Structural Parameters for Dense Temporal Graphs

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Abstract -

Temporal graphs provide a useful model for many real-world networks. Unfortunately the majority of algorithmic problems we might consider on such graphs are intractable. There has been recent progress in defining structural parameters which describe tractable cases by simultaneously restricting the underlying structure and the times at which edges appear in the graph. These all rely on the temporal graph being sparse in some sense. We introduce temporal analogues of three increasingly restrictive static graph parameters – cliquewidth, modular-width and neighbourhood diversity – which take small values for highly structured temporal graphs, even if a large number of edges are active at each timestep. The computational problems solvable efficiently when the temporal cliquewidth of the input graph is bounded form a subset of those solvable efficiently when the temporal modular-width is bounded, which is in turn a subset of problems efficiently solvable when the temporal neighbourhood diversity is bounded. By considering specific temporal graph problems, we demonstrate that (up to standard complexity theoretic assumptions) these inclusions are strict.

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1 Introduction

Temporal graphs, in which the set of active edges changes over time, are a useful formalism for modelling numerous real-world phenomena, from social networks in which friendships evolve over time to transport networks in which a particular connection is only available on particular days and times. This has inspired a large volume of research into the algorithmic aspects of such graphs in recent years [8, 26, 34], but unfortunately in many cases even problems which admit polynomial-time algorithms on static graphs become intractable in the temporal setting.

This has motivated the study of computational problems on restricted classes of temporal graphs, with mixed success: in a few cases, restricting the structure of the underlying static graph (e.g. to be a path or a tree) is effective, but numerous natural temporal problems remain intractable even when the underlying graph is very strongly restricted (e.g. when it is required to be a path [33] or a star [2]). Recently, several promising new parameters have been introduced that simultaneously restrict properties of the static underlying graph and the times at which edges are active in the graph; these include several analogues of treewidth for temporal graphs [18, 30], the temporal feedback edge/connection number [23], the timed vertex feedback number [7] and the (vertex-)interval-membership-width of the temporal

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graph [6]. However, all of these new temporal parameters are only small for temporal graphs that are, in some sense, sparse: none of them can be bounded on a temporal graph which has a superlinear (in the number of vertices) number of active edges at every timestep.

In this paper, we attempt to fill this gap in the toolbox of parameters for temporal graphs by introducing three new parameters which can take small values on temporal graphs which are dense but are sufficiently highly structured. Specifically, we define natural temporal analogues of cliquewidth, modular-width and neighbourhood diversity, all of which have proved highly effective in the design of efficient algorithms for static graphs.

Importantly, the neighbourhood diversity of a static graph upper bounds its modular-width, which upper bounds its cliquewidth. Both cliquewidth (introduced by Courcelle et al. [13]) and modular-width (introduced by Gajarský et al. [19], using the long-standing notion of modular decompositions [20]) can be defined in terms of width measures over composition trees allowing particular operations. Cliquewidth constructions have greater flexibility due to the fact we are allowed to use an additional "relabelling" operation; this makes it possible, for example, to build long induced paths, which cannot exist in graphs of small modular width. Courcelle et al. [11] show that any graph property expressible in monadic second order is solvable in linear time for graphs of bounded cliquewidth. Gajarský et al. [19] provide examples of problems (Hamilton Path and Colouring) that are hard with respect to cliquewidth but tractable with respect to the more restrictive parameter modular-width. Neighbourhood diversity is a highly restrictive parameter, introduced by Lampis [28], which requires that large sets of vertices have identical neighbourhoods. Cordasco [9] demonstrated that Equitable Colouring is hard with respect to modular-width but tractable with respect to neighbourhood diversity.

These three static parameters are the inspiration for our temporal parameters. Informally, our new parameters are defined as follows:

- \blacksquare A temporal graph has temporal neighbourhood diversity (TND) at most k if its vertices can be partitioned into at most k classes such that each class induces either an independent set or a clique in which all edges are active at exactly the same times, and two vertices in the same class have exactly the same neighbours outside the class at each timestep.
- \blacksquare Temporal modular-width (TMW) is a generalisation of TND: a temporal graph has TMW at most k if its vertices can be partitioned into modules such that two vertices in same module must have the exactly the same neighbours outside the class at each timestep, but now each module need only be a temporal graph which itself has TMW at most k, rather than a clique or independent set.
- Like the static version, temporal cliquewidth (TCW) is defined to be the minimum number of labels needed to construct a temporal graph using four operations (create a vertex with a new label; take a disjoint union of two graphs; add all edges between vertices of two specified labels; relabel all occurrences of one label to another); the difference from the static case is that when adding edges between two sets of vertices these edges must all be active at exactly the same times.

We note that in every case we will recover the corresponding static parameter if all edges are active at the same times. It is immediate that the TND of a temporal graph is an upper bound on its TMW, and it is straightforward to show (see Section 3) that the TMW is an upper bound on the TCW. Thus, the most general algorithmic result we hope to obtain is to show that a problem is tractable when the TCW of the input temporal graph is bounded, but we expect to be able to show tractability for more problems as we impose stronger restictions on the input by bounding respectively the TMW and the TND.

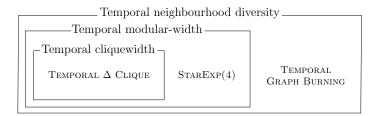


Figure 1 A diagram of our parameters and the problems we show to be tractable with respect to each. A problem is in a rectangle if it is tractable with respect to the parameter it is labelled with. Assuming $P \neq NP$, each problem is in the rectangle for the most general of the three parameters for which it is tractable.

To illustrate the value of considering this hierarchy of parameters, we provide examples of problems which can be solved efficiently when each of our three new parameters is bounded, but which (in the case of TND and TMW) remain intractable when we restrict only the next most restrictive parameter, as illustrated in Figure 1. Specifically, we prove that:

- TEMPORAL CLIQUE is solvable in linear time when a temporal cliquewidth decomposition of constant width is given (see Section 2).
- STAREXP(4) (the problem of deciding whether there is a closed temporal walk visiting all vertices in a star when each edge is active at no more than four times) remains NP-hard on graphs with TCW at most three, but is solvable in polynomial time when the TMW of the input graph is bounded by a constant; in fact we provide an fpt-algorithm with respect to TMW (see Section 3).
- Graph Burning is NP-hard on temporal graphs with constant TMW, but is solvable in polynomial time when the TND of the input graph is bounded by a constant; again this is an fpt-algorithm with respect to TND (see Section 4).

We also (in Section 4) provide an fpt-algorithm to solve SINGMINREACHDELETE parameterised by TND (when the number of appearances of each edge is bounded), in order to illustrate additional techniques that may be used when working with this new temporal parameter. We conjecture that SINGMINREACHDELETE is another example of a problem that is tractable with respect to TND but intractable when only the TMW is restricted.

The remainder of the paper is organised as follows. We conclude this section by introducing some key notation and definitions used throughout the paper. The following three sections are devoted to TCW, TMW and TND respectively, with each section containing the formal definition of a parameter as well as results about problems which can be solved efficiently when that parameter is bounded.

1.1 Notation and definitions

We use a number of standard notations for temporal graphs and related notions. A temporal graph $\mathcal{G} = (G, \lambda)$ consists of an underlying static graph $\mathcal{G}_{\downarrow} = G$, and a time-labeling function $\lambda : E \to 2^{\mathbb{N}}$, assigning to each edge a set of timesteps at which it is active. We refer to a pair (e, t) consisting of an edge $e \in E(G)$ and time $t \in \lambda(e)$ as a time-edge. The set of all time-edges of a temporal graph is denoted by $\varepsilon(\mathcal{G})$ and the lifetime Λ of a temporal graph refers to the final time at which any edge is active, i.e. $\Lambda = \max\{\max \lambda(e) : e \in E(G)\}$. The snapshot \mathcal{G}_t of a temporal graph \mathcal{G} at time t is the static graph $G = (V, E_t)$ where E is the set of edges active at time t.

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A temporal path on the temporal graph $\mathcal{G} = (G, \lambda)$ is a sequence of time, edge pairs $(e_1, t_1), ..., (e_\ell, t_\ell)$, such that $e_1, ..., e_\ell$ is path on G, and $t_i \in \lambda(e_i)$ for every $i \in [\ell]$, and $t_1, ..., t_\ell$ is a strictly increasing sequence of times. Given a temporal path $(e_1, t_1), ..., (e_\ell, t_\ell)$ we refer to the time t_1 as its departure time, and t_ℓ as its arrival time.

We refer the reader to [15] for background on parameterised complexity.

2 Tractability with respect to Temporal Cliquewidth

In this section we give the formal definition of the first of our new parameters, temporal cliquewidth, and demonstrate that the problem of finding a temporal clique admits an fpt-algorithm parameterised by temporal cliquewidth. Before defining temporal cliquewidth, we start by recalling the definition of cliquewidth in the static setting, as introduced by Courcelle and Olariu [14].

- ▶ **Definition 1** (Cliquewidth). The cliquewidth of a static graph G = (V, E) is the number of labels required to construct G using only the following operations:
- 1. Creating a new vertex with label i.
- 2. Taking the disjoint union of two labeled graphs.
- **3.** Adding edges to join all vertices labeled i to all vertices labeled j, where $i \neq j$.
- **4.** Renaming label i to label j.

We refer to an algorithm which constructs a graph G using the above operations as a cliquewidth construction of G.

Computing the cliquewidth of a graph is NP-hard [17], although there exists a polynomial-time algorithm to recognise graphs of cliquewidth at most three [10].

Translating this definition into the temporal setting, and preserving the idea that vertices with the same label should be indistinguishable when we add edges – which imposes additional restrictions on the times at which new edges are active – we obtain our definition of temporal cliquewidth.

- ▶ **Definition 2** (Temporal Cliquewidth). The temporal cliquewidth of a temporal graph $\mathcal{G} = (G, \lambda)$ is the number of labels required to construct \mathcal{G} using only the following operations:
- 1. Creating a new vertex with label i.
- 2. Taking the disjoint union of two labeled graphs.
- **3.** Adding edges to join all vertices labeled i to all vertices labeled j, where $i \neq j$, such that all the added edges are active at the same set of times T.
- **4.** Renaming label i to label j.

