

STRUCTURE OF A FOURTH-ORDER DISPERSIVE FLOW EQUATION THROUGH THE GENERALIZED HASIMOTO TRANSFORMATION

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ABSTRACT. This paper focuses on a one-dimensional fourth-order nonlinear dispersive partial differential equation for curve flows on a Kähler manifold. The equation arises as a fourth-order extension of the one-dimensional Schrödinger flow equation, with physical and geometrical backgrounds. First, this paper presents a framework that can transform the equation into a system of fourth-order nonlinear dispersive partial differential-integral equations for complex-valued functions. This is achieved by developing the so-called generalized Hasimoto transformation, which enables us to handle general higher-dimensional compact Kähler manifolds. Second, this paper demonstrates the computations to obtain the explicit expression of the derived system for three examples of the compact Kähler manifolds, dealing with the complex Grassmannian as an example in detail. In particular, the result of the computations when the manifold is a Riemann surface or a complex Grassmannian verifies that the expression of the system derived by our framework actually unifies the ones derived previously. Additionally, the computation when the compact Kähler manifold has a constant holomorphic sectional curvature, the setting of which has not been investigated, is also demonstrated.

1. INTRODUCTION

1.1. Setting of the problem and previous related results. Let N be a Kähler manifold of complex dimension $n \in \mathbb{N}$ with complex structure J and Kähler metric h , and let ∇ and R denote the Levi-Civita connection associated to h and the Riemann curvature tensor respectively.¹ This paper investigates a one-dimensional fourth-order nonlinear dispersive partial differential equation (PDE) for curve flows on N , which is formulated by

$$u_t = a J_u \nabla_x^3 u_x + \lambda J_u \nabla_x u_x + b R(\nabla_x u_x, u_x) J_u u_x + c R(J_u u_x, u_x) \nabla_x u_x \quad (1.1)$$

for a smooth map $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow N$. Here, $0 < T \leq \infty$, and $a \neq 0$, b, c, λ are real constants, $u_t = du\left(\frac{\partial}{\partial t}\right)$, $u_x = du\left(\frac{\partial}{\partial x}\right)$, ∇_t and ∇_x denote the covariant derivatives along u with respect to t and x respectively, and J_u denotes the complex structure at $u = u(t, x) \in N$. Geometrically, (1.1) describes the relationship among elements of $\Gamma(u^{-1}TN)$, where $\Gamma(u^{-1}TN)$ denotes the set of smooth sections of $u^{-1}TN$.

The equation (1.1) for curve flows on the canonical two-sphere \mathbb{S}^2 with additional assumptions $\lambda = 1$ and $c = 3(a - b)/2$ is the typical example with physical backgrounds, which is known to coincide with the following three-component system of PDEs

$$u_t = u \wedge \left\{ a u_{xxxx} + u_{xx} + (5a - b)(u_{xx}, u_x)u_x + \frac{5a - b}{2}(u_x, u_x)u_{xx} \right\} \quad (1.2)$$

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¹Throughout this paper, the definition of R is adopted to be $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$ for any vector fields X, Y, Z on N where $[X, Y] := XY - YX$.

for $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. Here, \wedge and (\cdot, \cdot) denote the exterior and the inner product in \mathbb{R}^3 respectively, $J_u = u \wedge$ on $T_u \mathbb{S}^2$, and h corresponds to the metric induced from (\cdot, \cdot) . The system (1.2) was proposed by Lakshmanan et al.([24, 33]), modeling the continuum limit of a one-dimensional isotropic Heisenberg ferromagnetic spin chain systems with nearest neighbor bilinear and bi-quadratic exchange interaction. It was proved in [24, 33] that (1.2) can be reduced (equivalently if $\gamma_2 = -\frac{5}{2}\gamma_1$) to the following fourth-order nonlinear Schrödinger equation

$$\begin{aligned} & \sqrt{-1}q_t + \gamma_1 q_{xxxx} + (q_{xx} + 2|q|^2 q) - 4\delta_1 |q|^2 q_{xx} - 4\delta_2 q^2 \bar{q}_{xx} \\ & - 4\delta_3 q |q_x|^2 - 4\delta_4 q_x^2 \bar{q} - 24\delta_5 |q|^4 q = 0 \end{aligned} \quad (1.3)$$

for $q = q(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathbb{C}$, where $\delta_1 = 3\gamma_1 + 2\gamma_2$, $\delta_2 = 2\gamma_1 + \gamma_2$, $\delta_3 = 9\gamma_1 + 4\gamma_2$, $\delta_4 = \frac{7}{2}\gamma_1 + 2\gamma_2$, $\delta_5 = \gamma_1 + \frac{1}{2}\gamma_2$, and $(\gamma_1, \gamma_2) = (a, -(5a-b)/2)$. It was also revealed in [24, 33] that (1.3) is completely integrable in the sense of Painlevé singularity structure analysis if and only if $\gamma_2 = -\frac{5}{2}\gamma_1$. Additionally, (1.3) with $\gamma_2 = -\frac{5}{2}\gamma_1$ arises in other contexts as well, such as the dynamics of a one-dimensional anisotropic Heisenberg ferromagnetic spin chain with octupole-dipole interaction in the continuum limit([8]) and the molecular excitations along the hydrogen bonding spine in an alpha helical protein with higher-order molecular interactions under specific parameter choices([9]).

The equation (1.1) for curve flows on \mathbb{S}^2 with $\lambda = 1$ and $c = 3(a-b)/2$ can be also derived from the so-called Fukumoto-Moffatt model for a vortex filament ([15, 16]). The equation proposed in [15, 16] describes the motion of an arc-length-parameterized space curve in \mathbb{R}^3 , denoted by $\vec{X} = \vec{X}(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathbb{R}^3$ here, which models the three-dimensional dynamics of a vortex filament in an incompressible viscous fluid, considering the deformation effect of the vortex core due to the self-induced strain. If \vec{X} satisfies the vortex filament equation, then the velocity vector $u := \vec{X}_x$ takes values in \mathbb{S}^2 and satisfies (1.2). Furthermore, it is proved by [15, 16] that the equation for \vec{X} is transformed to (1.3) via the so-called Hasimoto transformation([18]).

In the context of geometric dispersive PDEs, (1.1) is regarded as a fourth-order extension of the so-called one-dimensional Schrödinger flow equation

$$u_t = J_u \nabla_x u_x. \quad (1.4)$$

In fact, (1.1) can be regarded as the so-called generalized bi-Schrödinger flow equation if (N, J, h) is a locally Hermitian symmetric space and $c = 3(a-b)/2$: The generalized bi-Schrödinger flow equation was originally introduced by Ding and Wang in [11] as a PDE for time-dependent maps $u = u(t, x) : (-T, T) \times M \rightarrow N$ where M is a Riemannian manifold and N is a Kähler or para-Kähler manifold. When $M = \mathbb{R}$ with the Euclidean metric and (N, J, h) is a Kähler manifold, it is defined as the following Hamiltonian gradient flow equation

$$u_t = J_u \nabla E_{\alpha, \beta, \gamma}(u), \quad (1.5)$$

where $\beta \neq 0$ and α, β are real constants and $\nabla E_{\alpha, \beta, \gamma}(u)$ denotes the gradient (not the Levi-Civita connection only here) of the energy functional

$$\begin{aligned} E_{\alpha, \beta, \gamma}(u) & := \frac{\alpha}{2} \int_{\mathbb{R}} h(u_x, u_x) dx + \frac{\beta}{2} \int_{\mathbb{R}} h(\nabla_x u_x, \nabla_x u_x) dx \\ & + \gamma \int_{\mathbb{R}} h(R(u_x, J_u u_x) J_u u_x, u_x) dx. \end{aligned}$$

As is pointed out in [31], if (N, J, h) is a locally Hermitian symmetric space, then the explicit expression of (1.5) coincides with (1.1) under the setting

$$a = \beta, b = \beta + 8\gamma, c = \frac{3(a - b)}{2}, \lambda = -\alpha \quad (1.6)$$

Additionally, another fourth-order extension of (1.4), also generalizing (1.2), has been investigated in [6, 7, 28, 30], which is formulated by

$$u_t = a_1 J_u \nabla_x^3 u_x + a_2 J_u \nabla_x u_x + a_3 h(u_x, u_x) J_u \nabla_x u_x + a_4 h(\nabla_x u_x, u_x) J_u u_x \quad (1.7)$$

for curve flow $u = u(t, x)$ on a Kähler manifold (N, J, h) . As is shown in [31], (1.7) is also regarded as (1.5) provided that (N, J, h) is a Riemann surface with constant Gaussian curvature. However, the assumption seems to be a bit strong geometrically. It can be said that (1.1) modifies (1.7) to be geometrically more reasonable by considering some kind of symmetry and curvature on (N, J, h) as a Kähler manifold.

