Octahedral coordinates from the Wirtinger presentation

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Abstract

Let ρ be a representation of a knot group (or more generally, the fundamental group of a tangle complement) into $\mathrm{SL}_2(\mathbb{C})$ expressed in terms of the Wirtinger generators of a diagram D. In this note we give a direct algebraic formula for the geometric parameters of the octahedral decomposition of the knot complement associated to D. Our formula gives a new, explicit criterion for whether ρ occurs as a critical point of the diagram's Neumann-Zagier–Yokota potential function.

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Contents

1. Introduction

Studying representations of knot and 3-manifold groups into $\mathrm{SL}_2(\mathbb{C})$ is important in low-dimensional topology. The isometry group of hyperbolic 3-space is $\mathrm{PSL}_2(\mathbb{C})$, so this problem is closely related to the study of hyperbolic structures [Thu02]. While one can study representations algebraically, say via the Wirtinger presentation of the knot group, it is sometimes better to work more geometrically. For example, for geometric invariants like the hyperbolic volume it is usually better to work with the geometric parameters of an ideal triangulation of the knot complement than directly with ρ.

[Thu02] W. Thurston, *The geometry and topology of three-manifolds*

In this paper we are concerned with a particular family of ideal triangulations adapted to knot diagrams. Given a diagram D of a knot K the *octahedral decomposition* decomposes $S^3 \setminus K$ minus two points into a union of octahedra, with one for each crossing of the diagram. By further subdividing into tetrahedra one can obtain an ideal triangulation of a standard form. This works just as well for links and tangles. The geometric data of the tetrahedra are naturally described by the *octahedral coordinates*; these are implicit in work of Yokota [Yok00] and have been studied systematically by Kim, Kim, and Yoon [KKY18]. Our version of these coordinates is motivated by connections to the representation theory of quantum groups [McP22]. When studying geometric properties of a representation the octahedral coordinates are quite useful; for example, they enable a direct computation of the hyperbolic volume and Chern-Simons invariant.

[Yok00] Y. Yokota, *On the volume conjecture for hyperbolic knots.* arXiv

[KKY18] H. Kim, S. Kim, and S. Yoon, “Octahedral developing of knot complement. I: Pseudo-hyperbolic structure”. arXiv DOI

[McP22] C. McPhail-Snyder, *Hyperbolic structures on link complements, octahedral decompositions, and quantum $\mathfrak{sl}_2$.* arXiv

While the octahedral coordinates are geometrically natural they can be difficult to solve for in practice. Instead we might find representations of the knot group by some other method and then try to find the corresponding octahedral coordinates. Given a representation $\rho : \pi_1(S^3 \setminus K) \to \mathrm{SL}_2(\mathbb{C})$ described by its values on Wirtinger generators one can use the methods of Blanchet et al. [Bla+20] to find octahedral coordinates corresponding to ρ. This gives a somewhat complicated inductive method that does not usually produce simple formulas. In addition, some representations do not come from octahedral coordinates (one might have to conjugate first) and there is not a simple way to check when this occurs.

[Bla+20] C. Blanchet et al., “Holonomy braidings, biquandles and quantum invariants of links with $\mathsf{SL}_2(\mathbb{C})$ flat connections”. arXiv DOI

In this paper we give a new, elementary method (Theorem 1) for determining the octahedral coordinates of a tangle diagram directly from ρ. As an application we give an explicit criterion (Theorem 2) for whether a representation appears as a critical point of the potential function of the knot diagram. This question is relevant to the saddle-point method used in most approaches to the Volume Conjecture.

The Volume Conjecture proposes that the asymptotics of the Kashaev invariants of a hyperbolic knot recover geometric information like the hyperbolic volume [Kas97]. The Kashaev invariants are naturally associated with the octahedral decomposition, and most proofs of special cases of the conjecture proceed via studying an associated *potential function* [Yok00]. The potential function is defined on a space parametrizing the geometry of the ideal tetrahedra: it has singularities where they are degenerate and critical points at nondegenerate

[Kas97] R. M. Kashaev, “The hyperbolic volume of knots from quantum dilogarithm”. arXiv DOI solutions of the gluing equations of the triangulation. To apply the saddle-point method to the asymptotics of the Kashaev invariant it is important to use a diagram whose potential function has a smooth critical point at the complete hyperbolic structure. This question has been previously studied using other techniques [SY18; GMT16] and it is known that every hyperbolic knot that is alternating or has at most 12 crossings has such a diagram. We show (Theorem 3) that for knot diagrams this condition is equivalent to being *arc-faithful*, meaning that the Wirtinger generators of the over and under arcs at each crossing are always distinct. This new characterization may prove useful for showing that all hyperbolic knots have arc-faithful diagrams.

[SY18] M. Sakuma and Y. Yokota, "An application of non-positively curved cubings of alternating links". arXiv DOI

[GMT16] S. Garoufalidis, I. Moffatt, and D. P. Thurston, *Non-peripheral ideal decompositions of alternating knots*. arXiv DOI

Quantum topology gives other motivations for our results. For example, one can define [MR24] a geometrically twisted version of the Kashaev invariant that can also be understood as a quantization of the hyperbolic volume. This construction uses the octahedral coordinates in an essential way and we expect the results of this paper to be useful for computing and studying these invariants.

[MR24] C. McPhail-Snyder and N. Reshetikhin, "A quantization of the $\mathfrak{sl}_2$ Chern-Simons invariant of a link exterior"

This paper began as an attempt to re-derive and generalize a formula [Cho18, Theorem 3.12] of Cho for boundary-parabolic representations. Theorem 1 is similar to but distinct from the results of Cho, and our proof is quite different. Cho's result is more closely connected to the quandle $\mathcal{P}$ of parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$, and in particular to a presentation of it due to Inoue and Kabaya [IK14]. It would be interesting to better understand how our results relate to $\mathcal{P}$ and if this can be generalized to the boundary non-parabolic case.

[Cho18] J. Cho, "Quandle theory and the optimistic limits of the representations of link groups". arXiv DOI

[IK14] A. Inoue and Y. Kabaya, "Quandle homology and complex volume". arXiv DOI

Organization

- Sections 2 and 3 contain preliminary information on fundamental groups of tangle complements and octahedral colorings.
- Section 4 describes our formula and applies it to the existence of octahedral colorings.
- Section 5 uses our result to classify which octahedral colorings correspond to critical points of the potential function.

