On the weak reducing pairs in critical Heegaard splitting

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Abstract

A weak reducing pair in a Heegaard splitting $M = V \cup_S W$ is a pair of disjoint essential disks $D \in V$ and $E \in W$. The weakly reducible Heegaard splitting contains at least one weak reducing pair. Critical Heegaard splitting is a special case of weakly reducible Heegaard splitting which contains at least two weak reducing pairs satisfying some special conditions. In this paper, we discuss the properties of weak reducing pairs in a critical Heegaard splitting and give a necessary condition for Heegaard surface to be critical.

Keywords: 3-manifold; disk complex; critical Heegaard splitting; weak reducing pair
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1 Introduction

In [1], Casson-Gordon defined the weakly reducible and strongly irreducible Heegaard splitting. The assumption that a Heegaard splitting is strongly irreducible has proved to be much more useful than the assumption that it is of minimal genus. There have been some generalization of these definitions. For example, in

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[2] Hempel defined the distance of Heegaard splitting and it is easy to see that distance 0 means a Heegaard splitting is reducible, distance 1 means a Heegaard splitting is weakly reducible and distance greater than or equal to 2 means a Heegaard splitting is strongly irreducible.

In [3], Bachman defined a notion of critical surface, which can be regarded as a topological index 2 minimal surface, see [4]. It is easy to see that if a Heegaard surface is critical, then it is weakly reducible. Critical surface behave in some way similarly as incompressible surface and strongly irreducible surface do. Some results and properties of the critical Heegaard surface have been know, see for example [5]-[9].

Generalized Heegaard splitting was defined in [10]. It's easy to see that a weakly reducible Heegaard splitting has a generalized Heegaard splitting. So a critical Heegaard splitting also has a generalized Heegaard splitting. In [8], Lee gave a sufficient condition about when the amalgamation of two special Heegaard splittings must be critical. In the present work, we study the properties of weak reducing pairs of a critical Heegaard splitting, and then give a necessary condition for Heegaard surface to be critical which is related to the generalized Heegaard splitting.

The article is organized as follows. In Section 2, we introduce some basic definitions. In Section 3, we discuss the properties of weak reducing pairs in a critical Heegaard splitting, see Theorem 3.3. In Section 4, we give a necessary condition for Heegaard surface to be critical, see Theorem 4.4.

2 Preliminaries

In this section, we introduce some basic definitions and some useful results.

Throughout this paper, denote the intersection number of the objects A and B by $|A \cap B|$.

Let M be a closed orientable irreducible 3-manifold, S a closed orientable separating surfaces in M, dividing M into two submanifold V and W. If V and W are all handlebodies, then $V \cup_S W$ is called a Heegaard splitting of M and S is called the Heegaard surface. If there exist essential disks $D \in V$ and $E \in W$ with $\partial D = \partial E$, then the Heegaard splitting $V \cup W$ is called reducible. If there exist essential disks $D \in V$ and $E \in W$ with $D \cap E = \phi$, then the Heegaard splitting

$V \cup W$ is called weakly reducible. If for any essential disks $D \in V$ and $E \in W$, $D \cap E \neq \phi$, then the Heegaard splitting $V \cup W$ is called strongly irreducible. If there exist essential disks $D \in V$ and $E \in W$ with $|\partial D \cap \partial E| = 1$, then the Heegaard splitting $V \cup W$ is called unstabilized.

Let M be a closed orientable irreducible 3-manifold with Heegaard splitting $V \cup_S W$. Define the disk complex $\mathcal{D}_S$ of S as follows. Vertices of $\mathcal{D}_S$ are isotopy classes of compressing disks for S. A collecting of $k+1$ distinct vertices constitute a $k-$cell if there are pairwise disjoint representatives.

By an abuse of terminology, we sometimes identify a vertex with some representative compressing disk of the vertex. Let $\mathcal{D}_S(V)$ and $\mathcal{D}_S(W)$ be the subcomplexes of $\mathcal{D}_S$ spanned by compressing disks in V and W, respectively.

Definition 2.1. ([4])

A surface S is critical if vertices of $\mathcal{D}_S$ can be partitioned into two non-empty set $\mathcal{C}_0$ and $\mathcal{C}_1$:

(1) For each $i = 0, 1$, there is at least one pair of compressing disks $D_i \in \mathcal{D}_S(V) \cap \mathcal{C}_i$ and $E_i \in \mathcal{D}_S(W) \cap \mathcal{C}_i$ such that $D_i \cap E_i = \phi$.