This paper is concerned with correspondences between geometric dispersive PDEs for curve flows and systems of nonlinear PDEs for complex-valued functions (or equations for complex-matrix-valued functions), such as that between (1.2) for curve flows on \mathbb{S}^2 and (1.3). These correspondences have attracted much attention from researchers in mathematical physics, differential geometry, and theory of PDEs. Understanding them has the potential to promote the studies in both directions complementarily each other. In this connection, we next focus on two seemingly different methods developed in previous studies of the geometric dispersive PDEs (except of the fourth-order equations (1.1), (1.2), and (1.7) which is stated later in Introduction).

The first type of method is based on the development map acting on a space of smooth curves on N , embedding N as an adjoint orbit in an associated Lie algebra. Notably, the method essentially applies the properties of Hermitian symmetric spaces as (N, J, h) . It is to be stated first that Zakharov and Takhtadzhyan [40] showed that the Schrödinger flow equation (1.4) for curve flows on \mathbb{S}^2 is equivalent to the cubic nonlinear Schrödinger equation (NLS) for complex-valued functions. As the generalization, it was established by Terng and Uhlenbeck [39] that (1.4) for curve flows on a compact complex Grassmannian is equivalent to the matrix NLS which was first studied by Fordy and Kulish [14]. It is also pointed out in [39] that the existence of a time-global solution to the initial value problem of (1.4) for curve flows on the compact complex Grassmannian follows from the correspondence. The above equivalence with respect to (1.4) was investigated further by Terng and Thorbergsson [38] for the other three types of compact Hermitian symmetric spaces as (N, J, h) . Additionally, the equivalence with respect to (1.4) for curve flows on \mathbb{S}^2 in the periodic setting in x has been obtained by Liu [25] recently. The interested readers can also refer to, e.g., [1, 11, 12], for more details related to the method.

The second type of method, called the generalized Hasimoto transformation, is to transform a geometric dispersive PDE for curve flows into a nonlinear dispersive PDE for complex-valued functions or a system of them, by constructing a parallel (in x -direction) orthonormal frame along a curve flow $u = u(t, x)$ and then by expressing the equation satisfied by the components of u_x with respect to the frame. We expect that this method can handle Riemann surfaces or more general Kähler manifolds as (N, J, h) without imposing any symmetry or curvature conditions, and the expressions of the derived equations or systems are simpler, in that they are semilinear ones without any second-highest order derivatives in spatial variable. In addition, we expect that the derived expressions present insights on the essential structure

of the original geometric dispersive PDEs, although constructing the inverse of the transformation remains further discussion. Indeed, these insights have been sometimes applicable to the time-global solvability of the initial value problem for original geometric dispersive PDEs. See, e.g, the work by Chang et al. ([5]), Nahmod et al. ([26]), Rodnianski et al.([35]) for (1.4). Moreover, also for some analogous third-order geometric dispersive PDEs for curve flows on compact Riemann surfaces, equations derived from the generalized Hasimoto transformation have been investigated in [28] and in [36, 37], independently on whether any direct applications to the time-global solvability of their initial value problem exist or not. Notably, many results based on the method seem to handle “open” curve flows, where the spacial domain of x is the real line \mathbb{R} . On the other hand, if a geometric dispersive PDE for closed curve flows is considered, then the method requires some modifications, involving holonomy corrections along the closed curves to transform into PDEs for complex-valued functions that are periodic in x , which becomes rather complicated. Nonetheless, this case has been also investigated for (1.4). See [22, 23] for closed curve flows on locally Hermitian symmetric spaces, and [35] for those on Riemann surfaces.

In contrast, investigating our fourth-order PDE (1.1) in this context still remains unexplored. Related previous results on the correspondences are limited to when N is any one of G_{n_0, k_0} (including $\mathbb{S}^2 \cong G_{2,1}$), $G_{n_0}^{k_0}$, and a Riemann surface, which are stated more concretely in the next two paragraphs.

When N is either of G_{n_0, k_0} or $G_{n_0}^{k_0}$, our equation (1.1) with $c = 3(a - b)/2$ for $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow N$ is proved to be equivalent to a fourth-order matrix nonlinear (Schrödinger-like) differential-integral equation, which follows from the results by Ding and Wang ([11]): Taking the compact case $N = G_{n_0, k_0}$ of complex dimension $n = k_0(n_0 - k_0)$ as the example, the authors in [11] investigated (1.5) (not as (1.1)), and equivalently transformed it to

$$\begin{aligned} & \sqrt{-1}q_t - \alpha \left\{ q_{xx} + 2qq^*q \right\} + \beta \left\{ q_{xxxx} + 4q_{xx}q^*q + 2qq_{xx}^*q + 4qq^*q_{xx} \right. \\ & \quad \left. + 2q_xq_x^*q + 6q_xq^*q_x + 2qq_x^*q_x + 6qq^*qq^*q \right\} - 2(\beta + 8\gamma) \left\{ (qq^*q)_{xx} \right. \\ & \quad \left. + 2qq^*qq^*q + q \left(\int_0^x q^*(qq^*)_s q ds \right) + \left(\int_0^x q(q^*q)_s q^* ds \right) q \right\} = 0 \end{aligned} \quad (1.8)$$

for $q = q(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathcal{M}_{k_0 \times (n_0 - k_0)}$, where $\mathcal{M}_{k_0 \times (n_0 - k_0)}$ stands for the space of $k_0 \times (n_0 - k_0)$ complex-matrices and $q^* = \bar{q}^t$ is the transposed conjugate matrix-valued function of q . The proof basically employs the first type of method established by [39] based on the development map, combined with the idea of PDEs with given (non-zero) curvature representation in the category of Yang-Mills theory. They succeed to establish the results, even though the considered equation is not completely integrable unless $\beta + 8\gamma = 0$, revealing also that the nonlocal terms of integral type in (1.8) vanish under the integrable condition. It is also pointed out in [11] that, if $N = G_{2,1} \cong \mathbb{S}^2$, then the nonlocal terms of (1.8) vanish without the integrable condition, and (1.8) reduces to (1.3) under the setting (1.6).

When (N, J, h) is a Riemann surface with constant Gaussian curvature, our equation (1.1) with $c = 3(a - b)/2$ for $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow N$ is proved to be transformed to a fourth-order nonlinear dispersive PDE without non-local integral terms for complex-valued functions, which follows from the results by Ding and Zhong ([12]): The authors in

²Here and hereafter, G_{n_0, k_0} (resp. $G_{n_0}^{k_0}$) for integers n_0, k_0 satisfying $1 \leq k_0 < n_0$ denotes the compact (resp. noncompact) complex Grassmannian inherited with the structure as a Hermitian symmetric space.

[12] investigated (1.5) for $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow N$ where (N, J, h) is a Riemann surface. Assuming the existence of a time-independent edge point $\lim_{x \rightarrow \infty} u(t, x) \in N$, they employed the second type of method based on the generalized Hasimoto transformation, which transformed the equation into the following differential-integral equation

$$\begin{aligned} & \sqrt{-1}q_t - \alpha \left(q_{xx} + \frac{\kappa(u)}{2}|q|^2q \right) \\ & + \beta \left\{ q_{xxxx} - \frac{\kappa(u)}{2} (2|q_x|^2q - 2|q|^2q_{xx} - \bar{q}q_x^2) - \frac{(\kappa(u))_x}{2} (q^2\bar{q}_x - |q|^2q_x) \right\} \\ & - \gamma \left[3(\kappa(u))^2|q|^4q + \{3(\kappa(u))_x|q|^2q + 4\kappa(u)(|q|^2q)_x\}_x \right] - qW(t, x) = 0 \end{aligned} \quad (1.9)$$

for complex-valued function $q = q(t, x)$, where

$$W(t, x) = \frac{1}{2} \int_x^\infty (\kappa(u))_{\bar{x}} (\alpha|q|^2 - \beta(q\bar{q}_{\bar{x}\bar{x}} + \bar{q}q_{\bar{x}\bar{x}} - q_{\bar{x}}\bar{q}_{\bar{x}}) + 6\gamma\kappa(u)|q|^4) d\bar{x}$$

and $(\kappa(u))(t, x) := \kappa(u(t, x))$ is the Gaussian curvature at $u(t, x) \in N$. Notably, the expression is informative for our equation (1.1) with $c = 3(a - b)/2$ only if $\kappa(u)$ is constant and hence the non-local term vanishes. This is because (1.1) is not the same as (1.5) in general unless (N, J, h) is a locally Hermitian symmetric space. Additionally, the method of transformation has been developed to investigate (1.7) (not (1.1)) for closed curve flows on compact Riemann surfaces by Chihara ([6]), which gives essential insights on the structure of equations satisfied by higher-order x -directional covariant derivatives of a solution to (1.7). The obtained results are valid to our equation (1.1) only if the compact Riemann surface (N, J, h) has a constant Gaussian curvature.