We refer to an algorithm which constructs a temporal graph \mathcal{G} using the above operations as a temporal cliquewidth construction of \mathcal{G} .

We note that, if temporal graph \mathcal{G} has bounded temporal cliquewidth k, then the underlying graph G_{\downarrow} of \mathcal{G} has cliquewidth at most k. The construction of the underlying graph is found by adding a static edge (if one does not already exist) whenever a time-edge is added in the construction of the temporal graph. In addition, the snapshot of \mathcal{G} at any time t, G_t also has cliquewidth at most k. This follows from a similar argument. If we have a temporal graph where the edges all appear at the same times, the cliquewidth of the underlying graph is the same as the temporal cliquewidth and the cliquewidth of any snapshots where edges are

active.It follows immediately that it is NP-hard to compute temporal cliquewidth, as the NP-hard problem of computing the cliquewidth of a static graph is a special case.

The remainder of this section is devoted to proving that the problem TEMPORAL Δ CLIQUE is in FPT parameterised by the temporal cliquewidth of the input graph. This problem was introduced by Viard et al. [37] and asks, in an interval of times, whether there is a set of at least h vertices such that there is an appearance of an edge between every pair of vertices in every sub-interval of Δ times. Hermelin et al. [25] investigate the variant of this problem where the interval in question is the entire lifetime of the temporal graph. More formally, it asks if there exists a set V' of cardinality at least h such that for each pair of distinct vertices $u, v \in V'$ and each time $0 \le i \le \Lambda - \Delta$ where Λ is the lifetime of \mathcal{G} , there exists a time-edge at time $t' \in [i, \Delta + i]$.

Hermelin et al. note that this is the case if and only if there is a set of vertices of size at least k such that they form a clique on the static graph consisting of edges that appear in every interval of Δ timesteps. They name this static graph a Δ -association graph. It is more formally defined as $G = (V, \bigcap_{i=1}^{\Lambda(\mathcal{G})-\Delta+1} \bigcup_{j=i}^{i+\Delta-1} E_j)$ where E_j is the set of edges active at time j. This reformulates the problem as follows.

Temporal Δ Clique

Input: A temporal graph $\mathcal{G} = (V, E, \lambda)$ and two integers Δ and h where $\Delta \leq T(\mathcal{G})$.

Output: Is there a set $V' \subseteq V$ of vertices such that $|V'| \ge h$ and V' is a clique in the Δ -association graph G of G?

Note that a temporal graph \mathcal{G} with integers Δ and h is a yes-instance of Temporal Δ Clique if and only if its Δ -association graph G and h are a yes-instance of Clique.

▶ Proposition 3. If a temporal graph \mathcal{G} has temporal cliquewidth k, then the Δ -association graph G of \mathcal{G} has cliquewidth at most k.

Proof. Given a temporal cliquewidth construction of \mathcal{G} , we find a cliquewidth construction of its Δ -association graph G using the same number of labels. Suppose that, under such a construction of \mathcal{G} , there is a set of time-edges added between vertices with labels i and j, $i \neq j$ where at least one of these edges appears in the association graph G and at least one edge does not. That is, in the subgraph of G induced by the vertices with labels i and j at this point in the construction neither form an independent set nor a complete bipartite graph. Label the edges which are in this subgraph E_1 and those which are not E_2 . Then, for each edge e_2 in E_2 , there must be a time t such that all edges in E_1 appear at a time t' in $[t, t + \Delta]$ and e_2 does not.

We claim that the endpoints of edges in E_1 are labelled i and j before both endpoints of edges in E_2 . Let e_1 be an edge in E_1 with such an appearance t' in \mathcal{G} . Then the addition of the time-edge (e_1, t') must occur before the endpoints of e_1 and e_2 are labelled i and j. Else, e_2 must also be active at t' and thus there is a time in $[t, t + \Delta]$ at which e_2 appears. This contradicts our earlier assertion. Therefore, the endpoints of edges E_1 are labelled i and j before each endpoint of edges in E_2 is labelled the same.

Hence, the edges in G can be added at the point in the construction where the endpoints of the edges in E_1 have the same labels and those in E_2 do not. This implies that we can construct G as in Definition 1 with at most k labels if G can be constructed with k labels.

Gurski [22, Theorem 4.4] shows that CLIQUE is solvable in linear time in graphs of bounded cliquewidth if its construction is given. This gives us the following result.

▶ Theorem 4. Given a cliquewidth construction of the Δ -association graph of a temporal graph \mathcal{G} , Temporal Δ Clique can be solved in linear time.

3 Tractability with respect to Temporal Modular-width

We now introduce a more restrictive parameter, temporal modular-width, and show (in Section 3.1) that there exist problems which are efficiently solvable when this parameter is bounded even though they remain intractable on temporal graphs with constant temporal cliquewidth.

We begin with the formal definition of the parameter. Again, we start by recalling the definition of the corresponding static parameter on which our definition is based.

- ▶ **Definition 5** (Modular-width, Section 2.5 [19]). Suppose a static graph G can be constructed by the algebraic expression A which uses the following operations:
- 1. Creating an isolated vertex.
- **2.** Taking the disjoint union of two graphs.
- 3. Taking the complete join of two graphs. That is, for graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$ $(V_2, E_2), V(G_1 \otimes G_2) = V(G_1) \cup V(G_2) \text{ and } E(G_1 \otimes G_2) = E(G_1) \cup E(G_2) \cup \{(v, w) : v \in G_2\}$ $V(G_1)$ and $w \in V(G_2)$.
- **4.** The substitution of graphs G_1, \ldots, G_n into a graph G' with vertices v_1, \ldots, v_n . This gives the graph $G'(G_1,\ldots,G_n)$ with vertex set $\bigcup_{1\leq i\leq n}V(G_i)$ and edge set $\bigcup_{1\leq i\leq n}E(G_i)\cup I$ $\{(v, w) : v \in V(G_i), w \in V(G_j), \text{ and } (v_i, v_j) \in E(G')\}.$

The width of an expression A is the maximum number of operands in an occurrence of the operation 4 in A. The modular-width of G, written MW(G), is this minimum width of an expression A which constructs G.

We refer to the graphs G_1, \ldots, G_n which we substitute into G' as modules. It is known that for any graph G an algebraic expression of modular-width MW(G) can be be found in linear time [36]. Observe that operations 2 and 3 are special cases of operation 4. We note that the modular-width of a graph can be computed in linear time [31, 36].

We now define our temporal analogue of this parameter. For simplicity we do not explicitly include the disjoint union and complete join operations, noting that once again these are special cases of the substitution operation.

- **Definition 6** (Temporal Modular-width). Suppose a temporal graph \mathcal{G} can be constructed by the algebraic expression A which uses the following operations:
- 1. Creating an isolated vertex.
- **2.** The substitution of temporal graphs G_1, \ldots, G_n into a temporal graph G' with vertices v_1, \ldots, v_n . This gives the graph $\mathcal{G}'(\mathcal{G}_1, \ldots, \mathcal{G}_n)$ with vertex set $\bigcup_{1 \leq i \leq n} V(\mathcal{G}_i)$ and time-edge set $\bigcup_{1 \le i \le n} \varepsilon(\mathcal{G}_i) \cup \{(vw, t) : v \in V(\mathcal{G}_i), w \in V(\mathcal{G}_j), \text{ and } (v_i v_j, t) \in \varepsilon(\mathcal{G}')\}.$

The width of an expression A is the maximum number of operands in an occurrence of the operation 2 in A. The temporal modular-width of G is this minimum width of an expression A which constructs G.

As for temporal cliquewidth, we observe that the temporal modular-width of a temporal graph is equal to the modular-width of the underlying graph if all edges have the same temporal assignment. It follows that, as in the static case, the temporal modular-width bounds the length of the longest induced path in the underlying graph.

We now argue that, as claimed, bounding the temporal modular-width of a temporal graph is a strictly stronger restriction than bounding the temporal cliquewidth.

▶ **Theorem 7.** For any temporal graph $\mathcal{G} = (G, \lambda)$, the temporal cliquewidth is upper bounded by the temporal modular-width.

Proof. Assuming that \mathcal{G} has temporal modular-width at most k, we induct on the maximal temporal modular decomposition tree for \mathcal{G} to produce a temporal cliquewidth construction of \mathcal{G} using at most k labels. This is achieved by inductively producing a temporal cliquewidth construction corresponding to each child of the root node in the temporal modular decomposition tree, and then relabeling and combining these constructions to produce an overall expression for \mathcal{G} .

Any tree of depth one consists of only a single vertex and thus has temporal cliquewidth $1 \le k$.

Now, given a tree of depth d, consider the substitution $\mathcal{G}'(\mathcal{G}_1,...,\mathcal{G}_n)$ at the root node of the tree. Note that $n \leq k$ as \mathcal{G} has temporal modular-width at most k. Then each child \mathcal{G}_i has a decomposition tree of depth at most d-1, and therefore by induction has cliquewidth at most k. Therefore it is possible to find an expression A_i for every child \mathcal{G}_i using the operations of Definition 2, such that at most k labels are used. and furthermore in which every vertex in the resulting graph is relabeled to i. The graph \mathcal{G} can then be constructed by relabeling every vertex in each A_i to i, and then taking a disjoint union, before adding edges active at times $\lambda(\{v_i, v_j\})$ between all vertices labeled i and all vertices labeled j for every $\{v_i, v_j\} \in E(G)$, where $\mathcal{G}' = (G, \lambda)$.

We go on to show that the unique maximal temporal modular decomposition, and therefore the width, can be computed in polynomial time. Habib and Paul [24] describe a simple algorithm for finding the modular decomposition of a static graph. This operates by finding and repeatedly adding *splitters* to a candidate module. We use a similar method to find the temporal modular decomposition. We begin by defining key concepts.