(2) If $D \in \mathcal{D}_S(V) \cap \mathcal{C}_i$ and $E \in \mathcal{D}_S(W) \cap \mathcal{C}_{1-i}$, then $D \cap E \neq \phi$ for any representative disks. Namely, D and E are not joined by an edge.

If the Heegaard surface S is critical, then we say $V \cup_S W$ is a critical Heegaard splitting.

Definition 2.2. *A weak reducing pair, denoted by $\{D, E\}$, in a Heegaard splitting $V \cup_S W$ is a pair of essential disks $D \in V$ and $E \in W$ with $D \cap E = \phi$.*

By the definition, it is easy to see that if a Heegaard splitting $V \cup_S W$ is critical, then there exist at least two weak reducing pairs.

The definition of untelescoping was introduced in [10]. Let $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ be Heegaard splittings of 3-manifolds M_1 and M_2, respectively. Suppose $F_1 \subset \partial_- W_1$ and $F_2 \subset \partial_- W_2$ are two homeomorphic boundary components. Glue M_1 and M_2 together along F_1 and F_2. Let $M = M_1 \cup_F M_2$ and F be the image of F_1 and F_2 in M. Now collapse $(F_1 \cup F_2) \times [0, 1]$ to F and regard the 1-handles of W_1 and W_2 are attached to F. Let $V = V_1 \cup \{1 - \textit{handles in } W_2\}$ and $W = V_2 \cup \{1 - \textit{handles in } W_1\}$ and $S = V \cap W$. Then $V \cup_S W$ is called an amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$. It is easy to see that $V \cup_S W$ is a weakly reducible Heegaard splitting. Conversely, $(V_1 \cup_{S_1} W_1) \cup_F (V_2 \cup_{S_2} W_2)$ is called an untelescoping of $V \cup_S W$. Some times, $(V_1 \cup_{S_1} W_1) \cup_F (V_2 \cup_{S_2} W_2)$ is also called a generalized Heegaard splitting of M. See Fig.1.

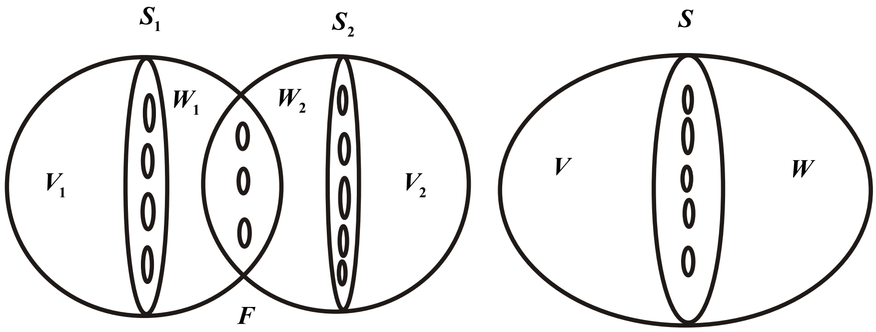

Figure 1: Amalgamation and untelescoping.

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3 The weak reducing pairs in Heegaard splitting

In this section, we discuss the properties of weak reducing pairs in Heegaard splitting.

Let V be a handlebody, $\{D_1, \cdots, D_t\} \subset V$ a set of disjoint essential disks in V. Then each component of $V \setminus (D_1 \cup \cdots \cup D_t)$, denoted it by $V_{m,n}$, is a handlebody. Here, the subscript means $V_{m,n}$ has genus m and n cutting sections of disks on its boundary.

Lemma 3.1. *Let $M = V \cup_S W$ be a weakly reducible unstabilized Heegaard splitting of a closed irreducible orientable 3-manifold M, $g(S) = g \geq 3$. Suppose $\{D_1, E_1\}$ and $\{D_2, E_2\}$ are two weak reducing pairs with $D_i \in \mathcal{D}_V (i = 1, 2)$, $E_i \in \mathcal{D}_W (i = 1, 2)$, $D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$, $D_1 \cap D_2 = \phi$ and $E_1 \cap E_2 \neq \phi$. Then there exist two weak reducing pairs, also denoted them by $\{D_1, E_1\}$ and $\{D_2, E_2\}$, such that $D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$, $D_1 \cap D_2 = \phi$ and $E_1 \cap E_2 = \phi$.*

Proof. Without loss of generality, we assume that each pair of essential disks with nonempty intersection intersect each other minimally among all isotopic disks. Using the usually cutting and pasting method, we know that $E_1 \cap E_2$ has no circle intersections, so $E_1 \cap E_2$ only has arc intersections. If $|E_1 \cap E_2| = 0$, then there is nothing to be proved. From now on, we assume that $|E_1 \cap E_2| > 0$.

