T \rightsquigarrow^{*} W}{L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W @ l}
$$

In this case, we simply apply IH. Then we know

$$
T \rightsquigarrow{ }^{*} W \text { and } T^{\prime} \rightsquigarrow^{*} W^{\prime}
$$

Our goal is to show $W \approx W^{\prime}$. But this is immediate from the determinacy lemma of multi-step reduction and the fundamental theorems.
Case

$$
\frac{L \mid \Psi \vdash_{i} \Gamma \quad x: T @ l \in \Gamma}{L \mid \Psi ; \Gamma \vdash_{i} x \stackrel{~}{\longleftrightarrow} x: T @ l}
$$

The goal is given by the equivalence between $\Gamma$ and $\Delta$.
Case

$$
\begin{aligned}
& L+l_{1} \approx l_{3} \text { : Level } \\
& L \vdash l_{2} \approx l_{4} \text { : Level } L\left|\Psi ; \Gamma \vdash_{i} S \Longleftrightarrow S^{\prime} @ l_{1} \quad L\right| \Psi ; \Gamma, x: S @ l_{1} \vdash_{i} T \Longleftrightarrow T^{\prime} @ l_{2} \\
& L\left|\Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S^{\prime \prime}\right) \cdot T^{\prime \prime} @ l_{1} \sqcup l_{2} \quad L\right| \Psi ; \Gamma \vdash_{i} s \Longleftrightarrow s^{\prime}: S @ l_{1} \\
& L \mid \Psi ; \Gamma \vdash_{i}\left(\mu: \Pi^{l_{1}, l_{2}}(x: S) . T\right) s \stackrel{\longleftrightarrow}{\longleftrightarrow}\left(\mu^{\prime}: \Pi^{l_{3}, l_{4}}\left(x: S^{\prime}\right) \cdot T^{\prime}\right) s^{\prime}: T[s / x] @ l_{2}
\end{aligned}
$$

By IH, we know from $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: \Pi^{l_{1}, l_{2}}\left(x: S^{\prime \prime}\right) \cdot T^{\prime \prime} @ l_{1} \sqcup l_{2}$ that there must be $W^{\prime}$, so that

$$
L \mid \Psi ; \Gamma \vdash_{i} W^{\prime} \approx \Pi^{l_{1}, l_{2}}\left(x: S^{\prime \prime}\right) \cdot T^{\prime \prime} @ l_{1} \sqcup l_{2}
$$

By the fundamental theorems, we know $W^{\prime}$ must be some $\Pi$ types. The return type is fixed so we do not have to do anything.

From the conversion lemma, we see that both $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} \mu \longleftrightarrow \mu^{\prime}$ : $W @ l$ return some equivalent types because they are in fact inference steps. Therefore, it makes sense when we instantiate $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T @ l$, we hide a syntactic equivalence judgment inside.

For the PER laws, we see that the difficulties primarily come from the convertibility checking of neutrals, because again they are inference steps. Moreover, due to their left-biased design, in general it is not true that the inferred types can be replaced by their equivalence. However, since we are hiding an equivalence judgment in $L \mid \Psi ; \Gamma \vdash_{i} \mu \sim \mu^{\prime}: T @ l$ during instantiation, following the same principle as the conversion lemma, we are able to erase the effect of the left bias and establish the PER laws.

The instantiation immediately gives us the convertibility lemma once we apply the escape lemma:
Theorem 8.4 (Convertibility).

- If $L \mid \Psi ; \Gamma \vdash_{i} T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} T \approx T^{\prime} @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime} @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \Longleftrightarrow t: T @ l$.
- If $L \mid \Psi ; \Gamma \vdash_{i} t \approx t^{\prime}: T @ l$, then $L \mid \Psi ; \Gamma \vdash_{i} t \stackrel{\wedge}{\Longleftrightarrow}: T @ l$.

In particular, we see that the convertibility checking algorithm is both sound and complete w.r.t. the syntactic equivalence judgments. The decidability of type checking requires the following lemma by attempting to relate two reflexively convertible types or terms.

## Lemma 8.5 (Decidability).

$\bullet$ if $L\left|\Phi ; \Delta \vdash_{i} T \Longleftrightarrow T @ l, L\right| \Psi ; \Gamma \vdash_{i} T^{\prime} \Longleftrightarrow T^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} T \Longleftrightarrow T^{\prime}$ @ $l$ is decidable.

