From the Lyndon factorization to the Canonical Inverse Lyndon factorization: back and forth

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Abstract

The notion of inverse Lyndon word is related to the classical notion of Lyndon word. More precisely, inverse Lyndon words are all and only the nonempty prefixes of the powers of the anti-Lyndon words, where an anti-Lyndon word with respect to a lexicographical order is a classical Lyndon word with respect to the inverse lexicographic order. Each word w admits a factorization in inverse Lyndon words, named the canonical inverse Lyndon factorization $\mathrm{ICFL}(w)$, which maintains the main properties of the Lyndon factorization of w. Although there is a huge literature on the Lyndon factorization, the relation between the Lyndon factorization CFL_{in} with respect to the inverse order and the canonical inverse Lyndon factorization ICFL has not been thoroughly investigated. In this paper, we address this question and we show how to obtain one factorization from the other via the notion of grouping, defined in [1]. This result naturally opens new insights in the investigation of the relationship between ICFL and other notions, e.g., variants of Burrows Wheeler Transform, as already done for the Lyndon factorization.

***K*eywords** Lyndon words · Lyndon factorization · Combinatorial algorithms on words

1 Introduction

A word w over a totally ordered alphabet Σ is a *Lyndon word* if for each nontrivial factorization $w = uv$, w is strictly smaller than vu for the lexicographical ordering. A well-known theorem of Lyndon asserts that any nonempty word factorizes uniquely into a nonincreasing product of Lyndon words, called its Lyndon factorization [2]. It can be efficiently computed. Linear-time algorithms for computing this factorization can be found in [3] whereas an $\mathcal{O}(\lg n)$-time parallel algorithm has been proposed in [4]. There are several results which give relations between Lyndon words, codes and combinatorics of words and there are many algorithmic applications of the Lyndon factorization [5]. For instance, bijective versions of the classical Burrows Wheeler Transform have been defined in [6, 7, 8] and they are based on combinatorial results on Lyndon words proved in [9] (see [10] for more recent related results). As another example, a main property proved in [11] is the *Compatibility Property*. Roughly, the compatibility property allows us to extend the mutual order between suffixes of products of the Lyndon factors to the suffixes of the whole word. Similar ideas have been further investigated to accelerate suffix sorting in practice [12].

More recently Lyndon words found a renewed theoretical interest and several variants of them have been studied [13, 14, 15, 16, 17]. In particular, inverse Lyndon words have been introduced in [1]. More precisely, an anti-Lyndon word with respect to a lexicographical order is a classical Lyndon word with respect to the inverse lexicographic order.

Then, inverse Lyndon words are all and only the nonempty prefixes of the powers of the anti-Lyndon words. Each word w admits a factorization in inverse Lyndon words, named the canonical inverse Lyndon factorization $\mathrm{ICFL}(w)$, which maintains the main properties of the Lyndon factorization of w: it is uniquely determined and can be computed in linear time. In addition, it maintains a similar Compatibility Property. Other combinatorial properties of $\mathrm{ICFL}(w)$ have been proved in [18].

In this paper we address a main theoretical question left open in [1]. More precisely, we have proved in [1] that the canonical inverse Lyndon factorization $\mathrm{ICFL}(w)$ of w is a grouping of the Lyndon factorization $\mathrm{CFL}_{in}(w)$ of w with respect to the inverse order. Roughly, this means that every element in $\mathrm{ICFL}(w)$ is the concatenation of factors of $\mathrm{CFL}_{in}(w)$ that are related by the prefix order. The proof of this result is not constructive and leaves open the problem of constructing this grouping, that is, how to obtain ICFL from CFL_{in}. Moreover, another open problem is how to obtain CFL_{in} from ICFL. The main result of this paper is to solve these two problems, by providing a full characterization of the relationship between CFL_{in} and ICFL via grouping.

In detail, we know that every element in $\mathrm{ICFL}(w)$ is the concatenation of factors of $\mathrm{CFL}_{in}(w)$ that are related by the prefix order. Then, in Proposition 9.1 we prove that $\mathrm{ICFL}(w)$ can be computed locally in the non-increasing maximal chains for the prefix order of $\mathrm{CFL}_{in}(w)$. There is a pair of words, the canonical pair associated with w, which is crucial in the construction of $\mathrm{ICFL}(w)$ (see Section 6 for the definition). In Proposition 9.8 we show how to determine such a pair in one of the aforementioned chains.

The procedure for obtaining $\mathrm{CFL}_{in}(w)$ from $\mathrm{ICFL}(w)$ is very simple. We know that every element in $\mathrm{ICFL}(w)$ is the concatenation of factors of $\mathrm{CFL}_{in}(w)$ that are related by the prefix order. Therefore, we first demonstrate that we can limit ourselves to handling the single factor m_i in $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$ (Corollary 8.1). Next, we show that each bordered word has a unique nonempty border which is unbordered (Proposition 8.2). This combinatorial property allows us to recursively construct a sequence of words uniquely associated with each m_i which will turn out to be $\mathrm{CFL}_{in}(m_i)$ (Definition 8.1, Corollary 8.2).

Propositions 5.2 and 7.2 were proved in [1] and [18], respectively. Both make use of Proposition 3.2 which was stated in a slightly incorrect form in [1, 18]. After having correctly stated Proposition 3.2, as minor results we demonstrate Propositions 5.2 and 7.2 again.

The paper is organized as follows. In Sections 2, 3, 4, 5, 6, 7, we gathered the basic definitions and known results we need. We show how to obtain the Lyndon factorization $\mathrm{CFL}_{in}(w)$ of w with respect to the inverse order from the canonical inverse Lyndon factorization $\mathrm{ICFL}(w)$ in Section 8 and we illustrate how to group factors of $\mathrm{CFL}_{in}(w)$ to obtain $\mathrm{ICFL}(w)$ in Section 9. We conclude in Section 10 with some main issues that follow on from the results proved in the paper.

2 Words

Throughout this paper we follow [19, 20, 21, 22, 23] for the notations. We denote by Σ^* the *free monoid* generated by a finite alphabet Σ and we set $\Sigma^+ = \Sigma^* \setminus 1$, where 1 is the empty word. For a word $w \in \Sigma^*$, we denote by $|w|$ its *length*. A word $x \in \Sigma^*$ is a *factor* of $w \in \Sigma^*$ if there are $u_1, u_2 \in \Sigma^*$ such that $w = u_1 x u_2$. If $u_1 = 1$ (resp. $u_2 = 1$), then x is a *prefix* (resp. *suffix*) of w. A factor (resp. prefix, suffix) x of w is *proper* if $x \neq w$. Two words x, y are *incomparable* for the prefix order, denoted as $x \bowtie y$, if neither x is a prefix of y nor y is a prefix of x. Otherwise, x, y are *comparable* for the prefix order. We write $x \leq_p y$ if x is a prefix of y and $x \geq_p y$ if y is a prefix of x. The notion of a pair of words comparable (or incomparable) for the suffix order is defined symmetrically.

We recall that, given a nonempty word w, a *border* of w is a word which is both a proper prefix and a suffix of w [24]. The longest proper prefix of w which is a suffix of w is also called *the border* of w [24, 22]. A word $w \in \Sigma^+$ is *bordered* if it has a nonempty border. Otherwise, w is *unbordered*. A nonempty word w is *primitive* if $w = x^k$ implies $k = 1$. An unbordered word is primitive. A *sesquipower* of a word x is a word $w = x^n p$ where p is a proper prefix of x and $n \geq 1$.

Two words x, y are called *conjugate* if there exist words u, v such that $x = uv, y = vu$. The conjugacy relation is an equivalence relation. A conjugacy class is a class of this equivalence relation.

Definition 2.1 *Let $(\Sigma, <)$ be a totally ordered alphabet. The lexicographic (or alphabetic order) $\prec$ on $(\Sigma^*, <)$ is defined by setting $x \prec y$ if*

- *x is a proper prefix of y, or*
- *$x = ras$, $y = rbt$, $a < b$, for $a, b \in \Sigma$ and $r, s, t \in \Sigma^*$.*

In the next part of the paper we will implicitly refer to totally ordered alphabets. For two nonempty words x, y, we write $x \ll y$ if $x \prec y$ and x is not a proper prefix of y [25]. We also write $y \succ x$ if $x \prec y$. Basic properties of the lexicographic order are recalled below.

Lemma 2.1 *For $x, y \in \Sigma^*$, the following properties hold.*

(1) $x \prec y$ if and only if $zx \prec zy$, for every word z.

(2) If $x \ll y$, then $xu \ll yv$ for all words u, v.

(3) If $x \prec y \prec xz$ for a word z, then $y = xy'$ for some word y' such that $y' \prec z$.

Let $\mathcal{S}_1, \ldots, \mathcal{S}_t$ be sequences, with $\mathcal{S}_j = (s_{j,1}, \ldots, s_{j,r_j})$. For abbreviation, we let

$$(\mathcal{S}_1, \ldots, \mathcal{S}_t)$$

stand for the sequence

$$(s_{1,1}, \ldots, s_{1,r_1}, \ldots, s_{t,1}, \ldots, s_{t,r_t})$$

3 Lyndon words

Definition 3.1 *A Lyndon word $w \in \Sigma^+$ is a word which is primitive and the smallest one in its conjugacy class for the lexicographic order.*

Example 3.1 Let $\Sigma = \{a, b\}$ with $a < b$. The words a, b, $aaab$, $abbb$, $aabab$ and $aababaabb$ are Lyndon words. On the contrary, $abab$, aba and $abaab$ are not Lyndon words. Indeed, $abab$ is a non-primitive word, $aab \prec aba$ and $aabab \prec abaab$.

Lyndon words are also called *prime words* and their prefixes are also called *preprime words* in [26].

Proposition 3.1 *Each Lyndon word w is unbordered.*

A class of conjugacy is also called a *necklace* and often identified with the minimal word for the lexicographic order in it. We will adopt this terminology. Then a word is a necklace if and only if it is a power of a Lyndon word. A *prenecklace* is a prefix of a necklace. Then clearly any nonempty prenecklace w has the form $w = (uv)^k u$, where uv is a Lyndon word, $u \in \Sigma^*$, $v \in \Sigma^+$, $k \geq 1$, that is, w is a sesquipower of a Lyndon word uv. The following result has been proved in [3]. It shows that the nonempty prefixes of Lyndon words are exactly the nonempty prefixes of the powers of Lyndon words with the exclusion of c^k, where c is the maximal letter and $k \geq 2$.

