Fibonacci and Lucas Sequences in Aperiodic Monotile Supertiles

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Abstract

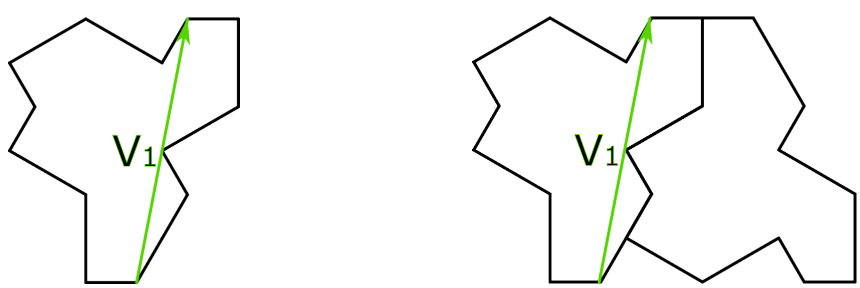
This paper first discusses the size and orientation of hat supertiles. Fibonacci and Lucas sequences, as well as a third integer sequence linearly related to the Lucas sequence are involved. The result is then generalized to any aperiodic tile in the hat family.

Introduction

2023 saw one of the most exciting discoveries in mathematics: the aperiodic monotiles [1][2], commonly known as *hat* and *spectre* tiles. One of their fascinating aspects is the tiling process, which can be generated using recursively defined supertiles. Although complex, each supertile carries a vector indicating its size and natural direction. In this paper, we calculate these vectors for the supertiles in the hat family, yielding a simple expression involving Fibonacci and Lucas sequences. Moreover, we discover an integer sequence linearly related to the Lucas sequence in the rotation of the supertiles.

Supertiles

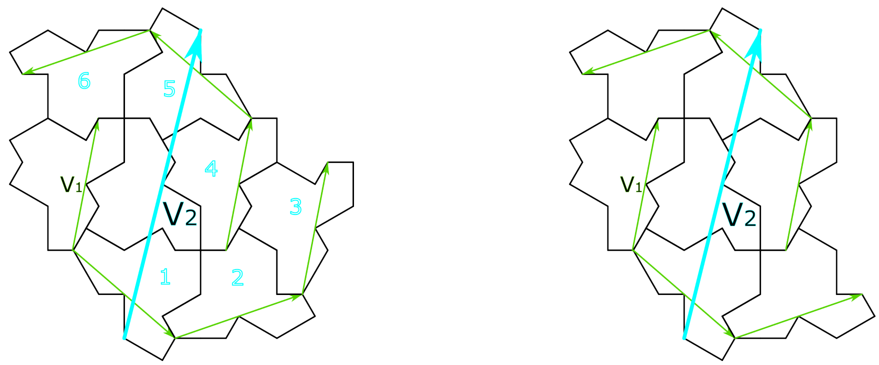
In this section, we will review the supertile forming rules. We start from the single hat and the two-hat compound (Figure 1), with a vector called V_1 marked in each piece. The two pieces are arranged in Figure 1 so that their V_1's are parallel. We define the head and tail of the hat piece as the head and tail of its V_1—the same for the two-hat compound. Let's call these tiles *hat-1* and *thc-1*, the *first generation of supertiles*.