Acknowledgements

I would like to thank Andy Putman for a comment on my MathOverflow question [McP23] about knot groups that led me to the proof of Theorem 3. I would also like to thank Seonhwa Kim for some useful comments on an early draft of the paper and for introducing me to relevant work of Sakuma and Yokota [SY18] and Garoufalidis, Moffatt, and Thurston [GMT16].

[McP23] C. McPhail-Snyder, *Can distinct meridians commute in a knot group?*

Conventions

All tangles are smooth and oriented. Our convention is that tangle diagrams go from left to right, so composition of tangles is horizontal and disjoint union is vertical. Composition of paths is also read left to right: fg means follow f then g. For this reason we use row vectors instead of column vectors. Our sign conventions the and typical labeling of the parts of a crossing are given in Figure 1.

2. The fundamental group(oid) of a tangle complement

Definition 2.1. The *complement* of a tangle T is its complement as a submanifold of $[0,1]^3$. We write $\pi(T)$ for the fundamental group of the complement. Suppose T has n incoming and

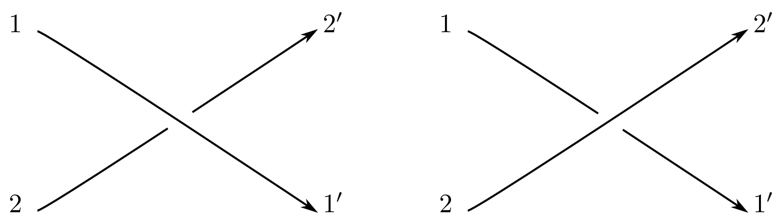


Figure 1: Positive (left) and negative (right) crossings with our standard labeling.

m outgoing boundary points and let D be a diagram of T. We can think of D as a decorated 4-valent graph G embedded in $[0,1] \times [0,1]$ with n edges intersecting $[0,1] \times \{0\}$ and m edges intersecting $[0,1] \times \{1\}$. We assign names to various parts of D:

- The *segments* of D are the edges[1] of G.
- An *arc* of D is a set of adjacent over-segments.
- A *component* of D is a set of adjacent segments; these are in bijection with connected components of T.
- A *region* of D is a connected component of the complement of G, equivalently a vertex of the dual graph of G. Regions are adjacent across segments, and our convention is to say that region j' is *below* region j when they are arranged as in Figure 2.

[1] Usually these are called the "edges" of the diagram, but we do not want to confuse them with edges of ideal polyhedra in the octahedral decomposition.

Theorem 2.2. Let D be a diagram of a tangle T. The choice of D gives a presentation for $\pi(T)$ with one generator w_i for each arc and one relation

$$w_{2'} = w_1^{-1} w_2 w_1 \text{ (positive)} \tag{1}$$

$$w_{1'} = w_2 w_1 w_2^{-1} \text{ (negative)} \tag{2}$$

at each crossing. (At a positive crossing 1 and $1'$ are the same arc.) We call this the *Wirtinger presentation* of $\pi(T)$ and denote it by $\pi(D)$. If i is a segment of D we write w_i for the associated Wirtinger generator. ⌟

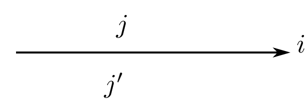


Figure 2: In this case we say that region j' is below region j across segment i. When crossing from j to j' we pick up a positive sign.

Our convention is to place the basepoint of $\pi(D)$ in the top region of the diagram. The generator w_i represents the homotopy class of a path that travels from the basepoint above any intervening strands, wraps around i, then returns above the diagram to the basepoint, as in Figure 5.

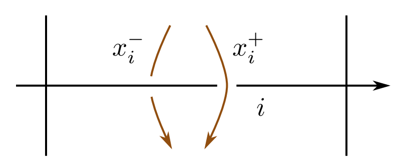


Figure 3: Generators of the fundamental groupoid $\Pi_1(D)$ of a tangle diagram.

To define the holonomy representation of an octahedral coloring and understand how it relates to a representation $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$ we need to introduce more basepoints and think of $\pi(T)$ as a groupoid instead of a group. Recall the characterization of a group G as a category with a single object $\bullet$ and an invertible morphism $g : \bullet \to \bullet$ for each $g \in G$. A *groupoid* is the natural generalization: there can be more than one object, but all morphisms are still invertible. For example, instead of the fundamental group of a topological space with a single basepoint, one can consider a fundamental groupoid with multiple basepoints whose morphisms $p \to q$ are homotopy classes of paths from p to q. Two paths (morphisms) f and g are composable only when the endpoint (codomain) of f is the startpoint (domain) of g. We consider a particular version for tangle complements:

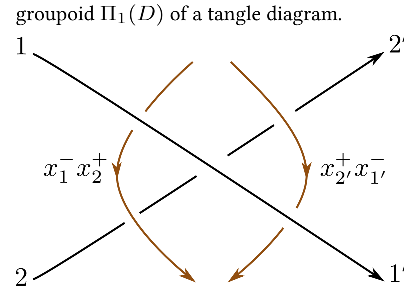


Figure 4: Deriving the middle relation at a crossing.

Definition 2.3. Let D be a tangle diagram. The *fundamental groupoid* $\Pi(D)$ of D has

- one object (i.e. basepoint) for each region of D,

- two generators $x_i^\pm$ for each segment i, representing paths above and below the segment (see Figure 3), and
- three relations for each crossing:

$$x_1^+ x_2^+ = x_{2'}^+ x_{1'}^+ \tag{3}$$

$$x_1^- x_2^- = x_{2'}^- x_{1'}^- \tag{4}$$

$$\begin{cases} x_1^- x_2^+ = x_{2'}^+ x_{1'}^- & \text{for a positive crossing, or} \\ x_1^+ x_2^- = x_{2'}^- x_{1'}^+ & \text{for a negative crossing.} \end{cases} \tag{5}$$

One can show that $\Pi(D)$ is equivalent to the fundamental group $\pi(T)$ and thus to to $\pi(D)$ by using the groupoid version [Bro06, Chapter 9] of the van Kampen theorem. We do not need this result so we omit the details, but we do use a functor $\mathcal{F} : \pi(D) \to \Pi(D)$ constructed using a natural family of paths in $\Pi(D)$:

[Bro06] R. Brown, *Topology and groupoids*

Definition 2.4. Let D be a tangle diagram with topmost region 0 and let j be a region of D. An *over path* s_j^+ is a path $0 \to j$ that passes over all strands of the tangle and similarly for an *under path* s_j^-. It is clear these are unique up to homotopy.