Proposition 3.2 *A word is a nonempty preprime word if and only if it is a sesquipower of a Lyndon word distinct of c^k, where c is the maximal letter and $k \geq 2$.*

The proof of Proposition 3.2 uses the following result which characterizes, for a given nonempty prenecklace w and a letter b, whether wb is still a prenecklace or not and, in the first case, whether wb is a Lyndon word or not [3, Lemma 1.6].

Theorem 3.1 *Let $w = (uav')^k u$ be a nonempty prenecklace, where uav' is a Lyndon word, $u, v' \in \Sigma^*$, $a \in \Sigma$, $k \geq 1$. For any $b \in \Sigma$, the word wb is a prenecklace if and only if $b \geq a$. Moreover $wb \in L$ if and only if $b > a$.*

4 Lyndon factorization

In the following $L = L_{(\Sigma^*, <)}$ will be the set of Lyndon words, totally ordered by the relation $\prec$ on $(\Sigma^*, <)$.

Theorem 4.1 *Any word $w \in \Sigma^+$ can be written in a unique way as a nonincreasing product $w = \ell_1 \ell_2 \cdots \ell_h$ of Lyndon words, i.e., in the form*

$$w = \ell_1 \ell_2 \cdots \ell_h, \text{ with } \ell_j \in L \text{ and } \ell_1 \succeq \ell_2 \succeq \ldots \succeq \ell_h \tag{4.1}$$

The sequence $\mathrm{CFL}(w) = (\ell_1, \ldots, \ell_h)$ in Eq. (4.1) is called the *Lyndon decomposition* (or *Lyndon factorization*) of w. It is denoted by $\mathrm{CFL}(w)$ because Theorem 4.1 is usually credited to Chen, Fox and Lyndon [2]. Uniqueness of the above factorization can be obtained as a consequence of the following result, proved in [3].

Lemma 4.1 *Let $w \in \Sigma^+$ and let $\mathrm{CFL}(w) = (\ell_1, \ldots, \ell_h)$. Then the following properties hold:*

(i) ℓ_h is the nonempty suffix of w which is the smallest with respect to the lexicographic order.

(ii) ℓ_h is the longest suffix of w which is a Lyndon word.

(iii) ℓ_1 is the longest prefix of w which is a Lyndon word.

A direct consequence is stated below and it is necessary for our aims.

Corollary 4.1 *Let $w \in \Sigma^+$, let ℓ_1 be its longest prefix which is a Lyndon word and let w' be such that $w = \ell_1 w'$. If $w' \neq 1$, then $\mathrm{CFL}(w) = (\ell_1, \mathrm{CFL}(w'))$.*

Sometimes we need to emphasize consecutive equal factors in CFL. We write $\mathrm{CFL}(w) = (\ell_1^{n_1}, \ldots, \ell_r^{n_r})$ to denote a tuple of $n_1 + \ldots + n_r$ Lyndon words, where $r > 0$, $n_1, \ldots, n_r \geq 1$. Precisely $\ell_1 \succ \ldots \succ \ell_r$ are Lyndon words, also named *Lyndon factors* of w. There is a linear time algorithm to compute the pair (ℓ_1, n_1) and thus, by iteration, the Lyndon factorization of w [22, 27]. Linear time algorithms may also be found in [3] and in the more recent paper [28].

5 Inverse Lyndon words

For the material in this section see [1, 29, 18]. Inverse Lyndon words are related to the inverse alphabetic order. Its definition is recalled below.

Definition 5.1 *Let $(\Sigma, <)$ be a totally ordered alphabet. The* inverse *$<_{in}$ of $<$ is defined by*

$$\forall a, b \in \Sigma \quad b <_{in} a \Leftrightarrow a < b$$

The inverse lexicographic *or* inverse alphabetic order *on $(\Sigma^*, <)$, denoted $\prec_{in}$, is the lexicographic order on $(\Sigma^*, <_{in})$.*

Example 5.1 Let $\Sigma = \{a, b, c, d\}$ with $a < b < c < d$. Then $dab \prec dabd$ and $dabda \prec dac$. We have $d <_{in} c <_{in} b <_{in} a$. Therefore $dab \prec_{in} dabd$ and $dac \prec_{in} dabda$.

Of course for all $x, y \in \Sigma^*$ such that $x \bowtie y$,

$$y \prec_{in} x \Leftrightarrow x \prec y.$$

Moreover, in this case $x \ll y$. This justifies the adopted terminology.

From now on, $L_{in} = L_{(\Sigma^*, <_{in})}$ denotes the set of the Lyndon words on Σ^* with respect to the inverse lexicographic order. Following [30], a word $w \in L_{in}$ will be named an *anti-Lyndon word*. Correspondingly, an *anti-prenecklace* will be a prefix of an *anti-necklace*, which in turn will be a necklace with respect to the inverse lexicographic order.

In the following, we denote by $\mathrm{CFL}_{in}(w)$ the Lyndon factorization of w with respect to the inverse order $<_{in}$.

Definition 5.2 *A word $w \in \Sigma^+$ is an inverse Lyndon word if $s \prec w$, for each nonempty proper suffix s of w.*

Example 5.2 The words a, b, $aaaaa$, $bbba$, $baaab$, $bbaba$ and $bbababbaa$ are inverse Lyndon words on $\{a, b\}$, with $a < b$. On the contrary, $aaba$ is not an inverse Lyndon word since $aaba \prec ba$. Analogously, $aabba \prec ba$ and thus $aabba$ is not an inverse Lyndon word.

The following result has been stated in [18].

Proposition 5.1 *A word $w \in \Sigma^+$ is an anti-Lyndon word if and only if it is an unbordered inverse Lyndon word.*

The following results have been proved in [1].

Lemma 5.1 *Any nonempty prefix of an inverse Lyndon word is an inverse Lyndon word.*

The proof of Proposition 5.2 was given in [1]. The reason we report it here is that this proof uses the statement in Proposition 3.2 but in [1] that statement was slightly incorrect. However, the proof still holds.

Proposition 5.2 *A word $w \in \Sigma^+$ is an inverse Lyndon word if and only if w is a nonempty anti-prenecklace.*

PROOF :

Let $w \in \Sigma^+$ be an inverse Lyndon word. The first letter of w is an anti-Lyndon word thus also a nonempty anti-prenecklace. Let p be the longest nonempty prefix of w which is an anti-prenecklace. By Theorem 3.1, if p were distinct from w, then we would have $p = (uav')^k u$, $w = (uav')^k ubt$, where $uav' \in L_{in}$, $u, v', t \in \Sigma^*$, $a, b \in \Sigma$, $a < b$, $k \geq 1$. Thus, $w \ll ubt$, in contradiction with Definition 5.2. Therefore, w is a nonempty anti-prenecklace. Conversely, let w be a nonempty anti-prenecklace, that is, a sesquipower of an anti-Lyndon word. If $w = a^n$, where a is the minimal letter in Σ and $n \geq 2$, then clearly w is an inverse Lyndon word. Otherwise, by Proposition 3.2, there is a word t such that $wt \in L_{in}$. By Proposition 5.1, wt is an inverse Lyndon word. If there existed a nonempty proper suffix s of w such that $w \prec s$, we clearly would have $w \ll s$. Hence, by item (2) in Lemma 2.1, $wt \ll st$, where st is a nonempty proper suffix of wt. This is in contradiction with Definition 5.2, thus w is an inverse Lyndon word. ■

6 Inverse Lyndon factorizations

For the material in this section see [1, 29, 18]. An inverse Lyndon factorization of a word $w \in \Sigma^+$ is a sequence $(m_1, \ldots, m_k)$ of inverse Lyndon words such that $m_1 \cdots m_k = w$ and $m_i \ll m_{i+1}$, $1 \leq i \leq k-1$. A word may have different inverse Lyndon factorizations (see Example 6.2) but it has a unique canonical inverse Lyndon factorization, denoted $\mathrm{ICFL}(w)$. If w is an inverse Lyndon word, then $\mathrm{ICFL}(w) = w$. Otherwise, $\mathrm{ICFL}(w)$ is recursively defined. The first factor of $\mathrm{ICFL}(w)$ is obtained by a special factorization of the shortest nonempty prefix z of w such that z is not an inverse Lyndon word defined below.

Definition 6.1 *[1] Let $w \in \Sigma^+$, let p be an inverse Lyndon word which is a nonempty proper prefix of $w = pv$. The* bounded right extension *$\overline{p}$ of p (relatively to w), if it exists, is a nonempty prefix of v such that:*

(1) $\overline{p}$ is an inverse Lyndon word,

(2) pz' is an inverse Lyndon word, for each proper nonempty prefix z' of $\overline{p}$,

(3) $p\overline{p}$ is not an inverse Lyndon word,

(4) $p \ll \overline{p}$.

Moreover, we set $\mathrm{Pref}_{bre}(w) = \{(p, \overline{p}) \mid p$ is an inverse Lyndon word which is a nonempty proper prefix of $w\}$.

It has been proved that $\mathrm{Pref}_{bre}(w)$ is empty if and only if w is an inverse Lyndon word (Proposition 4.2 in [1]). If w is not an inverse Lyndon word, then $\mathrm{Pref}_{bre}(w)$ contains only one pair and the description of this pair is given below (Propositions 4.1 and 4.3 in [1]). In the next, the unique pair $(p, \overline{p})$ in $\mathrm{Pref}_{bre}(w)$ will be named *the canonical pair* associated with w.

Proposition 6.1 *Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let z be the shortest nonempty prefix of w which is not an inverse Lyndon word. Then,*

(1) $z = p\overline{p}$, with $(p, \overline{p}) \in \mathrm{Pref}_{bre}(w)$.

(2) $p = ras$ and $\overline{p} = rb$, where $r, s \in \Sigma^*$, $a, b \in \Sigma$ and r is the shortest prefix of $p\overline{p}$ such that $p\overline{p} = rasrb$, with $a < b$.

Example 6.1 Let $\Sigma = \{a, b\}$ with $a < b$. Let us consider $w = babaaabb$ and the prefixes $p_1 = bab$ and $p_2 = babaaa$ of w. First, w is not an inverse Lyndon word. Thus, $\mathrm{Pref}_{bre}(w)$ contains only one pair. Moreover each proper nonempty prefix of w is an inverse Lyndon word. By item (1) in Proposition 6.1, we have $w = p\overline{p}$. By item (2) in Proposition 6.1, the bounded right extension of $p_1 = bab$ does not exist (we should have $\overline{p_1} = aaabb$ in contradiction with $p_1 \ll \overline{p_1}$). Since w starts with b, the shortest common prefix r of p and $\overline{p}$ has a positive length. Indeed, $p = p_2 = babaaa$ and $\overline{p} = \overline{p_2} = bb$.