(a) *Hat* (b) *Two-hat compound*

Figure 1: *Marked hat tiles.*

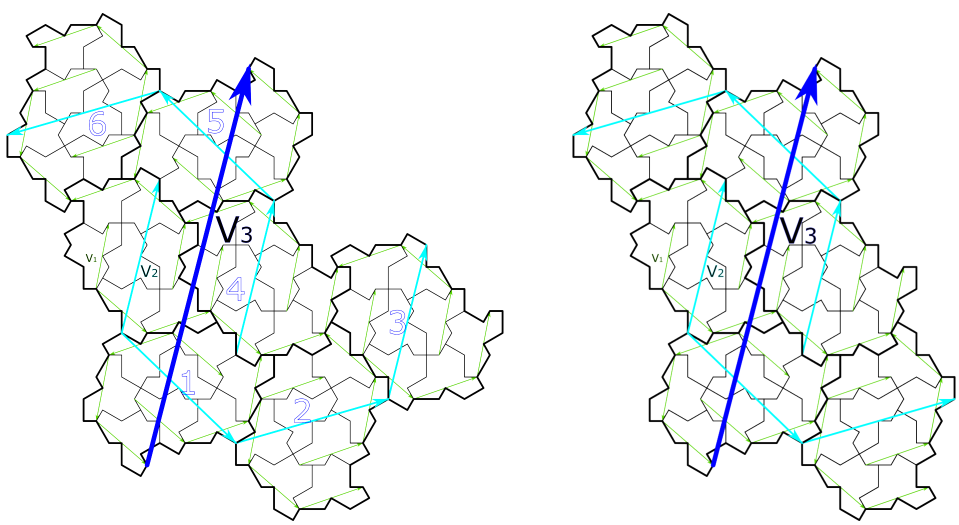
The next generation of supertiles is formed by the following pattern. There is a two-hat compound piece and a first hat piece with its V_1 120-degree clockwise from that of the two-hat compound, and the two pieces share the same tail. The next two hat pieces have their tails at the previous hat pieces' heads and rotated 60 degrees counterclockwise from the previous piece. The fourth is parallel to the third and is snapped in between the two-hat compound and the third hat piece. The next two hat pieces have their tails at the previous hat pieces' heads and rotated 60 degrees counterclockwise from the previous piece. This supertile has its *supervector* called V_2, shown in Figure 2. This forms the second generation supertile *hat-2* of the hat tile. The supertile of the two-hat compound *thc-2* is the same except for the missing third hat piece.



(a) *Hat supertile* hat-2 (b) *Two-hat compound supertile* thc-2

Figure 2: *The second generation supertiles.*

The previous paragraph also describes how the nth ($n \geq 3$) generation supertiles are formed using the prior supertiles. To see how the supervectors V_{n+1} are formed, we need to look at the case of $n = 3$. From Figure 3, we can see that thc-2 and the fourth hat-2 meet in the way that the thc-1 of the latter has its supervector exactly where that of the missing third hat-1 of thc-2 would be. The second observation is that V_3 starts from the head of the third hat-1 of the first hat-2 and ends at the head of the third hat-1 of the fifth hat-2. In general, these two facts work for all generations above this one.



(a) *Hat supertile* hat-3 (b) *Two-hat compound supertile* thc-3

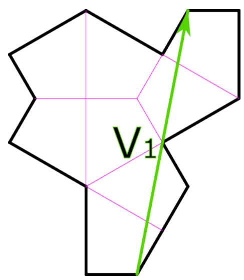
Figure 3: *The third generation supertiles.*

- In the nth supertile, thc-$(n-1)$ and the fourth hat-$(n-1)$ meet in the way that the thc-$(n-2)$ of the latter has its supervector where that of the missing third hat-$(n-2)$ of thc-$(n-1)$ would be.
- V_n starts from the head of the third hat-$(n-2)$ of the first hat-$(n-1)$ and ends at the head of the third hat-$(n-2)$ of the fifth hat-$(n-1)$.

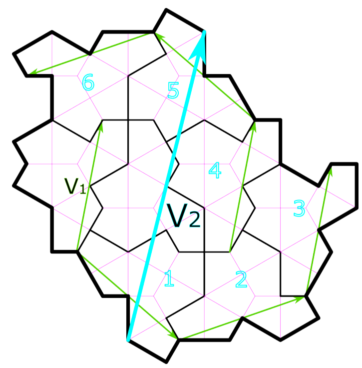
Supertile Rotaions

At first glance, V_1, V_2 and V_3 look suspiciously parallel. A careful calculation shows they are not. In Figure 4, the hexagonal grid is drawn for hat-1 and hat-2. Let the short edge of the hat tile be 1, then it's not hard to see that