- ▶ **Definition 8** (Splitter [24]). Given a set of vertices S, a vertex x is said to be a splitter for S if there exist vertices $u, v \in S$ such that x is adjacent to exactly one of u and v.
- ▶ **Definition 9** (Static Module [24]). A set of vertices M is a static module of the graph G, if and only if for every vertex $x \notin M$, x is not a splitter for M.
- ▶ **Definition 10** (Temporal Module). A set of vertices M is a temporal module of (G, λ) if and only if for every timestep $t \in [\Lambda]$ and vertex $x \notin M$, x is not a splitter for M in the snapshot graph $(V(G), \{e \in E(G) : t \in \lambda(e)\})$.

We now obtain a relationship between temporal modules and modules in the static graphs corresponding to each snapshot.

▶ **Lemma 11.** *M* is a temporal module of (G, λ) if and only if for every timestep $t \in [\Lambda]$, *M* is a module of the snapshot graph $(V(G), \{e \in E(G) : t \in \lambda(e)\})$.

Proof. If M is a temporal module of (G, λ) , then for any timestep t there are no splitters of M in $(V(G), \{e \in E(G) : t \in \lambda(e)\})$, and thus M is a module of $(V(G), \{e \in E(G) : t \in \lambda(e)\})$. Conversely, if for every timestep t, M is a module of $(V(G), \{e \in E(G) : t \in \lambda(e)\})$, it must have no splitters in $(V(G), \{e \in E(G) : t \in \lambda(e)\})$, and thus is a temporal module of (G, λ) .

We further demonstrate that, as in the static case, the set of maximal modules for a temporal graph is unique.

Lemma 12. The set of maximal temporal modules \mathcal{M} for the temporal graph (G,λ) is unique and partitions V(G).

Proof. Assume that such a set does not partition V(G), that is $\bigcup \mathcal{M} = V(G)$ but there exists $M_i, M_i \in \mathcal{M}$ such that $M_i \cap M_i \neq \emptyset$. By the previous lemma, we have that M_i and M_i are modules in every snapshot of (G,λ) , and hence $M_i \cup M_i$ is also a module in every snapshot of (G, λ) and is therefore a temporal module, but $M_i \cup M_j \supset M_i$, contradicting the maximality.

Now assume that such a set is not unique, that is there exists a set of maximal temporal modules \mathcal{P} with $\mathcal{P} \neq \mathcal{M}$. Now let $P_i \in \mathcal{P}$ be a module such that $P_i \notin \mathcal{M}$, and consider a vertex $v \in P_i$. As \mathcal{M} partitions V(G), there exists some module $M_j \in \mathcal{M}$ such that $v \in M_j$. Then $P_i \cup M_i$ is a module in every snapshot, and therefore a temporal module, but this contradicts the maximality of \mathcal{M} and \mathcal{P} .

A simple way to compute this unique maximal modular decomposition is then provided by the following observation.

▶ **Observation 13.** Let S be a subset of the vertices of a temporal graph (G, λ) . If S has a splitter x in any snapshot of (G,λ) , then any module of (G,λ) containing S also contains x.

Given a set \mathcal{M} of modules, repeatedly test if a non-trivial module containing a pair $M_1, M_2 \in \mathcal{P}$ exists, by considering $M_1 \cup M_2$ as a candidate module, and repeatedly adding any splitters from the remaining modules in \mathcal{M} . If a trivial module is obtained the pair is discounted, and otherwise the set of modules is updated with the newly found non-trivial module. We find the maximal modular decomposition by initialising the set of modules to all the singleton trivial modules, that is $\{\{v\}: v \in V(G)\}$.

▶ Theorem 14. We can find the maximal temporal modular decomposition in time $O(n^4\Lambda)$. where n is the number of vertices in the temporal graph.

Proof. The above algorithm begins with a set of n modules, and repeatedly considers pairs from this set. After each iteration the size of the list will either be reduced by one, or a pair will be discounted, thus the process must terminate after $O(n^2)$ iterations. On each iteration each remaining module is checked to see if it splits the current candidate module on any timestep, which is possible in $O((n+m)\Lambda) = O(n^2\Lambda)$, giving an overall runtime of $O(n^4\Lambda)$.

3.1 Star Exploration

In this section we consider the following problem, demonstrating that it remains NP-hard even on temporal graphs with temporal cliquewidth at most three, but that it is solvable in constant time on graphs with bounded temporal modular-width. This problem was first introduced by Akrida et al. [2].

STAREXP(4)

Input: A temporal star (S_n, λ) where $|\lambda(e)| \leq 4$ for every edge e in the star S_n .

Output: Is there a strict temporal walk, starting and finishing at the centre of the star, which visits every vertex of S_n ?

We begin with a simple observation about the temporal cliquewidth of temporal graphs whose underlying graph is a star.

▶ **Lemma 15.** A temporal star S_n has temporal cliquewidth at most 3.

Proof. Our proof is constructive. The temporal cliquewidth construction is as follows.

- 1. Order the leaves of the star l_1, \ldots, l_n arbitrarily.
- **2.** Introduce the central vertex c with label L_1 .
- 3. Introduce the vertex l_1 with label L_2 and add all time-edges between l_1 and c.
- **4.** For all remaining leaves $l_i \in l_2, \ldots, l_n$:
 - **a.** Introduce l_i with label L_3 and all time-edges between c and l_i .
 - **b.** Relabel l_i with label L_2 .

From this construction, we see that it requires at most 3 labels to construct a temporal star.

STAREXP(τ) is known to be NP-hard even for constant τ [2, 6]. Then, by Lemma 15, STAREXP(τ) is an example of a problem which is NP-hard on graphs of bounded temporal cliquewidth. We now show that STAREXP(τ) is tractable on graphs of bounded temporal modular-width. We begin with the following lemma.

▶ **Lemma 16.** If a temporal star S_n has temporal modular-width at most k, the leaves of S_n can be partitioned into k-1 subsets such that, if u and v are in the same subset and c is the central vertex in the star, the edges uc and cv are active at the same times.

Proof. We claim that the graph into which the final substitution is made in the maximal temporal modular decomposition is a star with at most k-1 leaves. Else, the graph constructed would either not be a star or not have temporal modular-width at most k. We begin by showing that, for this substitution, the central vertex c is the only vertex in its module. Suppose otherwise; following the substitution any other vertex in this module has a neighbour which is not c. This contradicts that S_n is a star.

In addition, we have that the other modules are independent sets, otherwise we would again have that two leaves are adjacent. Therefore, there are no edges in the graph before this final substitution is made. Since all edges between any two modules in a substitution are assigned the same time, all leaves in the same module must be adjacent to the central vertex by edges active at the same time. Thus, we have k-1 subsets such that, if u and v are in the same subset, the edges uc and v are active at the same times.

▶ **Lemma 17.** If there are strictly more than $\tau/2$ leaves of S_n whose incident edges are active at the same times, we have a no-instance of STAREXP (τ) .

Proof. By definition of the problem STAREXP(τ), each edge in the star is active at most τ times. Therefore, if there are $\lfloor \tau/2 \rfloor + 1$ vertices $u_1, \ldots, u_{\lfloor \tau/2 \rfloor + 1}$ such that $\lambda(u_1c) = \cdots = \lambda(u_{\lfloor \tau/2 \rfloor + 1}c)$ where c is the central vertex in the star, there is no temporal walk which starts at c and visits all of $u_1, \ldots, u_{\lfloor \tau/2 \rfloor + 1}$. To see this, note that visiting a leaf and returning requires the use of two distinct time-edges. Therefore, a walk visiting any $\lfloor \tau/2 \rfloor + 1$ leaves which departs from and returns to c must consist of at least $\tau + 1$ distinct time-edges. This is not possible if these vertices have incident edges are active at the same times and $|\lambda(uc)| \leq \tau$.

▶ **Theorem 18.** StarExp(τ) is solvable in $(k\tau)!(k\tau)^{O(1)}$ time when the temporal modular-width of the graph is at most k.

Proof. Given that the temporal modular-width of the graph is at most k, we check whether the number of leaves is more than $(k-1)\lfloor \tau/2 \rfloor$. If the number of leaves is at least $(k-1)\lfloor \tau/2 \rfloor + 1$ then, by the pigeon-hole principle and Lemma 16, there must be at least $\lfloor \tau/2 \rfloor + 1$ leaves whose edges to the central vertex are active at exactly the same times. In this case, by Lemma 17, we conclude that we have a no-instance of STAREXP (τ) .

Else, we have at most $(k-1)[\tau/2] < k\tau$ leaves. Note that, given an ordering of leaves to visit, we can check whether such a walk is valid in time polynomial in k and τ : we need only check for each leaf in order whether there are two appearances of its incident edge following the time-edges used to visit the previous leaf (and if so, greedily use the first two such appearances). Since there are fewer than $(k\tau)!$ possible orderings of the leaves, we can check each possibility in turn in time $(k\tau)!(k\tau)^{O(1)}$.

4 Tractability with respect to Temporal Neighbourhood Diversity

We now turn our attention to our final parameter, temporal neighbourhood diversity, which is the most restrictive and hence allows for the most problems to be solved efficiently. In Section 4.1 we demonstrate that TEMPORAL GRAPH BURNING is solvable in polynomial time when the temporal neighbourhood diversity is bounded by a constant (in fact we give an fpt-algorithm with respect to this parameterisation), even though the problem remains NP-hard when restricted to temporal graphs with constant temporal modular-width. To illustrate further techniques that may be used to design efficient algorithms on graphs of bounded temporal neighbourhood diversity, in Section 4.2 we also give an fpt-algorithm for the problem SINGMINREACHDELETE with a single source vertex.

We begin with the formal definition of temporal neighbourhood diversity. Once again, the definition is modelled on that for static graphs, which was first introduced by Lampis [28] and adpated by Ganian [21] to describe uncoloured graphs. In a static graph, we define the neighbourhood N(v) of a vertex v as the set of vertices which share an edge with v.

- ▶ **Definition 19** (Type, Definition 2.2 [21]). Two vertices v, v' have the same type if and only if $N(v) \setminus \{v'\} = N(v') \setminus \{v\}$.
- ▶ Definition 20 (Neighbourhood Diversity, Definition 2 [28]). A graph G = (V, E) has neighbourhood diversity at most k if and only if there exists a partition of V(G) into at most w sets where all sets have the same type. We refer to this partition as a neighbourhood partition.