- if $L\left|\Phi ; \Delta \vdash_{i} W \Longleftrightarrow W @ l, L\right| \Psi ; \Gamma \vdash_{i} W^{\prime} \Longleftrightarrow W^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} W \Longleftrightarrow W^{\prime} @ l$ is decidable.
- if $L\left|\Phi ; \Delta \vdash_{i} V \longleftrightarrow V @ l, L\right| \Psi ; \Gamma \vdash_{i} V^{\prime} \longleftrightarrow V^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} V \longleftrightarrow V^{\prime} @ l$ is decidable.
- if $L\left|\Phi \vdash_{i} \Gamma \Longleftrightarrow \Gamma, L\right| \Psi \vdash_{i} \Delta \Longleftrightarrow \Delta$ and $L \vdash \Phi \approx \Psi$, then whether $L \mid \Phi \vdash_{i} \Gamma \Longleftrightarrow \Delta$ is decidable.
$\bullet$ if $L\left|\Phi ; \Delta \vdash_{i} t \Longleftrightarrow t: T @ l, L\right| \Psi ; \Gamma \vdash_{i} t^{\prime} \Longleftrightarrow t^{\prime}: T @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} t \Longleftrightarrow t^{\prime}: T @ l$ is decidable.
- if $L\left|\Phi ; \Delta \vdash_{i} w \Longleftrightarrow w: W @ l, L\right| \Psi ; \Gamma \vdash_{i} w^{\prime} \Longleftrightarrow w^{\prime}: W @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} w \approx w^{\prime}: W$ @ $l$ is decidable.
- if $L\left|\Phi ; \Delta \vdash_{i} \mu \stackrel{\leftrightarrow}{\longleftrightarrow}: T @ l, L\right| \Psi ; \Gamma \vdash_{i} \mu^{\prime} \stackrel{\mu}{\longleftrightarrow} \mu^{\prime}: T^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} \mu \stackrel{\mu}{\longleftrightarrow}: T^{\prime \prime}$ @ l for some $T^{\prime \prime}$ is decidable.
- if $L \mid \Phi ; \Delta \vdash_{i} \mu \longleftrightarrow \mu: W$ @ l, $L \mid \Psi ; \Gamma \vdash_{i} \mu^{\prime} \longleftrightarrow \mu^{\prime}: W^{\prime} @ l, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Delta \approx \Gamma$, then whether $L \mid \Phi ; \Delta \vdash_{i} \mu \approx \mu^{\prime}: W^{\prime \prime} @ l$ is decidable.
- if $L\left|\Phi ; \Gamma^{\prime} \vdash_{i} \delta \Longleftrightarrow \delta: \Delta, L\right| \Psi ; \Gamma \vdash_{i} \delta^{\prime} \Longleftrightarrow \delta^{\prime}: \Delta, L \vdash \Phi \approx \Psi$ and $L \mid \Phi \vdash_{i} \Gamma^{\prime} \approx \Gamma$, then whether $L \mid \Phi ; \Gamma^{\prime} \vdash_{i} \delta \Longleftrightarrow \delta^{\prime}: \Delta$ is decidable.

Proof. We do a mutual induction on the first derivations and then invert the second ones. We can reject most cases when they have different root derivations. We consider a few cases.

Case

$$
\begin{array}{cl}
L \mid \Phi ; \Delta \vdash_{i} T \rightsquigarrow{ }^{*} W @ l \\
L \mid \Psi ; \Gamma \vdash_{i} W \Longleftrightarrow W @ l & \Longleftrightarrow T @ l
\end{array} \quad \begin{gathered}
L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} \rightsquigarrow^{*} W^{\prime} @ l \\
L \mid \Phi ; \Delta \vdash_{i} T \Longleftrightarrow W^{\prime} @ l
\end{gathered}
$$

In this case, we apply IH to decide whether $W$ and $W^{\prime}$ are convertible.
Case

$$
\begin{gathered}
\frac{L\left|\Phi ; \Delta \vdash_{i} S \Longleftrightarrow S @ l \quad L\right| \Phi ; \Delta, x: S @ l \vdash_{i} T \Longleftrightarrow T @ l^{\prime}}{L \mid \Phi ; \Delta \vdash_{i} \Pi^{l, l^{\prime}}(x: S) \cdot T \Longleftrightarrow \Pi^{l, l^{\prime}}(x: S) \cdot T @ l \sqcup l^{\prime}} \\
\frac{L\left|\Psi ; \Gamma \vdash_{i} S^{\prime} \Longleftrightarrow S^{\prime} @ l \quad L\right| \Psi ; \Gamma, x: S^{\prime} @ l \vdash_{i} T^{\prime} \Longleftrightarrow T^{\prime} @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} \Pi^{l, l^{\prime}}\left(x: S^{\prime}\right) \cdot T^{\prime} \Longleftrightarrow \Pi^{l, l^{\prime}}\left(x: S^{\prime}\right) \cdot T^{\prime} @ l \sqcup l^{\prime}}
\end{gathered}
$$

We can decide whether $S$ and $S^{\prime}$ are convertible by IH . When we decide $T$ and $T^{\prime}$, we must extend the equivalent local contexts.