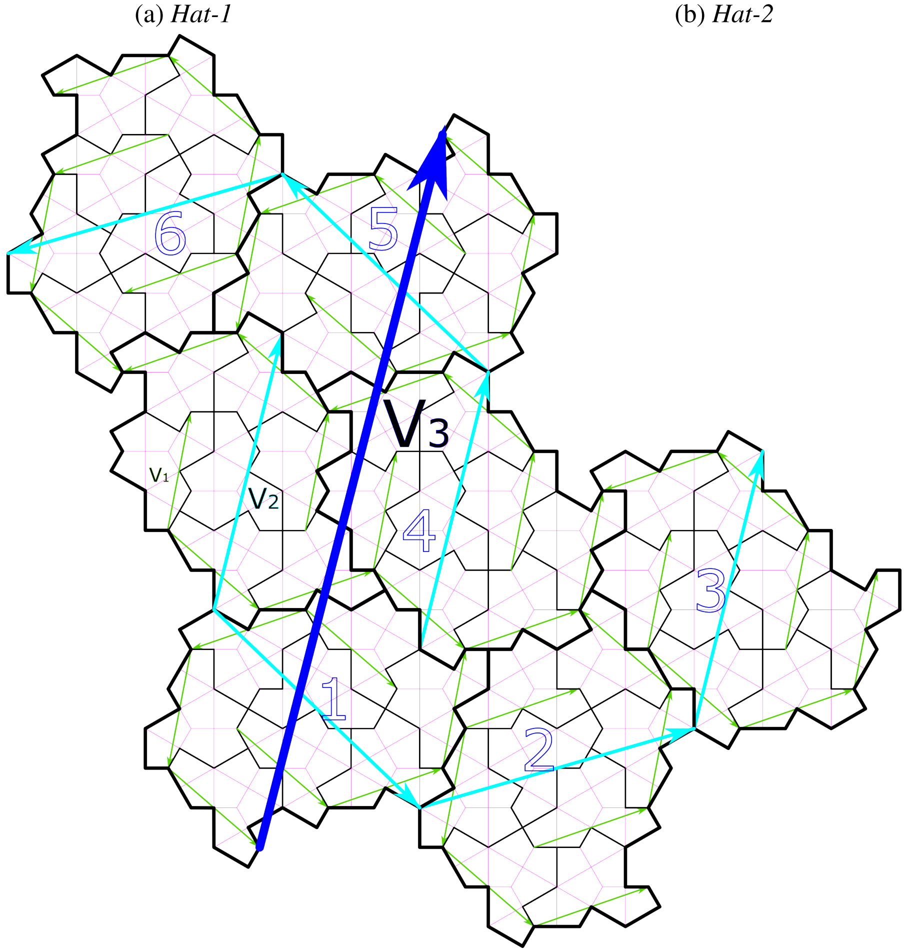
$$V_1 = (1, 3\sqrt{3}), \quad V_2 = (3, 7\sqrt{3}), \quad V_3 = (8, 18\sqrt{3}). \tag{1}$$



(a) *Hat-1*



(b) *Hat-2*



(c) *Hat-3*

Figure 4: *Hat-1, hat-2, and hat-3 with hexagonal grid.*

Since the rule of tiling is recursive, involving *two* previous generations, we expect there is a recurrence relation about V_n that involves two previous terms. Indeed, we have a very simple relation

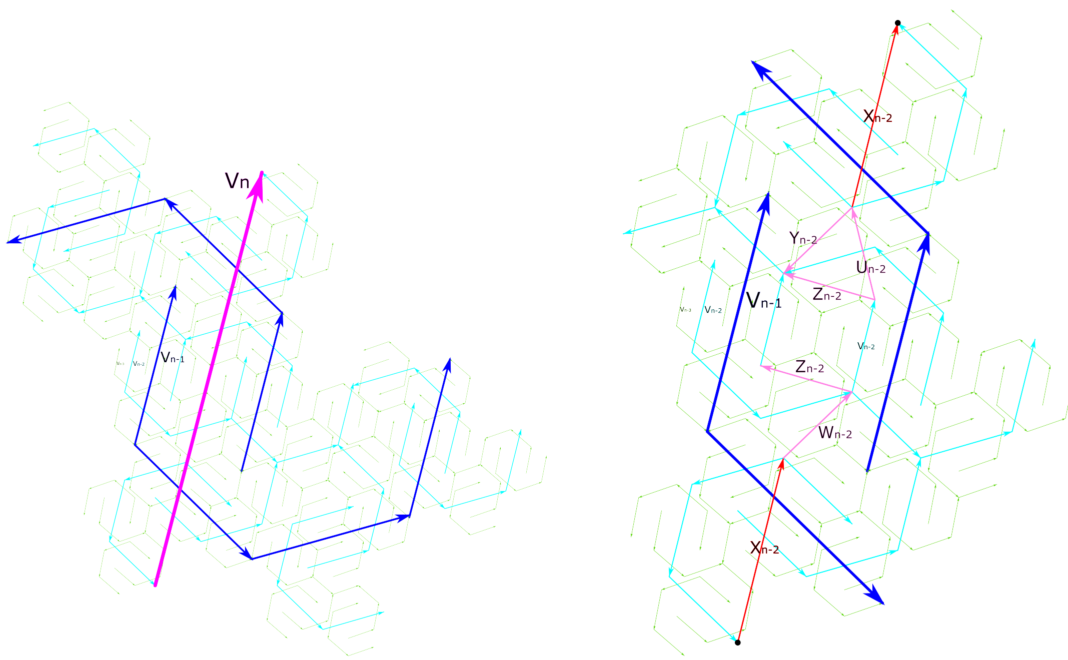
Lemma 1.