To write these paths as products of generators, choose a path of adjacent regions from 0 to j. If that path crosses segments $i_1, \dots, i_k$, then

$$s_j^+ = \left(s_{i_1}^+\right)^{\epsilon_1} \cdots \left(s_{i_k}^+\right)^{\epsilon_k} \tag{6}$$

where $\epsilon_i = +1$ if segment i is oriented left-to-right as we cross it from the top and -1 otherwise, as in Figure 2. Independence of equation (6) from the choice of path follows from equations (3) and (4).

Theorem 2.5. Let D be a tangle diagram. For each segment i write j for the region above i and set

$$\mathcal{F}(w_i) := s_j^+ \left[x_i^+ x_i^-\right] (s_j^+)^{-1}$$

where w_i is the Wirtinger generator associated to i. Then $\mathcal{F}$ is a well-defined homomorphism (i.e., functor) $\pi(D) \to \Pi(D)$. ⌟

An example is given in in Figure 5.

Proof. We need to check that $\mathcal{F}$ respects the relations of $\pi(D)$ coming from each crossing. Consider a positive crossing whose topmost region has over path s^+ and whose segments are labeled as in Figure 1. We have

$$\mathcal{F}(w_{2'}) = s^+ x_{2'}^+ \left(x_{2'}^-\right)^{-1} \left(s^+\right)^{-1}$$

while

$$\begin{aligned} &\mathcal{F}(w_1^{-1} w_2 w_1) \\ &= \left[s^+ x_1^- \left(x_1^+\right)^{-1} \left(s^+\right)^{-1}\right] s^+ x_1^+ x_2^+ \left(x_2^-\right)^{-1} \left(s^+ x_1^+\right)^{-1} \left[s^+ x_1^+ \left(x_1^-\right)^{-1} \left(s^+\right)^{-1}\right] \\ &= s^+ x_1^- x_2^+ \left(x_2^-\right)^{-1} \left(x_1^-\right)^{-1} \left(s^+\right)^{-1} \end{aligned}$$

and writing (5) and (4) as

$$x_{2'}^+ = x_1^- x_2^+ \left(x_{1'}^-\right)^{-1} \text{ and } \left(x_{2'}^-\right)^{-1} = x_{1'}^- \left(x_2^-\right)^{-1} \left(x_1^-\right)^{-1}$$

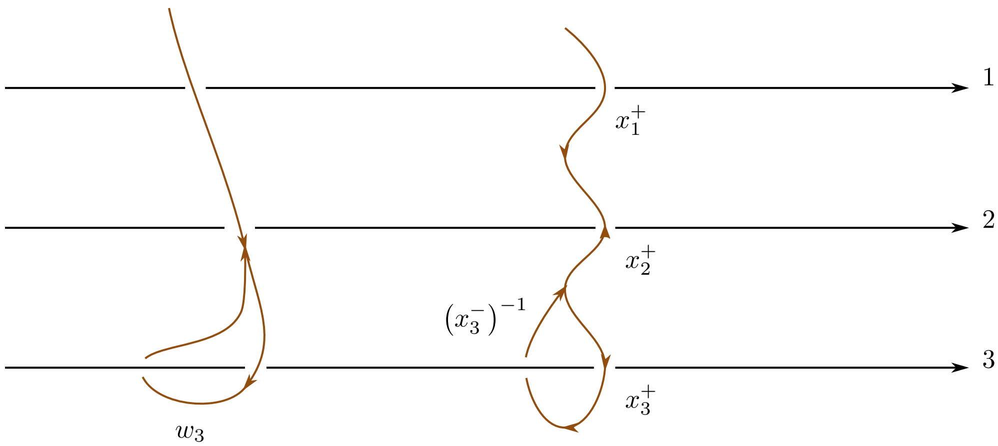


Figure 5: For this tangle the Wirtinger generator $w_3 \in \pi(D)$ of segment 3 is mapped to $\mathcal{F}(w_3) = s^+ x_3^+ (x_3^-)^{-1} (s^+)^{-1} = x_1^+ x_2^+ x_3^+ \left(x_3^-\right)^{-1} \left(x_2^+\right)^{-1} \left(x_1^+\right)^{-1} \in \Pi(D)$. Here s^+ is the over path of the region between segments 2 and 3.

gives the relation $\mathcal{F}(w_{2'}) = \mathcal{F}(w_1^{-1} w_2 w_1)$. The relation at a negative crossing follows from a similar computation. □

Corollary 2.6. Given a region j of D and a path to it as in equation (6), let

$$y_j = w_{i_k}^{\epsilon_k} \cdots w_{i_1}^{\epsilon_1}$$

be the right-to-left product of the Wirtinger generators of the segments crossed with signs recording the orientations.[2] Then

$$\mathcal{F}(y_j) = s_j^+ \left(s_j^-\right)^{-1}. \qquad \lrcorner$$

Proof. When $k = 1$ this is clear. Suppose $k = 2$ and all signs are positive. Then

$$\mathcal{F}(w_{i_2} w_{i_1}) = x_{i_1}^+ x_{i_2}^+ \left(x_{i_2}^-\right)^{-1} \left(x_{i_1}^+\right)^{-1} x_{i_1}^+ \left(x_{i_1}^-\right)^{-1} = x_{i_1}^+ x_{i_2}^+ \left(x_{i_1}^- x_{i_2}^-\right)^{-1}$$

and the general case follows by induction using the same idea. □

3. Octahedral colorings

Definition 3.1. An *octahedral color* is a triple $\chi = (a, b, m)$ of nonzero complex numbers. We denote the set of octahedral colors by X. When we work with an indexed set $\{\chi_i\}_{i \in I}$ of colors we typically write a_i, b_i, m_i for their components.

The *braiding* is the partially defined[3] map $B : \mathsf{X}^2 \to \mathsf{X}^2$ given by $B(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'})$,

[2] The elements y_j correspond to Dehn generators of $\pi(D)$ with the generator of the topmost region set to 1.