1.2. Main results in this paper. The main results in this paper are the following two.

First, for general compact Kähler manifold (N, J, h) of complex dimension $n \in \mathbb{N}$, we present a framework that can transform (1.1) for $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow N$ into an n -component system of fourth-order nonlinear dispersive partial differential-integral equations for complex-valued functions. To state our results precisely, let u^∞ be a fixed point in N , and let $C_{u^\infty}((-\infty, \infty) \times \mathbb{R}; N)$ denote the set of smooth maps $u = u(t, x) : (-T, T) \times \mathbb{R} \rightarrow N$ such that $\lim_{x \rightarrow -\infty} u(t, x) = u^\infty$ and $u_x(t, \cdot) : \mathbb{R} \rightarrow (u(t, \cdot))^{-1}TN$ is in the Schwartz class for any $t \in (-T, T)$. (The assumption on the set of maps is the same as that in [39].)

Theorem 1.1. *Under the above setting, (1.1) for $u \in C_{u^\infty}((-\infty, \infty) \times \mathbb{R}; N)$ is transformed into the system for $Q_1, \dots, Q_n : (-T, T) \times \mathbb{R} \rightarrow \mathbb{C}$ of the form*

$$\begin{aligned} & \sqrt{-1}\partial_t Q_j + (a\partial_x^4 + \lambda\partial_x^2)Q_j \\ & = d_1 \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x^2 Q_p \bar{Q}_q Q_r + d_2 \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \partial_x^2 \bar{Q}_q Q_r + d_3 \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \bar{\partial}_x \bar{Q}_q Q_r \\ & + d_4 \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \bar{Q}_q \partial_x Q_r + d_5 \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) \partial_x Q_p \bar{Q}_q Q_r \\ & + d_6 \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) Q_p \bar{\partial}_x \bar{Q}_q Q_r - \lambda \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \bar{Q}_q Q_r \end{aligned}$$

$$+ \sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^1(Q, \partial_x Q)(t, y) dy \right) Q_r + \sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^2(Q, \partial_x Q)(t, y) dy \right) Q_r \quad (1.10)$$

for each $j \in \{1, \dots, n\}$. Here, $d_1 = -a - b - 2c$, $d_2 = -a + b$, $d_3 = a + b - 2c$, $d_4 = -b - 2c$, $d_5 = a - b - 2c$, $d_6 = a + b$, and $S_{p,q,r}^j$ for $p, q, r, j \in \{1, \dots, n\}$ are complex-valued functions depending on u which is defined by (2.7), and

$$\begin{aligned} f_{j,r}^1(Q, \partial_x Q) &= -(b + 2c) \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j \overline{S_{\alpha,\beta,\gamma}^q} \partial_x Q_\alpha \overline{Q_\beta} \overline{Q_\gamma} Q_p \\ &\quad - (b + 2c) \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p \partial_x Q_\alpha \overline{Q_\beta} \overline{Q_\gamma} Q_q \\ &\quad + b \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j \overline{S_{\alpha,\beta,\gamma}^q} \overline{Q_\alpha} \partial_x Q_\beta \overline{Q_\gamma} Q_p \\ &\quad + b \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p Q_\alpha \overline{\partial_x Q_\beta} \overline{Q_\gamma} Q_q, \end{aligned} \quad (1.11)$$

$$\begin{aligned} f_{j,r}^2(Q, \partial_x Q) &= -a \sum_{p,q=1}^n \partial_x^2 (S_{p,q,r}^j) (\partial_x Q_p \overline{Q_q} + Q_p \overline{\partial_x Q_q}) \\ &\quad - 3a \sum_{p,q=1}^n \partial_x (S_{p,q,r}^j) \partial_x Q_p \overline{\partial_x Q_q} + \lambda \sum_{p,q=1}^n \partial_x (S_{p,q,r}^j) Q_p \overline{Q_q}. \end{aligned} \quad (1.12)$$

for each $j, r \in \{1, \dots, n\}$.

The proof of Theorem 1.1 is based on the generalized Hasimoto transformation, mainly following the argument of [5] and [35] to handle general compact Kähler manifolds as (N, J, h) and to work under the assumption of the existence of the fixed edge point $u^\infty \in N$. Importantly, investigating higher-order geometric PDEs such as (1.1) in this framework involves basic properties of the Riemann curvature tensor R in relation with the parallel (in x -direction) orthonormal frame along u . This paper develops an understanding of these properties by introducing $S_{p,q,r}^j$ for $p, q, r, j \in \{1, \dots, n\}$, and by deriving useful properties among them, which is a key ingredient of our proof. These functions are introduced by (2.7) and their properties are gathered in Section 2.1. Propositions 2.3, 2.6, and 2.7 are inherited from basic properties of R on a general Kähler manifold stated in Proposition 2.1, and using them enables us to arrive at the expression (1.10) comprehensively. These properties established in Section 2.1 are independent of the equation (1.1), and thus may be applicable in future for investigating other geometric PDEs. We should mention that our framework heavily relies on the Kählerity of N in order to construct the parallel moving frame.

Second, aiming at checking that (1.10) with (1.6) actually unifies (1.8) and (1.9) obtained previously by [11, 12], we take the following three examples of compact Kähler manifolds as (N, J, h) , and demonstrate the computations of (1.10):

- (1) compact Riemann surface
- (2) compact Kähler manifold with constant holomorphic sectional curvature
- (3) compact complex Grassmannian G_{n_0, k_0}

The results of the computations for these examples are respectively presented as (3.5), (3.15), and (4.47). What we find from the results is outlined in the following paragraphs just below.

If (N, J, h) is a compact Riemann surface with constant Gaussian curvature, then the computation of example (1) shows that (1.10) for $Q = Q_1$ under (1.6) coincides with (1.9) for q where $Q_1 = q$. Basically, no originality is claimed here because the orthonormal frame $\{e, Je\}$ we use is essentially the same as that used in [12]. If we were to add something, we note that a difference between them can be observed explicitly when the Gaussian curvature of N is not a constant. See Remark 3.1 for the detail.

If (N, J, h) is an n -dimensional compact Kähler manifold with constant holomorphic sectional curvature K and if $n \geq 2$, the setting of which has not been presented so far, then the computation of example (2) shows that nonlocal terms of integral type actually remain in the derived system of (3.15) unless $bK^2 = 0$. See Remark 3.2 also. Moreover, we point out that the computation of examples (2) is not so difficult. This is because the well-known formula (3.6) is available for associating R explicitly with h and the parallel orthonormal frame we use, the same of which is true for example (1). Notably, the typical example of N here is the n -dimensional complex projective space with the Fubini-Study metric. This example can be handled also in the framework of example (3), since it is identified with $G_{n+1,1}$. However, the computation of example (2) is worth demonstrating, because it is rather easier and faster compared with that of example (3).

If (N, J, h) is a compact complex Grassmannian, then the computation of example (3) shows the following.

Corollary 1.2. *Let (N, J, h) be the compact complex Grassmannian G_{n_0, k_0} of complex dimension $n = k_0(n_0 - k_0)$ as a Hermitian symmetric space. Then, under the same setting as in Theorem 1.1 with additional setting (1.6), the system of equations (1.10) for $Q_1 \dots, Q_n$ coincides with (1.8) for q up to a gauge transformation, where the (i, j) -component of q for $i \in \{1, \dots, k_0\}$ and $j \in \{1, \dots, n_0 - k_0\}$ is identified with $Q_{(j-1)k_0+i}$.*

This verifies that the first type of method based on the development map and the second one based on the parallel orthonormal frame are essentially same in this setting. We show Corollary 1.2 by a direct computation of (1.10) in Section 4.2. Interestingly, the computation of example (3) is more challenging than those of examples (1) and (2) especially in the case where G_{n_0, k_0} is a higher-rank symmetric space with $\min(k_0, n_0 - k_0) > 1$. (It is known that G_{n_0, k_0} in the case does not have a constant holomorphic sectional curvature, and thus does not fall into the scope of example (2).) Although a formula (4.13) to express R is available, it does not directly provide explicit relation between R and h , which is not compatible with the parallel orthonormal frame we use. To overcome the difficulty, we construct a parallel orthonormal frame concretely by taking a suitable orthogonal basis at a point in G_{n_0, k_0} , which makes (4.13) applicable for the computation. Additionally, we also provide more theoretical explanation for the reason why seemingly different two types of methods lead to the same result for G_{n_0, k_0} by comparing them in Section 4.3. More precisely, we show Proposition 4.7, which presents another proof of Corollary 1.2 without doing the computation in Section 4.2.

Finally, two additional comments on our results in this paper are in order.

First, recent studies on the initial value problem for (1.1) and (1.7) ([6, 7, 17, 29, 30, 31, 32]) have handled the case where (N, J, h) is any of the canonical \mathbb{S}^2 , a compact Riemann surface, a compact locally Hermitian symmetric space, or more general compact Kähler manifold. Through the studies, some results on time-local existence of a unique solution in a Sobolev space with high regularity have been established. This motivated the present author

to establish Theorem 1.1 in order to understand the structure of the geometric equations in the level of systems of nonlinear equations for complex valued functions. We expect that the derived expression in Theorem 1.1 will be informative in future work to establish fruitful results on the conditions for the solution to (1.1) to exist globally in time, or on the conditions for (1.1) to be completely integrable.

