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Abstract. Let G be a finite group. A contravariant functor from the category of finite free G-sets to vector spaces has an associated Hilbert series, which records the underlying sequence of G^n representations, $n \in \mathbb{N}$. We prove that this Hilbert series is rational with denominator given by linear polynomials with coefficients in the field generated by the character table of G.

1. Introduction

Recently, it has been observed that many naturally occurring sequences of group representations arising in algebra and topology admit extra structure: they assemble into a representation of category of a combinatorial nature [1, 2]. This has motivated the study of the representation theory of combinatorial categories, with a view towards determining the consequences of this additional structure. In this paper we will consider the particular case of the category of finite G-sets.

Let G be a finite group and let $\mathbf{FS}_G$ be the category of finite free G-sets and surjections between them. (Our results will also apply to the category of all maps). Let k be an algebraically closed field of characteristic p (including the case $p = 0$). An $(\mathbf{FS}_G)^{\mathrm{op}}$ module (or representation) is a functor $M : (\mathbf{FS}_G)^{\mathrm{op}} \to \mathrm{Vec}_k$. From M, we obtain a sequence of G^n representations $M_n := M([n] \times G)$ for $n \in \mathbb{N}$, by evaluating M on the free G set on $[n]$ elements. Our motivating question is:

- Which sequences of G^n representations can arise in this way?

Actually, in this generality, any sequence is possible. But when M is **finitely generated** (see Definition 3.1), there are strong restrictions on the possible sequence of representations. In this paper, we constrain the possible Hilbert series underlying a finitely generated $(\mathbf{FS}_G)^{\mathrm{op}}$ module, answering a question of Sam–Snowden.

1.1. Hilbert series and main result.

Following Sam–Snowden [3, §6.4] we define a Hilbert series that determines the class of M_n as a G^n representation in the (rationalized) Grothendieck group. (Equivalently, determines the Brauer character of M).

We let $\hat{G}$ be the set of simple representations of G. We let $\mathcal{R}(G)$ be the Grothendieck group of G tensored with $\mathbf{C}$. Given $S \in \hat{G}$ we write $x_S \in \mathcal{R}(G)$ for the class S so that additively $\mathcal{R}(G)$ has a basis $\{x_S\}_{S \in \hat{H}}$. Then $\mathcal{R}(G^n) = \mathcal{R}(G)^{\otimes n} = \mathbf{C}\{x_S\}^{\otimes n}_{S \in \hat{G}}$.

There is a projection map

$$\pi : \mathcal{R}(G^n) = \mathcal{R}(G)^{\otimes n} \to \mathrm{Sym}^n(\mathcal{R}(G)),$$

where we take Sym^n be defined by S_n coinvariants. Sam–Snowden defined the **enhanced Hilbert series** of M to be the element

$$H_M = \sum_{n\in\mathbb{N}} \pi([M_n]) \in \prod_{n\in\mathbb{N}} \mathrm{Sym}^n(\mathcal{R}(G)) = \mathbf{C}[[x_S \mid S \in \hat{G}]].$$

Example 1.1. The free $\mathbf{FS}_G{}^{\mathrm{op}}$ module on a generator in degree one has $(kG^n)_{n\in\mathbb{N}}$ as its underlying sequence of representations. Its Hilbert series is

$$\sum_{n\in\mathbb{N}} [kG^n] = \left(\sum_{S\in\hat{G}} \dim(P_S)x_S\right)^n = \frac{1}{1-\sum_{S\in\hat{G}}\dim(P_S)x_S},$$

where P_S is the minimal projective cover of S.

Example 1.2. The trivial $\mathbf{FS}_G$ module has $(k\mathrm{triv}_n)_{n\in\mathbb{N}}$ as its underlying sequence of representations. Its Hilbert series is

$$\sum_{n\in\mathbb{N}} x_{\mathrm{triv}}^n = \frac{1}{1-x_{\mathrm{triv}}},$$

where triv denotes the trivial representation.