[3] That is, B is a function $Y \to \mathsf{X}^2$ for a subset $Y \subset \mathsf{X}^2$.

where $\chi_{1'}$ and $\chi_{2'}$ are the colors defined by

$$\begin{aligned} a_{1'} &= a_1 A^{-1} \\ a_{2'} &= a_2 A \\ A &= 1 - \frac{m_1 b_1}{b_2}\left(1 - \frac{a_1}{m_1}\right)\left(1 - \frac{1}{m_2 a_2}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} b_{1'} &= \frac{m_2 b_2}{m_1}\left(1 - m_2 a_2 \left(1 - \frac{b_2}{m_1 b_1}\right)\right)^{-1} \\ b_{2'} &= b_1 \left(1 - \frac{m_1}{a_1}\left(1 - \frac{b_2}{m_1 b_1}\right)\right) \end{aligned} \tag{8}$$

$$m_{1'} = m_1 \quad m_{2'} = m_2 \tag{9}$$

It is elementary to check that B has a partially-defined inverse map $B^{-1}(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'})$ defined by

$$\begin{aligned} a_{1'} &= a_1 \tilde{A}^{-1} \\ a_{2'} &= a_2 \tilde{A} \\ \tilde{A} &= 1 - \frac{b_2}{m_1 b_1}(1 - m_1 a_1)\left(1 - \frac{m_2}{a_2}\right). \end{aligned} \tag{10}$$

$$\begin{aligned} b_{1'} &= \frac{m_2 b_2}{m_1}\left(1 - \frac{a_2}{m_2}\left(1 - \frac{m_1 b_1}{b_2}\right)\right) \\ b_{2'} &= b_1 \left(1 - \frac{1}{m_1 a_1}\left(1 - \frac{m_1 b_1}{b_2}\right)\right)^{-1} \end{aligned} \tag{11}$$

$$m_{1'} = m_1 \quad m_{2'} = m_2 \tag{12}$$

Definition 3.2. Let D be an oriented tangle diagram with set of segments S. An *octahedral coloring* of D is a function $S \to \mathsf{X}, i \mapsto \chi_i$ such that $B(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'})$ at every positive crossing and $B^{-1}(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'})$ at every negative crossing of D, where $1, 2, 1', 2'$ are the segments at the crossing arranged as in Figure 1.

For an octahedral color $\chi = (a, b, m)$ we write

$$\varphi^+(\chi) := \begin{bmatrix} a & 0 \\ (a - 1/m)/b & 1 \end{bmatrix} \tag{13}$$

$$\varphi^-(\chi) = \begin{bmatrix} 1 & (a - m)b \\ 0 & a \end{bmatrix} \tag{14}$$

and

$$\tau(\chi) := \varphi^+(\chi)\,\varphi^-(\chi)^{-1} \tag{15}$$

Definition 3.3. Let D be a tangle diagram with octahedral coloring $i \mapsto \chi_i$. The *holonomy representation* is the representation (i.e. functor) $\rho^\chi : \Pi(D) \to \mathrm{GL}_2(\mathbb{C})$ defined by[4]

$$\rho^\chi(x_i^+) = \varphi^+(\chi_i) \text{ and } \rho^\chi(x_i^-) = \varphi^-(\chi_i) \tag{16}$$

Via the functor $\mathcal{F}$ of Theorem 2.5 it induces a representation $\rho = \rho^\chi \mathcal{F}$ defined by

$$\rho(w_i) = \rho^\chi\left(s_i^+\right)\tau(\chi_i)\,\rho^\chi\left(s_i^+\right)^{-1}.$$

[4] One could instead obtain a representation $\Pi(D) \to \mathrm{PSL}_2(\mathbb{C})$ by dividing by an arbitrary square root of a. A lift $\pi(D) \to \mathrm{SL}_2(\mathbb{C})$ can be recovered from the m-decorations discussed below. This approach is geometrically more natural but algebraically inconvenient.

Lemma 3.4. The holonomy representation ρ^χ of an octahedral coloring is well-defined. The image of the induced representation ρ lies in $\mathrm{SL}_2(\mathbb{C})$. ⌟

Proof. For the first claim we can check the relations of $\Pi(D)$ crossing-by-crossing. Using (7–12) we can write the components of $\chi_{1'}$ and $\chi_{2'}$ in terms of the components of χ_1 and χ_2, and then checking the relations is a series of elementary computations.

For the second claim it suffices to compute the determinants of the generators:

$$\det \rho^\chi \mathcal{F}(w_i) = \det \rho^\chi \left(s^+\right) \det \tau(\chi) \det \rho^\chi \left(s^+\right)^{-1} = 1. \qquad \square$$

Remark 3.5. An octahedral coloring of D determines the geometric data of the octahedral decomposition associated to D. The face maps of this ideal triangulation give a representation $\pi(T) \to \mathrm{PSL}_2(\mathbb{C})$ that agrees with ρ^χ [McP22, Theorem 3.6].

Definition 3.6. Let D be an tangle diagram with set of regions R. Given an octahedral coloring χ of D we call the function $R \to \mathbb{C} \setminus \{0\}, j \mapsto c_j$ defined by

$$c_j = \det \rho^\chi(s_j^+)$$

the *region coloring*[5] of χ.

An equivalent definition in terms of a path from 0 to j is

$$c_j = a_{i_1}^{\epsilon_1} \cdots a_{i_k}^{\epsilon_k}$$

using the notation of equation (6). That is, the region color is the product of the a-coordinates of the segments crossed when traveling from the top region of the diagram, with signs to account for orientation.

Lemma 3.7. Any function from the regions of D to $\mathbb{C} \setminus \{0\}$ satisfying the following conditions agrees with the region coloring:

1. $f(0) = 1$, where 0 is the topmost region of D.
2. If region j' is below j across a segment with color χ_i,

$$f(j') = a_i f(j)$$ ⌟

Proof. These rules clearly determine $f(j)$ for each j. f is well-defined because the region colors are (in turn, because ρ^χ is well-defined). $\square$

One motivation for equations (13) and (14) is the factorization of elements of $\mathrm{SL}_2(\mathbb{C})$. Let $g \in \mathrm{SL}_2(\mathbb{C})$ be a matrix with nonzero $1,1$ entry. Then there are $a, e, f \in \mathbb{C}$ with

$$g = \begin{bmatrix} a & -e \\ f & (1-ef)/a \end{bmatrix} = \begin{bmatrix} a & 0 \\ f & 1 \end{bmatrix} \begin{bmatrix} 1 & e \\ 0 & a \end{bmatrix}^{-1}$$

and we extend our earlier notation by defining

$$\varphi^+(g) := \begin{bmatrix} a & 0 \\ f & 1 \end{bmatrix} \tag{17}$$

$$\varphi^-(g) := \begin{bmatrix} 1 & e \\ 0 & a \end{bmatrix} \tag{18}$$

[5] The c_i are the region variables of Kim, Kim, and Yoon [KKY18] and [McP22], but with the convention that the variable of the top region is always 1.

so that $g = \varphi^+(g)\,\varphi^-(g)^{-1}$. Similarly $\varphi^+(\tau(\chi)) = \varphi^+(\chi)$ and $\varphi^-(\tau(\chi)) = \varphi^-(\chi)$ for any octahedral color χ.