$$V_n = 3V_{n-1} - V_{n-2}, \quad \forall n \geq 3.$$

Proof. We will prove by induction. For $n = 3$, it is true. Assume it is so for some $n - 1 \geq 3$. In the layout for hat-n, if we only focus on the supervectors of generations $n - 3$ to n (Figure 5), then we can see

$$V_n = 2X_{n-2} + W_{n-2} + U_{n-2} + V_{n-2} = 5V_{n-2} + W_{n-2} + U_{n-2},$$

where we've used the fact that $X_{n-2} = 2V_{n-2}$ from the half-hexagon shape. And because the fourth hat-$(n-1)$ rotates 120 degrees counterclockwise with respect to the first hat-$(n-1)$, Z_{n-2} is W_{n-2} rotating 120 degrees counterclockwise. Similarly, because the fifth hat-$(n-1)$ rotates 120 degrees counterclockwise with respect to the fourth hat-$(n-1)$, Y_{n-2} is Z_{n-2} rotating 120 degrees counterclockwise. Therefore, U_{n-2} is W_{n-2} rotating 60 degrees counterclockwise.



(a) *Supervectors in the hat-n layout* (b) *Decomposition of V_n*

Figure 5: *Supervectors in the hat-n layout.*

For simplicity, we will use complex numbers for vector manipulation. Each V_n has its corresponding complex number v_n, and likewise for the other vectors. Rotating $m \times 60$ degrees counterclockwise is simply multiplication by $e^{im\pi/3}$. We then have

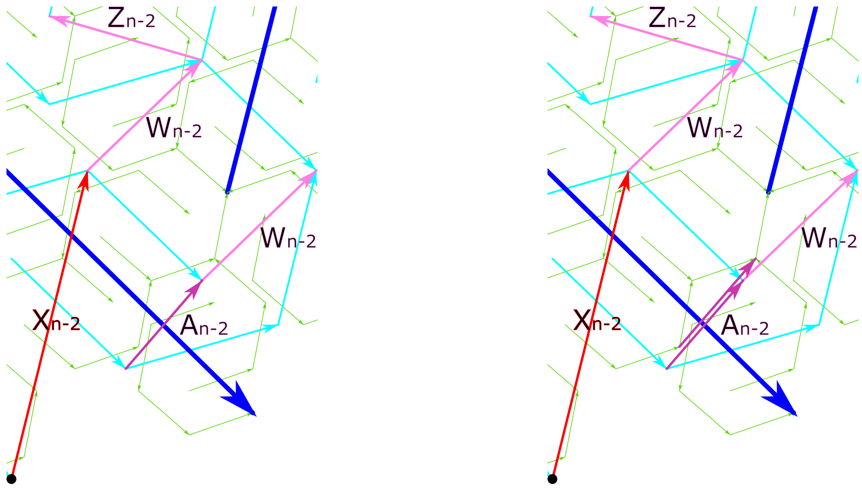
$$v_n = 5v_{n-2} + (1 + e^{i\pi/3})w_{n-2}.$$