Second, as is stated above, Ding and Wang investigated (1.5) in [11]. In fact, they proposed some unclear questions in their paper, one of which is commented as follows:

Except the Hermitian symmetric spaces $G_{n,k}$ of compact type (i.e, A III), there are C I, D III and BD I-types of symmetric spaces (and also two exceptional Hermitian symmetric spaces E III and E VII) (refer to [23, 26]). Can we establish similar results on these symmetric spaces? ([11], p. 192)

Our Theorem 1.1 handles general compact Kähler manifolds as (N, J, h) , including these symmetric spaces. Moreover, we expect that the computation of example (3) for G_{n_0, k_0} in Section 4.2 can be proceeded in similar way also for these other symmetric spaces, only by modifying the setting in Section 4.1 suitably. If this is true, then the result of the computation will provide a partial answer to the above proposed question, although the approach may not be what was intended by the authors in [11] and the way to construct the inverse of the transformation which verifies the equivalence is still unclear in this approach.

The organization of this paper is as follows: In Section 2, the parallel orthonormal frame is constructed and associated basic properties are provided in Section 2.1, and then Theorem 1.1 is proved in Section 2.2. In Section 3, the computations of (1.10) for examples (1) and (2) of (N, J, h) are demonstrated. In Section 4, the setting of G_{n_0, k_0} are reviewed in Section 4.1, and then the computation of (1.10) for example (3) of (N, J, h) is demonstrated to show Corollary 1.2 in Section 4.2, with additional explanation in Section 4.3. Supplemental materials are stated in Appendix.

2. REDUCTION TO A SYSTEM OF PDES

In this section, suppose that (N, J, h) is a compact Kähler manifold of complex dimension n and $u \in C_{u^\infty}((-T, T) \times \mathbb{R}; N)$ is a solution to (1.1) as in Theorem 1.1.

2.1. Parallel moving frame and the associated properties of the Riemann curvature tensor. For fixed $u^\infty \in N$, let $\{e_1^\infty, \dots, e_n^\infty, J_{u^\infty}e_1^\infty, \dots, J_{u^\infty}e_n^\infty\}$ be an orthonormal basis for $T_{u^\infty}N$ with respect to h . Following [5] and [35], we take the orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ for $u^{-1}TN$ that satisfies

$$\nabla_x e_p(t, x) = 0, \quad (2.1)$$

$$\lim_{x \rightarrow -\infty} e_p(t, x) = e_p^\infty, \quad (2.2)$$

and $e_{p+n} = J_u e_p$ for any $p \in \{1, \dots, n\}$. The Kählerity $\nabla J = 0$ ensures $\nabla_x e_{p+n} = 0$ for any $p \in \{1, \dots, n\}$. (In what follows, $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ is written by $\{e_j, J e_j\}_{j=1}^n$ for simplicity.) In the setting, we can write

$$\nabla_t e_p = \sum_{j=1}^n a_j^p e_j + \sum_{j=1}^n b_j^p J e_j \quad (\forall p \in \{1, \dots, n\}), \quad (2.3)$$

where for each $p, j \in \{1, \dots, n\}$, $a_j^p = a_j^p(t, x)$ and $b_j^p = b_j^p(t, x)$ denote real-valued functions of (t, x) . We will seek conditions on a_j^p and b_j^p later. (See (2.61).) The components of R

associated with $\{e_j, J e_j\}_{j=1}^n$ are expressed by the following:

$$R(e_p, e_q)e_r = \sum_{j=1}^n R_{p,q,r}^j e_j + \sum_{j=1}^n R_{p,q,r}^{j+n} J e_j, \quad (2.4)$$

$$R(e_p, J e_q)e_r = \sum_{j=1}^n R_{p,q+n,r}^j e_j + \sum_{j=1}^n R_{p,q+n,r}^{j+n} J e_j \quad (2.5)$$

for $p, q, r \in \{1, \dots, n\}$, where $R_{p,q,r}^j, R_{p,q,r}^{j+n}, R_{p,q+n,r}^j, R_{p,q+n,r}^{j+n}$ are real-valued functions of (t, x) . Furthermore, let us set

$$R_{p,q,r}^{A,j} := R_{p,q,r}^j + \sqrt{-1} R_{p,q,r}^{j+n}, \quad R_{p,q,r}^{B,j} := R_{p,q+n,r}^j + \sqrt{-1} R_{p,q+n,r}^{j+n}, \quad (2.6)$$

$$S_{p,q,r}^j := \frac{1}{2} (R_{p,q,r}^{A,j} + \sqrt{-1} R_{p,q,r}^{B,j}), \quad T_{p,q,r}^j := \frac{1}{2} (-R_{p,q,r}^{A,j} + \sqrt{-1} R_{p,q,r}^{B,j}) \quad (2.7)$$

for $p, q, r, j \in \{1, \dots, n\}$, all of which are then complex-valued functions of (t, x) . In fact, these functions at (t, x) depend on $u(t, x)$. In particular, by definition, it follows that

$$\begin{aligned} S_{p,q,r}^j &= \frac{1}{2} \{h(R(e_p, e_q)e_r, e_j) + \sqrt{-1}h(R(e_p, e_q)e_r, J e_j)\} \\ &\quad + \frac{\sqrt{-1}}{2} \{h(R(e_p, J e_q)e_r, e_j) + \sqrt{-1}h(R(e_p, J e_q)e_r, J e_j)\}. \end{aligned} \quad (2.8)$$

Here, we recall the following basic properties for R on the Kähler manifold (N, J, h) :

Proposition 2.1. *For any $Y_1, \dots, Y_4 \in \Gamma(u^{-1}TN)$, the following properties hold:*

- (i) $R(Y_1, Y_2) = -R(Y_2, Y_1)$,
- (ii) $R(Y_1, Y_2)Y_3 + R(Y_2, Y_3)Y_1 + R(Y_3, Y_1)Y_2 = 0$,
- (iii) $h(R(Y_1, Y_2)Y_3, Y_4) = h(R(Y_3, Y_4)Y_1, Y_2) = h(R(Y_4, Y_3)Y_2, Y_1)$,
- (iv) $R(Y_1, Y_2)J e Y_3 = J e R(Y_1, Y_2)Y_3$,
- (v) $R(J e Y_1, Y_2)Y_3 = -R(Y_1, J e Y_2)Y_3, \quad R(J e Y_1, Y_2)Y_3 = R(J e Y_2, Y_1)Y_3$.

The properties (i)-(iii) follow from the definition of R . The property (iv) holds since (N, J, h) is a Kähler manifold. The property (v) follows from (i) and $J e^2 = -\text{id}$ on $\Gamma(u^{-1}TN)$.

The following properties for $R_{p,q,r}^{A,j}$ and $R_{p,q,r}^{B,j}$ are inherited from Proposition 2.1 for R :

Proposition 2.2. *The following properties hold:*

$$R_{p,q,r}^{A,j} = -R_{q,p,r}^{A,j} \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.9)$$

$$R_{p,q,r}^{B,j} = R_{q,p,r}^{B,j} \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.10)$$

$$R_{p,q,r}^{A,j} + R_{q,r,p}^{A,j} + R_{r,p,q}^{A,j} = 0 \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.11)$$

$$R_{p,q,r}^{A,j} = \sqrt{-1}(R_{q,r,p}^{B,j} - R_{r,p,q}^{B,j}) \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.12)$$

$$\text{Re}[R_{p,q,r}^{A,j}] = \text{Re}[R_{r,j,p}^{A,q}] = \text{Re}[R_{j,r,q}^{A,p}] \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.13)$$

$$\text{Im}[R_{p,q,r}^{B,j}] = \text{Im}[R_{r,j,p}^{B,q}] = \text{Im}[R_{j,r,q}^{B,p}] \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.14)$$

$$\text{Im}[R_{p,q,r}^{A,j}] = \text{Re}[R_{r,j,p}^{B,q}] = -\text{Re}[R_{j,r,q}^{B,p}] \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.15)$$

In particular, if $\nabla R = 0$ holds, that is, (N, J, h) is locally Hermitian symmetric, the following also holds:

Proposition 2.3. *Under the additional assumption $\nabla R = 0$, the following also holds:*

$$\partial_x R_{p,q,r}^{A,j} = \partial_x R_{p,q,r}^{B,j} = 0 \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.16)$$

Proof of Proposition 2.3. Let $p, q, r \in \{1, \dots, n\}$ be any given. From $\nabla R = 0$ and (2.1),

$$\begin{aligned} & \nabla_x \{R(e_p, e_q)e_r\} \\ &= (\nabla_x R)(e_p, e_q)e_r + R(\nabla_x e_p, e_q)e_r + R(e_p, \nabla_x e_q)e_r + R(e_p, e_q)\nabla_x e_r \\ &= 0. \end{aligned}$$