We note that the neighbourhood diversity of a graph can be computed in linear time [28]. We now define the analogous temporal parameter, where we require that the edges between sets are all active at the same times.

- ▶ **Definition 21** (Temporal Neighbourhood). The temporal neighbourhood of a vertex v in a temporal graph (G, λ) is the set TN(v) of vertex time pairs (w, t) where $(w, t) \in TN(V)$ if and only if $vw \in E(G)$ and $t \in \lambda(vw)$.
- ▶ **Definition 22** (Temporal Type). Two vertices u, v have the same temporal type if and only if $\{(w,t) \in TN(v) : w \neq v\} = \{(w,t) \in TN(v') : w \neq u\}$.
- ▶ Definition 23 (Temporal Neighbourhood Diversity). A graph \mathcal{G} has temporal neighbourhood diversity at most k if and only if there exists a partition of $V(\mathcal{G})$ into at most k sets where all sets have the same temporal type. We refer to this partition as a temporal neighbourhood partition.

It is immediate from this definition that, when all edges are assigned the same times, the temporal neighbourhood diversity of the graph is the same as the neighbourhood diversity of the underlying graph. We now argue that, as in the static case, the subgraph induced by any class must form a clique or independent set; moreover, in the temporal setting, this must be true at every timestep.

▶ **Lemma 24.** At any snapshot \mathcal{G}_t of \mathcal{G} , the subgraph induced by the vertices in a class X of a temporal neighbourhood partition of \mathcal{G} either forms an independent set or a clique.

Proof. Let u and v be vertices in X a class of a temporal neighbourhood partition of \mathcal{G} . Suppose the time-edge (uv,t) exists in \mathcal{G} , then for any other vertex w in X, the time-edges (wu,t) and (wv,t) must exist. Thus, the vertices in X form a clique at time t.

Similarly, suppose the time-edge (uv,t) does not exist in \mathcal{G} . Then, for any other $w \in X$, the time-edges (wu,t) and (wv,t) cannot exist. Otherwise, w has a different temporal neighbourhood to v and u respectively. Therefore, the vertices in X must form an independent set at time t.

As stated, temporal neighbourhood diversity is the most restrictive parameter in our hierarchy. We observe that for any temporal graph \mathcal{G} the temporal modular-width is upper bounded by the temporal neighbourhood diversity. Each class in the temporal neighbourhood partition forms a module, and it follows from Lemma 24 that each of these modules can be constructed by repeatedly substituting into a complete graph or independent set of two vertices.

Finally, we argue that we can compute the temporal neighbourhood diversity efficiently.

▶ Proposition 25. The temporal neighbourhood diversity of a temporal graph (G, λ) can be calculated in $O(\Lambda n^3)$ time.

Proof. Observe that we can check time $O(n\Lambda)$ whether two vertices are in the same class of a maximal temporal neighbourhood partition. There are at most $O(n^2)$ pairs of vertices to compare. Therefore, dividing vertices of a temporal graph into these equivalence classes can be done in $O(\Lambda n^3)$ time.

4.1 Temporal Graph Burning

In this section we define Temporal Graph Burning, a temporal analogue of the static Graph Burning problem first proposed by Bonato et al.[5]. Static Graph Burning is NP-hard on general graphs [4], was recently shown to be in FPT parameterised by static modular width [27]. In contrast, we prove that Temporal Graph Burning remains NP-hard on graphs with constant temporal modular-width. This difference arises from the fact that, in the static setting, the length of a longest induced path in the graph (which is upper bounded by the modular-width) gives an upper bound on the time taken to burn the graph. In the temporal setting, on the other hand, the times assigned to edges mean that even graphs with small diameter may take many steps to burn. In contrast, we show that Temporal Graph Burning can be solved in time $O(n^5 \Lambda k! 4^k)$ on temporal graphs with n vertices, lifetime Λ and temporal neighbourhood diversity k.

The TEMPORAL GRAPH BURNING problem asks how quickly a fire can be spread over the vertices of a temporal graph in the following discrete time process, where a fire is placed at a vertex of a graph on each timestep.

1. At time t = 0 a fire is placed at a chosen vertex. All other vertices are unburnt.

- 2. At all times $t \ge 1$, the fire spreads, burning all vertices u adjacent to an already burning vertex v where the edge between u and v is active at time t. Then, another fire is placed at a chosen vertex.
- 3. This process ends once all vertices are burning.

We refer to a sequence of vertices at which fires are placed as a strategy.

▶ **Definition 26** (Burning Strategy). A burning strategy for a temporal graph (G, λ) is a sequence of vertices $S = s_1, s_2, ..., s_\ell$ such that $s_i \in V(G)$ for all $i \leq \ell$, and on each timestep i a fire is placed at s_i .

For convenience, we allow for strategies that place fires at already burning vertices, although it is worth noting that such moves may be omitted. If every vertex in the graph is burning after a strategy is played, we say that strategy is successful.

▶ **Definition 27** (Successful Burning Strategy). A burning strategy $S = s_1, s_2, ..., s_\ell$ for a temporal graph (G, λ) is successful if every vertex in G is burning on timestep ℓ when the moves from S are played.

The decision problem asks how many timesteps it takes to burn a given temporal graph.

TEMPORAL GRAPH BURNING

Input: A temporal graph (G, λ) and an integer h.

Output: Does there exist a successful burning strategy for (G, λ) of length less than or equal to h?

This problem is in NP, with a strategy providing a certificate. Given a strategy it can be checked in polynomial time if it is successful and of length less than or equal to h by simulating temporal graph burning on the input graph.

We show that TEMPORAL GRAPH BURNING is NP-hard even on graphs of bounded temporal modular-width. This is achieved by reducing from (3, 2B)-SAT, an NP-hard variant of the Boolean satisfiability problem in which each variable appears exactly twice both positively and negatively [3], defined formally as follows.

(3, 2B)-SAT

Input: A pair (B, C) where B is a set of Boolean variables, and $C = C_1 \wedge ... \wedge C_m$ is a set of clauses over B in CNF, each containing 3 literals $C_j^1 \vee C_j^2 \vee C_j^3$, such that each variable appears exactly twice negatively and exactly twice positively.

Output: Is there a truth assignment to the variables such all of the clauses in C are satisfied?

Our reduction produces a graph where each edge is active on exactly one timestep, and furthermore every connected component has a bounded temporal neighbourhood diversity, and hence the graph overall has bounded temporal modular-width.

▶ Theorem 28. Temporal Graph Burning is NP-hard even when restricted to graphs with constant temporal modular-width.

Proof. We reduce from (3, 2B)-SAT. As such we begin by describing how to construct an instance $((G, \lambda), h)$ of Temporal Graph Burning, given an instance (B, C) of (3, 2B)-SAT with $C = C_1 \wedge \cdots \wedge C_m$, and |B| = n. We then go on to show that (B, C) is a yes-instance of (3, 2B)-SAT if and only if $((G, \lambda), h)$ is a yes-instance of Temporal Graph Burning.

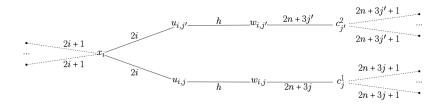


Figure 2 The connected component corresponding to the literal x_i , where x_i is the 1st literal in clause c_i , and the second literal in clause $c_{i'}$.

To construct $((G, \lambda), h)$, begin by setting h = 2n + 3m + 1. Now let the vertex set of G be given by the union of the following sets:

```
■ the set of literal vertices: \{x_i, \neg x_i : i \in [n]\},

■ the set of clause vertices: \{c_j^1, c_j^2, c_j^3 : j \in [m]\},

■ the set of appearance vertices: \{u_{i,j}, w_{i,j} : x_i \text{ or } \neg x_i \text{ appears in } C_j\},

■ the set of leaf vertices: \{y_{i,d}, \neg y_{i,d} : i \in [n], d \in [h+1]\} \cup \{z_{j,d}^1, z_{j,d}^2, z_{j,d}^3 : j \in [m], d \in [h+1]\}.
```

We represent the set of time-edges as a set of pairs of edges and a single timestep, such that if $(\{u,v\},t)$ is a time-edge then $\{u,v\} \in E(G)$ and $\lambda(\{u,v\}) = \{t\}$. This set is then given by the union of the following sets:

```
 = \{(\{x_i, y_{i,d}\}, 2i+1), (\{\neg x_i, \neg y_{i,d}\}, 2i+1) : i \in [n], d \in [h+1]\}, 
 = \{(\{c_j^1, z_{j,d}^1\}, 2n+3j+1), (\{c_j^2, z_{j,d}^2\}, 2n+3j+1), (\{c_j^3, z_{j,d}^3\}, 2n+3j+1) : j \in [m], d \in [h+1]\}, 
 = \{(\{x_i, u_{i,j}\}, 2i), (\{u_{i,j}, w_{i,j}\}, h), (\{w_{i,j}, c_j^{\ell}\}, 2n+3j) : x_i \text{ is the } \ell^{\text{th}} \text{ literal in } C_j\}, 
 = \{(\{\neg x_i, u_{i,j}\}, 2i), (\{u_{i,j}, w_{i,j}\}, h), (\{w_{i,j}, c_j^{\ell}\}, 2n+3j) : \neg x_i \text{ is the } \ell^{\text{th}} \text{ literal in } C_j\}.
```

Note that this graph consists of 2n connected components, each corresponding to a literal x_i or $\neg x_i$. Each of these connected components contain exactly two clause vertices, as every variable appears exactly twice positively and exactly twice negatively. We denote by H_i , and $\neg H_i$ the connected components containing x and $\neg x_i$ respectively. One of these connected components can be seen in Figure 2.

In order to show that it is possible to burn this graph in h timesteps if and only if there is a satisfying truth assignment for (B, C), we first state the following claims, whose proofs are omitted due to space constraints.

 \triangleright Claim 29. In order to burn (G, λ) in h or fewer timesteps, for each pair of literal vertices x_i and $\neg x_i$, a fire must be placed at or adjacent to one on timestep 2i - 1, and at the other on timestep 2i.