Now there are three cases needed to be considered.

Case 1: Both D_1 and D_2 are separating in V. See Fig.2.

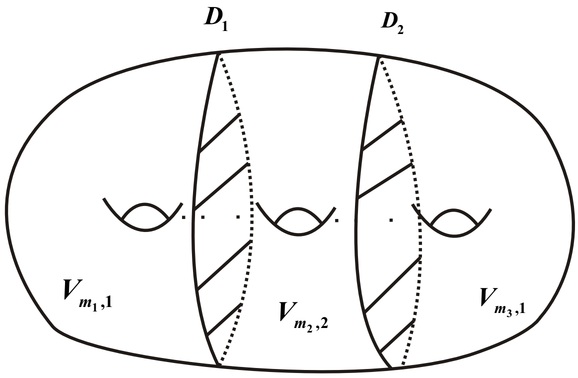

Figure 2: Both D_1 and D_2 are separating.

At this time, $V \backslash (D_1 \cup D_2) = V_{m_1,1} \cup V_{m_2,2} \cup V_{m_3,1}$, where $m_i \geq 1 (i = 1, 2, 3)$, $m_1 + m_2 + m_3 = g$. Without loss of generality, assume $V_{m_1,1}$ contains one cutting section of D_1 on its boundary, $V_{m_3,1}$ contains one cutting section of D_2 on its boundary and $V_{m_2,2}$ contains both cutting sections of D_1 and D_2 on its boundary. By the conditions of the theorem, we know $\partial E_1 \in V_{m_2,2} \cup V_{m_3,1}$ and $\partial E_2 \in V_{m_1,1} \cup V_{m_2,2}$.

Now consider ∂E_2. Since $D_1 \cap E_2 \neq \phi$, we can choose an arc of $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by α, which intersects $\partial V_{m_1,1}$ nonempty. Then we choose an arc of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by β, such that after isotopy, $\alpha \cup \beta$ bounds an essential disk E_2' in the handlebody W which is disjoint from E_1 and E_2. Isotopy E_2' such that $\mid E_2' \cap D_1 \mid$ is minimal. If $\mid E_2' \cap D_1 \mid \neq 0$, then replace E_2 by E_2' and we get the conclusion. If $\mid E_2' \cap D_1 \mid = 0$, then it is easy to see that at this time, $\partial E_2' \subset \partial V_{m_1,1}$. So there exists an essential nonseparating disk D_1' in $V_{m_1,1}$ which intersects $\partial E_2'$ nonempty. Now substitude D_1' and E_2' for D_1 and E_2, respectively. Thus, we get the conclusion.

Case 2: One of D_1 and D_2, without loss of generality, assume D_1, is separating in V, and D_2 is nonseparating in V. See Fig.3.

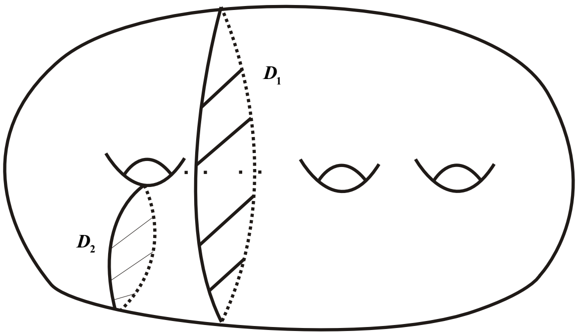

Figure 3: D_1 is separating and D_2 is nonseparating.

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At this time, $V \setminus D_1 = V_{m_1,1} \cup V_{m_2,1}$ with $m_i \geq 1 (i = 1, 2)$ and $m_1 + m_2 = g$. Without loss of generality, assume $D_2 \in V_{m_1,1}$. Now consider ∂E_2. We can choose an essential arc of $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by α, which intersects $\partial V_{m_2,1}$ nonempty, and an essential arc of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by β, such that after isotopy, $\alpha \cup \beta$ bounds an essential disk E_2' in W with $E_2' \cap (E_1 \cup E_2) = \phi$. If $|E_2' \cap D_1| \neq 0$, then replace E_2 by E_2' and we get the conclusion. If $|E_2' \cap D_1| = 0$, then we can choose an essential nonseparating disk D_1' in $V_{m_2,1}$ such that $| D_1' \cap E_2' | \neq 0$. Then substitude D_1' and E_2' for D_1 and E_2, respectively. Thus, we get the conclusion.