On the other hand, from (2.4), the Kählerity $\nabla J = 0$, and (2.1),

$$\nabla_x \{R(e_p, e_q)e_r\} = \sum_{j=1}^n (\partial_x R_{p,q,r}^j) e_j + \sum_{j=1}^n (\partial_x R_{p,q,r}^{j+n}) J_u e_j.$$

Comparing both equations, and noting $\{e_j, J e_j\}_{j=1}^n$ is the orthonormal frame, we have

$$0 = \partial_x R_{p,q,r}^j = \operatorname{Re}[\partial_x R_{p,q,r}^{A,j}] \quad \text{and} \quad 0 = \partial_x R_{p,q,r}^{j+n} = \operatorname{Im}[\partial_x R_{p,q,r}^{A,j}]$$

for any $j \in \{1, \dots, n\}$, which shows $\partial_x R_{p,q,r}^{A,j} = 0$ for any $p, q, r, j \in \{1, \dots, n\}$. In the same way as above, we use (2.5) and then obtain

$$0 = \nabla_x \{R(e_p, J_u e_q)e_r\} = \sum_{j=1}^n (\partial_x R_{p,q+n,r}^j) e_j + \sum_{j=1}^n (\partial_x R_{p,q+n,r}^{j+n}) J_u e_j.$$

From this, it follows that

$$0 = \partial_x R_{p,q+n,r}^j = \operatorname{Re}[\partial_x R_{p,q,r}^{B,j}] \quad \text{and} \quad 0 = \partial_x R_{p,q+n,r}^{j+n} = \operatorname{Im}[\partial_x R_{p,q,r}^{B,j}],$$

and thus $\partial_x R_{p,q,r}^{B,j} = 0$ for any $p, q, r, j \in \{1, \dots, n\}$. \square

Proof of Proposition 2.2. The property (2.9) follows from (i) in Proposition 2.1: Indeed, (i) for $(Y_1, Y_2) = (e_p, e_q)$ and (2.4) shows

$$\begin{aligned} 0 &= R(e_p, e_q)e_r + R(e_q, e_p)e_r \\ &= \sum_{j=1}^n (R_{p,q,r}^j + R_{q,p,r}^j) e_j + \sum_{j=1}^n (R_{p,q,r}^{j+n} + R_{q,p,r}^{j+n}) J_u e_j, \end{aligned}$$

which implies

$$R_{p,q,r}^j = -R_{q,p,r}^j, \quad R_{p,q,r}^{j+n} = -R_{q,p,r}^{j+n} \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.17)$$

Recalling (2.6), we see (2.17) is equivalent to (2.9).

The property (2.10) follows from (i) and the second part of (v) in Proposition 2.1: Indeed, $R(e_p, J_u e_q)e_r = R(e_q, J_u e_p)e_r$ follows from them. This combined with (2.5) yields

$$R_{p,q+n,r}^j = R_{q,p+n,r}^j, \quad R_{p,q+n,r}^{j+n} = R_{q,p+n,r}^{j+n} \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.18)$$

Recalling (2.6), we see (2.18) is equivalent to (2.10).

The properties (2.11) and (2.12) follow from (ii), (iv) and (v) in Proposition 2.1: Indeed, (ii) for $(Y_1, Y_2, Y_3) = (e_p, e_q, e_r)$ shows

$$0 = \sum_{j=1}^n (R_{p,q,r}^j + R_{q,r,p}^j + R_{r,p,q}^j) e_j + \sum_{j=1}^n (R_{p,q,r}^{j+n} + R_{q,r,p}^{j+n} + R_{r,p,q}^{j+n}) J_u e_j,$$

which implies

$$R_{p,q,r}^j + R_{q,r,p}^j + R_{r,p,q}^j = 0, \quad R_{p,q,r}^{j+n} + R_{q,r,p}^{j+n} + R_{r,p,q}^{j+n} = 0 \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.19)$$

Taking the summation of the first equality of (2.19) and the second one multiplied by $\sqrt{-1}$, we obtain (2.11). On the other hand, using (ii) for $(Y_1, Y_2, Y_3) = (e_p, e_q, J_u e_r)$, (iv) and the first part of (v), we deduce

$$\begin{aligned} 0 &= R(e_p, e_q)J_u e_r + R(e_q, J_u e_r)e_p + R(J_u e_r, e_p)e_q \\ &= J_u R(e_p, e_q)e_r + R(e_q, J_u e_r)e_p - R(e_r, J_u e_p)e_q \\ &= \sum_{j=1}^n J_u (R_{p,q,r}^j e_j + R_{p,q,r}^{j+n} J_u e_j) + \sum_{j=1}^n (R_{q,r+p,n}^j e_j + R_{q,r+p,n}^{j+n} J_u e_j) \\ &\quad - \sum_{j=1}^n (R_{r,p+n,q}^j e_j + R_{r,p+n,q}^{j+n} J_u e_j) \\ &= \sum_{j=1}^n \{ (-R_{p,q,r}^{j+n} + R_{q,r+p,n}^j - R_{r,p+n,q}^j) + (R_{p,q,r}^j + R_{q,r+p,n}^{j+n} - R_{r,p+n,q}^{j+n}) J_u \} e_j. \end{aligned}$$

From this, for any $p, q, r, j \in \{1, \dots, n\}$, it follows that

$$-R_{p,q,r}^{j+n} + R_{q,r+p,n}^j - R_{r,p+n,q}^j = 0, \quad R_{p,q,r}^j + R_{q,r+p,n}^{j+n} - R_{r,p+n,q}^{j+n} = 0. \quad (2.20)$$

Two equalities in (2.20) can be written respectively as follows:

$$\operatorname{Im}[R_{p,q,r}^{A,j}] = \operatorname{Re}[R_{q,r,p}^{B,j}] - \operatorname{Re}[R_{r,p,q}^{B,j}], \quad (2.21)$$

$$\operatorname{Re}[R_{p,q,r}^{A,j}] = -\operatorname{Im}[R_{q,r,p}^{B,j}] + \operatorname{Im}[R_{r,p,q}^{B,j}]. \quad (2.22)$$

Using (2.21) and (2.22), we obtain the desired (2.12) as follows:

$$\begin{aligned} R_{p,q,r}^{A,j} &= -\operatorname{Im}[R_{q,r,p}^{B,j}] + \operatorname{Im}[R_{r,p,q}^{B,j}] + \sqrt{-1}(\operatorname{Re}[R_{q,r,p}^{B,j}] - \operatorname{Re}[R_{r,p,q}^{B,j}]) \\ &= \sqrt{-1} \{ (\operatorname{Re}[R_{q,r,p}^{B,j}] + \sqrt{-1} \operatorname{Im}[R_{q,r,p}^{B,j}]) - (\operatorname{Re}[R_{r,p,q}^{B,j}] + \sqrt{-1} \operatorname{Im}[R_{r,p,q}^{B,j}]) \} \\ &= \sqrt{-1}(R_{q,r,p}^{B,j} - R_{r,p,q}^{B,j}). \end{aligned}$$

The properties (2.13)-(2.15) follow from (iii) and (iv) in Proposition 2.1: First, noting $R_{p,q,r}^j = h(R(e_p, e_q)e_r, e_j)$ and using (iii) for $(Y_1, Y_2, Y_3, Y_4) = (e_p, e_q, e_r, e_j)$, we obtain

$$R_{p,q,r}^j = R_{r,j,p}^q = R_{j,r,q}^p \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.23)$$

which by (2.6) is equivalent to (2.13). Second, noting $R_{p,q+n,r}^{j+n} = h(R(e_p, J_u e_q)e_r, J_u e_j)$ and using (iii), we obtain

$$R_{p,q+n,r}^{j+n} = h(R(e_r, J_u e_j)e_p, J_u e_q) = h(R(J_u e_j, e_r)J_u e_q, e_p).$$

Here, $h(R(e_r, J_u e_j)e_p, J_u e_q) = R_{r,j+n,p}^{q+n}$ follows from (2.5), and

$$h(R(J_u e_j, e_r)J_u e_q, e_p) = h(R(e_j, J_u e_r)e_q, J_u e_p) = R_{j,r+n,q}^{p+n}$$

follows from (iii), the first part of (v), and (2.5). Combining them, we obtain

$$R_{p,q+n,r}^{j+n} = R_{r,j+n,p}^{q+n} = R_{j,r+n,q}^{p+n} \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.24)$$

which is equivalent to (2.14). Third, noting $R_{p,q,r}^{j+n} = h(R(e_p, e_q)e_r, J_u e_j)$ and using (iii) for $(Y_1, Y_2, Y_3, Y_4) = (e_p, e_q, e_r, J_u e_j)$, we have

$$R_{p,q,r}^{j+n} = h(R(e_r, J_u e_j)e_p, e_q) = h(R(J_u e_j, e_r)e_q, e_p).$$

Here, $h(R(e_r, J_u e_j)e_p, e_q) = R_{r,j+n,p}^q$ follows from (2.5), and

$$h(R(J_u e_j, e_r)e_q, e_p) = -h(R(e_j, J_u e_r)e_q, e_p) = -R_{j,r+n,q}^p$$

follows from the first part of (v) and (2.5). Combining them, we obtain

$$R_{p,q,r}^{j+n} = R_{r,j+n,p}^q = -R_{j,r+n,q}^p \quad (\forall p, q, r, j \in \{1, \dots, n\}), \quad (2.25)$$

which is equivalent to (2.15). \square

Remark 2.4. In the above proof, the conditions (2.11) and (2.12) are obtained by using (ii) in Proposition 2.1 for $(Y_1, Y_2, Y_3) = (e_p, e_q, e_r)$ and $(e_p, e_q, J_u e_r)$. Additionally, no other conditions can be obtained from (ii) even if we choose $(e_p, J_u e_q, e_r)$, $(e_p, J_u e_q, J_u e_r)$, $(J_u e_p, e_q, e_r)$, $(J_u e_p, e_q, J_u e_r)$, $(J_u e_p, J_u e_q, e_r)$, or $(J_u e_p, J_u e_q, J_u e_r)$ as (Y_1, Y_2, Y_3) , which is due to the properties of R stated above.