To calculate w_{n-2}, let's move W_{n-2} to a new location (Figure 6(a)). From this new location, we get

$$w_{n-2} + a_{n-2} = (1 + e^{-i\pi/3})v_{n-2}.$$

A_{n-2} can be moved in parallel to a new location (Figure 6(b)), from which we get

$$a_{n-2} = (1 + e^{-i\pi/3})v_{n-3}.$$



(a) W_{n-2} (b) A_{n-2}

Figure 6: *Parallel movement of vectors.*

Combining all facts, we get

$$v_n = 5v_{n-2} + (1 + e^{i\pi/3})(1 + e^{-i\pi/3})(v_{n-2} - v_{n-3}) = 8v_{n-2} - 3v_{n-3}.$$

By induction,

$$v_{n-1} = 3v_{n-2} - v_{n-3}.$$

Therefore,

$$v_n = 3v_{n-1} - v_{n-2}. \tag{2}$$

□

We can start the sequence $\{V_n\}$ from $V_0 = 3V_1 - V_2 = (0, 2\sqrt{3})$. Figures in this paper are oriented so that V_0 points up.

A Hidden Sequence

From Lemma 1 and the initial values V_0 and V_1, one can get all V_n. Once again we will work with complex numbers. The characteristic polynomial of Equation (2) has solutions

$$\lambda_{1,2} = \frac{1}{2}\left(3 \pm \sqrt{5}\right) = \varphi^{\pm 2},$$

where φ is the golden ratio. Hence

$$v_n = a\varphi^{2n} + b\varphi^{-2n}.$$

Since $v_0 = 2\sqrt{3}i$ and $v_1 = 1 + 3\sqrt{3}i$, we have

$$a = \frac{1}{\sqrt{5}} + \sqrt{3}i, \quad b = -\frac{1}{\sqrt{5}} + \sqrt{3}i,$$

and

$$v_n = \left(\frac{1}{\sqrt{5}} + \sqrt{3}i\right)\varphi^{2n} + \left(-\frac{1}{\sqrt{5}} + \sqrt{3}i\right)\varphi^{-2n} = F_{2n} + \sqrt{3}L_{2n}i, \tag{3}$$

where $\{F_n\}$ is the Fibonacci sequence and $\{L_n\}$ is the Lucas sequence. The angle V_n rotates from V_0 clockwise is

$$\theta_n = \arctan\frac{\varphi^{2n} - \varphi^{-2n}}{\sqrt{15}(\varphi^{2n} + \varphi^{-2n})} = \arctan\left(\frac{F_{2n}}{\sqrt{3}L_{2n}}\right), \quad \forall n \geq 0. \tag{4}$$

We have the total rotation of all generations

$$\lim_{n\to\infty}\theta_n = \operatorname{arccot}(\sqrt{15}) = \arcsin\frac{1}{4}. \tag{5}$$

The angle each V_n rotates from the previous V_{n-1} can be calculated as

$$\alpha_n = \theta_n - \theta_{n-1} = \operatorname{arccot}\left(\sqrt{3}\frac{8L_{4n-2} + 21}{15}\right), \quad \forall n \geq 1.$$

Define

$$G_n = \frac{8L_{4n-2} + 21}{15}, \quad \forall n \geq 1. \tag{6}$$

From the recurrence relation of $\{L_n\}$, it can be calculated that

$$G_n = 7G_{n-1} - G_{n-2} - 7, \quad \forall n \geq 3,$$

with $G_1 = 3$ and $G_2 = 11$. $\{G_n\}$ is an integer sequence.

Propsition 2. *Let the short edge of the original hat tile be 1, the supervector*

$$V_n = (F_{2n}, \sqrt{3}L_{2n}), \quad \forall n \geq 0.$$

The n-th generation of supertiles rotates from the $(n-1)$*-th by an angle of*

$$\operatorname{arccot}(G_n\sqrt{3}),$$

with G_n *defined in Equation (6).*

The first few terms of G_n are

3, 11, 67, 451, 3083, 21123, 144771, 992267, 6801091, 46615363, 319506443, 2189929731, 15010001667.