To state our main result, we introduce some notation. We fix an embedding from the roots of unity of k to $\mathbf{C}$, and write φ_S for the Brauer character of S and Φ_S for the Brauer character of the minimal projective cover of S. Given a p-regular conjugacy class c of G, let $x_c := \frac{|c|}{|G|}\sum_{S\in\hat{G}} \overline{\Phi_S(c)}x_S$.

Theorem 1.3. *Let M be a subquotient of an $\mathbf{FS}_G$ module generated in degree $\le d$. Then H_M is rational with denominator a product of degree 1 polynomials of the form*

$$1 - \sum_c a_c x_c,$$

where the sum is over p-regular conjugacy classes c of G, and $a_c \in \mathbb{N}, a_c \le |G|d$.

Both Example 1.1 and 1.2 are generated in degree 1. These have denominator $1-|G|x_e$ and $1-\sum_c x_c$ respectively.

Remark 1.4. Sam–Snowden [3] proved that H_M was rational with denominator of the form: $1+\sum_S b_S x_S$ for b_S algebraic integers in the field $\mathbb{Q}(\zeta_N)$ where N is the exponent of G (or more generally the smallest number such that G is N-good). They asked whether it was possible to improve this class of denominators, in particular by replacing $\mathbb{Q}(\zeta_N)$ by the field generated by the Brauer character table of $\hat{G}$. Theorem 1.3 answers this question in the affirmative, and further restricts the form of the denominators. □

1.2. **Proof Strategy.** We prove Theorem 1.3 using the results of our previous paper [4] on $\mathbf{FS}^{\mathrm{op}}$ modules. A finitely generated $(\mathbf{FS}_G)^{\mathrm{op}}$ module restricts to a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module, along the functor $X \mapsto G\times X$. In [4], for every $\mathbf{FS}^{\mathrm{op}}$ module M and $d\in\mathbb{N}$ we constructed a sort of Koszul complex $\mathbf{K}_d(M)$. Further we proved that if M is finitely generated then, many of the iterated Koszul complexes

$$\mathbf{K}_{d_1}\circ\cdots\circ\mathbf{K}_{d_r}(M)$$

have vanishing cohomology. Because the Hilbert series of $\mathbf{K}_d(M)$ is related to the Hilbert series of M in a predictable way, this vanishing implies that the Hilbert series H_M satisfies nontrivial relations. In terms of an associated exponential generating function E_M, these relations take the form of a system of linear differential equations. We diagonalize and solve this system of differential equations. Because E_M lies in the space of solutions, we obtain the form of E_M and consequently H_M.

1.3. **Notation.**

- $[n]$ denotes the set $\{1, 2, \ldots, n\}$.
- k is an algebraically closed field of characteristic $p \geq 0$, $\mathbf{C}$ is the complex numbers
- G is a finite group, we will also write G for the category with one object with automorphism group G
- $\hat{G}$ is the set of isomorphism classes of simple k representations.
- $\mathbf{FS}_G$ denotes the category of finite free G sets and surjections
- $\sqcup_n G^n$ denotes the category with objects natural numbers $n \in \mathbb{N}$, and automorphism group G^n.
- Given a category $\mathcal{C}$, we write $\mathrm{Rep}(\mathcal{C})$ for the category of functors $\mathcal{C} \to \mathrm{Vec}_k$, from $\mathcal{C}$ to the category of k vector spaces.
- φ_S is the Brauer character of a simple module S and Φ_S is the Brauer character of its projective cover.
- $\mathcal{R}(G)$ is the Grothendieck group of G tensored with $\mathbf{C}$, for a G representation $[M]$ denotes its class in $\mathcal{R}(G)$. If S is a simple representation, we write $x_S = [S]$.
- $\mathrm{tr}(g, -) : \mathcal{R}(G) \to \mathbf{C}$ is the Brauer trace, for $g \in G$ a p-regular element.

2. Variants of Hilbert series and differential equations

There are two other generating functions which record the same data as H_M. We define the exponential generating function $E_M \in \mathbf{C}[[x_S \mid S \in \hat{G}]]$ by

$$E_M := \sum_{n \in \mathbb{N}} \frac{\pi([M_n])}{n!},$$

so that E_M is obtained from H_M by applying the transform

$$\prod_S x_S^{i_S} \mapsto \frac{\prod_S x_S^{i_S}}{(\sum_S i_S)!}.$$

Similarly we can define $\widetilde{H}_M$ by applying the transform $\prod_S x_S^{i_S} \mapsto \prod_S \frac{x_S^{i_S}}{i_S!}$.