Remark 2.5. In the above proof, the conditions (2.13)-(2.15) are obtained by investigating

$$h(R(K(p)e_p, K(q)e_q)K(r)e_r, K(j)e_j) \quad (2.26)$$

where $(K(p), K(q), K(r), K(j)) = (I_d, I_d, I_d, I_d)$, (I_d, J_u, I_d, J_u) or (I_d, I_d, I_d, J_u) . Although there are seemingly 2^4 -types of expression of (2.26) depending on the choice of I_d or J_u as $K(\cdot)$, no other conditions can be obtained even if we investigate the rest $(2^4 - 3)$ -types: The most curious may be the case where $(K(p), K(q), K(r), K(j)) = (I_d, J_u, I_d, I_d)$, in that $R_{p,q+n,r}^j = h(R(e_p, J_u e_q)e_r, e_j)$ appearing in (2.5) is not investigated in this context. As for the case, we use (iii) in the same way as above to see

$$R_{p,q+n,r}^j = h(R(e_r, e_j)e_p, J_u e_q) = h(R(e_j, e_r)J_u e_q, e_p).$$

Here, $h(R(e_r, e_j)e_p, J_u e_q) = R_{r,j,p}^{q+n}$ follows from (2.4), and

$$h(R(e_j, e_r)J_u e_q, e_p) = -h(R(e_j, e_r)e_q, J_u e_p) = -R_{j,r,q}^{p+n}$$

follows from (iii) combined with the first part of (v) and (2.4). Combining them, we get

$$R_{p,q+n,r}^j = R_{r,j,p}^{q+n} \quad \text{and} \quad R_{p,q+n,r}^j = -R_{j,r,q}^{p+n} \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.27)$$

However, by replacing indexes $(p, q, r, j) \rightarrow (r, j, p, q)$ for the first part of (2.27) and $(p, q, r, j) \rightarrow (j, r, q, p)$ for the second part, (2.27) becomes

$$R_{r,j+n,p}^q = R_{p,q,r}^{j+n} \quad \text{and} \quad R_{j,r+n,q}^p = -R_{p,q,r}^{j+n} \quad (\forall p, q, r, j \in \{1, \dots, n\}),$$

which is nothing but (2.25). Hence, (2.27) is now meaningless. In the same way, it turns out that the conditions obtained from the rest $(2^4 - 4)$ -expressions of (2.26) are reduced to either of (2.23), (2.24), (2.25), and (2.27). We omit the detail.

Next, let us see the functions $S_{p,q,r}^j$ play the crucial role to express R .

Proposition 2.6. *Under the same assumption as that in Proposition 2.2,*

$$\langle R(U, V)W \rangle_j = \sum_{p,q,r=1}^n S_{p,q,r}^j \left(\langle U \rangle_p \overline{\langle V \rangle_q} - \langle V \rangle_p \overline{\langle U \rangle_q} \right) \langle W \rangle_r \quad (\forall j \in \{1, \dots, n\}) \quad (2.28)$$

holds for any $U, V, W \in \Gamma(u^{-1}TN)$. Here, for any $\Xi \in \Gamma(u^{-1}TN)$ and $j \in \{1, \dots, n\}$, $\langle \Xi \rangle_j$ denotes a complex-valued function defined by

$$\langle \Xi \rangle_j := h(\Xi, e_j) + \sqrt{-1} h(\Xi, J_u e_j). \quad (2.29)$$

If we write $\Xi = \sum_{k=1}^n (\Xi_k^R + J_u \Xi_k^I) e_k$ for $\Xi \in \Gamma(u^{-1}TN)$ by using real-valued functions Ξ_k^R and Ξ_k^I and substitute it into (2.29), then we see

$$\langle \Xi \rangle_j = \Xi_j^R + \sqrt{-1} \Xi_j^I. \quad (2.30)$$

Proof of Proposition 2.6. To begin with, let us write $U, V, W \in \Gamma(u^{-1}TN)$ as

$$U = \sum_{p=1}^n (U_p^R + J_u U_p^I) e_p, \quad V = \sum_{q=1}^n (V_q^R + J_u V_q^I) e_q, \quad W = \sum_{r=1}^n (W_r^R + J_u W_r^I) e_r, \quad (2.31)$$

where $U_p^R, U_p^I, V_q^R, V_q^I, W_r^R, W_r^I$ for all $p, q, r \in \{1, \dots, n\}$ are real-valued functions of (t, x) . As R is trilinear,

$$\begin{aligned} & R(U, V)W \\ &= \sum_{p,q,r=1}^n R(U_p^R e_p + J_u U_p^I e_p, V_q^R e_q + J_u V_q^I e_q) (W_r^R e_r + J_u W_r^I e_r) \\ &= \sum_{p,q,r=1}^n \left\{ \begin{array}{l} U_p^R V_q^R W_r^R R(e_p, e_q) e_r + U_p^R V_q^R W_r^I R(e_p, e_q) J_u e_r \\ + U_p^R V_q^I W_r^R R(e_p, J_u e_q) e_r + U_p^R V_q^I W_r^I R(e_p, J_u e_q) J_u e_r \\ + U_p^I V_q^R W_r^R R(J_u e_p, e_q) e_r + U_p^I V_q^R W_r^I R(J_u e_p, e_q) J_u e_r \\ + U_p^I V_q^I W_r^R R(J_u e_p, J_u e_q) e_r + U_p^I V_q^I W_r^I R(J_u e_p, J_u e_q) J_u e_r \end{array} \right\}. \end{aligned}$$

By (iv) and the first part of (v) in Proposition 2.1, this becomes

$$R(U, V)W = \sum_{p,q,r=1}^n \left\{ \begin{array}{l} (U_p^R V_q^I - U_p^I V_q^R) W_r^R R(e_p, J_u e_q) e_r \\ + (U_p^R V_q^I - U_p^I V_q^R) W_r^I J_u R(e_p, J_u e_q) e_r \\ + (U_p^R V_q^R + U_p^I V_q^I) W_r^R R(e_p, e_q) e_r \\ + (U_p^R V_q^R + U_p^I V_q^I) W_r^I J_u R(e_p, e_q) e_r \end{array} \right\}.$$

Substitution of (2.4) and (2.5) into the above yields

$$\begin{aligned} & R(U, V)W \\ &= \sum_{p,q,r,k=1}^n (U_p^R V_q^I - U_p^I V_q^R) (W_r^R + J_u W_r^I) (R_{p,q+n,r}^k + J_u R_{p,q+n,r}^{k+n}) e_k \\ &+ \sum_{p,q,r,k=1}^n (U_p^R V_q^R + U_p^I V_q^I) (W_r^R + J_u W_r^I) (R_{p,q,r}^k + J_u R_{p,q,r}^{k+n}) e_k. \end{aligned}$$

By (2.30) and (2.6), this expression yields

$$\begin{aligned} & \langle R(U, V)W \rangle_j \\ &= \sum_{p,q,r=1}^n (U_p^R V_q^I - U_p^I V_q^R) \left\langle \sum_{k=1}^n (W_r^R + J_u W_r^I) (R_{p,q+n,r}^k + J_u R_{p,q+n,r}^{k+n}) e_k \right\rangle_j \end{aligned}$$

$$\begin{aligned}
& + \sum_{p,q,r=1}^n (U_p^R V_q^R + U_p^I V_q^I) \langle \sum_{k=1}^n (W_r^R + J_u W_r^I) (R_{p,q,r}^k + J_u R_{p,q,r}^{k+n}) e_k \rangle_j \\
& = \sum_{p,q,r=1}^n (U_p^R V_q^I - U_p^I V_q^R) (W_r^R + \sqrt{-1} W_r^I) (R_{p,q+n,r}^j + \sqrt{-1} R_{p,q+n,r}^{j+n}) \\
& \quad + \sum_{p,q,r=1}^n (U_p^R V_q^R + U_p^I V_q^I) (W_r^R + \sqrt{-1} W_r^I) (R_{p,q,r}^j + \sqrt{-1} R_{p,q,r}^{j+n}) \\
& = \sum_{p,q,r=1}^n R_{p,q,r}^{B,j} (U_p^R V_q^I - U_p^I V_q^R) (W_r^R + \sqrt{-1} W_r^I) \\
& \quad + \sum_{p,q,r=1}^n R_{p,q,r}^{A,j} (U_p^R V_q^R + U_p^I V_q^I) (W_r^R + \sqrt{-1} W_r^I).
\end{aligned}$$