Case 3: Both D_1 and D_2 are nonseparating in V.

Then there are two subcases needed to be considered.

Subcase 3.1: $D_1 \cup D_2$ is nonseparating in V. See Fig.4.

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At this time, consider one arc of $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$ which intersects ∂D_1 nonempty. Denote it by α. Since $E_1 \cap D_1 = \phi$, we can choose an essential arc of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by β, such that $\alpha \cup \beta$ bounds an essential disk E_2' with $E_2' \cap (E_1 \cup E_2) = \phi$ and $E_2' \cap D_1 \neq \phi$. Now replace E_2 by E_2' and we get the conclusion.

Subcase 3.2: $D_1 \cup D_2$ is separating in V. See Fig.5.

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Let $V \setminus (D_1 \cup D_2) = V_{m_1,2} \cup V_{m_2,2}$ with $m_i \geq 1 (i = 1, 2)$ and $m_1 + m_2 = g$.

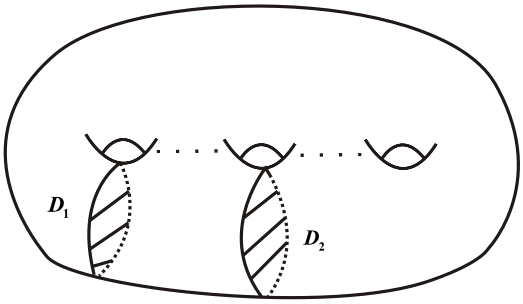

Figure 4: $D_1 \cup D_2$ is nonseparating.

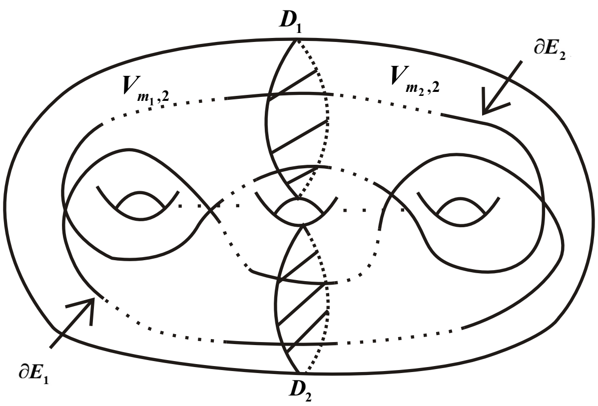

Figure 5: $D_1 \cup D_2$ is separating.

Since D_1 and D_2 are nonseparating in V and $D_1 \cup D_2$ is separating in V, we know that $\mid \partial D_1 \cap \partial E_2 \mid$ and $\mid \partial D_2 \cap \partial E_1 \mid$ are all even.

Now consider the arcs of $\partial E_2 \cap V_{m_1,2}$ and $\partial E_1 \cap V_{m_1,2}$. Since $\mid E_1 \cap E_2 \mid$ is minimal, we can choose one arc of $\partial E_2 \setminus (E_1 \cap E_2)$, say α, in $\partial V_{m_1,2}$ and one arc of $\partial E_1 \setminus (E_1 \cap E_2)$, say β, in $\partial V_{m_1,2}$ such that $\alpha \cup \beta$ bounds an essential disk E_2' in W with $E_2' \cap (E_1 \cup E_2) = \phi$. Since $\alpha \cup \beta$ is essential in $\partial V_{m_1,2}$, we can choose an nonseparating essential disk D_1' in $V_{m_1,2}$ such that $D_1' \cap E_2' \neq \phi$. Now substitude D_1' and E_2' for D_1 and E_2, respectively. Similarly, we can choose one arc of $\partial E_2 \setminus (E_1 \cap E_2)$, say α', in $\partial V_{m_2,2}$ and one arc of $\partial E_1 \setminus (E_1 \cap E_2)$, say β', in $\partial V_{m_2,2}$ such that $\alpha' \cup \beta'$ bounds an essential disk E_1' in W with $E_1' \cap (E_1 \cup E_2) = \phi$. Since $\alpha' \cup \beta'$ is essential in $\partial V_{m_2,2}$, we can choose an nonseparating essential disk D_2' in $V_{m_2,2}$ such that $D_2' \cap E_1' \neq \phi$. Now substitude D_2' and E_1' for D_2 and E_1, respectively. Thus, we get the conclusion.