Case

$$
\begin{array}{cc}
T \rightsquigarrow{ }^{*} W \quad L \mid \Phi ; \Delta \vdash_{i} t \rightsquigarrow{ }^{*} w: T @ l & \\
L \mid \Phi ; \Delta \vdash_{i} w \Longleftrightarrow w: W @ l & \\
L \mid \Phi ; \Delta \vdash_{i} t \Longleftrightarrow t: T @ l &
\end{array}
$$

By fundamental theorems, we know $W \approx W^{\prime}$. Then by IH , we can decide whether $w$ and $w^{\prime}$ are convertible. Case

$$
\begin{gathered}
L\left|\Phi ; \Delta \vdash_{i} S @ l \quad L\right| \Phi ; \Delta \vdash_{i} w: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
\frac{L \mid \Phi ; \Delta, x: S @ l \vdash_{i}\left(w: \Pi^{l, l^{\prime}}(x: S) . T\right) x \Longleftrightarrow\left(w: \Pi^{l, l^{\prime}}(x: S) . T\right) x: T @ l^{\prime}}{L \mid \Phi ; \Delta \vdash_{i} w \Longleftrightarrow w: \Pi^{l, l^{\prime}}(x: S) \cdot T @ l \sqcup l^{\prime}} \\
L\left|\Psi ; \Gamma \vdash_{i} S @ l \quad L\right| \Psi ; \Gamma \vdash_{i} w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime} \\
\frac{L \mid \Psi ; \Gamma, x: S @ l \vdash_{i}\left(w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T\right) x \stackrel{ }{\Longleftrightarrow}\left(w^{\prime}: \Pi^{l, l^{\prime}}(x: S) \cdot T\right) x: T @ l^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} w^{\prime} \Longleftrightarrow w^{\prime}: \Pi^{l, l^{\prime}}(x: S) . T @ l \sqcup l^{\prime}}
\end{gathered}
$$

Again, it is quite immediate by IH .
Case

$$
\frac{L \mid \Phi ; \Delta \vdash_{i} \mu \longleftrightarrow \mu: W @ l}{L \mid \Phi ; \Delta \vdash_{i} \mu \Longleftrightarrow \mu: V @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} \mu^{\prime} \longleftrightarrow \mu^{\prime}: W^{\prime} @ l}{L \mid \Psi ; \Gamma \vdash_{i} \mu^{\prime} \Longleftrightarrow \mu^{\prime}: V @ l}
$$

By IH, we know $L \mid \Phi ; \Delta \vdash_{i} \mu \longleftrightarrow \mu^{\prime}: W^{\prime \prime} @ l$ for some $W^{\prime \prime}$.
Case

$$
\frac{L \mid \Phi ; \Delta \vdash_{i} \mu \stackrel{\longleftrightarrow}{\longleftrightarrow} \cdot T @ l \quad T \rightsquigarrow^{*} W}{L \mid \Phi ; \Delta \vdash_{i} \mu \longleftrightarrow \mu: W @ l} \quad \frac{L \mid \Psi ; \Gamma \vdash_{i} \mu^{\prime} \stackrel{\mu^{\prime}}{\longleftrightarrow} T^{\prime} @ l \quad T^{\prime} \rightsquigarrow^{*} W^{\prime}}{L \mid \Psi ; \Gamma \vdash_{i} \mu^{\prime} \longleftrightarrow \mu^{\prime}: W^{\prime} @ l}
$$

By IH, we have $L \mid \Psi ; \Gamma \vdash_{i} \mu^{\prime} \stackrel{ }{\longleftrightarrow} \mu^{\prime}: T^{\prime \prime} @ l$ for some $T^{\prime \prime}$. $T^{\prime \prime}$ will reduce to some normal form by the fundamental theorems.

Theorem 8.6 (Decidability of Convertibility).

- If $L \mid \Psi ; \Gamma \vdash_{i} T$ @ l and $L \mid \Psi ; \Gamma \vdash_{i} T^{\prime} @ l$, then whether $L \mid \Psi ; \Gamma \vdash_{i} T \Longleftrightarrow T^{\prime}$ @l is decidable.
- If $L \mid \Psi ; \Gamma \vdash_{i} t: T @ l$ and $L \mid \Psi ; \Gamma \vdash_{i} t^{\prime}: T @ l$, then whether $L \mid \Psi ; \Gamma \vdash_{i} t \stackrel{t^{\prime}}{ }: T$ @ $l$ is decidable.

Proof. First we use the fundamental theorems from the second instantiation to show that both types (or terms, resp.) are reflexively convertible. Then we use the decidability lemma above.

At this point, we have justified the decidability of convertibility checking of DeLAM and therefore conclude our investigations.

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