Proof. Begin by observing that in order for (G, λ) to burn in h or fewer timesteps, every literal vertex x_i or $\neg x_i$ must be burning by the end of timestep 2i, as every literal vertex has h+1 adjacent leaves with edges active at timestep 2i+1. Observe that the fire can only spread to a literal vertex x_i or $\neg x_i$ by the end of timestep 2i if it originates at one of the two adjacent non-leaf vertices $u_{i,j}$ and $u_{i,j'}$. As a result, for each literal vertex x_i or $\neg x_i$ a fire must either be placed at the literal vertex by the end of timestep 2i, or at one of the vertices $u_{i,j}$ and $u_{i,j'}$ by the end of timestep 2i-1. We now continue by induction on the variable index i. For the base case it follows immediately that a fire must be placed at x_1 on timestep 1 or 2, or the adjacent vertex $u_{1,j}$ on timestep 1. The same is true of the vertex $\neg x_1$. Therefore on timestep 1 a fire must be placed at or adjacent to one of these vertices,

and on timestep 2 a fire must be placed at the other vertex. For the inductive step assume that the claim is true for all literal vertices x_a and $\neg x_a$ with a < i. Then, on timestep 2i - 1, there must be no fires already placed either at or adjacent to x_i or $\neg x_i$. Then, a fire must be placed at x_i on timestep 2i - 1 or 2i, or at the adjacent vertex $u_{i,j}$ on timestep 2i. Again, the same is true of vertex $\neg x_i$, and therefore a fire must be placed at one of these vertices on timestep 2i, and at or adjacent to the other on timestep 2i - 1.

 \triangleright Claim 30. In order to burn (G, λ) in h or fewer timesteps, for each triple of clause vertices c_j^1, c_j^2, c_j^3 there must exist a permutation π of $\{1, 2, 3\}$, such that a fire is placed at or adjacent to $c_j^{\pi(1)}$ on timestep 2n + 3j - 2, at or adjacent to $c_1^{\pi(2)}$ on timestep 2n + 3j - 1, and at $c_1^{\pi(3)}$ on timestep 2n + 3j.

Proof. Begin by observing that in order for (G, λ) to burn in h or fewer timesteps, every clause vertex c_i^{ℓ} must be burning by the end of timestep 2n + 3j, as every clause vertex has k+1 adjacent leaves with edges active at timestep 2n+3j+1. Observe that the fire can only spread to a clause vertex c_i^{ℓ} by the end of timestep 2n+3j if it originates at the adjacent non-leaf vertex $w_{i,j}$. As a result, for every clause vertex c_j^{ℓ} a fire must either be placed at the clause vertex by the end of timestep 2n+3j, or at $w_{i,j}$ before the end of timestep 2n + 3j - 1. We now continue by induction on the clause index j. For the base case we see from Claim 29 that no fire can be placed at either c_1^1 or $w_{i,1}$ prior to timestep 2n+1, and it therefore follows that a fire must be placed at c_i^1 on one of the timesteps 2n+1, 2n+2, or 2n+3, or the adjacent vertex $w_{i,1}$ on timesteps 2n+1 or 2n+2. The same is true of the two other clause vertices c_1^2 and c_1^3 . Therefore there must exist a permutation π of $\{1,2,3\}$, such that a fire is placed at or adjacent to $c_1^{\pi(1)}$ on timestep 2n+1, at or adjacent to $c_1^{\pi(2)}$ on timestep 2n+2, and at $c_1^{\pi(3)}$ on timestep 2n+3. For the inductive step assume that the claim is true for all clause vertices c_a^1, c_a^2, c_a^3 with a < j. Then on timestep 2n + 3j - 2, there must be no fires already placed either at or adjacent to c_j^1 . Then, a fire must be placed at c_i^1 on one of the timesteps 2n+3j, 2n+3j-1, 2n+3j-2, or at the adjacent vertex $w_{i,j}$ on one of the timesteps 2n + 3j - 1, 2n + 3j - 2. The same is true of the vertices c_j^2 and c_j^3 , and therefore there must exist a permutation π of $\{1,2,3\}$, such that a fire is placed at or adjacent to $c_j^{\pi(1)}$ on timestep 2n+3j-2, at or adjacent to $c_1^{\pi(2)}$ on timestep 2n+3j-1, and at $c_1^{\pi(3)}$ on timestep 2n + 3j.

 \triangleright Claim 31. Suppose that there is a successful strategy for (G, λ) of length at most h, and consider the connected component H_i or $\neg H_i$ at which a fire is placed on timestep 2i. Let c_j^{ℓ} be a clause vertex in this connected component. Then a fire must be placed at or adjacent to c_j^{ℓ} prior to timestep 2n + 3j.

Proof. If a fire is placed in H_i or $\neg H_i$ on timestep 2i, then it must be placed at the corresponding literal vertex x_i or $\neg x_i$ by Claim 29. The fire cannot spread to the adjacent non-leaf vertex $u_{i,j}$ from the literal vertex, as the edge between $u_{i,j}$ and the literal vertex is only active at time 2i. Thus, the fire must spread to $u_{i,j}$ from $w_{i,j}$. And, by Claim 30, a fire is either placed at c_j^{ℓ} or at $w_{i,j}$. This fire cannot be placed at c_j^{ℓ} on timestep 2n+3j, as otherwise it would not reach $w_{i,j}$. In every other case the fire is placed prior to timestep 2n+3j as required.

We are now ready to prove the correctness of our reduction, and begin by showing that if $((G, \lambda), h)$ is a yes-instance, then so is (B, C).

Assume that $((G, \lambda), h)$ is a yes-instance, and consider a strategy that burns the graph in h or fewer timesteps. We then assign a value to each variable x_i according to when the

strategy places fires in the connected components H_i or $\neg H_i$. If on timestep 2i-1 a fire is placed in H_i , then x_i is assigned true, otherwise, if a fire is placed in $\neg H_i$ then x_i is assigned false. Now consider any clause $C_j \in C$, and see that by Claim 30 one of the corresponding vertices c_j^1, c_j^2 , or c_j^3 must have a fire placed at it on timestep 2n+3j. Then, by Claim 31, this vertex must be in the connected component in which a fire was placed on timestep 2i-1. This literal is therefore assigned true, and thus the clause is satisfied, as required.

Now assume that (B,C) is a yes-instance, and consider a satisfying assignment of the variables in B. We now describe a burning strategy for (G, λ) , and show that it burns the graph in h or fewer timesteps. We place fires such that a fire is placed at the vertex corresponding to the truthful literal in $\{x_i, \neg x_i\}$ on timestep 2i-1, and a fire is placed at the vertex corresponding to the falseful literal in $\{x_i, \neg x_i\}$ on timestep 2i. As we have a satisfying assignment, each clause $C_j = C_j^1 \vee C_j^2 \vee C_j^3$ must contain a literal C_j^ℓ that evaluates to true, and we place a fire at the corresponding clause vertex c_i^{ℓ} on timestep 2n+3j. Fires are placed at the other two clause vertices $\{c_j^1,c_j^2,c_j^3\} \setminus \{c_j^\ell\}$ on timesteps 2n+3j-2 and 2n + 3j - 1. In this strategy, every literal vertex x_i or $\neg x_i$ is burnt by the end of timestep 2i, and therefore all literal leaves $y_{i,d}$ burn by the end of timestep h. Every clause vertex c_i^{ℓ} is burnt by the end of timestep 2n + 3j, and therefore all clause leaves $z_{j,d}^{\ell}$ burn by the end of timestep h. Finally, if a vertex pair $u_{i,j}$ and $w_{i,j}$ belongs to a connected component H_i or $\neg H_i$ in which a fire is placed on timestep 2i-1, the fire spreads from the literal vertex to $u_{i,j}$ on timestep 2i, and then from $u_{i,j}$ on timestep h. Otherwise, the vertex pair must be on a path with a clause vertex at which a fire is placed by timestep 2n + 3j - 1, then the fire spreads from the clause vertex to $w_{i,j}$ on timestep 2n+3j, and from $w_{i,j}$ to $u_{i,j}$ on timestep h. Thus, in either case, both of these vertices will burn by the end of timestep h as required.

We now show that TEMPORAL GRAPH BURNING is solvable efficiently when the temporal neighbourhood diversity of the input graph is bounded. Throughout we assume that the lifetime Λ of the input temporal graph is at most the number of vertices n, as it is possible to burn any temporal graph in n timesteps by placing a fire at every vertex in turn.

We begin by defining notation for the burning set of vertices on a given timestep when a strategy is played.

▶ **Definition 32** (Burning Set). Given a strategy S the burning set $B_t(S)$ at timestep t is the set of vertices immediately after a fire is placed on timestep t when S is played.

We now prove a lemma that shows that if for two strategies S and R and some timestep t_1 we have $B_{t_1}(S) \subseteq B_{t_1}(R)$, then we are able to use an initial segment from strategy R and find remaining moves from times $t_1 + 1$ onwards to obtain a strategy that will burn the graph at least as fast as S.

Thus throughout, in order to show that the existence of a strategy that burns the graph at least as fast as another, we need argue only about the existence of such a timestep t and the first t moves made by the strategies.

▶ Lemma 33. Let S be a successful strategy for (G, λ) . Suppose that there is some timestep $t_1 < |S|$ and strategy R with |R| = |S| such that $B_{t_1}(S) \subseteq B_{t_1}(R)$ and on every timestep after t_1 , R places a fire at the same vertex as S. Then R is also successful.

Proof. We prove by induction on $t_2 - t_1$ that for any timestep $t_2 \ge t_1$ we have that $B_{t_2}(S) \subseteq B_{t_2}(R)$.

Our base case is given, as R is defined such that $B_{t_1}(S) \subseteq B_{t_1}(R)$. We now assume that $B_t(S) \subseteq B_t(R)$, for some $t_1 \le t < |S|$, and show that $B_{t+1}(S) \subseteq B_{t+1}(R)$.

Now let $v \in B_{t+1}(S)$. If $v \in B_t(S)$ then $v \in B_t(R)$ and therefore $v \in B_{t+1}(R)$.