If $g \neq \pm 1$ then its eigenspaces are all 1-dimensional. We write

$$L(g; m) \in \mathbb{C}P^1$$

for the m-eigenspace of g, $(z_1 : z_2)$ for homogeneous coordinates on $\mathbb{C}P^1$, and

$$h(z_1 : z_2) = \frac{z_1}{z_2}$$

for the Hopf map $h : \mathbb{C}P^1 \to \mathbb{C} \cup \{\infty\}$. We think of $(z_1 : z_2)$ as a *row* vector so that for any $A \in \mathrm{GL}_2(\mathbb{C})$,

$$L(A^{-1} g A; m) = L(g; m) A. \tag{19}$$

If $g = \tau(\chi)$ for some octahedral color χ, i.e. if there is a $b \in \mathbb{C} \setminus \{0\}$ with

$$\varphi^+(g) = \begin{bmatrix} a & 0 \\ (a - 1/m)/b & 1 \end{bmatrix} \text{ and } \varphi^-(g) = \begin{bmatrix} 1 & (a-m)b \\ 0 & a \end{bmatrix}$$
$$\text{so } g = \begin{bmatrix} a & -(a-m)b \\ (a-1/m)/b & m + m^{-1} - a \end{bmatrix}$$

then

$$L(g; 1/m) = (1 : -b) \tag{20}$$
$$L(g; m) = (a - 1/m : -b(a-m)) \tag{21}$$

when $m \neq m^{-1}$, while in the parabolic case $m = 1$ the single eigenspace is

$$L(g; 1) = (1 : -b)$$

and similarly for $m = -1$. The point is that presenting g as $\tau(\chi)$ is essentially[6] equivalent to the choice of a preferred eigenspace of g. When g is the image of a meridian this extra choice occurs naturally in the study of representations of link (and 3-manifold) groups into $\mathrm{SL}_2(\mathbb{C})$.

[6] We also want b to not be 0 or ∞, which imposes some extra conditions.

Definition 3.8. We can extend the usual definition of meridian for knots to tangles, in which case the Wirtinger generator of a segment of D represents a meridian of the corresponding tangle component. All the Wirtinger generators for a given component of T are conjugate, so we can talk about the *meridian eigenvalues* of ρ for a component of T without ambiguity.

Let $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$ be a representation. We say that ρ is *meridian-nontrivial* if $\rho(w) \neq \pm 1$ for any meridian w of ρ. An *m-decoration* of a $\mathrm{SL}_2(\mathbb{C})$-structure is a choice of eigenvalue for all the images of the meridians. Since the meridians of each connected component are all conjugate an m-decoration is determined by a choice of m for each component of T.

The holonomy representation of an octahedral coloring is naturally m-decorated: by Lemma 3.4 the meridian of a segment with color (a, b, m) has eigenvalues m and m^{-1}, and our convention is that the distinguished eigenvalue is m.

Remark 3.9. Usually for a link group with peripheral subgroups $W_i \cong \mathbb{Z} \times \mathbb{Z}$ a *decoration* of ρ is a choice of eigenspaces for the $\rho(W_i)$, up to equivalence [GTZ15, Section 4]. For meridian-nontrivial representations this can be recovered from an m-decoration. One can generalize this to tangles but we do not need the details here.

[GTZ15] S. Garoufalidis, D. P. Thurston, and C. K. Zickert, "The complex volume of $\mathrm{SL}(n,\mathbb{C})$-representations of 3-manifolds". arXiv DOI

4. A formula for octahedral colorings

Using the definitions of the proceeding sections we can precisely phrase our problem as:

Given a tangle diagram D and an m-decorated representation $\rho : \pi(D) \to \mathrm{SL}_2(\mathbb{C})$, find an octahedral coloring χ of D with $\rho^\chi \mathcal{F} = \rho$.

It turns out that in general this is not possible: there can be singular ρ for which no octahedral coloring exists. However, these are sparse enough that one can always avoid them by gauge transformation (conjugation).

Definition 4.1. Two representations ρ and ρ' are *gauge-equivalent* if there is a $g \in \mathrm{SL}_2(\mathbb{C})$ with

$$\rho'(x) = g^{-1}\rho(x)g$$

for every $x \in \pi(T)$.

Definition 4.2. Fix a tangle diagram D of a tangle T. We say an m-decorated representation $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$ is *admissible* if there is an octahedral coloring whose holonomy representation ρ^χ agrees with ρ in the sense that

$$\rho = \rho^\chi \mathcal{F}$$

as homomorphisms $\pi(D) \to \mathrm{SL}_2(\mathbb{C})$.

It is easy to find inadmissible representations: for example, one can choose a Wirtinger generator for an arc touching the top region of D to have zero $1, 1$ entry. By using the formula below it is easy to check whether a representation is admissible:

Theorem 1. Let $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$ be a meridian-nontrivial representation. Choose a diagram D of T and an m-decoration of ρ.

Suppose i is a segment of D below region j (as in Figure 2). Write m_i for the eigenvalue of $\rho(w_i)$ determined by the m-decoration. Recall our notation $y_j = s_j^+ \left(s_j^-\right)^{-1} \in \pi(D)$. For a matrix A write $e_{11}(A)$ for its $1, 1$ (upper-left) entry and define

$$\chi_i = \left(\frac{e_{11}(\rho(w_i y_i))}{e_{11}(\rho(y_i))}, \frac{-1}{h\left(L(\rho(w_i); m_i^{-1})\varphi^+(\rho(y_j))\right)}, m_i \right) \tag{22}$$

Then

1. ρ is m-admissible for D if and only if the components above lie in $\mathbb{C} \setminus \{0\}$ for every segment i, and in this case
2. $i \mapsto \chi_i$ defines an octahedral coloring of D whose holonomy representation ρ^χ induces ρ. ⌟

Remark 4.3. If a segment is meridian-trivial then its entire component is as well. The representations of a tangle $T = T' \cup T_0$ with T_0 meridian-trivial are in bijection with representations of T' so we do not lose any generality by excluding them.