□

Lemma 3.2. *Let $M = V \cup_S W$ be a weakly reducible unstabilized Heegaard splitting of a closed irreducible orientable 3-manifold M, $g(S) = g \geq 3$. Suppose $\{D_1, E_1\}$ and $\{D_2, E_2\}$ are two weak reducing pairs with $D_i \in \mathcal{D}_V (i = 1, 2)$, $E_i \in \mathcal{D}_W (i = 1, 2)$, $D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$, $D_1 \cap D_2 \neq \phi$ and $E_1 \cap E_2 \neq \phi$. Then there exist two weak reducing pairs, also denoted them by $\{D_1, E_1\}$ and $\{D_2, E_2\}$, with $D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$ satisfying that either $D_1 \cap D_2 = \phi$ or $E_1 \cap E_2 = \phi$.*

Proof. Without loss of generality, we assume that each pair of essential disks with nonempty intersection intersect each other minimally among all isotopic disks. Using the usually cutting and pasting method, we know that both $D_1 \cap D_2$ and $E_1 \cap E_2$ have no circle intersections, so they only have arc intersections. From now on, we assume that $|D_1 \cap D_2| > 0$ and $|E_1 \cap E_2| > 0$.

Without loss of generality, we consider D_1 and D_2. Now there are three cases needed to be considered.

Case 1: Both D_1 and D_2 are separating in V. See Fig.6.

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Suppose $V \setminus D_1 = V_{m_1,1} \cup V_{m_2,1}$. Since $D_1 \cap E_1 = \phi$, we can assume $\partial E_1 \subset \partial V_{m_1,1}$. For $D_1 \cap D_2 \neq \phi$, it is easy to see that D_2 cuts $V_{m_1,1}$ into several parts. Consider ∂E_2. Since D_2 is separating in V, $D_2 \cap E_2 = \phi$ and $E_1 \cap E_2 \neq \phi$,

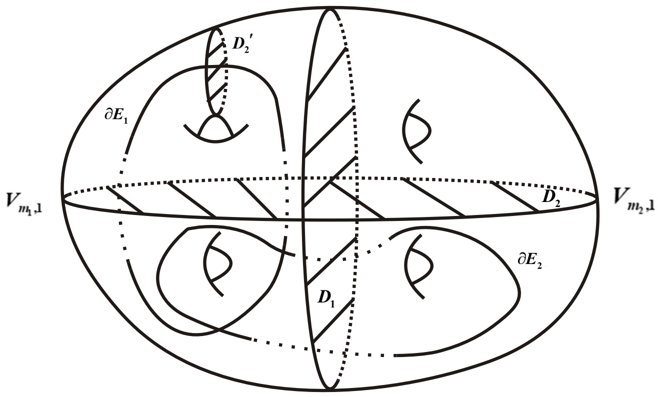

Figure 6: Both D_1 and D_2 are separating.

there exists one component of $V_{m_1,1} \setminus D_2$ which has no intersection with ∂E_2 but intersects ∂E_1 nonempty. Now choose an essential disk D_2' in this component with $D_2' \cap \partial E_1 \neq \phi$, $D_2' \cap (D_1 \cup D_2) = \phi$. So $D_2' \cap E_2 = \phi$. Replace D_2 by D_2'. Then we get the conclusion.

Case 2: One of D_1 and D_2, without loss of generality, assume D_1, is separating in V, and D_2 is nonseparating in V. See Fig.7.

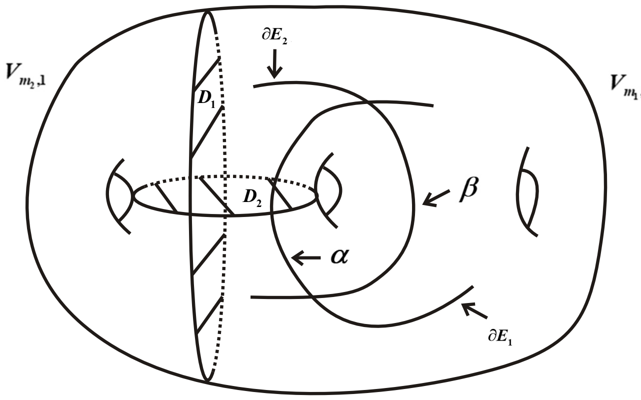

Figure 7: D_1 is separating and D_2 is nonseparating.