For these Hilbert series, we will establish the following version of Theorem 1.3. Given a p-regular conjugacy class c of G, let $x_c := \frac{|c|}{|G|} \sum_{S \in \hat{G}} \overline{\Phi_S(c)} x_S$.

Theorem 2.1. *Let M be a subquotient of an $(\mathbf{FS}_G)^{\mathrm{op}}$ module generated in degree $\leq d$. Then*

1. *E_M is a linear combination of products of the form $\prod_c (x_c)^{r_c} \exp(a_c x_c)$, for $r_c, a_c \in \mathbb{N}$ and $a_c \leq d|G|$.*
2. *$\widetilde{H}_M$ is rational, with denominator a product of polynomials of the form*

$$P_{c,a}(t) := \prod_{S \in \hat{G}} (1 - a\ \frac{|c|\ \overline{\Phi_S(c)}}{|G|} x_S)$$

for $a \in \mathbb{N}$, $a \leq d|G|$.

We note that Theorem 1.3 and part (2) of Theorem 2.1 are immediate consequences of part (1), and the fact that the transform $x^n \mapsto n!x^n$ of formal power series acts by

$$x^r \exp(jx) \mapsto \left(\frac{d}{dx}\right)^r \frac{1}{1-jx}$$

which is proportional to $\frac{1}{(1-jx)^r}$.

To establish Theorem 1.3 we will show that E_M satisfies a system of differential equations. Given a p-regular element $g \in G$, we let ∂_g denote the differential operator

$$\partial_g := \sum_{S \in \hat{G}} \varphi_S(g) \partial_S.$$

Note that ∂_g only depends on the conjugacy class of g.

Theorem 2.2. *Let M be an $(\mathbf{FS}_G)^{\mathrm{op}}$ module that is a subquotient of one generated in degree d. Then there exists an $r \in \mathbb{N}$ such that E_M satisfies the differential equation $\left(\frac{\partial_g}{d|G|+1}\right)^r E_M = 0$ for every p-regular $g \in G$.*

Assuming Theorem 2.2 we now prove part (1) of Theorem 1.3. Given a p-regular conjugacy class c, we have

$$\partial_g(x_c) = \frac{|c|}{|G|} \sum_{S \in \hat{G}} \varphi_S(g) \overline{\Phi_S(c)} = \begin{cases} 1 & \text{if } g \in c \\ 0 & \text{otherwise} \end{cases}$$

by the orthogonality relations between simple and projective Brauer characters. In single variable power series, we have that the solution space of $\prod_{a=0}^{d|G|} (\partial_x - a)^r$ is spanned by $x^i \exp(ax)$ for $a \leq d|G|$ and $i < r$, $a, i \in \mathbb{N}$. From the orthogonality relation, we obtain that the solution space of differential equations of Theorem 2.2 is precisely the span of the functions in part (1) of Theorem 1.3.

3. $\mathbf{FS}^{\mathrm{op}}$ modules and Koszul complexes

In this section we recall the Koszul complexes introduced in [4] and use them to establish Theorem 2.2.

Given an $\mathbf{FS}^{\mathrm{op}}$ module, M, and $d \in \mathbb{N}, d \geq 1$ there is an associated Koszul complex $\mathbf{K}_d(M)$ of $\mathbf{FS}^{\mathrm{op}}$ modules [4, §3.2]. We let $\Sigma^k M$ be the $\mathbf{FS}^{\mathrm{op}}$ module defined by $\Sigma^k M(X) := M([k] \sqcup X)$. Then the complex $\mathbf{K}_d(M)$ takes the form

$$\Sigma^d M \leftarrow \Sigma^{d-1} M^{\oplus \binom{d}{2}} \leftarrow \cdots \leftarrow \Sigma^k M^{\oplus s(d,k)} \leftarrow \cdots \leftarrow \Sigma M^{\oplus (d-1)!},$$

where $s(d,k)$ is the unsigned Stirling number of the first kind which is the coefficient of x^k in $\prod_{i=0}^{d-1}(1+ix)$.