General Aperiodic Monotiles

For a general aperiodic Tile(a, b), given the same alignment as the hat tile, following the red lines in Figure 7, we can calculate

$$V_1 = (s, 3t), \quad V_2 = V_1 + (2s, 4t) = (3s, 7t),$$

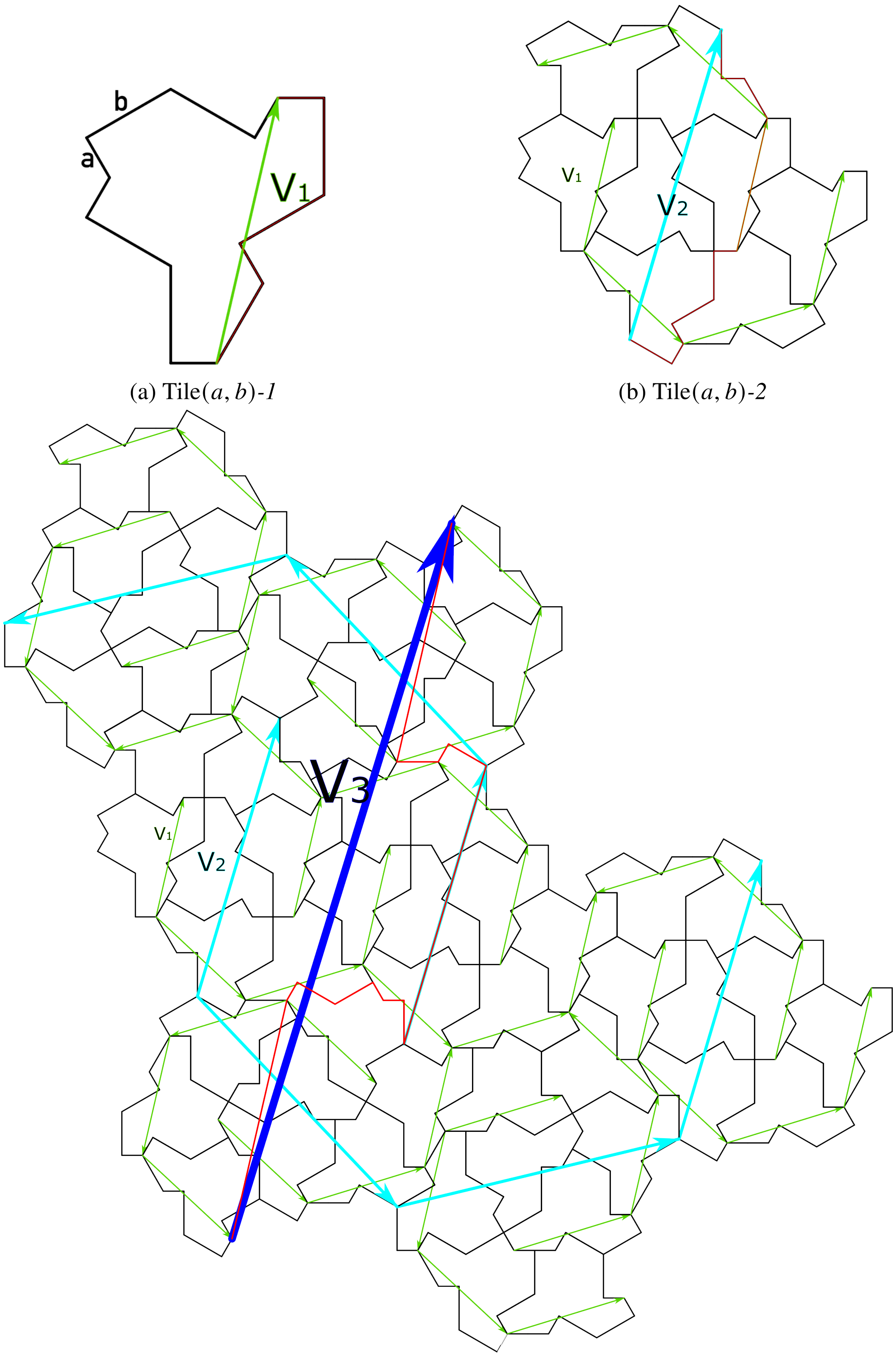
and

$$V_3 = V_2 + 4V_1 + (s, -t) = (8s, 18t),$$

where $s = (\sqrt{3}b - a)/2$ and $t = (\sqrt{3}a + b)/2$. It's straightforward to check

$$V_3 = 3V_2 - V_1.$$

Since the proof of Lemma 1 only depends on this initial condition and the hexagonal structure of the supervectors, it is valid for Tile(a, b) as well.