Otherwise if $v \notin B_t(S)$ then either the fire spreads to v on timestep t+1 when S is played, or S places a fire at v on timestep t+1.

In the former case v is temporally adjacent to a vertex in u in $B_t(S)$ on timestep t+1. It must be the case that u is also in $B_t(R)$, and therefore the fire will spread to v on t+1when R is played if v has not already burnt.

In the latter case either v is burning before R places a fire a on timestep t+1 when R is played, or $r_{t+1} = s_{h+1} = v$.

Therefore $B_{t+1}(S) \subseteq B_{t+1}(R)$. We then have that $V(G) = B_{|S|}(S) \subseteq B_{|S|}(R)$, and therefore R is successful.

We now show that we can delay placing a fire at a vertex belonging to a class in which the fire is already burning, without causing a large effect on the set of vertices that will burn at each timestep.

Lemma 34. Let (G,λ) be a temporal graph with temporal neighbourhood partition $(X_i)_{i\in I}$. Now let S be a strategy that burns this graph, and let u be a vertex at which S places a fire on a timestep t_1 , and X_i be the class to which this vertex belongs. Let S' be a strategy which plays as follows until t_2 , for any timestep $t_2 > t_1$:

$$s'_{t} = \begin{cases} s_{t} & \text{if } t < t_{1} \\ s_{t+1} & \text{if } t_{1} \le t < t_{2} \\ u & \text{if } t = t_{2} \end{cases}$$

Providing there exists a vertex $w \in X_i$ which is burning before the end of timestep t_1 when S' is played, we have that $B_{t_2}(S) \subseteq B_{t_2}(S')$.

Proof. We will show that for any vertex $x \in B_{t_2}(S)$ we have that $x \in B_{t_2}(S')$.

Consider the case where x is a vertex at which S places a fire. If $x \neq u$ then S' will place a fire at x on the same timestep or earlier than S, and thus $x \in B_{t_2}(S)$. Otherwise, if x = uthen S' places a fire at it on timestep t_2 , and again $x \in B_{t_2}(S)$.

Now, consider the case where S does not place a fire at x. It must then burn because the fire spreads to it. Note, as $x \in B_{t_2}(S)$, there must exist a temporal path that traverses the vertices $y_1, ..., y_h$ such that $y_h = x$, and y_1 is a vertex at which S places a fire prior to the departure time t_{α} , and the arrival time t_{β} is less than or equal to t_2 . Let P be the shortest such path.

If $y_1 \neq u$ then $x = y_h$ will still burn before the end of timestep $t_\beta \leq t_2$ when S' is played, as S' will also place a fire at y_1 prior to timestep t_{α} . Thus $x \in B_{t_2}(S')$.

Otherwise if $y_1 = u$ then u is temporally adjacent to y_2 on timestep $t_\alpha > t_1$. As $w, u \in X_i$, w is also temporally adjacent to y_2 on this timestep, and there is also a temporal path $P' = w, y_2, ..., y_h$ starting at time $t_\alpha > t_1$, and identical to P in all but the first vertex. Therefore in this case when S' is played $x = y_h$ will burn before the end of timestep t, as w burns by the end of timestep $t_1 < t_{\alpha}$ when S' is played. Thus, again, $x \in B_{t_2}(S')$.

We go on to show that any time we place a fire at a vertex, we may instead place a fire at another unburnt vertex in the same class, if such a vertex exists, and obtain a strategy that burns the graph in the same time.

Lemma 35. Let (G,λ) be a temporal graph with temporal neighbourhood partition $(X_i)_{i\in I}$.

Let S and S' both be strategies with |S'| = |S|, such that on every timestep, S and S' both place fires in the same class, that is, for any $i \leq \min(|S|, |S'|)$ we have that $\{s_i, s_i'\} \subseteq X_j$. Furthermore, S' places a fire at an already burning vertex on a timestep i if and only if S also places a fire at an already burning vertex on timestep i.

S is successful if and only if S' is.

Proof. We show that on each timestep t, the number of burning vertices in each class from the temporal neighbourhood partition is the same when S and S' are played. We continue by induction on the timestep.

On the first timestep only s_1 is burning when S is played, and only s'_1 is burning when S' is played. By the definitions of S and S' these two vertices are both in the same class in the temporal neighbourhood partition.

Then, assume that on any timestep t the number of burning vertices in each class from the temporal neighbourhood partition is the same when S and S' are played. Now given any class X_i from the temporal neighbourhood partition, let b_i be the number of vertices burning in X_i at the end of timestep t+1 when S is played. The number of vertices burning in X_i at the end of timestep t+1 when S' is played is then the number of vertices that were burning on timestep t, plus the number of vertices to which the fire spreads, plus one if a fire was placed in X_i by S' on timestep t+1. All of the vertices in X_i will be burning by the end of timestep t+1 if the fire spreads to any vertex $v \in X_i$, as any burning vertex u adjacent to von t+1 is also adjacent to all other vertices in X_i . Furthermore, such a vertex $u \in X_i$ exists if and only if there is a burning vertex in X_j when S is played, as the number of vertices burning in X_j at the end of timestep t is the same when both S and S' are played. Then, all of the vertices of X_i are burning on timestep t+1 when S is played if and only if all of the vertices of X_i are burning on timestep t+1 when S' is played. Furthermore, S' places a fire in X_i if and only if S does also. Therefore on timestep t+1 either the number of burning vertices in X_i does not change when either S or S' is played, increases by one when each strategy is played, or all of the vertices are burning when S or S' is played.

We now show that we can reorder any successful burning strategy S, so that initially one fire is placed in every *placement class* for S. This reordering gives a new strategy which is still successful.

- ▶ **Definition 36** (Placement Classes). The placement classes for a strategy S denoted C(S) is the set of classes from the temporal neighbourhood partition in which S places fires.
- ▶ **Lemma 37.** Given a temporal graph (G, λ) , let S be any successful strategy. There is then a successful strategy S' with |S'| = |S|, and C(S') = C(S), such that the first |C(S)| burns are in distinct equivalence classes in the temporal neighbourhood partition.

Proof. Assume that (G, λ) is a counterexample. Then let S be successful strategy minimal in the timestep t_c , such that at the end of timestep t_c there is a fire placed in every class in C(S). Now, let $u \in X_i$ be a vertex at which S places a fire on timestep $t_1 \leq |C(S)|$, such that a fire has already been placed at a vertex $w \in X_i$ prior to t_1 .

Consider the strategy S' which makes moves as follows:

$$s'_t = \begin{cases} s_t & \text{if } t < t_1 \text{ or } t > t_c \\ s_{t+1} & \text{if } t_1 \le t < t_c \\ u & \text{if } t = t_c \end{cases}$$

By Lemma 34, we have that $B_{t_c}(S) \subseteq B_{t_c}(S')$. Then, by Lemma 33, S' burns (G, λ) in the same or less time as S. This contradicts the assumption that S was minimal in the number

of moves played until a fire is placed in every class in C(S), so no such counterexample (G, λ) can exist.

Finally we show that, given a strategy S that places fires only in distinct classes for the first |C(S)| moves, we can arbitrarily reorder all subsequent moves made after timestep |C(S)|.

▶ Lemma 38. Let (G, λ) be a temporal graph, and S a successful strategy such that the first |C(S)| fires placed by S are placed in distinct classes from the temporal neighbourhood partition. Let $f: [|C(S)|+1, |S|] \to [|C(S)|+1, |S|]$ be any bijection.

Then the strategy S' given by

$$s'_{t} = \begin{cases} s_{t} & \text{if } t \leq |C(S)| \\ s_{f(t)} & \text{otherwise} \end{cases}$$

is successful, and burns the graph in the same or less time as S.

Proof. Let (G, λ) , along with a strategy S and bijection f be a counterexample.

Then let R be a successful strategy with $|R| \leq |S|$, C(R) = C(S), and $r_t = s_t$ for all $t \leq |C(S)|$. Furthermore, assume that R is the strategy minimal in the timestep t_2 such that for every timestep $t \geq t_2$ $r_t = s_{f(t)}$. (Note that it is possible that $t_2 = |R| + 1$, and there is no terminal sub-sequence on which R agrees with the permutation of S.) Let t_1 be the timestep on which R places a fire at $s_{f(t_2-1)} = s'_{t_2-1}$.

Now let R' be the strategy that makes moves as follows:

$$r'_t = \begin{cases} r_t & \text{if } t < t_1 \text{ or } t \ge t_2 \\ r_{t+1} & \text{if } t_1 \le t < t_2 - 1 \\ s_{f(t_2 - 1)} & \text{if } t = t_2 - 1 \end{cases}$$

Then, as R' places a fire at a vertex in every class in C(S) prior to timestep t_1 , by Lemma 34 we have that $B_{t_2-1}(R') \subseteq B_{t_2-1}(R)$. Then, by Lemma 33, R' burns (G, λ) in the same or less time as R. This contradicts the assumption that R was minimal in the timestep t_2 , so no such counterexample (G, λ) , strategy S, and bijection f can exist.

We now present an algorithm for TEMPORAL GRAPH BURNING, and show that this algorithm is an fpt-algorithm with respect to temporal neighbourhood diversity.

Algorithm 1 TND Graph Burning Algorithm

Input: A temporal graph \mathcal{G} , and an integer k.

Output: True if and only if there exists a successful burning strategy of length at most h.

```
1: Compute the temporal neighbourhood partition \Theta of (G, \lambda).
```

2: for all possible subsets $A \subseteq \Theta$ do

```
3: for all possible orderings of A do
4: for all possible subsets B ⊆ A do
5: Compute a strategy that first places a fire in order in every class from A, and then places fires at every unburnt vertex in B in any order.
6: if this strategy is successful and consists of k or fewer moves then
7: Let possible value at possible value and consists of A do
6: if this strategy is successful and consists of k or fewer moves then
7: Let possible value at possible value at possible value at possible value at possible orderings of A do
6: return true.
```

8: If no such strategy is found, return false.

We now prove correctness of this algorithm, using the following lemmas.