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Suppose $V \setminus D_1 = V_{m_1,1} \cup V_{m_2,1}$. Since $D_1 \cap E_1 = \phi$, we can assume $\partial E_1 \subset \partial V_{m_1,1}$. Consider the arcs of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$ and $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$. Choose one arc of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$ which intersects ∂D_2 nonempty. Denote it by α.

At this time, we can choose one arc of $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by β, such that β contains in $\partial V_{m_1,1}$ and $\alpha \cup \beta$ bounds an essential disks E_1' in W with $E_1' \cap (E_1 \cup E_2) = \phi$, $E_1' \cap D_2 \neq \phi$. Replace E_1 by E_1'. Thus, we get the conclusion.

Case 3: Both D_1 and D_2 are nonseparating in V. See Fig.8.

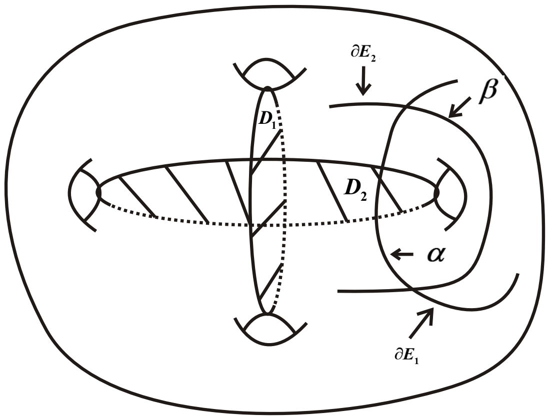

Figure 8: Both D_1 and D_2 are nonseparating.

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Consider the arcs of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$ and $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$. Choose one arc of $\partial E_1 \setminus (\partial E_1 \cap \partial E_2)$ which intersects ∂D_2 nonempty. Denote it by α. Since $E_2 \cap D_2 = \phi$, we can choose one arc of $\partial E_2 \setminus (\partial E_1 \cap \partial E_2)$, denoted it by β, such that $\alpha \cup \beta$ bounds an essential disks E_1' in W with $E_1' \cap (E_1 \cup E_2) = \phi$, $E_1' \cap D_2 \neq \phi$. Replace E_1 by E_1'. Thus, we get the conclusion.

□

Using the above lemmas, we have the following main theorem.

Theorem 3.3. *Let $M = V \cup_S W$ be a weakly reducible unstabilized Heegaard splitting of a closed irreducible orientable 3-manifold M, $g(S) = g \geq 3$. If S is critical with $\mathcal{D}_S = \mathcal{C}_0 \cup \mathcal{C}_1$, then there exists two weak reducing pairs $\{D_1, E_1\} \subset \mathcal{C}_0$ and $\{D_2, E_2\} \subset \mathcal{C}_1$ with $D_i \in \mathcal{D}_V (i = 1, 2)$, $E_i \in \mathcal{D}_W (i = 1, 2)$, $D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$ such that $D_1 \cap D_2 = \phi$ and $E_1 \cap E_2 = \phi$.*

Proof. Since S is critical, there exist two weak reducing pairs $\{D_1, E_1\} \subset \mathcal{C}_0$ and $\{D_2, E_2\} \subset \mathcal{C}_1$ with $D_i \in \mathcal{D}_V (i = 1, 2)$, $E_i \in \mathcal{D}_W (i = 1, 2)$, $D_1 \cap E_2 \neq \phi$ and $D_2 \cap E_1 \neq \phi$. If $D_1 \cap D_2 \neq \phi$ and $E_1 \cap E_2 \neq \phi$, then by Lemma 3.2, there exist two weak reducing pairs, also denoted them by $\{D_1, E_1\}$ and $\{D_2, E_2\}$, with $D_1 \cap D_2 = \phi$ or $E_1 \cap E_2 = \phi$. Then by Lemma 3.1, we get the conclusion.

□

4 A necessary condition for Heegaard surface to be critical

In this section, we give a necessary condition for Heegaard surface to be critical.