The above results suggest the following characterization of the canonical pair $(p, \overline{p})$ associated with w (Proposition 6.2 in [18]).

Proposition 6.2 *Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. A pair of words $(p, \overline{p})$ is the canonical pair associated with w if and only the following conditions are satisfied.*

(1) $z = p\overline{p}$ is the shortest nonempty prefix of w which is not an inverse Lyndon word.

(2) $p = ras$ and $\overline{p} = rb$, where $r, s \in \Sigma^*$, $a, b \in \Sigma$ and r is the shortest prefix of $p\overline{p}$ such that $p\overline{p} = rasrb$, with $a < b$.

(3) $\overline{p}$ is an inverse Lyndon word.

Given a word w which is not an inverse Lyndon word, Proposition 6.2 suggests a method to identify the canonical pair $(p, \overline{p})$ associated with w: just find the shortest nonempty prefix z of w which is not an inverse Lyndon word and then a factorization $z = p\overline{p}$ such that conditions (2) and (3) in Proposition 6.2 are satisfied.

The canonical inverse Lyndon factorization has been also recursively defined.

Definition 6.2 *Let $w \in \Sigma^+$.*
(Basis Step) *If w is an inverse Lyndon word, then* $\mathrm{ICFL}(w) = (w)$.
(Recursive Step) *If w is not an inverse Lyndon word, let $(p, \overline{p})$ be the canonical pair associated with w and let $v \in \Sigma^*$ such that $w = pv$. Let $\mathrm{ICFL}(v) = (m'_1, \ldots, m'_k)$ and let $r, s \in \Sigma^*$, $a, b \in \Sigma$ such that $p = ras$, $\overline{p} = rb$ with $a < b$.*

$$\mathrm{ICFL}(w) = \begin{cases} (p, \mathrm{ICFL}(v)) & \text{if } \overline{p} = rb \leq_p m'_1 \\ (pm'_1, m'_2, \ldots, m'_k) & \text{if } m'_1 \leq_p r \end{cases}$$

Example 6.2 *[1].* Let $\Sigma = \{a, b, c, d\}$ with $a < b < c < d$, $w = dabadabdabdadac$. We have $\mathrm{CFL}_{in}(w) = (daba, dab, dab, dadac)$, $\mathrm{ICFL}(w) = (daba, dabdab, dadac)$. Another inverse Lyndon factorizations of w is $(dabadab, dabda, dac)$. Consider $z = dabdadacddbdc$. It is easy to see that $(dab, dadacd, db, dc)$, $(dabda, dac, ddbdc)$, $(dab, dadac, ddbdc)$ are all inverse Lyndon factorizations of z. The first factorization has four factors whereas the others have three factors. Moreover $\mathrm{ICFL}(z) = \mathrm{CFL}_{in}(z) = (dab, dadac, ddbdc)$.

7 Groupings

In this section we recall a special property of ICFL proved in [1], needed here for our aims.

Let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$, where $\ell_1 \succeq_{in} \ell_2 \succeq_{in} \ldots \succeq_{in} \ell_h$. Consider the partial order $\geq_p$, where $x \geq_p y$ if y is a prefix of x. Recall that a *chain* is a set of a pairwise comparable elements. We say that a chain is maximal if it is not strictly contained in any other chain. A non-increasing *(maximal) chain* in $\mathrm{CFL}_{in}(w)$ is the sequence corresponding to a (maximal) chain in the multiset $\{\ell_1, \ldots, \ell_h\}$ with respect to $\geq_p$. We denote by $\mathcal{PMC}$ a non-increasing maximal chain in $\mathrm{CFL}_{in}(w)$. Looking at the definition of the (inverse) lexicographic order, it is easy to see that a $\mathcal{PMC}$ is a sequence of consecutive factors in $\mathrm{CFL}_{in}(w)$. Moreover $\mathrm{CFL}_{in}(w)$ is the concatenation of its $\mathcal{PMC}$. The formal definitions are given below.

Definition 7.1 *Let $w \in \Sigma^+$, let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$ and let $1 \leq r < s \leq h$. We say that $\ell_r, \ell_{r+1}, \ldots, \ell_s$ is a non-increasing* maximal chain for the prefix order *in $\mathrm{CFL}_{in}(w)$, abbreviated $\mathcal{PMC}$, if $\ell_r \geq_p \ell_{r+1} \geq_p \ldots \geq_p \ell_s$. Moreover, if $r > 1$, then $\ell_{r-1} \not\geq_p \ell_r$, if $s < h$, then $\ell_s \not\geq_p \ell_{s+1}$. Two $\mathcal{PMC}$ $\mathcal{C}_1 = \ell_r, \ell_{r+1}, \ldots, \ell_s$, $\mathcal{C}_2 = \ell_{r'}, \ell_{r'+1}, \ldots, \ell_{s'}$ are* consecutive *if $r' = s + 1$ (or $r = s' + 1$).*

Definition 7.2 *Let $w \in \Sigma^+$, let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$. We say that $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ is* the decomposition *of $\mathrm{CFL}_{in}(w)$ into its non-increasing maximal chains for the prefix order if the following holds*

(1) Each $\mathcal{C}_j$ is a non-increasing maximal chain in $\mathrm{CFL}_{in}(w)$.

(2) $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ are consecutive, $1 \leq j \leq s - 1$.

(3) $\mathrm{CFL}_{in}(w)$ is the concatenation of the sequences $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s$.

Example 7.1 *[1].* Let $\Sigma = \{a, b, c, d\}$ with $a < b < c < d$, $w = dabadabdabdadac$. In Example 6.2, we observed that $\mathrm{CFL}_{in}(w) = (daba, dab, dab, dadac)$. This sequence has two $\mathcal{PMC}$, namely $(daba, dab, dab)$, $(dadac)$. The decomposition of $\mathrm{CFL}_{in}(w)$ into its $\mathcal{PMC}$ is $((daba, dab, dab), (dadac))$. Let $z = dabdadacddbdc$. Then $\mathrm{CFL}_{in}(z) = (dab, dadac, ddbdc)$ has three $\mathcal{PMC}$: (dab), $(dadac)$, $(ddbdc)$. The decomposition of $\mathrm{CFL}_{in}(w)$ into its $\mathcal{PMC}$ is $((dab), (dadac), (ddbdc))$.

A *grouping* of $\mathrm{CFL}_{in}(w)$ is an inverse Lyndon factorization $(m_1, \ldots, m_k)$ of w such that any factor is a product of consecutive factors in a $\mathcal{PMC}$ of $\mathrm{CFL}_{in}(w)$. Formally, the definition of a grouping of $\mathrm{CFL}_{in}(w)$ is given below in two steps. We first define the grouping of a $\mathcal{PMC}$. Then a grouping of $\mathrm{CFL}_{in}(w)$ is obtained by changing each $\mathcal{PMC}$ with one of its groupings.

Definition 7.3 *Let $\ell_1, \ldots, \ell_h$ be words in L_{in} such that ℓ_i is a prefix of ℓ_{i-1}, $1 < i \leq h$. We say that $(m_1, \ldots, m_k)$ is a* grouping *of $(\ell_1, \ldots, \ell_h)$ if the following conditions are satisfied.*

(1) m_j is an inverse Lyndon word,

(2) $\ell_1 \cdots \ell_h = m_1 \cdots m_k$. More precisely, there are $i_0, i_1, \ldots, i_k$, $i_0 = 0$, $1 \leq i_j \leq h$, $i_k = h$, such that $m_j = \ell_{i_{j-1}+1} \cdots \ell_{i_j}$, $1 \leq j \leq k$,

(3) $m_1 \ll \ldots \ll m_k$.

We now extend Definition 7.3 to $\mathrm{CFL}_{in}(w)$.

Definition 7.4 *Let $w \in \Sigma^+$ and let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$. We say that $(m_1, \ldots, m_k)$ is a* grouping *of $\mathrm{CFL}_{in}(w)$ if it can be obtained by replacing any $\mathcal{PMC}$ $\mathcal{C}$ in $\mathrm{CFL}_{in}(w)$ by a grouping of $\mathcal{C}$.*

Groupings of $\mathrm{CFL}_{in}(w)$ are inverse Lyndon factorizations of w but there are inverse Lyndon factorizations which are not groupings. Surprisingly enough, $\mathrm{ICFL}(w)$ is a grouping of $\mathrm{CFL}_{in}(w)$ but it is not always its unique grouping.

Proposition 7.1 *[1] Let $(\Sigma, <)$ be a totally ordered alphabet. For any $w \in \Sigma^+$, $\mathrm{ICFL}(w)$ is a grouping of $\mathrm{CFL}_{in}(w)$.*

Example 7.2 *[1].* Let $\Sigma = \{a, b, c, d\}$, $a < b < c < d$, and $w = dabadabdabdadac$. We have $\mathrm{CFL}_{in}(w) = (daba, dab, dab, dadac)$, $\mathrm{ICFL}(w) = (daba, dabdab, dadac)$ (see Example 6.2). $\mathrm{ICFL}(w)$ is a grouping of $\mathrm{CFL}_{in}(w)$ but $(dabadab, dabda, dac)$ is not a grouping. Next, let $y = dabadabdabdabdadac$. We have $\mathrm{CFL}_{in}(y) = (daba, dab, dab, dab, dadac)$ and $\mathrm{ICFL}(y) = (daba, (dab)^3, dadac)$. The inverse Lyndon factorization $(dabadab, (dab)^2, dadac)$ is another grouping of $\mathrm{CFL}_{in}(y)$.

Lemma 7.1 has been proved in [18]. It is needed in the proof of Proposition 7.2. In turn, Proposition 7.2 shows that the word p in the pair $(p, \bar{p}) \in \mathrm{Pref}_{bre}(w)$ has a grouping-like property. Again, the proof of Proposition 7.2 was given in [18] but in [18] uses a slightly incorrect statement of Proposition 3.2. However, the above-mentioned proof, also given below, still holds because Proposition 3.2 is in fact unnecessary for it.

Lemma 7.1 *Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $\mathrm{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with $h > 0$, $n_1, \ldots, n_h \geq 1$. For all $z \in \Sigma^+$ and $b \in \Sigma$ such that z is an anti-prenecklace, zb is not an anti-prenecklace and zb is a prefix of w, there is an integer g such that*

$$zb = (u_1 v_1)^{n_1} \cdots (u_g v_g)^{n_g} u_g b,$$

where $u_j v_j = u_j a_j v'_j = \ell_j, 1 \leq j \leq g, a_j < b$ and $u_g b$ is an anti-prenecklace.