▶ **Lemma 39.** The TND GRAPH BURNING ALGORITHM returns true for a temporal graph (G, λ) and integer h if and only if there exists a strategy S that burns the graph in h or fewer timesteps.

Proof. If there exists a strategy that burns (G, λ) in h or fewer timesteps then by Lemma 37 and Lemma 35 there exists a strategy S that first places a fire at an arbitrary vertex from every class in C(S). The remaining moves must then place fires at every other vertex of any class to which the fire will not spread before the graph is burnt. These classes must be some subset of the classes from C(S), and from Lemma 38 we know that these can be made in any order, and therefore in particular in the order described in the algorithm.

The algorithm exhaustively checks every such strategy, and thus will return true if any strategy exists that burns (G, λ) in h or fewer moves, and false otherwise.

This allows us to obtain fixed parameter tractability, as bounding the temporal neighbourhood diversity bounds the number of such strategies that we have to check.

▶ **Theorem 40.** TEMPORAL GRAPH BURNING is solvable in time $O(n^5\Lambda k!4^k)$, where n is the size of the input temporal graph \mathcal{G} , Λ the lifetime, and k the temporal neighbourhood diversity. If the temporal neighbourhood partition is given, we obtain a runtime of $O(n^2\Lambda k!4^k)$.

Proof. The TND Graph Burning Algorithm solves TEMPORAL GRAPH BURNING. It begins by computing the temporal neighbourhood partition, which we know from Proposition 25 that we can do in time $O(n^3)$. Furthermore there are 2^k subsets A of the classes in the temporal neighbourhood partition Θ , and then at most k! possible orderings of any subset A, and at most and at most 2^k sets B. We then simulate temporal graph burning on the graph, which is possible in $O(n^2\Lambda)$ time giving us an overall running time of $O(n^5\Lambda k!4^k)$, where n is the size of the input graph, and Λ the lifetime. If the decomposition is given, we instead obtain a runtime of $O(n^2\Lambda k!4^k)$, as we may drop the $O(n^3)$ factor needed to compute it.

4.2 Minimum Reachability Edge Deletion

Here we give another problem which is tractable with respect to temporal neighbourhood diversity. Given a specified source vertex, we seek a minimum set of edge appearances that can be deleted to limit the number of vertices reachable from that source.

We say a vertex v is temporally reachable from a vertex u in \mathcal{G} if there exists a temporal path from u to v. We say a vertex v is temporally reachable from a set S if there is a vertex in S from which v is temporally reachable. The reachability set reach(v) of a vertex v is the set of vertices temporally reachable from v. We can now give the formal problem definition; this is a special case of the problem MINREACHDELETE studied by Molter et al. [35] in which multiple sources are allowed.

SINGLETON MINIMUM TEMPORAL REACHABILITY EDGE DELETION (SINGMINREACHDELETE)

Input: A temporal graph $\mathcal{G} = (G, \lambda)$, a vertex $v_s \in V(G)$ and positive integer r.

Output: What is the cardinality of the smallest set of time-edges E such that the vertex v_s has temporal reachability at most r after their deletion from \mathcal{G} ?

SINGMINREACHDELETE was shown by Enright et al. [16] to be NP-hard (and W[1]-hard parameterised by the maximum number of vertices that are allowed to be reached following deletion) even when the lifetime of the input temporal graph is 2 and every edge is active at exactly one timestep. While the result of [16] is for a version of the problem when the source

set S is the entire vertex set, it is clear from the construction that hardness also holds with a single source vertex.

We show that this problem is in FPT when parameterised by temporal neighbourhood diversity and the temporality of the input graph $\tau(\mathcal{G})$, which was defined by Mertzios et al. [32] to be the maximum number of times an edge appears. When the temporal graph in question is clear from context, we just refer to τ . We note that it remains open whether the problem belongs to FPT parameterised by temporal neighbourhood diversity alone, or indeed parameterised by temporal modular width or temporal cliquewidth; the techniques we use here do not extend naturally to these less restrictive settings.

We now give a formal statement of our result.

▶ **Theorem 41.** SINGMINREACHDELETE is solvable in time $g(k,\tau)\log^{O(1)}r + \Lambda n^3$, where g is a computable function. If a temporal neighbourhood decomposition is given, we can solve the problem in time $g(k,\tau)\log^{O(1)}r$.

This uses a result by Lokshtanov [29] which gives an FPT algorithm for the following problem.

INTEGER QUADRATIC PROGRAMMING

Input: A $n \times n$ integer matrix Q, an $m \times m$ matrix B and an m-dimensional vector \mathbf{b} . Parameter: $n + \alpha$ where α is the maximum absolute value of any entry in B or Q. Output: Find a vector $\mathbf{x} \in \mathbb{Z}^n$ which minimises $\mathbf{x}^T Q \mathbf{x}$, subject to $B \mathbf{x} \leq \mathbf{b}$.

Their result is as follows.

▶ Theorem 42 (Theorem 1, [29]). There exists an algorithm that, given an instance of INTEGER QUADRATIC PROGRAMMING, runs in time $f(n,\alpha)L^{O(1)}$ (for some computable function f), and outputs a vector $x \in \mathbb{Z}^n$. Here L is the total number of bits required to encode the input integer quadratic program. If the input IQP has a feasible solution then x is feasible, and if the input to the IQP is not unbounded, then x is an optimal solution.

Let \mathcal{G} be the temporal graph in the input of SINGMINREACHDELETE. We denote by \mathcal{G}_s the temporal neighbourhood partition graph of \mathcal{G} . This is a temporal graph where the classes of the temporal neighbourhood partition form the vertex set. We refer to nodes of \mathcal{G}_s and vertices of \mathcal{G} to differentiate between the two. An edge exists at time t between nodes A and B in \mathcal{G}_s if and only if there exist edges in \mathcal{G} at time t between (all) vertices in A and (all) vertices in B. Note that, given two vertices of the same temporal type, their reachability sets must consist of the same vertices except for the vertices themselves; as a result, the reachability sets of vertices in a given class all have the same cardinality. Moreover, all vertices of the same type are first reached from the source at the same time (where we say a vertex is "first reached" at time t if the final time-edge in an earliest-arriving temporal path from the source is at time t).

Our strategy for solving SINGMINREACHDELETE is as follows. We partition each class according to the time at which the vertices are first reached after the deletion of time-edges, and consider all possibilities for which of these subclasses are non-empty. Given a function ϕ telling us which subclasses are non-empty, we argue that we can determine efficiently whether there is indeed a deletion such that precisely these subclasses are non-empty, and if so compute exactly the pairs of subclasses between which we must delete time-edges to achieve this. For a fixed ϕ , we then encode the problem as an instance of INTEGER QUADRATIC PROGRAMMING, where the variables are the sizes of the subclasses and the objective function seeks to minimise the number of time-edges we must delete.

From now on we denote by A_t the subclass of vertices in class A that are first reached at time t following the deletion of time-edges, with an additional subclass A_{∞} consisting of those vertices in A not temporally reachable from the source.

We begin by making some assertions about these subclasses.

- ▶ Lemma 43. For each class A in the temporal neighbourhood partition, the number of indices t such that A_t can be non-empty following a deletion is at most $\tau \cdot k$ where k is the temporal neighbourhood diversity of \mathcal{G} .
- **Proof.** It suffices to show that there are at most $\tau(k-1)$ possibilities for the earliest arrival time at any given vertex, since the set of possible earliest arrival times will be the same for all vertices in the same class; we then have one additional subclass for the "unreached" subclass.

Suppose that v is first reached from the source at time t in $\mathcal{G} \setminus E$, for some set of time-edges E. Then there is a temporal path from the source to v whose last time-edge is at time t, so in particular we know that there is an edge incident with v in G that is active at time t. The number of distinct times at which there is an edge incident with any vertex is at most $\tau(k-1)$, since there can be at most τ distinct times at which edges to each other class are active, giving the result.

In total, therefore, we need to consider at most $k^2\tau$ subclasses. Let $\phi(x_i^A)$ be a function on this set of subclasses which maps x_i^A to empty to indicate that A_i should be empty, and to non-empty otherwise. We now make some observations about which deletions of time-edges can reduce the cardinality of the temporal reachability set of the source.

- ▶ **Lemma 44.** If both endpoints u, v of an edge uv are reached from the source by time t_2 , deletion of any appearances at or after time t_2 will not change the reachability set of the source. Further, if neither u nor v is reached by time t_1 from the source, deletion of any appearances at or before t_1 cannot change the reachability set of the source.
- **Proof.** Suppose there is a vertex u reachable from the source by a path using the edge e at time t_1 as described. This contradicts the assumption that neither endpoint is reached before time t_1 from the source. Therefore there is no temporal path from the source using e at time t_1 . Hence deletion of this appearance cannot affect reachability of the source.

Now consider a vertex w reachable from the source by a path p which uses an edge uv at time t_2 where t_2 is as described in the lemma statement. Suppose without loss of generality that, in p, the edge is traversed from u to v at time t_2 . Deletion of the appearance (uv, t_2) does not change the reachability of w from the source as we can reach p from the source by another path p' at a time strictly before p. Therefore, we can still reach p from the source without using the edge p at time p and deletion of p does not change the reachability set of the source.

- ▶ Corollary 45. An optimal deletion will never include the deletion of a time-edge with both endpoints in the same subclass.
- ▶ **Lemma 46.** Let \mathcal{G}' be the temporal graph obtained from making an optimal deletion of time-edges in \mathcal{G} . Fix v to be any vertex, and A to be any class of the temporal neighbourhood partition of \mathcal{G}' which is reached by the source before v is reached. Then, in any snapshot of \mathcal{G}' , v is either adjacent to all vertices in A or none of them.
- **Proof.** Suppose for contradiction that, after an optimal deletion, the number of edges between v and A at time t is |E(v,A,t)|, where 0 < |E(v,A)| < |A|. That is, it is neither complete nor empty.

Let w and w' be vertices in A such that the time-edge (vw',t) is deleted and (vw,t) is not after an optimal deletion. Then, since all of the vertices in A are reached at a time before t by the source, reinstating the time-edge (vw',t) does not increase the reachability of the source. Thus, the deletion is not optimal; a contradiction. Hence, in an optimal deletion, the edges at some time t are either complete or empty between a vertex and every subclass reached before its own.