Proof of Theorem 1. First suppose we have an admissible representation ρ so there is an octahedral coloring $i \mapsto \chi_i$ whose holonomy representation induces ρ. By equation (20) the b-coordinate of segment i satisfies

$$-b_i^{-1} = h(L(\tau(\chi_i)\,; m_i^{-1}))$$

and by the definition of $\mathcal{F}$

$$\tau(\chi_i) = \rho^\chi(s_j^+)^{-1}\rho(w_i)\rho^\chi(s_j^+)$$

so by equation (19)

$$-b_i^{-1} = h(L(\rho(w_i); m_i^{-1})\rho^\chi(s_i^+)).$$

Our claim follows from the observation that $\rho^\chi(s_j^+) = \varphi^+(\rho(r_j))$.

For the a-coordinates, consider the function

$$f(j) = e_{11}[\rho(y_j)] = e_{11}\left[\rho\left(s_j^+\left(s_j^-\right)^{-1}\right)\right]$$

from regions j of D to $\mathbb{C}$. If i is a segment between regions j and j' as in Figure 2,

$$\rho(y_{j'}) = \rho\left(x_i^+ s_j^+\left(s_j^- x_i^-\right)^{-1}\right) = \varphi^+(\chi_i)\,\rho\left(s_j^+\right)\varphi^-(\chi_i)^{-1}\rho\left(s_j^-\right)^{-1}.$$

Observe that for any $\chi_1 = (a_1, b_1, m_1)$ and $\chi_2 = (a_2, b_2, m_2)$

$$\begin{aligned} a_1 a_2 &= e_{11}\left[\varphi^+(\chi_1)\,\varphi^+(\chi_2)\,\varphi^-(\chi_2)^{-1}\varphi^-(\chi_1)^{-1}\right] \\ &= e_{11}\left[\varphi^+(\chi_1)\,\varphi^-(\chi_1)^{-1}\right] e_{11}\left[\varphi^+(\chi_2)\,\varphi^-(\chi_2)^{-1}\right]. \end{aligned}$$

Therefore

$$e_{11}\left[\varphi^+(\chi_i)\,\rho\left(s_j^+\right)\varphi^-(\chi_i)^{-1}\rho\left(s_j^-\right)^{-1}\right] = a_i e_{11}\left[\rho\left(s_j^+\left(s_j^-\right)^{-1}\right)\right].$$

By Lemma 3.7 $f(j)$ is the region color of j for all j, so the ratio in equation (22) is the a-coordinate of χ_i as claimed.

We conclude that when ρ is admissible the components of the corresponding octahedral coloring must be given by equation (22). Conversely if the components of equation (22) are never 0 or ∞ they define a χ-coloring with holonomy ρ, so ρ is admissible as claimed. □

Corollary 4.4. Suppose χ is an octahedral coloring of a diagram D with holonomy representation ρ. Choose a component of D and define a new coloring $\tilde{\chi}$ by replacing

$$(a_i, b_i, m_i) \mapsto \left(a_i, b_i\frac{a_i - m_i}{a_i - 1/m_i}, 1/m_i\right)$$

for each segment i of this component and leaving the other colors unchanged. Then $\tilde{\chi}$ is a coloring of D that induces the same holonomy representation as χ but with the opposite m-decoration on the selected component. ⌟

Proof. Equation (22) only uses the value of m_i in the formula for the b-coordinate. Apply equation (21). □

With a slightly weaker definition of admissibility Blanchet et al. [Bla+20] prove that every ρ is gauge-equivalent to an admissible representation. Here we strengthen their result and make it more explicit; our theorem was also shown by Yoon [Yoo21, Theorem 1.2] by a different method.

[Yoo21] S. Yoon, "On the potential functions for a link diagram". arXiv DOI

Definition 4.5. An octahedral coloring is *strongly admissible* if it is admissible and no color is of the form (m, b, m) or $(m, b, 1/m)$.

A coloring is strongly admissible if and only if it occurs as a solution of the region equations [McP22, Section 6.3] of D, so a corollary of the theorem below is that one can find all representations of $\pi(D)$ up to gauge equivalence by solving these equations.

Theorem 4.6. If ρ is meridian-nontrivial then for any choice of m-decoration it is gauge-equivalent to a strongly admissible representation. ⌟

Proof. For each segment i of our diagram, consider the following subsets of $\mathrm{SL}_2(\mathbb{C})$:

$$X_0(i) = \left\{g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, e_{11}(g^{-1}\rho(y_j)g) = 0\right\} \cup \left\{g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, e_{11}(g^{-1}\rho(y_{j'})g) = 0\right\} \tag{23}$$

$$X_1(i) = \left\{g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, L(\rho(w_i; 1/m)g\varphi^+\left(g^{-1}\rho(y_j)g\right)) = (1:0)\right\} \tag{24}$$

$$X_2(i) = \left\{g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, L(\rho(w_i; 1/m)g\varphi^+\left(g^{-1}\rho(y_j)g\right)) = (0:1)\right\} \tag{25}$$

$$X_3(i) = \left\{g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, L(\rho(w_i; m)g\varphi^+\left(g^{-1}\rho(y_j)g\right)) = (1:0)\right\} \tag{26}$$

$$X_4(i) = \left\{g \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, L(\rho(w_i; m)g\varphi^+\left(g^{-1}\rho(y_j)g\right)) = (0:1)\right\} \tag{27}$$

where we write j and j' for the regions above and below i. For $k = 0, \ldots, 4$ let $X_k = \cup_i X_k(i)$ be the union over all segments. If $g \notin X_0 \cup X_1 \cup X_2$ then $\rho' = g^{-1}\rho g$ is admissible. If also $g \notin X_3 \cup X_4$ then it is strongly admissible. Thus it suffices to show that $X = X_0 \cup X_1 \cup X_3 \cup X_4 \cup X_5$ is not all of $\mathrm{SL}_2(\mathbb{C})$.

Consider the Zariski topology on $\mathrm{SL}_2(\mathbb{C})$ (the closed sets are zero sets of systems of polynomial equations). Because $\mathrm{SL}_2(\mathbb{C})$ is irreducible as an algebraic variety every open nonempty subset is dense, and in particular the $\mathrm{SL}_2(\mathbb{C}) \setminus X_k(i)$ are all open and dense. The complement

$$\mathrm{SL}_2(\mathbb{C}) \setminus X = \mathrm{SL}_2(\mathbb{C}) \setminus \bigcup_{k,i} X_k(i) = \bigcap_{k,i} (\mathrm{SL}_2(\mathbb{C}) \setminus X_k(i))$$

is a finite intersection of these, so it is open and dense as well, and in particular is nonempty. □

One can obtain a more constructive proof of this result by first conjugating by a g that avoids X_1 and X_3 then conjugating by a suitable lower-triangular matrix.