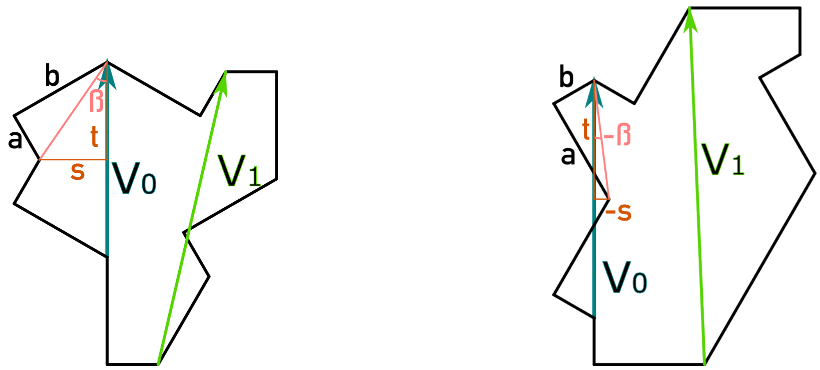


(a) Tile(a, b)-*1*

(b) Tile(a, b)-*2*

(c) Tile(a, b)-*3*

Figure 7: *The first three generations of supertiles for* Tile(a, b).



(a) $s, \beta > 0$ (b) $s, \beta < 0$

Figure 8: *Marked* Tile(a, b) *tiles.*

Lemma 3. *For any aperiodic* Tile(a, b)*, with the supertiles and supervectors defined the same way as for the hat tile, we have*

$$V_n = 3V_{n-1} - V_{n-2}, \quad \forall n \geq 3.$$

Let $V_0 = 3V_1 - V_2 = (0, 2t)$. Figure 8 shows drawings of two generic cases where $s > 0$ and $s < 0$, respectively. Let $\beta = \arctan(s/t)$. β is positive (negative resp.) when V_1 tilts to the right (left resp.) of V_0. We are considering the cases where $a > 0, b > 0$ and $a \neq b$. This implies $\beta \in (-\pi/6, \pi/12) \cup (\pi/12, \pi/3)$. When $\beta = 0$, V_1 align with V_0, hence all $\{V_n\}$ are parallel.

The calculation of $\{V_n\}$ follows the same line as in the hat tile, and the result is

Propsition 4. *For any aperiodic* Tile(a, b)*, with the supertiles and supervectors defined the same way as for the hat tile, we have*

$$V_n = (F_{2n}\, s, L_{2n}\, t), \quad \forall n \geq 0.$$

The n-th generation of supertiles rotates clockwise from the $(n-1)$*-th by an angle of*