By using an elementary calculation for complex numbers, (2.30) for U, V, W , and using (2.7), we deduce

$$\begin{aligned}
& \langle R(U, V)W \rangle_j \\
& = \sum_{p,q,r=1}^n R_{p,q,r}^{A,j} \operatorname{Re} \left[\overline{(U_p^R + \sqrt{-1} U_p^I)} (V_q^R + \sqrt{-1} V_q^I) \right] (W_r^R + \sqrt{-1} W_r^I) \\
& \quad + \sum_{p,q,r=1}^n R_{p,q,r}^{B,j} \operatorname{Im} \left[\overline{(U_p^R + \sqrt{-1} U_p^I)} (V_q^R + \sqrt{-1} V_q^I) \right] (W_r^R + \sqrt{-1} W_r^I) \\
& = \sum_{p,q,r=1}^n \left\{ R_{p,q,r}^{A,j} \operatorname{Re} \left[\overline{\langle U \rangle_p} \langle V \rangle_q \right] + R_{p,q,r}^{B,j} \operatorname{Im} \left[\overline{\langle U \rangle_p} \langle V \rangle_q \right] \right\} \langle W \rangle_r \\
& = \frac{1}{2} \sum_{p,q,r=1}^n \left\{ (R_{p,q,r}^{A,j} + \sqrt{-1} R_{p,q,r}^{B,j}) \langle U \rangle_p \overline{\langle V \rangle_q} - (-R_{p,q,r}^{A,j} + \sqrt{-1} R_{p,q,r}^{B,j}) \overline{\langle U \rangle_p} \langle V \rangle_q \right\} \langle W \rangle_r \\
& = \sum_{p,q,r=1}^n \left\{ S_{p,q,r}^j \langle U \rangle_p \overline{\langle V \rangle_q} - T_{p,q,r}^j \overline{\langle U \rangle_p} \langle V \rangle_q \right\} \langle W \rangle_r. \tag{2.32}
\end{aligned}$$

Here, note that

$$T_{p,q,r}^j = S_{q,p,r}^j \quad (\forall p, q, r, j \in \{1, \dots, n\}), \tag{2.33}$$

which immediately follows from (2.9), (2.10) in Proposition 2.2. Applying (2.33) to (2.32), we derive

$$\begin{aligned}
\langle R(U, V)W \rangle_j & = \sum_{p,q,r=1}^n \left\{ S_{p,q,r}^j \langle U \rangle_p \overline{\langle V \rangle_q} - T_{q,p,r}^j \overline{\langle U \rangle_q} \langle V \rangle_p \right\} \langle W \rangle_r \\
& = \sum_{p,q,r=1}^n S_{p,q,r}^j \left(\langle U \rangle_p \overline{\langle V \rangle_q} - \langle V \rangle_p \overline{\langle U \rangle_q} \right) \langle W \rangle_r, \tag{2.34}
\end{aligned}$$

which is the desired result. \square

The next proposition also follows from Proposition 2.2.

Proposition 2.7. *Under the same assumption as that in Proposition 2.2,*

$$S_{p,q,r}^j = S_{r,q,p}^j \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.35)$$

Proof of Proposition 2.7. For any $p, q, r, j \in \{1, \dots, n\}$, it follows that

$$\begin{aligned} -\sqrt{-1}(R_{p,q,r}^{A,j} - R_{p,r,q}^{A,j}) &= -\sqrt{-1}(R_{p,q,r}^{A,j} + R_{r,p,q}^{A,j}) \quad (\because (2.9)) \\ &= \sqrt{-1}R_{q,r,p}^{A,j} \quad (\because (2.11)) \\ &= -(R_{r,p,q}^{B,j} - R_{p,q,r}^{B,j}) \quad (\because (2.12)) \\ &= R_{p,q,r}^{B,j} - R_{p,r,q}^{B,j} \quad (\because (2.10)) \end{aligned} \quad (2.36)$$

By multiplying both sides of (2.36) by $\sqrt{-1}$ and by transposing the terms, (2.36) reads

$$R_{p,q,r}^{A,j} - \sqrt{-1}R_{p,q,r}^{B,j} = R_{p,r,q}^{A,j} - \sqrt{-1}R_{p,r,q}^{B,j}. \quad (2.37)$$

This shows

$$T_{p,q,r}^j = T_{p,r,q}^j \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.38)$$

By combining (2.38) and (2.33), we obtain

$$S_{p,q,r}^j = T_{q,p,r}^j = T_{q,r,p}^j = S_{r,q,p}^j \quad (\forall p, q, r, j \in \{1, \dots, n\}). \quad (2.39)$$

□

Propositions 2.6 and 2.7 and sometimes Proposition 2.3 will be sufficient to show the claims in Section 2.2 and Section 3.

2.2. Proof of Theorem 1.1. In this subsection, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\{e_j, J e_j\}_{j=1}^n$ be the orthonormal frame for $u^{-1}TN$ introduced in Section 2.1. We represent $u_x, u_t \in \Gamma(u^{-1}TN)$ by

$$u_x = \sum_{p=1}^n (\xi_p + \eta_p J u) e_p, \quad u_t = \sum_{p=1}^n (\mu_p + \nu_p J u) e_p, \quad (2.40)$$

where $\xi_p, \eta_p, \mu_p, \nu_p$ for $p \in \{1, \dots, n\}$ are real-valued functions of (t, x) . Set $Q_j := \langle u_x \rangle_j$ and $P_j := \langle u_t \rangle_j$ for $j \in \{1, \dots, n\}$. By (2.30), they satisfy

$$Q_j = \langle u_x \rangle_j = \xi_j + \sqrt{-1}\eta_j, \quad P_j = \langle u_t \rangle_j = \mu_j + \sqrt{-1}\nu_j. \quad (2.41)$$

Substitution of (1.1) into $P_j = \langle u_t \rangle_j$ yields

$$\begin{aligned} P_j &= \langle (a J u \nabla_x^3 + \lambda J u \nabla_x) u_x \rangle_j \\ &\quad + b \langle R(\nabla_x u_x, u_x) J u u_x \rangle_j + c \langle R(J u u_x, u_x) \nabla_x u_x \rangle_j. \end{aligned} \quad (2.42)$$

We compute the right hand side of (2.42). First, it follows from (2.40)

$$\begin{aligned} (a J u \nabla_x^3 + \lambda J u \nabla_x) u_x &= (a J u \nabla_x^3 + \lambda J u \nabla_x) \sum_{j=1}^n (\xi_j + J u \eta_j) e_j \\ &= \sum_{j=1}^n (a J u \partial_x^3 \xi_j - a \partial_x^3 \eta_j + \lambda J u \partial_x \xi_j - \lambda \partial_x \eta_j) e_j, \end{aligned}$$

which by (2.30) shows

$$\langle (a J u \nabla_x^3 + \lambda J u \nabla_x) u_x \rangle_j = a \sqrt{-1} \partial_x^3 \xi_j - a \partial_x^3 \eta_j + \lambda \sqrt{-1} \partial_x \xi_j - \lambda \partial_x \eta_j$$

$$= \sqrt{-1}(a\partial_x^3 + \lambda\partial_x)Q_j. \quad (2.43)$$

Second, noting $\langle \nabla_x u_x \rangle_p = \partial_x Q_p$, $\langle u_x \rangle_q = Q_q$, and $\langle J_u u_x \rangle_r = \sqrt{-1}Q_r$, we apply (2.34) in Proposition 2.6 for $(U, V, W) = (\nabla_x u_x, u_x, J_u u_x)$ to deduce

$$\begin{aligned} \langle R(\nabla_x u_x, u_x) J_u u_x \rangle_j &= \sum_{p,q,r=1}^n S_{p,q,r}^j (\partial_x Q_p \overline{Q_q} - Q_p \overline{\partial_x Q_q}) \sqrt{-1} Q_r \\ &= \sqrt{-1} \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{Q_q} Q_r - \sqrt{-1} \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x Q_q} Q_r. \end{aligned} \quad (2.44)$$