5. Smoothness of the potential function

One way to subdivide the octahedral decomposition is to split the octahedron at each crossing into four tetrahedra, one for each region touching the crossing. We can then geometrize them by placing their vertices in the boundary at infinity of hyperbolic space, i.e. $\mathbb{C}P^1$. Such geometric ideal tetrahedra are determined up to congruence by the cross-ratio of their vertices, usually called the *shape parameter*. In this case these parameters are given by equation (28) as in Figure 6. (For details see [McP22]. Note that we are using a different convention on negatively oriented tetrahedra.) When a shape parameter is 0, 1, or ∞ it represents a geometrically degenerate tetrahedron whose vertices coincide.

Definition 5.1. An octahedral coloring of a crossing is *pinched*[7] if any of the shape parameters

$$z_N = \frac{b_{2'}}{b_1}, z_W = \frac{b_2}{m_1 b_1}, z_S = \frac{m_2 b_2}{m_1 b_{1'}}, z_E \frac{m_2 b_{2'}}{b_{1'}} \tag{28}$$

are 1. (These quantities can never be 0 or ∞ as part of the definition of octahedral coloring.) An octahedral coloring of a diagram is pinched if any of its crossings are.

[7] This term is due to Kim, Kim, and Yoon [KKY18]. At a pinched crossing the ideal points of the tetrahedra are "pinched" together.

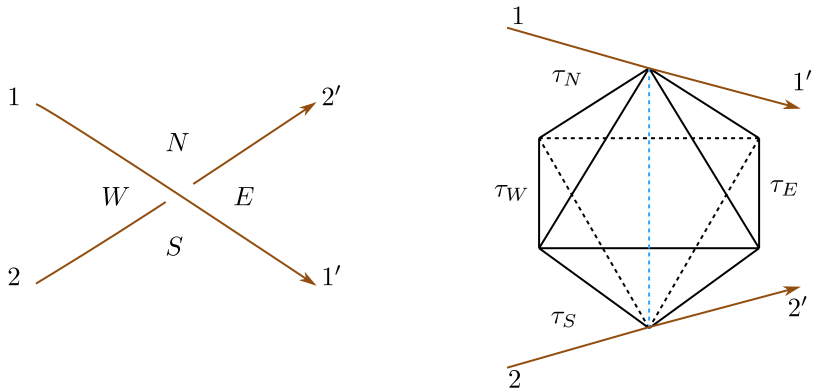


Figure 6: One can divide the octahedron at a crossing into four tetrahedra, one for each region touching the crossing. The shape parameters are then given as ratios of the b and m-coordinates, as in equation (28).

It is useful to understand when this can be avoided, in particular in the context of the Volume Conjecture. To explain the connection to we introduce a version of the Neumann-Zagier–Yokota potential function of the diagram defined using the dilogarithm

$$\mathrm{Li}_2(z) := \int_0^z \frac{-\log(1-t)}{t} dt.$$

Li_2 has a branch point at 1, where it is continuous but not differentiable. It can be used to compute the volumes [Zag07] and Chern-Simons invariants [Neu04] of hyperbolic 3-manifolds. Set

[Zag07] D. Zagier, "The dilogarithm function". DOI
[Neu04] W. D. Neumann, "Extended Bloch group and the Cheeger-Chern-Simons class". arXiv DOI

$$\mathrm{L}(\zeta) := \frac{\mathrm{Li}_2(e^{2\pi i \zeta})}{2\pi i}. \tag{29}$$

Let D be a link with ℓ components. Fix a choice $\mu = (\mu_1, \dots, \mu_\ell)$ of complex number for each component of ℓ and introduce a variable $\beta_1, \dots, \beta_n$ for each segment of D. We think of the β_i and μ_j as logarithms of the b and m coordinates of some octahedral coloring. For each crossing c (as in Figure 6) of D set

$$\Phi_c = \mathrm{L}(\beta_{2'} - \beta_1) - \mathrm{L}(\beta_2 - \beta_1 - \mu_1) + \mathrm{L}(\beta_2 - \beta_{1'} + \mu_2 - \mu_1) - \mathrm{L}(\beta_{2'} - \beta_{1'} + \mu_2)$$

The arguments are logarithms of the shape parameters (28).

Definition 5.2. The *potential function* of D with respect to μ is

$$\Phi_{D,\mu}(\beta_1, \dots, \beta_n) = \sum_{\text{crossings } c} \epsilon(c) \Phi_c \tag{30}$$

where $\epsilon(c) = +1$ for positive crossings and -1 for negative crossings. Because Li_2 has a branch point at 1 this function is analytic when the arguments of the functions L (i.e. the logarithms of the shape parameters) are not in $2\pi i\mathbb{Z}$.

Theorem 5.3. Let $\mathcal{C}_{D,\mu}$ be the set of generalized critical points of $\Phi_{D,\mu}$, i.e. points $\beta = (\beta_1, \dots, \beta_n)$ where

$$\frac{\partial \Phi_{D,\mu}}{\partial \beta_j} \equiv 0 \pmod{2\pi i} \text{ for } j = 1, \dots, n.$$

Then there is a bijection between $\mathcal{C}_{D,\mu}$ and the set of non-pinched octahedral colorings of D with m-decoration matching μ (i.e. where the eigenvalue of component j is $e^{2\pi i \mu_j}$). ⌟

Proof. By [McP22, Theorem 6.3] non-pinched colorings are in bijection with solutions of the *segment equations* of the diagram; these are a set of rational equations in the b-variables, with one for each segment. They have solutions if and only if the gluing equations of the four-term decomposition have nondegenerate solutions. Because

$$\frac{d}{d\zeta}\,\mathrm{L}(\zeta) = -\log(1 - e^{2\pi i \zeta})$$

we can work out that

$$\exp \frac{\partial \Phi_{D,\mu}}{\partial \beta_j} = 1$$

is the segment equation of segment j. □

One can easily handle pinched crossings by working directly with octahedral colorings. However, the smoothness of $\Phi_{D,\mu}$ is still significant for the Volume Conjecture. Most proofs follow Yokota [Yok00] by showing that the large N asymptotics of the Kashaev invariant are given by a contour integral of the form

$$\int_\Gamma e^{N\Phi_{D,\mu}(\beta_1,\ldots,\beta_n)}\, d\beta_1 \cdots d\beta_n$$

as $N \to \infty$. The usual saddle-point approximations only work near smooth critical points of $\Phi_{D,\mu}$. More significantly the proof usually requires moving the contour Γ through the parameter space which requires $\Phi_{D,\mu}$ to be complex-analytic. We can use the formula below to study whether a representation corresponds to a pinched octahedral coloring:

Theorem 5.4. The shape parameters at a crossing are

$$z_N = \frac{h(L(w_1, 1/m_1)\varphi^+(y_N))}{h(L(w_{2'}, 1/m_2)\varphi^+(y_N))} \tag{31}$$

$$z_W = \frac{h(L(w_1, 1/m_1)\varphi^+(y_W))}{h(L(w_2, 1/m_2)\varphi^+(y_W))} \tag{32}$$

$$z_S = \frac{h(L(w_{1'}, 1/m_1)\varphi^+(y_S))}{h(L(w_2, 1/m_2)\varphi^+(y_S))} \tag{33}$$

$$z_E = \frac{h(L(w_{1'}, 1/m_1)\varphi^+(y_E))}{h(L(w_{2'}, 1/m_2)\varphi^+(y_E))} \tag{34}$$

where y_N is the group element associated to the north region of the crossing (as in Figure 6) and so on. ⌟

Proof. We show how to compute the formula for z_W. By Theorem 1 we have

$$-\frac{1}{b_1} = h(L(\rho(w_1; 1/m_1)\varphi^+(y_N)))$$
$$-\frac{1}{b_2} = h(L(\rho(w_2; 1/m_2)\varphi^+(y_W)))$$

Because $\varphi^+(y_W) = \varphi^+(\chi_1)\,\varphi^+(y_N)$ and

$$(1 : -b)\begin{bmatrix} a & 0 \\ (a - 1/m)/b & 1 \end{bmatrix} = (1/m : -b)$$

we see that

$$-\frac{1}{m_1 b_1} = h(L(\rho(w_1; 1/m_1)\varphi^+(y_W)))$$

as required. The other cases follow from similar computations. □

As an immediate corollary we have:

Theorem 2. Let $\rho : \pi(D) \to \mathrm{SL}_2(\mathbb{C})$ be an admissible m-decorated representation and χ an octahedral coloring with holonomy ρ. A crossing is pinched if and only if

$$L(\rho(w_1); 1/m_1^{-1}) = L(\rho(w_2); 1/m_2^{-1}).$$ ⌟

Proof. One can use equations (8) and (11) to show that $z_W = 1$ if and only if the crossing is pinched. $\varphi^+(y_W)$ is invertible, so $L(w_1; 1/m_1)\varphi^+(y_W) = L(w_2; 1/m_2)\varphi^+(y_W)$ iff $L(w_1; 1/m_1) = L(w_2; 1/m_2)$. □

In other words, χ is pinched if and only if at some crossing the Wirtinger generators have the same eigenspace, equivalently the same fixed point when acting on $\mathbb{C}P^1$ by fractional linear transformations. Note that this condition above depends only on ρ, and in fact only on the image of ρ in $\mathrm{PSL}_2(\mathbb{C})$.

If we restrict our attention to hyperbolic knots then we can be more specific. Recall that the holonomy of the complete hyperbolic structure is a faithful, discrete representation $\rho : \pi(K) \to \mathrm{PSL}_2(\mathbb{C})$. It is boundary-parabolic (the meridians have eigenvalues ± 1) and always lifts to $\mathrm{SL}_2(\mathbb{C})$.

Definition 5.5. A tangle diagram D is *arc-faithful* if at each crossing the Wirtinger generators of the over and under arcs are distinct elements of $\pi(T)$.[8]

[8] Since $w_1 = w_2$ if and only if $w_1 = w_1^{-1} w_2 w_1$ it does not matter which under arc is used.

Theorem 3. Let D be a diagram of a hyperbolic knot. The holonomy ρ of its complete hyperbolic structure comes from a non-pinched octahedral coloring of D if and only if D is arc-faithful. ⌟

Proof. The only if is obvious: if two adjacent arcs have the same Wirtinger generators then their images always share a fixed point.

Conversely, suppose ρ comes from a pinched octahedral coloring of D, so there are Wirtinger generators w_1 and w_2 for which $\rho(w_1)$ and $\rho(w_2)$ have the same fixed point. As $\rho(w_1)$ and $\rho(w_2)$ are parabolic this implies they commute, hence w_1 and w_2 commute as ρ is faithful. By the classification of finitely abelian subgroups of 3-manifold groups [Hem76, Theorem 9.13] two commuting infinite-order elements generate a subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. As w_1 and w_2 map to the same element of the first homology they must be the same element. □

[Hem76] J. Hempel, *3-manifolds*

The existence of smooth critical points of $\Phi_{D,\mu}$ has previously been studied, and we can express our results in this context:

Definition 5.6. An edge of an ideal triangulation of a knot complement is (homtopically) *peripherial* if it is homotopic to a curve in the boundary of a regular neighborhood of the knot.

Corollary 5.7. Let D be a diagram of a hyperbolic knot.

(a) The edges of the four-term octahedral decomposition associated to D are homotopically non-peripheral if and only if D is arc-faithful.

(b) If D is reduced and alternating then it is arc-faithful. ⌟

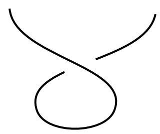
Proof. (a) If there are any nondegenerate solutions to the gluing equations of the triangulation then its edges must be non-peripheral [DG12, Lemma 3.5] and in this case there is always a solution corresponding to the complete hyperbolic structure.

(b) Garoufalidis, Moffatt, and Thurston [GMT16] and Sakuma and Yokota [SY18] independently showed that the edges of the four-term decomposition of a reduced alternating diagram are non-peripheral. Part (a) implies such a diagram is arc-faithful. □

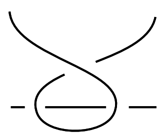
[DG12] N. M. Dunfield and S. Garoufalidis, "Incompressibility criteria for spun-normal surfaces". arXiv DOI

It is easy to find diagrams that are not arc-faithful. However, by avoiding these obvious bad configurations it also seems easy to find arc-faithful diagrams.

Example 5.8. Any diagram with a kink



is never-arc faithful. More generally any piece of a diagram that looks like



will prevent arc-faithfulness.

According to Seonhwa Kim's database [Kim] of boundary-parabolic representations every hyperbolic knot with at most 12 crossings has an arc-faithful diagram. It seems that this should be true for all knots, and as discussed above this seems relevant to a general proof of the Volume Conjecture.

[Kim] S. Kim, *Database of boundary-parabolic representations of knot groups*

Conjecture 5.9. Every hyperbolic knot has an arc-faithful diagram. ⌟

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