$$\arctan\left(s/(G_n\, t)\right) = \arctan\left(\tan\beta / G_n\right).$$

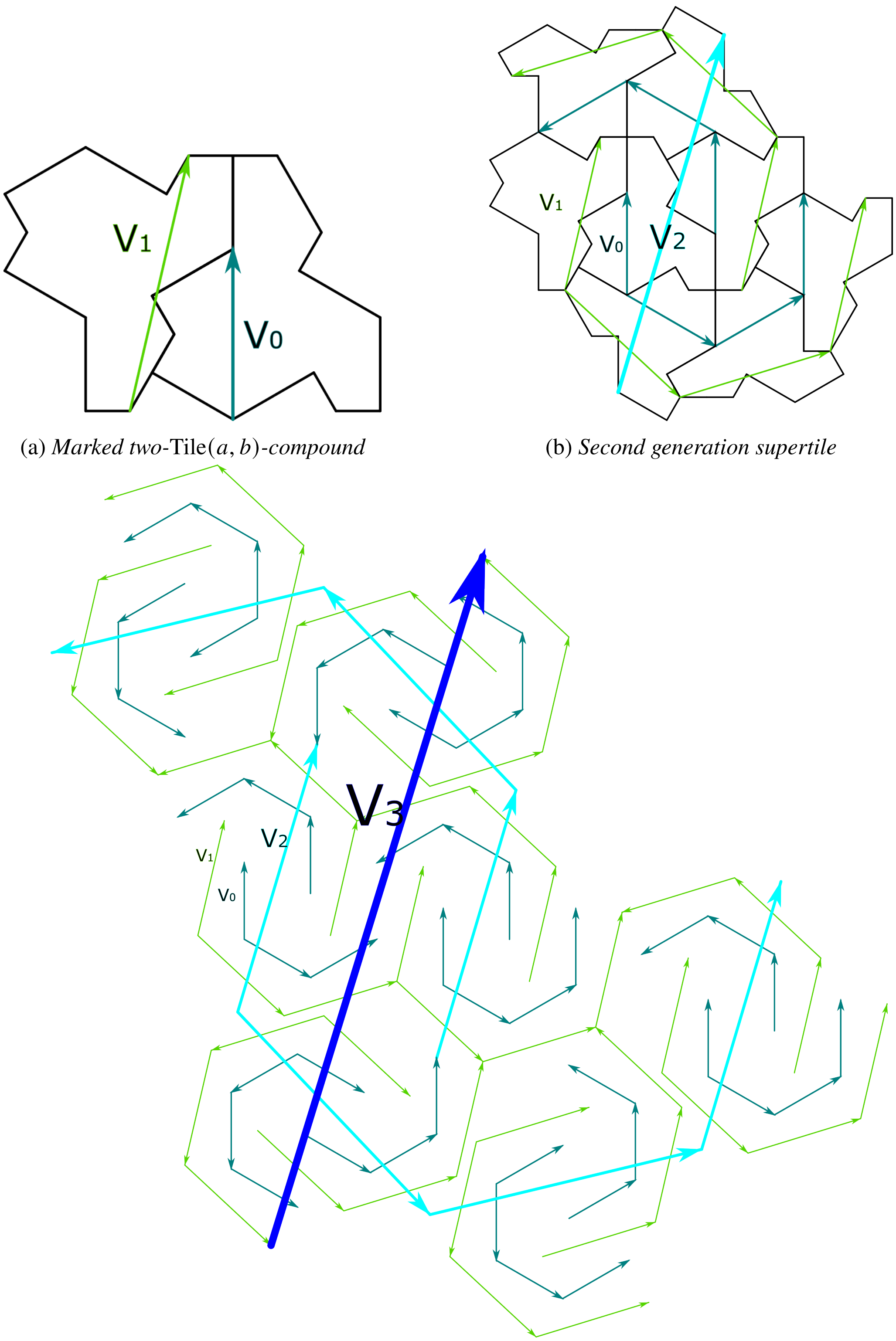
The total rotation of all generations is

$$\arctan\left(s/(\sqrt{5}\, t)\right) = \arctan\left(\tan\beta/\sqrt{5}\right).$$

If we add V_0 to the two-tile-compound as in Figure 9(a), then Figure 9(b) shows that V_0 form the same hexagonal structure as V_1 in the second generation supertile. However, the structure of V_0 is not exactly the same as all other V_n's, as Figure 9(c) shows.

Conclusions

The supervectors of the supertiles take an elegant form involving Fibonacci and Lucas sequences, indicating yet another deep combinatorial meaning of aperiodic monotiles in the hat family. A new integer sequence linearly related to the Lucas sequence is in the rotational angles of the supertiles. In the fractal limit, the supertiles scale by a factor of φ^2 after each iteration, and the tangent of the incremental rotational angles scales by a factor of φ^{-4}. The meaning of the tangent of the such angles need to be explored.



(a) *Marked two-*Tile(a, b)*-compound*

(b) *Second generation supertile*

(c) *Supervectors in the third generation supertile*

Figure 9: *Hexagonal structure for* V_0.

Acknowledgments

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References

[1] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. "An aperiodic monotile." 2023. https://arxiv.org/abs/2303.10798

[2] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. "A chiral aperiodic monotile." 2023. https://arxiv.org/abs/2305.17743