Remark 7.1 *[18] Let x, y two different borders of a same word $w \in \Sigma^+$. If x is shorter than y, then x is a border of y.*

Proposition 7.2 *Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $(p, \bar{p}) \in \mathrm{Pref}_{bre}(w)$ and let $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$. Let $\mathrm{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with $h > 0$, $n_1, \ldots, n_h \geq 1$ and let $(\ell_1^{n_1}, \ldots, \ell_q^{n_q})$ be a $\mathcal{PMC}$ in $\mathrm{CFL}_{in}(w)$, $1 \leq q \leq h$. Then the following properties hold.*

(1) $p = \ell_1^{n_1} \cdots \ell_g^{n_g}$, for some g, $1 \leq g \leq q$.

(2) $\ell_g = u_g v_g = u_g a_g v'_g$, $\bar{p} = u_g b$, $a_g < b$.

PROOF :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $(p, \bar{p}) \in \mathrm{Pref}_{bre}(w)$. Let $\mathrm{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with $h > 0$, $n_1, \ldots, n_h \geq 1$ and let $(\ell_1^{n_1}, \ldots, \ell_q^{n_q})$ be a $\mathcal{PMC}$ in $\mathrm{CFL}_{in}(w)$, $1 \leq q \leq h$. Let $r, s \in \Sigma^*$, $a', b \in \Sigma$ be such that $p = ra's$, $\bar{p} = rb$, $a' < b$.

By Proposition 5.2, the word $p\bar{p} = prb$ is not an anti-prenecklace but its longest proper prefix is an anti-prenecklace. Thus, by Lemma 7.1 there is an integer g such that

$$p\bar{p} = (u_1 v_1)^{n_1} \cdots (u_g v_g)^{n_g} u_g b,$$

where $u_j v_j = u_j a_j v'_j = \ell_j, 1 \leq j \leq g, a_j < b$ and $u_g b$ is an anti-prenecklace. Moreover $u_g b$ is a prefix of $\ell_{g+1}^{n_{g+1}} \cdots \ell_h^{n_h}$. Let

$$\beta = (u_1 v_1)^{n_1} (u_2 v_2)^{n_2} \cdots (u_g v_g)^{n_g}.$$

The word β is a nonempty proper prefix of $p\bar{p}$ thus, by Definition 6.1, β is an inverse Lyndon word. Therefore $g \leq q$ (otherwise $g \geq q+1$, hence ℓ_q would be a prefix of β and there would be a word z' such that $\ell_{q+1}z'$ is a suffix of β, a contradiction since $\ell_q \ll \ell_{q+1}$ implies $\beta \ll \ell_{q+1}z'$). Moreover $\beta \ll u_g b$.

Since $p = ra's$, $\bar{p} = rb$, $a' < b$, then $p\bar{p} = \beta u_g b = ra'srb$. By Proposition 6.1, r is a suffix of u_g. If $r = u_g$, then $p = \beta$ and the proof is ended. By contradiction, assume that r is a proper suffix of u_g.

Since r is a proper suffix of u_g, we have $|\beta| < |p| \leq |\beta u_g|$. Hence, $\ell_g = u_g v_g$ and u_g are nonempty prefixes of p. By Lemma 5.1 u_g is an inverse Lyndon word, thus, by Proposition 5.2, u_g is a nonempty anti-prenecklace. Therefore, u_g and $u_g b$ are both nonempty anti-prenecklaces. By Theorem 3.1, there are $x, y \in \Sigma^*$, an integer $t \geq 1$, $c \in \Sigma$ such that xy is an anti-Lyndon word, $u_g = (xy)^t x$, $y = cy'$ with $c \geq b$.

Observe that $m_1 = pr'$, for a prefix r' of r. Thus again $|\beta| < |m_1| \leq |\beta u_g|$. Moreover, the words ℓ_{g+1} and $u_g b = (xy)^t xb$ are both prefixes of the same word $\gamma = \ell_{g+1}^{n_{g+1}} \cdots \ell_h^{n_h}$, hence they are comparable for the prefix order. Since ℓ_{g+1} is the longest anti-Lyndon prefix of γ, we have $|\ell_{g+1}| \geq |xy|$ and since ℓ_{g+1} is unbordered, either $\ell_{g+1} = xy$ is a prefix of ℓ_g and $g+1 \leq q$, or the word $u_g b = (xy)^t xb$ is a prefix of ℓ_{g+1}. By Proposition 7.1, the first case holds, otherwise m_1 would not be a product of anti-Lyndon words because m_1 is a prefix of βu_g longer than β.

Recall that r is a proper suffix of u_g. Moreover r and ra' are also prefixes of u_g because ra' and u_g are both prefixes of p and r is shorter than u_g. Therefore r is a border of u_g and u_g starts with ra'. Of course $r \neq x$ because u_g starts with ra' and also with xc, with $c \geq b > a'$. Now r and x are two different borders of u_g. If r were shorter than x, then r would be a border of x by Remark 7.1. This is impossible because $rcy'(xy)^{t-1}x$ would be a suffix of the inverse Lyndon word u_g and u_g starts with ra', with $c \geq b > a'$. Thus $|r| > |x| \geq 0$. Since r is a nonempty border of $u_g = (xy)^t x$ and $|r| > |x| \geq 0$, one of the following three cases holds:

$$r = (xy)^{t'}x, \quad 0 < t' < t \tag{7.1}$$

$$r = y_1(xy)^{t'}x, \quad y_1 \text{ nonempty suffix of } y, \quad 0 \leq t' < t \tag{7.2}$$

$$r = x_1(yx)^{t'}, \quad x_1 \text{ nonempty suffix of } x, \quad 0 < t' \leq t \tag{7.3}$$

Assume that Eq. (7.1) holds. Then p starts with $ra' = (xy)^{t'}xa'$, $a' < b$, and p also starts with $u_g = (xy)^t x$. Since $t' < t$, the letter a' should be the first letter of $y = cy'$, $c \geq b > a'$. Therefore, Eq. (7.1) cannot hold.

Assume that Eq. (7.2) holds. Recall that r is a prefix of u_g. Therefore y_1 is a prefix of xy and y_1 is also a nonempty suffix of xy. The word xy is an anti-Lyndon word, thus xy is umbordered. Consequently, $y_1 = xy$, hence $x = 1$ and $y_1 = y$. By Eq. (7.2), we have $r = y^{t'+1}$, with $0 \leq t' < t$. Moreover, $t'+1 < t$ since r is a proper suffix of $u_g = y^t$. As above, p starts with $ra' = y^{t'+1}a'$, $a' < b$, and p also starts with $u_g = y^t$. Since $t'+1 < t$, the letter a' should be the first letter of $y = cy'$, $c \geq b > a'$. Therefore, Eq. (7.2) cannot hold.

Finally, assume that Eq. (7.3) holds. If $x_1 \neq x$, then $x_1 y$ would be both a proper nonempty suffix and a prefix of xy, hence a nonempty border of xy, which is impossible since xy is an anti-Lyndon word. Therefore $x_1 = x$. If $t' < t$, then r satisfies Eq. (7.1) and we proved that this is impossible. Thus $t' = t$, which implies $r = u_g$, a contradiction.

■

The following result was proved in [1] and will be used in Section 9.

Proposition 7.3 *Let $(\Sigma, <)$ be a totally ordered alphabet. Let $w \in \Sigma^+$ and let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$. If w is an inverse Lyndon word, then either w is unbordered or $\ell_1, \ldots, \ell_h$ is a $\mathcal{PMC}$ in $\mathrm{CFL}_{in}(w)$. In both cases $\mathrm{ICFL}(w) = (w)$ is the unique grouping of $\mathrm{CFL}_{in}(w)$.*

8 From ICFL to CFL_{in}

In what follows, $(\Sigma, <)$ denotes a totally ordered alphabet. Let $w \in \Sigma^+$ and let $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$, where $m_i \ll m_{i+1}$, $1 \leq i \leq k-1$. In this section we give an algorithm to construct $\mathrm{CFL}_{in}(w)$ starting from $\mathrm{ICFL}(w)$ (Definition 8.1). To this aim, we first demonstrate that we can limit ourselves to handling the single factor m_i in $\mathrm{ICFL}(w)$ (Corollary 8.1). This result is a direct consequence of the following proposition.

Proposition 8.1 *Let $w \in \Sigma^+$. If $(m_1, \ldots, m_k)$ is a grouping of $\mathrm{CFL}_{in}(w)$, then*

$$\mathrm{CFL}_{in}(w) = (\mathrm{CFL}_{in}(m_1), \ldots, \mathrm{CFL}_{in}(m_k))$$

PROOF :

Let $w \in \Sigma^+$ and let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$. We prove our statement by induction on $|w|$. If $|w| = 1$, then there is only one grouping of $\mathrm{CFL}_{in}(w)$, namely (w). Obviously $\mathrm{CFL}_{in}(w) = (w)$ and we are done.

Assume $|w| > 1$. Let $(m_1, \ldots, m_k)$ be a grouping of $\mathrm{CFL}_{in}(w)$ and let $w' \in \Sigma^*$ be such that $w = m_1 w'$. According to Definition 7.4 there are words in $\mathrm{CFL}_{in}(w)$, say $\ell_1, \ldots, \ell_v$, $1 \leq v \leq h$, such that ℓ_i is a prefix of ℓ_{i-1}, $1 < i \leq v$ and $m_1 = \ell_1 \cdots \ell_v$. Moreover, by Theorem 4.1

$$\mathrm{CFL}_{in}(m_1) = (\ell_1, \ldots, \ell_v) \tag{8.1}$$

If $w' = 1$, then $w = m_1$, $v = h$, hence $\mathrm{CFL}_{in}(w) = \mathrm{CFL}_{in}(m_1)$ and we are done.

Otherwise, $w' = m_2 \cdots m_k \in \Sigma^+$, $v < h$ and, by Theorem 4.1, $\mathrm{CFL}_{in}(w') = (\ell_{v+1}, \ldots, \ell_h)$. In addition, $(m_2, \ldots, m_k)$ is a grouping of $\mathrm{CFL}_{in}(w')$ and $|w'| < |w|$. By using the induction hypothesis, we have

$$\mathrm{CFL}_{in}(w') = (\ell_{v+1}, \ldots, \ell_h) = (\mathrm{CFL}_{in}(m_2), \ldots, \mathrm{CFL}_{in}(m_k)) \tag{8.2}$$

Finally, by Eqs. (8.1), (8.2), we obtain

$$\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h) = (\mathrm{CFL}_{in}(m_1), \ldots, \mathrm{CFL}_{in}(m_k)) \tag{8.3}$$

■

Corollary 8.1 *Let $w \in \Sigma^+$. If $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$, then*

$$\mathrm{CFL}_{in}(w) = (\mathrm{CFL}_{in}(m_1), \ldots, \mathrm{CFL}_{in}(m_k))$$

PROOF :

Let $w \in \Sigma^+$ and let $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$. By Proposition 7.1, $\mathrm{ICFL}(w)$ is a grouping of $\mathrm{CFL}_{in}(w)$, hence our claim follows by Proposition 8.1. ■

In the following proposition, we show that each bordered word has a unique nonempty border which is unbordered.