We will not fully describe the differentials of $\mathbf{K}_d(M)$ here; for our purposes, it suffices to know the following. Given a surjection $f : [k] \twoheadrightarrow [k-1]$ there is an associated a map of $\mathbf{FS}^{\mathrm{op}}$ modules $f^* : \Sigma^{k-1} M \to \Sigma^k M$ given by

$$(f \sqcup \mathrm{id}_X)^* : M([k-1] \sqcup X) \to M([k] \sqcup X).$$

The differentials of the complex of $\mathbf{K}_d(M)$ are linear combinations of maps of this form.

We may iterate the construction of $\mathbf{K}_d$, by taking the total complex. The following theorem asserts that for a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module, the result of this iteration is often exact. Before stating it, we recall the definition of finite generation for ${\mathbf{FS}_G}^{\mathrm{op}}$ modules.

Definition 3.1. An ${\mathbf{FS}_G}^{\mathrm{op}}$ module M is finitely generated in degree $\leq d$ if one of the following equivalent conditions holds.

(1) There are elements $x_1 \in M([n_1] \times G), \ldots, x_r \in M([n_r] \times G)$ with $n_i \leq d$ such that every ${\mathbf{FS}_G}^{\mathrm{op}}$ submodule containing $x_1, \ldots, x_r$ is equal to M.
(2) There is a surjection of ${\mathbf{FS}_G}^{\mathrm{op}}$ modules $\bigoplus_{i=1}^r \mathbf{P}_{\mathbf{FS}_G, n_i} \to M$ where $n_i \leq d$. and $\mathbf{P}_{\mathbf{FS}_G, n_i}$ is the principal projective ${\mathbf{FS}_G}^{\mathrm{op}}$ module defined by $\mathbf{P}_{\mathbf{FS}_G, n_i}(X) := k\mathbf{FS}_G(X, [n] \times G)$

Theorem 3.2. *[4, Theorem 1.2] Let M be an $\mathbf{FS}^{\mathrm{op}}$ module which is a subquotient of one that is finitely generated in degree $\leq d$. Then there exists $r \in \mathbb{N}$ such that $\mathbf{K}_{d+1}^{\circ r}(M)$ is exact.*

3.1. **Restricting $(\mathbf{FS}_G)^{\mathrm{op}}$ modules.** We write i for the embedding of $\mathbf{FS}$ into $\mathbf{FS}_G$ given by $i([n]) = [n] \times G$. There is a restriction functor $i^* : \mathrm{Rep}({\mathbf{FS}_G}^{\mathrm{op}}) \to \mathrm{Rep}(\mathbf{FS}^{\mathrm{op}})$.

There is a functor $G^- : \mathbf{FS}^{\mathrm{op}} \to \mathrm{Set}$ given by $[n] \mapsto G^n$. If M is an ${\mathbf{FS}_G}^{\mathrm{op}}$ module, there is a natural action $G^- \times i^*M \to i^*M$ given by the action of multiplication

$$G^n \times M([n] \times G) \to M([n] \times G).$$

From this construction, we obtain the following.

Proposition 3.3. *The category of ${\mathbf{FS}_G}^{\mathrm{op}}$ modules is equivalent to the category of $\mathbf{FS}^{\mathrm{op}}$ modules equipped with an action the $\mathbf{FS}^{\mathrm{op}}$ group G^- by natural transformations.*

Proof. This is a straightforward consequence of the fact that every map of G-sets $[n] \times G \to [m] \times G$ factors uniquely as multiplication by G^n followed by a map of the form $f \times \mathrm{id}$ where $f : [m] \to [n]$. $\square$

Furthermore, we have that finitely generated ${\mathbf{FS}_G}^{\mathrm{op}}$ modules restrict to finitely generated $\mathbf{FS}^{\mathrm{op}}$ modules.