Let V be a handlebody with $g(V) \geq 2$, D_1 and D_2 two disjoint essential disks in V, α an essential arc in ∂V with two points of $\partial\alpha$ belong to ∂D_1 and ∂D_2, respectively. Then the boundary curve of regular neighborhood of $D_1 \cup N(\alpha) \cup D_2$ bounds a disk E in V. If E is essential in V, then E is called the band-sum of D_1 and D_2 along α.

Definition 4.1. *Let V be a handlebody with $g(V) \geq 2$, D_1 and D_2 essential disks in V. If one of them, say D_1, is a band-sum of D_2 and a copy of D_2 along some essential arc α, then D_1 and D_2 are called related. See Fig.9.*

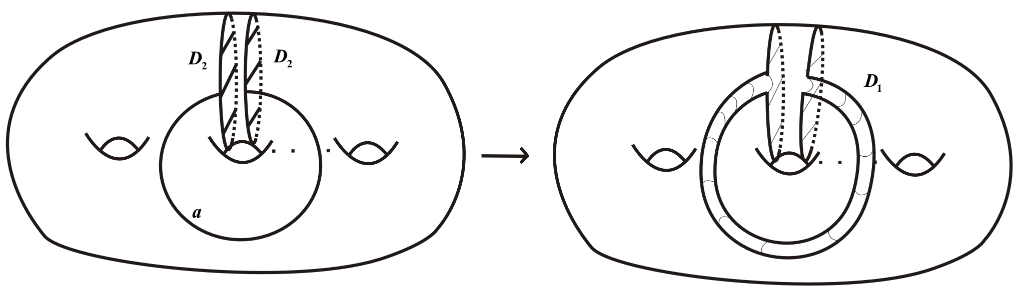

Figure 9: Band-sum of essential disk D along essential arc α.

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Remark 4.2. *(1) It is easy to see that if D_1 is a band-sum of an essential disk D_2 and a copy of D_2 along an essential arc α in handlebody V, then D_1 is separating in V. In fact, the separating disk D_1 cuts the handlebody V into two handlebodies* *and one of them is a solid torus containing the essential disk D_2 as its meridian disk.*

(2) If D_1 is a band-sum of an essential D_2 and a copy of D_2 along an essential arc α in a handlebody V, then $V \setminus D_2$ is a handlebody with $g(V \setminus D_2) = g(V) - 1$ and D_1 is boundary parallel in $V \setminus D_2$.

Lemma 4.3. *Let $M = V \cup_S W$ be a weakly reducible unstabilized Heegaard splitting of a closed irreducible orientable 3-manifold M, $g(S) = g \geq 3$. Suppose $\{D_1, E_1\}$ and $\{D_2, E_2\}$ are two weak reducing pairs with $D_i \in \mathcal{D}_V(i = 1, 2)$, $E_i \in \mathcal{D}_W(i = 1, 2)$, $D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$, $D_1 \cap D_2 = \phi$ and $E_1 \cap E_2 = \phi$. Then D_1 and D_2 are not related, and E_1 and E_2 are not related, too.*

Proof. If not the case. Without loss of generality, assume D_2 is a band-sum of D_1 along some essential arc. See Fig.10.

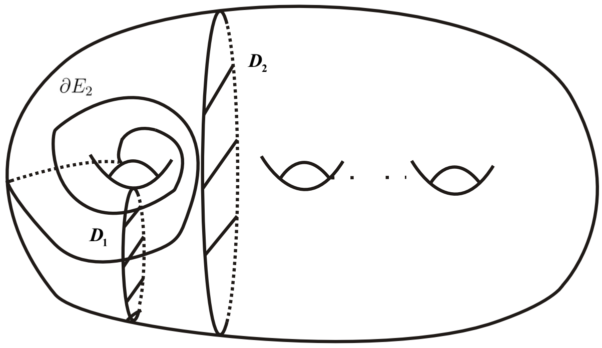

Figure 10: Two essential disks are related.

.

Suppose $V \setminus D_2 = V_{m_1,1} \cup V_{m_2,1}$ with $m_1 = 1$. So D_1 is a meridian disk in $V_{m_1,1}$. By the assumption, $\partial E_2 \in \partial V_{m_1,1}$ and $\partial E_2 \cap \partial D_1 \neq \phi$. Now the component of $\partial V \setminus (\partial D_1 \cup \partial D_2)$ which intersects $V_{m_1,1}$ nonempty is a pair of pants P with $\partial P = e_1 \cup e_2 \cup e_3$. Suppose e_1 is the cutting section of ∂D_2. Then the arcs of $\partial E_2 \setminus (\partial E_2 \cap \partial D_1)$ have end points on e_2 and e_3, and these arcs cut the pants P into some disks and an annulus A with $e_1 \subset \partial A$.