Proposition 8.2 *Each bordered word has a unique nonempty border which is unbordered.*

PROOF :

We first prove that each bordered word $w \in \Sigma^+$ has a nonempty border z which is unbordered, by induction on the length of w. Indeed, the shortest bordered word is $w = aa$, where a is a letter and $z = a$.

Let $|w| > 2$ and let y be a nonempty border of w. Thus, there are nonempty words x, x' such that

$$w = xy = yx' \tag{8.4}$$

If y is unbordered, we are done. Otherwise, since y is a bordered word shorter than w, by induction hypothesis, y has a nonempty border z which is unbordered and there are $t, t' \in \Sigma^+$ such that

$$y = tz = zt' \tag{8.5}$$

By using Eqs. (8.4), (8.5), we get

$$xtz = xy = w = yx' = zt'x'$$

hence z is a nonempty border of w which is unbordered.

Now we prove that there is only one unbordered nonempty border of w. Suppose that y, v are two nonempty unbordered borders of w. Thus there are $x, x', u, u' \in \Sigma^+$ such that

$$w = xy = yx' = uv = vu' \tag{8.6}$$

Assume $|y| \geq |v|$ (a similar argument holds if $|y| \leq |v|$). By Eq. (8.6), there are $t, t' \in \Sigma^*$ such that

$$y = tv = vt'$$

Since y is unbordered we get $t = t' = 1$, hence $y = v$. ■

Example 8.1 Let $\Sigma = \{a, b\}$. The word $ababa$ has the nonempty borders a, aba and a is unbordered. The word $aaaa$ has the nonempty borders a, aa, aaa and a is unbordered.

Definition 8.1 below provides a recursive construction of a sequence $\mathcal{NB}(x)$ of suffixes of a word x. Proposition 8.3 proves that every nonempty word x is the concatenation of the elements of $\mathcal{NB}(x)$ and that there is only one sequence $\mathcal{NB}(x)$ associated with x according to Definition 8.1. Subsequently Proposition 8.4 proves that if $w \in \Sigma^+$ and x is a factor of a grouping of of $\mathrm{CFL}_{in}(w)$, then $\mathrm{CFL}_{in}(x) = \mathcal{NB}(x)$. According to Corollary 8.1, our aim, that is, to construct $\mathrm{CFL}_{in}(w)$ from $\mathrm{ICFL}_{in}(w)$, is achieved (Corollary 8.2).

Definition 8.1 *Let $x \in \Sigma^+$.*

(Basis Step) *If x is unbordered, then $\mathcal{NB}(x) = (x)$.*

(Recursive Step) *If x is bordered, let z be the unique nonempty border of x which is unbordered and let $y \in \Sigma^+$ be such that $x = yz$. Then, $\mathcal{NB}(x) = (\mathcal{NB}(y), z)$.*

The length $|\mathcal{NB}(x)|$ *of $\mathcal{NB}(x)$ is the number of elements in $\mathcal{NB}(x)$.*

Proposition 8.3 *For any word $x \in \Sigma^+$, there is a unique sequence $\ell'_1, \ldots, \ell'_v$ of words over Σ such that $\mathcal{NB}(x) = (\ell'_1, \ldots, \ell'_v)$. If $\ell'_1, \ldots, \ell'_v$ is such that $\mathcal{NB}(x) = (\ell'_1, \ldots, \ell'_v)$, then*

$$x = \ell'_1 \cdots \ell'_v$$

PROOF :

Notice that if x is unbordered, then, by Definition 8.1, $\mathcal{NB}(x) = (x)$ is uniquely determined and we are done. The proof is by induction on $|x|$. By the above reasoning if $|x| = 1$, then the conclusion follows immediately and the same holds for any unbordered word x.

Hence, let x be a bordered word such that $|x| > 1$. Let

$$\mathcal{NB}(x) = (\ell'_1, \ldots, \ell'_v), \quad \mathcal{NB}(x) = (\mu_1, \ldots, \mu_p) \tag{8.7}$$

with $v > 1$, $p > 1$. Let z be the unique nonempty border of x which is unbordered and let $y \in \Sigma^+$ be such that $x = yz$. By Definition 8.1, we have

$$\mathcal{NB}(x) = (\mathcal{NB}(y), z) = (\ell'_1, \ldots, \ell'_v), \quad \mathcal{NB}(x) = (\mathcal{NB}(y), z) = (\mu_1, \ldots, \mu_p) \tag{8.8}$$

Hence, $\ell'_v = z = \mu_p$ and $\mathcal{NB}(y) = (\ell'_1, \ldots, \ell'_{v-1})$, $\mathcal{NB}(y) = (\mu_1, \ldots, \mu_{p-1})$. Obviously $|y| < |x|$. By using induction hypothesis, we have $p = v$, $\ell'_i = \mu_i$, $1 \leq i \leq v$,

$$x = yz = \ell'_1 \cdots \ell'_{v-1} z = \ell'_1 \cdots \ell'_v$$

and the proof is complete. ∎

The following example provides an interesting observation.

Example 8.2 Let $\Sigma = \{a, b, c, d\}$, $a < b < c < d$, and $w = dabadabdabdadac$. We have $\mathrm{ICFL}(w) = (daba, dabdab, dadac)$ (see Example 6.2). Then, $\mathcal{NB}(daba) = daba$, $\mathcal{NB}(dabdab) = (dab, dab)$, $\mathcal{NB}(dadac) = dadac$. Notice that

$$\begin{aligned} \mathrm{CFL}_{in}(w) &= (daba, dab, dab, dadac) = (\mathcal{NB}(daba), \mathcal{NB}(dabdab), \mathcal{NB}(dadac)) \\ &\neq (dabadabdabdadac) = (\mathcal{NB}(dabadabdabdadac)) = (\mathcal{NB}(w)) \end{aligned}$$

Next, let $y = dabadabdabdabdadac$. We have $\mathrm{ICFL}(w) = (daba, (dab)^3, dadac)$ and

$$\mathrm{CFL}_{in}(y) = (daba, dab, dab, dab, dadac) = (\mathcal{NB}(daba), \mathcal{NB}((dab)^3), \mathcal{NB}(dadac))$$

Proposition 8.4 *Let $(\Sigma, <)$ be a totally ordered alphabet. Let $m, \ell_1, \ldots, \ell_h$ be words in Σ^+ such that*

(1) m is an inverse Lyndon word.

(2) $\ell_1, \ldots, \ell_h$ are words in L_{in} such that ℓ_i is a prefix of ℓ_{i-1}, $1 < i \leq h$.

(3) $m = \ell_1 \cdots \ell_h$.

Then the two sequences $\mathrm{CFL}_{in}(m)$, $\mathcal{NB}(m)$ are equal, that is, $\mathrm{CFL}_{in}(m) = \mathcal{NB}(m)$.

PROOF :

The proof is by induction on $|\mathcal{NB}(m)|$. If $|\mathcal{NB}(m)| = 1$, then by Definition 8.1, the inverse Lyndon word m is unbordered and $\mathcal{NB}(m) = (m)$. Thus, by Proposition 5.1, the word m is is an anti-Lyndon word. Consequently, $\mathrm{CFL}_{in}(m) = (m) = \mathcal{NB}(m)$.

Otherwise, let $|\mathcal{NB}(m)| > 1$. Let y, z be such that z is the unique unbordered border of m and $m = yz$. Then, by Definition 8.1, $\mathcal{NB}(m) = (\mathcal{NB}(y), z)$. Let $\ell_1, \ldots, \ell_h$ be as in the statement. Of course $\mathrm{CFL}_{in}(m) = (\ell_1, \ldots, \ell_h)$. Let $\mathcal{NB}(y) = (\ell'_1, \ldots, \ell'_v)$. By item (3) and by Proposition 8.3 applied to y, we have

$$m = yz = \ell'_1 \cdots \ell'_v z = \ell_1 \cdots \ell_h \tag{8.9}$$

Now, ℓ_h is a border of m, because ℓ_h is a prefix of ℓ_1, hence of m. Moreover, by Proposition 3.1, ℓ_h is an unbordered word. By Proposition 8.2 and looking at Eq. (8.9), we have

$$z = \ell_h, \quad y = \ell'_1 \cdots \ell'_v = \ell_1 \cdots \ell_{h-1} \tag{8.10}$$

The word y is an inverse Lyndon word because it is a nonempty prefix of the inverse Lyndon word m (Lemma 5.1). Moreover $\mathrm{CFL}_{in}(y) = (\ell_1, \ldots, \ell_{h-1})$ and $y, \ell_1, \ldots, \ell_{h-1}$ satisfy the hypothesis of the statement with $|\mathcal{NB}(y)| < |\mathcal{NB}(m)|$. Thus, by using the induction hypothesis and Eq. (8.10), $\mathrm{CFL}_{in}(y)$ and $\mathcal{NB}(y)$ are equal, that is,

$$(\ell_1, \ldots, \ell_{h-1}) = \mathrm{CFL}_{in}(y) = \mathcal{NB}(y) = (\ell'_1, \ldots, \ell'_v) \tag{8.11}$$

and consequently

$$h - 1 = v, \quad \ell_i = \ell'_i, \quad 1 \leq i \leq h-1 \tag{8.12}$$

Finally, by Eqs. (8.10), (8.11), (8.12), we have

$$\mathrm{CFL}_{in}(m) = (\ell_1, \ldots, \ell_h) = (\ell'_1, \ldots, \ell'_v, \ell_h) = (\ell'_1, \ldots, \ell'_v, z) = (\mathcal{NB}(y), z) = \mathcal{NB}(m)$$

■

Proposition 8.5 *Let $w \in \Sigma^+$. If $(m_1, \ldots, m_k)$ is a grouping of $\mathrm{CFL}_{in}(w)$, then for each i, $1 \leq i \leq v$, $\mathrm{CFL}_{in}(m_i) = \mathcal{NB}(m_i)$.*

PROOF :