Proposition 3.4. *Let M be an ${\mathbf{FS}_G}^{\mathrm{op}}$ module. If M is a subquotient of a module generated in degree $\leq d$, then i^*M is of a subquotient of one generated in degree $\leq |G|d$.*

Proof. We write $\mathbf{P}_{{\mathbf{FS}_G}^{\mathrm{op}}}(n)$ for the principal ${\mathbf{FS}_G}^{\mathrm{op}}$ module generated in degree n. An ${\mathbf{FS}_G}^{\mathrm{op}}$ module is The theorem follows from the identity $i^*\mathbf{P}_{{\mathbf{FS}_G}^{\mathrm{op}}, n} = \mathbf{P}_{\mathbf{FS}^{\mathrm{op}}, |G|n}$ and passing to quotients. $\square$

Given an ${\mathbf{FS}_G}^{\mathrm{op}}$ module, we have that $\mathbf{K}_d(i^*M)$ carries the structure of a $G \times {\mathbf{FS}_G}^{\mathrm{op}}$ module, where G acts on

$$\Sigma^k i^*M(X) = M([k] \times G \sqcup X \times G)$$

by the constant action on $[k] \times G$ and G^X acts on $\Sigma^k i^*M(X)$ by the action of G^X on $X \times G$. (The differentials of $\mathbf{K}_d$ preserve this structure because the maps $f^* : \Sigma^{k-1} i^*M \to \Sigma^k i^*M$ associated above to a surjection $f : [k] \twoheadrightarrow [k-1]$ do).

3.2. **Proof of Theorem 2.2.** Forgetting the differentials, the restriction of $\mathbf{K}_d(M)$ to $G \times \sqcup_n G^n$ only depends on the restriction of M to $\sqcup_n G^n$. Since functor $M \mapsto \mathbf{K}_d(M)$ is exact we obtain an associated operator

$$K_d : \prod_{n \in \mathbb{N}} \mathcal{R}(G^n)^{S_n} \to \prod_{n \in \mathbb{N}} \mathcal{R}(G) \otimes \mathcal{R}(G^n)^{S_n},$$

which takes $[M]$ to the class of $[\mathbf{K}_d(M)]$. (The class of a finite complex $C_\bullet$ is defined to be $\sum_i (-1)^i [C_i] = \sum_i (-1)^i H_i(C_\bullet)$). Further, pairing with the trace operator $\mathrm{tr}(g, -) : \mathcal{R}(G) \to \mathbf{C}$ and conjugating by the isomorphism

$$\pi/n! : \prod_{n \in \mathbb{N}} \mathcal{R}(G^n)^{S_n} \cong \mathbf{C}[[x_S | S \in \hat{G}]]$$

given by projecting and dividing by $n!$ we obtain an operator

$$\mathrm{tr}(g, K_d) : \mathbf{C}[[x_S | S \in \hat{G}]] \to \mathbf{C}[[x_S | S \in \hat{G}]].$$

Similiarly there is an operator $\mathrm{tr}(g, \Sigma^k) : \mathbf{C}[[x_S \mid S \in \hat{G}] \to \mathbf{C}[[x_S \mid S \in \hat{G}]$ for $k \in \mathbb{N}$ associated to the functor $\Sigma^k : \mathrm{Rep}(\mathbf{FS}_G{}^{\mathrm{op}}) \to \mathrm{Rep}(\mathbf{FS}_G{}^{\mathrm{op}})$ and we have that $\mathrm{tr}(g, \Sigma^k) = \mathrm{tr}(g, \Sigma)^{\circ k}$

Proposition 3.5. *There is an identity of operators* $\mathrm{tr}(g, K_d) = \prod_{j=0}^{d-1} (\partial_g - j)$.