Since $\partial E_1 \cap \partial D_2 \neq \phi$, $\partial E_1 \cap \partial D_1 = \phi$ and $\partial E_1 \cap \partial E_2 = \phi$, the arcs of $\partial E_1 \setminus (\partial E_1 \cap \partial D_2)$ which contains in $\partial V_{m_1,1}$ must contain in annulus A with the end points of these arcs containing in e_1. But essential arcs in annulus A must have end points on different boundaries of A. So the arcs of $\partial E_1 \setminus (\partial E_1 \cap \partial D_2)$ are boundary parallel in annulus A. After isotopy, we have $E_1 \cap D_2 = \phi$. But this contradicts the assumption.

$\square$

Based on the above discussion, we have the following theorem.

Theorem 4.4. *Let M be a closed irreducible orientable 3-manifold, $V \cup_S W$ a weakly reducible unstabilized Heegaard splitting of M with genus $g(S) \geq 3$. Suppose the Heegaard surface S is critical with $\mathcal{D}_S = \mathcal{C}_0 \cup \mathcal{C}_1$, $D_i \in \mathcal{D}_V(i = 1, 2)$, $E_i \in \mathcal{D}_W(i = 1, 2)$, $\{D_1, E_1\} \subset \mathcal{C}_0$, $\{D_2, E_2\} \subset \mathcal{C}_1$, $D_i \cap E_i = \phi(i = 1, 2)$, $D_1 \cap E_2 \neq \phi$, and $D_2 \cap E_1 \neq \phi$. Then $V \cup_S W$ has a generalized Heegaard splitting $(V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2)$, which is obtained from weak reduction along weak reducing pair $\{D_1, E_1\}$ and $D_2 \in \mathcal{D}_{V_1}$, $E_2 \in \mathcal{D}_{V_2}$.*

Proof. By Theorem 3.3, we can choose two weak reducing pairs, also denoted by $\{D_1, E_1\}$, $\{D_2, E_2\}$, in $\mathcal{D}_S$ with $D_i \in \mathcal{D}_V(i = 1, 2)$, $E_i \in \mathcal{D}_W(i = 1, 2)$, $\{D_1, E_1\} \subset \mathcal{C}_0$, $\{D_2, E_2\} \subset \mathcal{C}_1$,$D_1 \cap E_2 \neq \phi$, $D_2 \cap E_1 \neq \phi$, $D_1 \cap D_2 = \phi$, $E_1 \cap E_2 = \phi$. Let $V_1 = V \setminus D_1$, $W_1 = (\partial_+ V_1 \times I) \cup (2-handle\ of\ E_1)$, $W_2 = (\partial_- W_1 \times I) \cup (1-handle\ of\ D_2)$, $V_2 = M \setminus (V_1 \cup W_1 \cup W_2)$. Then V_2 is isotopic to $W \setminus E_1$.

By Lemma 4.3, D_1 and D_2 are not related, and E_1 and E_2 are not related. Then by the Remark 4.2, D_2 is an essential disk in V_1 and E_2 is an essential disk in V_2. Then we get the conclusion.

$\square$

Remark 4.5. *In Theorem 4.4, a critical Heegaard splitting has a generalized Heegaard splitting $(V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2)$ with two disjoint essential disk D_2 and E_2 persist into V_1 and V_2, respectively. Although essential disk $E_1 \in W_1$ intersects the essential disk $D_1 \in V_1$, it is not sure that $V_1 \cup_{S_1} W_1$ is strongly irreducible. The situation with $W_2 \cup_{S_2} V_2$ is the same. In [8], a sufficient condition for a critical Heegaard splitting is that the critical Heegaard splitting is an amalgamation of two strongly irreducible Heegaard splittings $V_1 \cup_{S_1} W_1$ and $W_2 \cup_{S_2} V_2$ along two homeomorphic boundary components of $\partial_- W_1$ and $\partial_- W_2$ with two essential disks in V_1 and V_2 persist into disjoint essential disks in $V \cup_S W$, respectively.*

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