Let $w \in \Sigma^+$, let $(m_1, \ldots, m_k)$ be a grouping of $\mathrm{CFL}_{in}(w)$. Looking at Definitions 7.3, 7.4, the conclusion follows easily from Proposition 8.4 applied to the word m_i, for each i, $1 \leq i \leq v$. ■

Proposition 8.6 *Let $w \in \Sigma^+$. If $(m_1, \ldots, m_k)$ is a grouping of $\mathrm{CFL}_{in}(w)$, then $\mathrm{CFL}_{in}(w) = (\mathcal{NB}(m_1), \ldots, \mathcal{NB}(m_k))$.*

PROOF :

Let $w \in \Sigma^+$, let $(m_1, \ldots, m_k)$ be a grouping of $\mathrm{CFL}_{in}(w)$. By Propositions 8.1, 8.5, we have

$$\mathrm{CFL}_{in}(w) = (\mathrm{CFL}_{in}(m_1), \ldots, \mathrm{CFL}_{in}(m_k)) = (\mathcal{NB}(m_1), \ldots, \mathcal{NB}(m_k))$$

■

Corollary 8.2 *Let $w \in \Sigma^+$. If $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$, then*

$$\mathrm{CFL}_{in}(w) = (\mathcal{NB}(m_1), \ldots, \mathcal{NB}(m_k))$$

PROOF :

The conclusion follows easily from Propositions 7.1 and 8.6. ■

Example 8.3 Let $\Sigma = \{a, b, c, d\}$, $a < b < c < d$, and let $y = dabadabdabdabdadac$. We know that $\mathrm{ICFL}(y) = (daba, (dab)^3, dadac)$ (see Example 7.2). By Corollaries 8.1 and 8.2, we have

$$\begin{aligned}
\mathrm{CFL}_{in}(y) &= (\mathrm{CFL}_{in}(daba), \mathrm{CFL}_{in}((dab)^3), \mathrm{CFL}_{in}(dadac)) \\
&= (\mathcal{NB}(daba), \mathcal{NB}((dab)^3), \mathcal{NB}(dadac)) \\
&= (daba, dab, dab, dab, dadac)
\end{aligned}$$

The inverse Lyndon factorization $(dabadab, (dab)^2, dadac)$ is another grouping of $\mathrm{CFL}_{in}(y)$. By Propositions 8.1 and 8.6, we have

$$\begin{aligned}
\mathrm{CFL}_{in}(y) &= (\mathrm{CFL}_{in}(dabadab), \mathrm{CFL}_{in}((dab)^2), \mathrm{CFL}_{in}(dadac)) \\
&= (\mathcal{NB}(dabadab), \mathcal{NB}((dab)^2), \mathcal{NB}(dadac)) \\
&= (daba, dab, dab, dab, dadac)
\end{aligned}$$

9 From CFL_{in} to ICFL

In what follows, $(\Sigma, <)$ denotes a totally ordered alphabet. Let $w \in \Sigma^+$. The aim of this section is to show how to get $\mathrm{ICFL}(w)$ from $\mathrm{CFL}_{in}(w)$. As we know, $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$ is a grouping of $\mathrm{CFL}_{in}(w)$ (Proposition 7.1). We also know that the word p in the pair $(p, \bar{p}) \in \mathrm{Pref}_{bre}(w)$ has a grouping-like property (Proposition 7.2). These properties allow us to prove that $\mathrm{ICFL}(w)$ can be computed locally in the non-increasing maximal chains for the prefix order of its decomposition (Proposition 9.1).

Remark 9.1 Let $(\ell'_1, \ldots, \ell'_v)$ be a non-increasing chain for the prefix order of anti-Lyndon words. Let $y = \ell'_1 \cdots \ell'_v$. It is easy to see that $\mathrm{CFL}_{in}(y) = (\ell'_1, \ldots, \ell'_v)$.

Proposition 9.1 *Let $w \in \Sigma^+$, let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$ and let $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ be the decomposition of $\mathrm{CFL}_{in}(w)$ into its non-increasing maximal chains for the prefix order. Let $w_1, \ldots, w_s$ be words such that $\mathrm{CFL}_{in}(w_j) = \mathcal{C}_j$, $1 \leq j \leq s$. Then $\mathrm{ICFL}(w)$ is the concatenation of the sequences $\mathrm{ICFL}(w_1), \ldots, \mathrm{ICFL}(w_s)$, that is,*

$$\mathrm{ICFL}(w) = (\mathrm{ICFL}(w_1), \ldots, \mathrm{ICFL}(w_s)) \tag{9.1}$$

PROOF :

Let $w \in \Sigma^+$, let $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$ and let $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s)$ be the decomposition of $\mathrm{CFL}_{in}(w)$ into its non-increasing maximal chains for the prefix order. Let $w_1, \ldots, w_s$ be words such that $\mathrm{CFL}_{in}(w_j) = \mathcal{C}_j$, $1 \leq j \leq s$. Let $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$. The proof is by induction on $|w|$. If $|w| = 1$, then w is an inverse Lyndon word and by Proposition 7.3 we are done. Therefore assume $|w| > 1$. If $k = 1$, by Definition 6.2, $w = m_1$ is an inverse Lyndon word and by Proposition 7.3 we are done. Assume $k > 1$, thus w is a word which is not an inverse Lyndon word.

By Proposition 7.1, there are indexes $i_2, \ldots, i_s$, $1 < i_2 \ldots < i_s \leq k$ such that

$$w_1 = m_1 \cdots m_{i_2 - 1},\ w_2 = m_{i_2} \cdots m_{i_3 - 1}, \ldots, w_s = m_{i_s} \cdots m_k$$

We actually prove more, namely we prove Eq. (9.2) below:

$$\mathrm{ICFL}(w_1) = (m_1, \ldots, m_{i_2 - 1}),\ \mathrm{ICFL}(w_2) = (m_{i_2}, \ldots, m_{i_3 - 1}), \ldots, \mathrm{ICFL}(w_s) = (m_{i_s}, \ldots, m_k) \tag{9.2}$$

Notice that if w_1 is an inverse Lyndon word, then $w_1 = m_1$. Otherwise, since $m_1 \ll m_2$ and by item (2) in Lemma 2.1, we would have $w_1 = m_1 \cdots m_{i_2 - 1} \ll m_2 \cdots m_{i_2 - 1}$, a contradiction. By a similar argument, if w_1 is an inverse Lyndon word, then $w_1 w_2$ is not an inverse Lyndon word. Indeed, let ℓ the last element in $\mathcal{C}_1$ and ℓ' the first element in $\mathcal{C}_2$. The word ℓ is a prefix of w_1 and by $\ell \ll \ell'$ we get $w_1 w_2 \ll w_2$.

Let $(p, \overline{p})$ be the canonical pair associated with w and let $v \in \Sigma^*$ be such that $w = pv$. If w_1 is not an inverse Lyndon word, let $(q, \overline{q})$ be the canonical pair associated with w_1. The word $q\overline{q}$ is a prefix of w which is not an inverse Lyndon word. Since $p\overline{p}$ is the shortest prefix of w which is not an inverse Lyndon word, we have $|p\overline{p}| \leq |q\overline{q}|$, hence $p\overline{p}$ is a prefix of w_1. In turn, since $q\overline{q}$ is the shortest prefix of w_1 which is not an inverse Lyndon word, we have $|q\overline{q}| \leq |p\overline{p}|$. In conclusion, $p\overline{p} = q\overline{q}$ and, looking at Proposition 6.1, we see that $p = q$ and $\overline{p} = \overline{q}$. If w_1 is an inverse Lyndon word, the same reasoning applies to the canonical pair $(q, \overline{q})$ associated with $w_1 w_2$.

Let $\mathrm{ICFL}(v) = (m'_1, \ldots, m'_k)$ and let $r, s \in \Sigma^*$, $a, b \in \Sigma$ such that $p = ras$, $\overline{p} = rb$ with $a < b$. By Definition 6.2 one has

$$\mathrm{ICFL}(w) = \begin{cases} (p, \mathrm{ICFL}(v)) & \text{if } \overline{p} = rb \leq_p m'_1 \\ (pm'_1, m'_2, \ldots, m'_k) & \text{if } m'_1 \leq_p r \end{cases} \tag{9.3}$$

By Proposition 7.2, there is g, $1 \leq g \leq h$, such that $p = \ell_1 \cdots \ell_g$. Let $w'_1 \in \Sigma^*$ be such that $w_1 = pw'_1$. If $w'_1 \neq 1$, let $\mathcal{C}'_1 = \mathrm{CFL}_{in}(w'_1)$. Since $\mathcal{C}'_1$ is $\mathcal{C}_1$ after erasing $\ell_1, \ldots, \ell_g$, it is easy to see that $\mathcal{C}'_1$ is a non-increasing maximal chain for the prefix order. Hence the decomposition $\mathcal{D}$ of $\mathrm{CFL}_{in}(v)$ into its non-increasing maximal chains for the prefix order is

$$\mathcal{D} = \begin{cases} (\mathcal{C}'_1, \mathcal{C}_2, \ldots, \mathcal{C}_s) & \text{if } w'_1 \neq 1 \\ (\mathcal{C}_2, \ldots, \mathcal{C}_s) & \text{if } w'_1 = 1 \end{cases}$$

Of course $|v| < |w|$, thus by induction hypothesis, we have

$$\mathrm{ICFL}(v) = \begin{cases} (\mathrm{ICFL}(w'_1), \ldots, \mathrm{ICFL}(w_s)) & \text{if } w'_1 \neq 1 \\ (\mathrm{ICFL}(w_2), \ldots, \mathrm{ICFL}(w_s)) & \text{if } w'_1 = 1 \end{cases} \tag{9.4}$$

More specifically,

$$\mathrm{ICFL}(w_2) = (m_{i_2}, \ldots, m_{i_3 - 1}), \ldots, \mathrm{ICFL}(w_s) = (m_{i_s}, \ldots, m_k) \tag{9.5}$$

Moreover, if $w_1' \neq 1$ one has

$$\mathrm{ICFL}(w_1') = \begin{cases}(m_2, \ldots, m_{i_2-1}) & \text{if } m_1 = p \\ (m_1', m_2, \ldots, m_{i_2-1}) & \text{if } m_1 = pm_1'\end{cases} \tag{9.6}$$

Notice that if $m_1 = p$ and $w_1' \neq 1$, then w_1 is not an inverse Lyndon word (because $w_1 \neq m_1$) and, by Definition 6.2, $(p, \mathrm{ICFL}(w_1')) = \mathrm{ICFL}(w_1)$ (because the canonical pair associated with w_1 is equal to the canonical pair associated with w). Furthermore, by Eqs. (9.3), (9.4), (9.5), $\mathrm{ICFL}(w) = (p, \mathrm{ICFL}(v)) = (p, \mathrm{ICFL}(w_1'), \ldots, \mathrm{ICFL}(w_s)) = (\mathrm{ICFL}(w_1), \ldots, \mathrm{ICFL}(w_s))$ Thus, Eqs. (9.1), (9.2) hold. Analogously, by Definition 6.2, if $m_1 = p$ and $w_1' = 1$, then $w_1 = p$ and clearly $\mathrm{ICFL}(w_1) = (p) = (m_1)$. Thus, by Eqs. (9.4), (9.5), $\mathrm{ICFL}(w) = (p, \mathrm{ICFL}(v)) = (p, \mathrm{ICFL}(w_2), \ldots, \mathrm{ICFL}(w_s)) = (\mathrm{ICFL}(w_1), \mathrm{ICFL}(w_2), \ldots, \mathrm{ICFL}(w_s))$ and Eqs. (9.1), (9.2) hold.