Proof. In homological degree i, $\mathbf{K}_d(M) = \Sigma^{s(d,i)} M$, where $s(d, i)$ is Stirling number defined to be the coefficient of x^d in $\prod_{j=0}^{d-1}(x + j)$. We have that $\mathrm{tr}(g, \Sigma^k) = \mathrm{tr}(g, \Sigma)^{\circ k}$, so it suffices to prove that $\mathrm{tr}(g, \Sigma) = \partial_g$. We have that

$$\mathrm{tr}(g, \Sigma) x_{S_1} \dots x_{S_m} = \frac{1}{(m-1)!} \pi(\mathrm{tr}(g, \Sigma) \sum_{\sigma \in \mathbf{S}_m} x_{S_{\sigma(1)}} \otimes \cdots \otimes x_{S_{\sigma(m)}})$$

$$= \frac{1}{(m-1)!} \sum_{\sigma \in \mathbf{S}_m} \varphi_{S_{\sigma(1)}}(g) x_{\sigma(2)} \dots x_{\sigma(m)}.$$

Here π denotes the projection $\mathcal{R}(G)^{\otimes n} \to \mathrm{Sym}^n(\mathcal{R}(G))$. This agrees with the action of $\sum_{S \in \hat{G}} \varphi_S(g) \partial_S$. ∎

Given the calculation of Proposition 3.5 , Theorem 2.2 is an immediate consequence of the following.

Proposition 3.6. *If M is an $\mathbf{FS}_G$ module which is a subquotient of one generated in degree d, exists an r such that* $\mathrm{K}^{\circ r}_{d|G|+1}(E_M) = 0$.

Proof. Apply Proposition 3.4, and Theorem 3.2. Because the homology of $\mathbf{K}^{\circ r}_{d|G|+1}(i^* M)$ vanishes, it follows that the class $\mathrm{K}^{\circ r}_{d|G|+1}(E_M) = 0$. ∎

4. Further examples and computations

In this section, we include some additional examples of Hilbert series of $\mathbf{FS}_G{}^{\mathrm{op}}$ modules.

Example 4.1. Let W be a projective representation of G. Then there is a $\mathbf{FS}_G{}^{\mathrm{op}}$ representation $\mathbf{Q}_W$ with $\mathbf{Q}_W(X) = \mathrm{Hom}_G(W, k\mathbf{Fin}_G(X, G))$. Then

$$E_{\mathrm{P}_W} = \sum_c \frac{|c|\overline{\chi_W(c)}}{|G|} \exp(|G|x_c)$$

and

$$H_{P_W} = \sum_c \frac{|c|\overline{\chi_W(c)}}{|G|} \frac{1}{1 - |G|x_c},$$

where χ_W denotes the Brauer character of W and the sum is over p-regular conjugacy classes of G.

There is a Day convolution tensor product $\circledast$ on the category of $\mathbf{FS}_G{}^{\mathrm{op}}$ modules defined by

$$(M \circledast N)(X) = \bigoplus_{X = A \sqcup B} M(A) \otimes N(B).$$

Then $E_{M \circledast N} = E_M E_N$. Combined with the previous example, for any p-regular conjugacy class c and any $d \in \mathbb{N}$ we may construct an $\mathbf{FS}_G{}^{\mathrm{op}}$ modules L generated in degree d such that $\exp(d|G|x_c)$ appears in the expansion of E_L with nonzero coefficient.

Example 4.2. Let M be a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module. Then M pulls back to a finitely generated $(\mathbf{FS}_G)^{\mathrm{op}}$ module along the map $p : \mathbf{FS}_G \to \mathbf{FS}$ defined by $p(X) = X/G$. The enhanced Hilbert series of p^*M is given by substituting x_{triv} into the ordinary Hilbert series of M, hence H_{p^*M} has denominator of the form $\prod_{a=1}^r (1 - a x_{\mathrm{triv}})^{j_a}$.

Finally, we record Proposition 4.3 we have that $x_S = \sum_c \varphi_S(c) x_c$ or more generally that $[M] = \sum_c \mathrm{tr}(c, M) x_c$.

Proposition 4.3. *Let g be p-regular element of G. Then*

$$\mathrm{tr}(g, x_c) = \begin{cases} 1 & \text{if } g \in c \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from the orthogonality relations between the Brauer characters of simple and projective modules

$$\sum_{S \in \hat{G}} \overline{\Phi_S(g)} \varphi_S(h) = \frac{|G|}{|C(g)|} \delta_{g,h}$$

□

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