▶ Lemma 47. Let \mathcal{G}' be the temporal graph obtained from making an optimal deletion of time-edges in \mathcal{G} , and let A_i and B_j be two subclasses. In each snapshot of \mathcal{G}' , the graph is either complete or empty between A_i and B_j . Therefore, all vertices in the same subclass have the same temporal neighbourhood in \mathcal{G}' .

Proof. Suppose for a contradiction that there is a time t at which the graph between A_i and B_j is neither complete nor empty. Let i and j be the respective times at which A_i and B_j are first reached in \mathcal{G}' . Denote by E the set of edges deleted between A_i and B_j at time t under this deletion. Without loss of generality, assume $j \geq i$. If $t \leq i$ or $t \geq j$, then by Lemma 44, the edges between A_i and B_j at time t are not deleted in an optimal deletion.

Therefore, we can assume that i < t < j. If there is at least one edge remaining between A_i and B_j then at least one vertex in B_j must first be reached from the source at time t < j by a path traversing an edge from a vertex in A_i to a vertex in B_j at time t. This contradicts our assumption that the vertices in B_j are first reached at time j.

The previous three lemmas describe which deletions are useful for reducing the reachability of the source and achieving some assignment ϕ . We now specify the time needed to determine whether a given assignment ϕ is feasible and, if so, find the unique minimal set of times at which we must delete edges between each pair of subclasses in order to realise ϕ .

▶ Lemma 48. Given an assignment ϕ of subclasses to empty and non-empty, we can check in time $O(k^5\tau^3)$ whether it is possible to delete time-edges from $\mathcal G$ so that, for each subclass A and index t, A_t is empty if and only if $\phi(x_i^A) = \text{empty}$. If such a deletion is possible, the algorithm outputs the unique minimal set of times at which we must delete edges between each pair of subclasses in order to realise ϕ .

Proof. Given ϕ , we create a temporal graph \mathcal{H} consisting of a node for each subclass of \mathcal{G} . Two nodes of \mathcal{H} are adjacent at time t if and only if the corresponding classes of \mathcal{G} are adjacent at time t. Note that if two nodes of \mathcal{H} are subclasses of the same class, they are adjacent at time t if and only if there exist edges at that time in that class. Each node of \mathcal{H} is labelled with the time at which it is first reached from the source. In the case of an "unreached" subclass, this time label is ∞ . In the algorithm where we check whether an assignment ϕ is valid, we will refer to the nodes as pairs (a,t) where a is the name of the node and t is the time at which it is first reached. By Lemma 43, there are at most $k^2\tau$ nodes in \mathcal{H} .

Recall that ϕ is a function which takes each subclass to either an empty or non-empty marker. We check whether an assignment ϕ is possible by a variation of breadth-first search (BFS) on the graph \mathcal{H} . We begin by finding the subgraph induced by removing any of the subclasses which are empty under ϕ from \mathcal{H} . Call this graph \mathcal{H}' . If the resulting graph is disconnected and there is a node with time $t \neq \infty$ in a different connected component to the source, we reject.

We keep track of two values for each node a in \mathcal{H} ; t_{target}^a and t_{first}^a . We initialise t_{first}^a with the value ∞ for all nodes a in \mathcal{H}' and set t_{target}^a to be the time at which vertices in node a should be first reached by the source, as prescribed by ϕ . The value t_{first}^a will be the earliest

time of arrival from the source once the algorithm finishes. Therefore we accept if, for all nodes $a, t_{\text{target}}^a = t_{\text{first}}^a$ when the algorithm has finished executing. We initialise an empty set E of edges which we will mark as deleted. We begin by adding the node containing the source to the queue. After dequeuing a node a, for each of its neighbours b, we check if the edge ab is labelled with a time t after t_{first}^a and before t_{first}^b . For all t after t_{first}^a such that, (ab,t) is in \mathcal{H}' , (ab,t) is not in E, and t is before t_{target}^b , we add (ab,t) to E. Then we are left with two cases, either there is no time t^* such that ab is active at time t^* where t^* is after t_{first}^a and before t_{first}^b , or such a time exists. If the former is true, we simply consider the next node in the queue. Else, t^* must be equal to or after t_{target}^b . Let t^* be the earliest time after t_{first}^a such that ab is active. Then, we update t_{first}^b to t^* and add b to the queue. The algorithm terminates when the queue is empty.

We note that a node can be added to the queue at most as many times at which it could be reached from the source. This is $\tau(k-1)$ times. Furthermore, the operations done when considering a dequeued node take linear time in the number of nodes. By Lemma 43, there are at most $k^2\tau$ nodes in \mathcal{H} . Therefore, the algorithm runs in $O(k^5\tau^3)$ time.

Once the algorithm terminates, $t_{\rm first}^a$ is the earliest time of arrival of a path from the source to the node a given the deletions made. We make a deletion if and only if otherwise there is a temporal path from the source that arrives at a node a before t_{target}^a . Therefore, $t_{\text{target}}^a = t_{\text{first}}^a$ for all nodes a (and the algorithm accepts) if and only if there is a set of deletions E such that the earliest time of arrival at a node in \mathcal{H} is as prescribed by ϕ . Observe that (ab,t) is added to E only if all edges at time t between the subclasses corresponding to a and b must be deleted in order to realise ϕ . Therefore E describes the minimal set of times at which we must delete edges between each pair of subclasses.

We can now transform our edge deletion problem given some ϕ into an instance of IQP and use the algorithm of Theorem 42 to solve it efficiently.

▶ **Lemma 49.** Given a fixed, feasible ϕ we can find in time $f(k,\tau)\log^{O(1)}r$ the cardinalities of the subclasses which minimise the number of deleted edges such that the source reaches r vertices.

Proof. Given a ϕ , we can express the division of vertices among non-empty subclasses such that at most r vertices are temporally reachable from the source and the minimum number of deletions are made as an instance of Integer Quadratic Programming. We arbitrarily order the subclasses. Then, the vector of variables is $\mathbf{x} = (x_1, \dots, x_\ell)^T$, where the *i*th variable is the number of vertices in the ith subclass. For a given assignment ϕ , we can find the minimum deletion between the subclasses such that the assignment holds by Lemma 48. Then, the matrix Q_{ϕ} is a $\ell \times \ell$ triangular matrix where there is a y in position $q_{i,j}$ if we have determined that there are exactly y timesteps at which we must delete all time-edges between subclass i and subclass j to match the assignments of ϕ . Therefore, x^TQx gives the number of edges we need to delete. We write $x^{A,t}$ for the entry of **x** corresponding to the subclass of A that is first reached from the source at time t.

We employ the following linear constraints

$$\forall A, t, x^{A,t} \ge 0$$

$$\sum_{t \ne \infty} x^{A,t} \le r \tag{1}$$

$$\forall A, \sum_{t} x^{A,t} = |A|. \tag{2}$$

$$\forall A, \sum_{t} x^{A,t} = |A|. \tag{2}$$

The constraints ensure that every $x^{A,t}$ is non-negative and

- 1. reach $(v_s) \leq r$, and
- 2. the sum of cardinalities of the subclasses of A is the cardinality of A.

We can express these conditions in the form $B\mathbf{x} = \mathbf{b}$, where B is a $(2\ell + 1 + (k^2\tau)) \times \ell$ matrix and b is a $(2\ell + 1 + (k^2\tau))$ -dimensional vector. Note that the largest absolute value in Q_{ϕ} is at most τ and the largest absolute value in B is 1. The number of variables ℓ is at most $k^2\tau$. It therefore follows from Theorem 42 that we can solve this instance of IQP in time $f(k,\tau)\log^{O(1)}r$.

We now give the proof of Theorem 41.

Proof. We solve SINGMINREACHDELETE as follows. For the input temporal graph \mathcal{G} , we find the temporal neighbourhood partition graph. This can be found in time $O(\Lambda n^3)$ by Proposition 25. We make a subclass for each time a class in the partition could be reached from the source, with an additional subclass for vertices in this class which are not temporally reachable from the source following a deletion; by Lemma 43, there are at most $k^2\tau$ subclasses. For each possible assignment ϕ of subclasses to empty/non-empty, we find the sets of time-edges between two subclasses which must be deleted by Lemma 48; there are at most $2^{k^2\tau}$ possible functions ϕ to consider. Given the sets of time-edges we must delete and a function ϕ , we then apply Lemma 49 to find the number of vertices in each subset such that the number of vertices reached from the source is at most r and the number of time-edges deleted is minimised.

5 Conclusion and Open Questions

We have described three temporal parameters that form a hierarchy mirroring the one formed by their static analogues, and that can all be small when the temporal graph is dense at every timestep. We provide examples of problems demonstrating that there is a separation between the classes of problems admitting efficient algorithms when each of the parameters is bounded. As is the case for the corresponding static parameters, we expect that there will be many problems for which these temporal parameters give fixed-parameter tractability, and suggest exploration of temporal extensions of static problems for which there are known fpt-algorithms as future work. From a practical perspective, it would also be interesting to investigate the values of these new parameters on dense real-world temporal networks.

One of the most celebrated results involving static cliquewidth is a metatheorem due to Courcelle et al. [11] which guarantees the existence of a linear-time algorithm for any problem expressible in a suitable fragment of logic (MSO₁) on graphs of bounded cliquewidth. It is a natural question whether an analogous metatheorem exists for temporal cliquewidth. A promising approach might be to encode a temporal graph as an arbitrary relational structure (as has been done for a temporal version of treewidth [18]). A major challenge here, however, is that to the best of our knowledge there is no single notion of cliquewidth for relational structures: several alternatives have been introduced [1, 12], but none has all of the desirable properties. Moreover, we believe that any encoding of a temporal graph of bounded temporal cliquewidth as a relational structure that preserves all the information in the original is unlikely to have bounded width for any cliquewidth-style measure unless we also bound the lifetime of the temporal graph. Nevertheless, this general direction merits further investigation, and there is potential for a useful metatheorem even if it is necessary to further restrict the fragment of logic considered or the structure of the temporal graph.

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