Otherwise, if $m_1 = pm_1'$, then $w_1' \neq 1$. By Definition 6.2 and Eq. (9.6), we have $\mathrm{ICFL}(w_1) = (pm_1', m_2, \ldots m_{i_2-1}) = (m_1, m_2, \ldots m_{i_2-1})$. Thus, by Eqs. (9.3), (9.4), (9.5), (9.6), we have

$$\begin{aligned}\mathrm{ICFL}(w) &= (m_1, m_2, \ldots, m_k) \\ &= (pm_1', m_2, \ldots m_{i_2-1}, m_{i_2} \ldots m_{i_3-1}, \ldots, m_{i_s} \cdots m_k \\ &= (pm_1', m_2, \ldots m_{i_2-1}, \mathrm{ICFL}(w_2), \ldots, \mathrm{ICFL}(w_s)) \\ &= (\mathrm{ICFL}(w_1), \mathrm{ICFL}(w_2), \ldots, \mathrm{ICFL}(w_s))\end{aligned}$$

and Eqs. (9.1), (9.2) hold. This ends the proof. ■

Proposition 9.1 shows that to obtain $\mathrm{ICFL}(w)$ from $\mathrm{CFL}_{in}(w)$ we can limit ourselves to the case in which $\mathrm{CFL}_{in}(w)$ is a chain with respect to the prefix order. Thus in the results that follow we will focus on these chains. In Proposition 9.2 we will prove that some products of consecutive elements in such a chain form an inverse Lyndon word. Then we will prove some properties of such products in Propositions 9.3, 9.4. We will use these properties to establish which of the aforementioned products can be elements of an inverse Lyndon factorization (Proposition 9.5).

Proposition 9.2 *Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,*

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, j, $1 \leq i < j \leq h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

and $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1. Then, the word $\ell_1 \ell_2 \cdots \ell_{i+1} \cdots \ell_j$ is an inverse Lyndon word.

PROOF :

Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, j, $1 \leq i < j \leq h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

and $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1. Let u, v be words such that

$$u = \ell_{i+1} \cdots \ell_j, \quad \ell_1 = uv$$

Thus

$$\ell_1 \ell_2 \cdots \ell_{i+1} \cdots \ell_j = (uv)^i u$$

Therefore the word $\ell_1 \ell_2 \cdots \ell_{i+1} \cdots \ell_j$ is a sesquipower of the anti-Lyndon word ℓ_1, hence, by Proposition 5.2, $\ell_1 \ell_2 \cdots \ell_{i+1} \cdots \ell_j$ is an inverse Lyndon word. ■

Proposition 9.3 *Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,*

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, $1 \leq i < h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

If there is j, $i < j \leq h$, such that $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1, then $|\ell_1| > |\ell_{i+1} \cdots \ell_j|$.

PROOF :

Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, j, $1 \leq i < j \leq h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

and $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1. Thus $|\ell_1| \geq |\ell_{i+1} \cdots \ell_j|$. By contradiction, assume $|\ell_1| = |\ell_{i+1} \cdots \ell_j|$. Hence $\ell_1 = \ell_{i+1} \cdots \ell_j$. In addition $|\ell_t| < |\ell_1|$, $i+1 \leq t \leq j$, because ℓ_t is a prefix of ℓ_{t-1} and $\ell_1 = \ell_i \neq \ell_{i+1}$. Consequently, ℓ_j would be a nonempty border of ℓ_1, in contradiction with Proposition 3.1. ∎

Proposition 9.4 *Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,*

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, $1 \leq i < h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

If there is j, $i < j \leq h$, such that $\ell_{i+1} \cdots \ell_j$ is a prefix of $\ell_1 \ell_2 \cdots \ell_i$, then $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1.

PROOF :

Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, j, $1 \leq i < j \leq h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

and $\ell_{i+1} \cdots \ell_j$ is a prefix of $\ell_1 \ell_2 \cdots \ell_i$. If $\ell_{i+1} \cdots \ell_j$ were not a prefix of ℓ_1, then $i > 1$, $j > i+1$ and there would exist q, $i+1 < q \leq j$, such that

$$\ell_1 = \ell_{i+1} \cdots \ell'_q, \quad \ell_q = \ell'_q \ell''_q, \quad \ell'_q \neq 1$$

If $\ell''_q = 1$, then $\ell_q = \ell'_q$ would be a proper suffix of ℓ_1 which is also a nonempty prefix of ℓ_1, hence ℓ_q would be a nonempty border of ℓ_1, in contradiction with Proposition 3.1. Consequently $\ell''_q \neq 1$ would be a proper suffix of ℓ_q which is also a nonempty prefix of ℓ_2 and therefore of ℓ_q. Hence ℓ''_q would be a nonempty border of ℓ_q, in contradiction with Proposition 3.1. ∎

Proposition 9.5 *Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,*

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, j, $1 \leq i < j < h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

and $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1. If $\ell_{i+1} \cdots \ell_{j+1}$ is not a prefix of ℓ_1, then one has

$$\ell_1 \ll \ell_{i+1} \cdots \ell_{j+1}$$

More specifically, there are words $r, s, s' \in \Sigma^*$ and $a, b \in \Sigma$, $a < b$ such that

$$\ell_1 = \ell_{i+1} \cdots \ell_j ras, \quad \ell_{j+1} = rbs'$$

PROOF :

Let $\ell_1, \ldots, \ell_h$ be anti-Lyndon words that form a non-increasing chain for the prefix order, that is,

$$\ell_1 \geq_p \ell_2 \geq_p \ldots \geq_p \ell_h$$

Let $h \geq 2$ and let i, j, $1 \leq i < j < h$, be such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$$

and $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1.

By Proposition 9.3 one has $|\ell_1| > |\ell_{i+1} \cdots \ell_j|$, hence there is a nonempty word x such that $\ell_1 = \ell_{i+1} \cdots \ell_j x$. Notice that x cannot be a prefix of ℓ_{j+1} because ℓ_{j+1} is a proper prefix of ℓ_1 and ℓ_1 is unbordered (Proposition 3.1). If $\ell_{i+1} \cdots \ell_{j+1}$ is not a prefix of ℓ_1, then ℓ_{j+1} cannot be a prefix of x either.

Thus, there are words $r, s, s' \in \Sigma^*$ and $a, b \in \Sigma$, $a \neq b$ such that

$$\ell_1 = \ell_{i+1} \cdots \ell_j ras, \quad \ell_{j+1} = rbs'$$

The word ℓ_{j+1} is a prefix of ℓ_{i+1}, hence ℓ_{j+1} is a prefix of ℓ_1 such that $|\ell_{j+1}| \leq |\ell_{i+1}|$. Therefore, there is $z \in \Sigma^*$ such that $\ell_1 = rbs'zras$ which yields $a < b$ because ℓ_1 is an inverse Lyndon word. Consequently,

$$\ell_1 = \ell_{i+1} \cdots \ell_j ras \ll \ell_{i+1} \cdots \ell_j rbs' = \ell_{i+1} \cdots \ell_{j+1}$$

■

In Propositions 9.6 - 9.8 we prove some properties of the factors in CFL_{in}. At the end of this section we briefly discuss how Propositions 9.7, 9.8 can be used to obtain ICFL from CFL_{in}.

Proposition 9.6 *Let $w \in \Sigma^+$, let $(\ell_1, \ldots, \ell_h)$ be a chain for the prefix order in $\mathrm{CFL}_{in}(w)$. The word $\ell_1 \cdots \ell_h$ is not an inverse Lyndon word if and only if $h > 2$ and there are i, j, $1 \leq i < j < h$, such that*

(1) $\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$,

(2) $\ell_{i+1} \cdots \ell_j$ *is a prefix of* ℓ_1

(3) $\ell_{i+1} \cdots \ell_{j+1}$ *is not a prefix of* ℓ_1

(4) We have

$$\ell_1 \ll \ell_{i+1} \cdots \ell_{j+1}$$

More specifically, there are words $r, s, s' \in \Sigma^*$ and $a, b \in \Sigma$, $a < b$ such that

$$\ell_1 = \ell_{i+1} \cdots \ell_j ras, \quad \ell_{j+1} = rbs'$$

PROOF :

Let $w \in \Sigma^+$, let $(\ell_1, \ldots, \ell_h)$ be a chain for the prefix order in $\mathrm{CFL}_{in}(w)$. Let $h > 2$ and let i, j, $1 \leq i < j < h$, be such that items (1)-(4) in the statement are satisfied. We prove that $\ell_1 \cdots \ell_h$ is not an inverse Lyndon word. Indeed, by item (4)

$$\ell_1 \ll \ell_{i+1} \cdots \ell_{j+1}$$

Thus, by item (2) in Lemma 2.1,

$$\ell_1 \ell_2 \cdots \ell_h \ll \ell_{i+1} \cdots \ell_{j+1} \ell_{j+2} \cdots \ell_h$$

therefore $\ell_1 \ell_2 \cdots \ell_h$ does not satisfy Definition 5.2.

Conversely, if $\ell_1 \cdots \ell_h$ is not an inverse Lyndon word, then there is i, $1 \leq i < h$, such that

$$\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1},$$

otherwise $\ell_1 \cdots \ell_h = (\ell_1)^h$ would be a sesquipower of the anti-Lyndon word ℓ_1, hence, by Proposition 5.2, $\ell_1 \cdots \ell_h$ would be an inverse Lyndon word. Furthermore, there is j, $i < j < h$, such that $\ell_{i+1} \cdots \ell_j$ is a prefix of ℓ_1 but $\ell_{i+1} \cdots \ell_{j+1}$ is not a prefix of ℓ_1 since otherwise $\ell_{i+1} \cdots \ell_h$ would be a prefix of ℓ_1 and $\ell_1 \cdots \ell_h$ would be an inverse Lyndon word by Proposition 9.2. Thus, by Proposition 9.5,

$$\ell_1 \ll \ell_{i+1} \cdots \ell_{j+1}$$

More specifically, there are words $r, s, s' \in \Sigma^*$ and $a, b \in \Sigma$, $a < b$ such that

$$\ell_1 = \ell_{i+1} \cdots \ell_j ras, \quad \ell_{j+1} = rbs'$$

and the proof is complete. ■

Proposition 9.7 *Let $w \in \Sigma^+$, let $(\ell_1, \ldots, \ell_h)$ be a non-increasing maximal chain for the prefix order ($\mathcal{PMC}$) in $\mathrm{CFL}_{in}(w)$, where ℓ_1 is the first element in $\mathrm{CFL}_{in}(w)$, and let $\mathrm{ICFL}(w) = (m_1, \ldots, m_k)$. If $\ell_1 \cdots \ell_h$ is an inverse Lyndon word, then*

$$m_1 = \ell_1 \cdots \ell_h$$

PROOF :

Let $w \in \Sigma^+$, let $(\ell_1, \ldots, \ell_h)$ be a non-increasing maximal chain for the prefix order ($\mathcal{PMC}$) in $\mathrm{CFL}_{in}(w)$, where ℓ_1 is the first element in $\mathrm{CFL}_{in}(w)$. If $\ell_1 \cdots \ell_h$ is an inverse Lyndon word, then $m_1 = \ell_1 \cdots \ell_h$. Otherwise, by Proposition 7.1, there would exist j, t, $1 \leq j < t \leq h$, such that

$$\ell_1 \cdots \ell_j = m_1 \ll m_2 = \ell_{j+1} \cdots \ell_t$$

and this would imply, by item (2) in Lemma 2.1,

$$\ell_1 \cdots \ell_h = (\ell_1 \cdots \ell_j)\ell_{j+1} \cdots \ell_h \ll \ell_{j+1} \cdots \ell_h$$

in contradiction with Definition 5.2. ∎

Proposition 9.8 *Let $w \in \Sigma^+$, let $(\ell_1, \ldots, \ell_h)$ be a non-increasing maximal chain for the prefix order ($\mathcal{PMC}$) in $\mathrm{CFL}_{in}(w)$, where ℓ_1 is the first element in $\mathrm{CFL}_{in}(w)$. If $\ell_1 \cdots \ell_h$ is not an inverse Lyndon word, then $h > 2$ and there are i, j, $1 \leq i < j < h$, such that*

(1) $\ell_1 = \ell_2 = \ldots = \ell_i \neq \ell_{i+1}$,

(2) $\ell_{i+1} \cdots \ell_j$ *is a prefix of* ℓ_1

(3) $\ell_{i+1} \cdots \ell_{j+1}$ *is not a prefix of* ℓ_1

(4) We have

$$\ell_1 \ll \ell_{i+1} \cdots \ell_{j+1} \tag{9.7}$$

More specifically, there are words $r, s, s' \in \Sigma^*$ and $a, b \in \Sigma$, $a < b$ such that

$$\ell_1 = \ell_{i+1} \cdots \ell_j ras, \quad \ell_{j+1} = rbs' \tag{9.8}$$

(5) We have

$$p\bar{p} = \ell_1 \cdots \ell_i \ell_{i+1} \cdots \ell_j rb$$

where $(p, \overline{p})$ is the canonical pair associated with w.

PROOF :

Let $w \in \Sigma^+$, let $(\ell_1, \ldots, \ell_h)$ be a non-increasing maximal chain for the prefix order ($\mathcal{PMC}$) in $\mathrm{CFL}_{in}(w)$, where ℓ_1 is the first element in $\mathrm{CFL}_{in}(w)$.

If $\ell_1 \cdots \ell_h$ is not an inverse Lyndon word, then by Proposition 9.6, items (1)-(4) are satisfied. Set $p_1 = \ell_1 \cdots \ell_i$, $p_2 = \ell_{i+1} \cdots \ell_j rb$. We claim that

$$p_1 p_2 = \ell_1 \cdots \ell_i \ell_{i+1} \cdots \ell_j rb = p\bar{p}$$

where $(p, \overline{p})$ is the canonical pair associated with w.

Indeed, by Eq. (9.8), we have

$$p_1 = \ell_1 \cdots \ell_i \ll \ell_{i+1} \cdots \ell_j rb = p_2$$

which implies

$$p_1 p_2 \ll \ell_{i+1} \cdots \ell_j rb$$

where $\ell_{i+1} \cdots \ell_j rb$ is a suffix of $p_1 p_2$, that is, $p_1 p_2$ is not an inverse Lyndon word.

Moreover, for each proper nonempty prefix u of p_2, u is also a proper prefix of ℓ_1, hence there is $v \in \Sigma^*$ such that $p_1 u = (\ell_1)^i u = (uv)^i u$ is a sesquipower of the anti-Lyndon word ℓ_1, hence, by Proposition 5.2, $p_1 u$ is an inverse Lyndon word. Moreover, by Lemma 5.1 and Proposition 9.2, the word p_1 and all its nonempty prefixes are inverse Lyndon words. This shows that $p_1 p_2$ is the shortest prefix of $\ell_1 \cdots \ell_h$ which is not an inverse Lyndon word, hence, by Proposition 6.1, $p_1 p_2 = p\bar{p}$, where $(p, \overline{p})$ is the canonical pair associated with w. This finishes the proof. ∎

Proposition 9.1 shows that to obtain $\mathrm{ICFL}(w)$ from $\mathrm{CFL}_{in}(w)$ we can limit ourselves to the case in which $\mathrm{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$ is a chain with respect to the prefix order. Now, if $\ell_1 \cdots \ell_h$ is an inverse Lyndon word, then $\mathrm{ICFL}(w)$ can be easily obtained from $\mathrm{CFL}_{in}(w)$ by Proposition 9.7. Otherwise, Proposition 9.8 allows us to determine the canonical pair $(p, \overline{p})$ associated with w from $\mathrm{CFL}_{in}(w)$ and then, recursively, $\mathrm{ICFL}(w)$. Examples 9.1 and 9.2 should clarify this procedure.

Example 9.1 Let $\Sigma = \{a, b\}$ with $a < b$. Let us consider $w = babaababaababab$. The word w is not an inverse Lyndon word because $w = babaababaababab \ll babab$. Moreover

$$\mathrm{CFL}_{in}(w) = (babaa, babaa, ba, ba, b) = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$$

Of course, $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$ is a non-increasing maximal chain for the prefix order in $\mathrm{CFL}_{in}(w)$, where ℓ_1 is the first element in $\mathrm{CFL}_{in}(w)$. Items (1)-(3) in Proposition 9.8 are satisfied with $i = 2$ and $j = 4$. As for Eq. (9.8) in Proposition 9.8, we have

$$\ell_1 = babaa = \ell_3\ell_4 a, \quad \ell_5 = b$$

By item (5) in Proposition 9.8, for the canonical pair $(p, \overline{p})$ associated with w, we have

$$p\bar{p} = \ell_1\ell_2\ell_3\ell_4\ell_5$$

We consider Proposition 6.2 to determine p and $\bar{p}$. By item (2) in Proposition 6.2, $\bar{p}$ is different from b because w does not start with a, hence $\bar{p}$ ends with ab and we have to look for the occurrences of the factor aa in $p\overline{p}$. There are two occurrences of aa as a factor of $p\bar{p}$ and, by applying Proposition 6.2, we see that

$$p = babaababaa = \ell_1\ell_2, \quad \bar{p} = babab = \ell_3\ell_4\ell_5$$

Finally, by Proposition 9.2, $\bar{p} = babab$ is an inverse Lyndon word and, by Definition 6.2, we have

$$\mathrm{ICFL}(w) = (p, \overline{p}) = (babaababaa, babab)$$

Example 9.2 Let $\Sigma = \{a, b, c, d\}$, $a < b < c < d$, and $y = dabadabdabdabdadac$ (see Example 7.2). We have

$$\mathrm{CFL}_{in}(y) = (daba, dab, dab, dab, dadac)$$

By Proposition 9.1,

$$\mathrm{ICFL}(y) = (\mathrm{ICFL}(dabadabdabdab), \mathrm{ICFL}(dadac))$$

The word $z = dabadabdabdab$ is not an inverse Lyndon word and, by Proposition 9.8, for the canonical pair $(p, \overline{p})$ associated with w, we have $p\bar{p} = dabadabd$. Thus $p = daba$, $\bar{p} = dabd$. Then, we compute $\mathrm{ICFL}(dabdabdab)$. By Propositions 9.2 and 9.7, $\mathrm{ICFL}(dabdabdab) = ((dab)^3)$. Therefore, $m_1 = p$ and

$$\mathrm{ICFL}(dabadabdabdab) = (daba, \mathrm{ICFL}(dabdabdab)) = (daba, (dab)^3)$$

Of course $\mathrm{ICFL}(dadac) = dadac$, hence $\mathrm{ICFL}(y) = (daba, (dab)^3, dadac)$.

10 Conclusions

The Lyndon factorization and the canonical inverse Lyndon factorization play a crucial role in various applications of sequence comparison and combinatorial pattern matching [6, 31, 32]. Although the two types of factorization are related by the notion of the anti-Lyndon word, their connections still remained unexplored. This paper addresses this open problem. More precisely, the main contribution is to show how to obtain the Lyndon factorization $\mathrm{CFL}_{in}(w)$ of w with respect to the inverse order from the canonical inverse Lyndon factorization $\mathrm{ICFL}(w)$ and vice versa how to group factors of $\mathrm{CFL}_{in}(w)$ to obtain $\mathrm{ICFL}(w)$. This result on the connection between the classical Lyndon factorization and the unexplored inverse Lyndon factorization opens up new research directions. For instance, the search for new bijective variants of the Burrows Wheeler Transform, based on multisets of inverse words instead of multisets of Lyndon words. On the other hand the characterizations provided in the paper can be used to better explore whether many results already known for $\mathrm{CFL}(w)$ can be easily extended to $\mathrm{ICFL}(w)$. Finally, an interesting question is whether it is possible to find an alternative definition of the canonical inverse Lyndon factorization based directly only on the grouping of $\mathrm{CFL}_{in}(w)$. The current definition is based on the use of the pair $(p, \bar{p})$ and is not immediate to understand.

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