Third, in the same way as above, noting $\langle J_u u_x \rangle_p = \sqrt{-1}Q_p$, $\langle u_x \rangle_q = Q_q$, and $\langle \nabla_x u_x \rangle_r = \partial_x Q_r$, we apply (2.34) for $(U, V, W) = (J_u u_x, u_x, \nabla_x u_x)$ to deduce

$$\begin{aligned} \langle R(J_u u_x, u_x) \nabla_x u_x \rangle_j &= \sum_{p,q,r=1}^n S_{p,q,r}^j \left(\sqrt{-1} Q_p \overline{Q_q} - Q_p \overline{\sqrt{-1} Q_q} \right) \partial_x Q_r \\ &= 2\sqrt{-1} \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{Q_q} \partial_x Q_r. \end{aligned}$$

Furthermore, replacing indexes $(p, q, r) \rightarrow (r, q, p)$ in the summation and using (2.35) in Proposition 2.7, we find

$$\langle R(J_u u_x, u_x) \nabla_x u_x \rangle_j = 2\sqrt{-1} \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{Q_q} Q_r. \quad (2.45)$$

Substituting (2.43), (2.44), and (2.45) into (2.42), we have

$$\begin{aligned} P_j &= \sqrt{-1}(a\partial_x^3 + \lambda\partial_x)Q_j + (b+2c)\sqrt{-1} \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{Q_q} Q_r \\ &\quad - b\sqrt{-1} \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x Q_q} Q_r. \end{aligned} \quad (2.46)$$

Next, we seek the condition obtained from the fact $\nabla_x u_t = \nabla_t u_x$. By (2.40) and (2.1),

$$\langle \nabla_x u_t \rangle_j = \left\langle \sum_{p=1}^n (\partial_x \mu_p + J_u \partial_x \nu_p) e_p \right\rangle_j = \partial_x \mu_j + \sqrt{-1} \partial_x \nu_j = \partial_x P_j. \quad (2.47)$$

On the other hand, by (2.40) and (2.3),

$$\nabla_t u_x = \sum_{p=1}^n (\partial_t \xi_p + J_u \partial_t \eta_p) e_p + \sum_{p,r=1}^n (\xi_r + J_u \eta_r) (a_p^r + J_u b_p^r) e_p,$$

which shows

$$\begin{aligned} \langle \nabla_t u_x \rangle_j &= \partial_t \xi_j + \sqrt{-1} \partial_t \eta_j + \sum_{r=1}^n \{ (a_j^r \xi_r - b_j^r \eta_r) + \sqrt{-1} (a_j^r \eta_r + b_j^r \xi_r) \} \\ &= \partial_t Q_j + \sum_{r=1}^n (a_j^r + \sqrt{-1} b_j^r) Q_r. \end{aligned} \quad (2.48)$$

Since $\nabla_x u_t = \nabla_t u_x$ holds, (2.47) and (2.48) show

$$\partial_t Q_j = \partial_x P_j - \sum_{r=1}^n (a_j^r + \sqrt{-1}b_j^r)Q_r \quad (j \in \{1, \dots, n\}). \quad (2.49)$$

Next, we seek the condition obtained from the fact $\nabla_x \nabla_t e_r = \nabla_t \nabla_x e_r + R(u_x, u_t)e_r = R(u_x, u_t)e_r$. By (2.1) and (2.3),

$$\langle \nabla_x \nabla_t e_r \rangle_j = \left\langle \sum_{p=1}^n \partial_x (a_p^r + J_u b_p^r) e_p \right\rangle_j = \partial_x (a_j^r + \sqrt{-1}b_j^r). \quad (2.50)$$

On the other hand, noting $\langle u_x \rangle_p = Q_p$, $\langle u_t \rangle_p = P_p$ and $\langle e_r \rangle_p = \delta_{pr}$, we apply (2.34) for $(U, V, W) = (u_x, u_t, e_r)$, which yields

$$\langle R(u_x, u_t)e_r \rangle_j = \sum_{p,q,r'=1}^n S_{p,q,r'}^j (Q_p \overline{P_q} - P_p \overline{Q_q}) \delta_{rr'} = \sum_{p,q=1}^n S_{p,q,r}^j (Q_p \overline{P_q} - P_p \overline{Q_q}). \quad (2.51)$$

Comparing (2.50) and (2.51), we have

$$\partial_x (a_j^r + \sqrt{-1}b_j^r) = \sum_{p,q=1}^n S_{p,q,r}^j (Q_p \overline{P_q} - P_p \overline{Q_q}) \quad (j, r \in \{1, \dots, n\}). \quad (2.52)$$

Furthermore, using (2.46) with the replacement of indexes $(p, q, r) \rightarrow (\alpha, \beta, \gamma)$ in the summation, we have

$$\begin{aligned} Q_p \overline{P_q} &= -\sqrt{-1}a Q_p \overline{\partial_x^3 Q_q} - \sqrt{-1}\lambda Q_p \overline{\partial_x Q_q} \\ &\quad - (b+2c)\sqrt{-1} \sum_{\alpha,\beta,\gamma=1}^n \overline{S_{\alpha,\beta,\gamma}^q} \partial_x \overline{Q_\alpha} Q_\beta \overline{Q_\gamma} Q_p \\ &\quad + b\sqrt{-1} \sum_{\alpha,\beta,\gamma=1}^n \overline{S_{\alpha,\beta,\gamma}^q} \overline{Q_\alpha} \partial_x Q_\beta \overline{Q_\gamma} Q_p \\ &= \partial_x (-\sqrt{-1}a Q_p \overline{\partial_x^2 Q_q}) + \sqrt{-1}a \partial_x Q_p \overline{\partial_x^2 Q_q} - \sqrt{-1}\lambda Q_p \overline{\partial_x Q_q} \\ &\quad - (b+2c)\sqrt{-1} \sum_{\alpha,\beta,\gamma=1}^n \overline{S_{\alpha,\beta,\gamma}^q} \partial_x \overline{Q_\alpha} Q_\beta \overline{Q_\gamma} Q_p \\ &\quad + b\sqrt{-1} \sum_{\alpha,\beta,\gamma=1}^n \overline{S_{\alpha,\beta,\gamma}^q} \overline{Q_\alpha} \partial_x Q_\beta \overline{Q_\gamma} Q_p \end{aligned}$$

for any $p, q \in \{1, \dots, n\}$. Substituting this into (2.52) multiplied by $\sqrt{-1}$, we deduce

$$\begin{aligned} &\sqrt{-1} \partial_x (a_j^r + \sqrt{-1}b_j^r) \\ &= a \sum_{p,q=1}^n S_{p,q,r}^j \partial_x \left(\partial_x^2 Q_p \overline{Q_q} + Q_p \overline{\partial_x^2 Q_q} \right) \\ &\quad - a \sum_{p,q=1}^n S_{p,q,r}^j \left(\partial_x^2 Q_p \overline{\partial_x Q_q} + \partial_x Q_p \overline{\partial_x^2 Q_q} \right) \end{aligned}$$

$$\begin{aligned}
& + \lambda \sum_{p,q=1}^n S_{p,q,r}^j \left(\partial_x Q_p \overline{Q_q} + Q_p \overline{\partial_x Q_q} \right) \\
& + (b+2c) \sum_{p,q,\alpha,\beta,\gamma=1}^n \left(S_{p,q,r}^j \overline{S_{\alpha,\beta,\gamma}^q} \partial_x Q_\alpha \overline{Q_\beta} \overline{Q_\gamma} Q_p + S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p \partial_x Q_\alpha \overline{Q_\beta} \overline{Q_\gamma} \overline{Q_q} \right) \\
& - b \sum_{p,q,\alpha,\beta,\gamma=1}^n \left(S_{p,q,r}^j \overline{S_{\alpha,\beta,\gamma}^q} Q_\alpha \partial_x Q_\beta \overline{Q_\gamma} Q_p + S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p Q_\alpha \partial_x \overline{Q_\beta} \overline{Q_\gamma} \overline{Q_q} \right) \\
& =: \partial_x \left\{ a \sum_{p,q=1}^n S_{p,q,r}^j \left(\partial_x^2 Q_p \overline{Q_q} + Q_p \overline{\partial_x^2 Q_q} \right) \right\} - \partial_x \left\{ a \sum_{p,q=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} \right\} \\
& + \partial_x \left\{ \lambda \sum_{p,q=1}^n S_{p,q,r}^j Q_p \overline{Q_q} \right\} - S_{j,r}^R - S_{j,r}^{\nabla R} \tag{2.53}
\end{aligned}$$

for any $j, r \in \{1, \dots, n\}$, where

$$S_{j,r}^R := -(b+2c)(S_1 + S_2) + b(S_3 + S_4), \tag{2.54}$$

$$S_1 := \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j \overline{S_{\alpha,\beta,\gamma}^q} \partial_x Q_\alpha \overline{Q_\beta} \overline{Q_\gamma} Q_p, \tag{2.55}$$

$$S_2 := \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p \partial_x Q_\alpha \overline{Q_\beta} \overline{Q_\gamma} \overline{Q_q}, \tag{2.56}$$

$$S_3 := \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j \overline{S_{\alpha,\beta,\gamma}^q} Q_\alpha \partial_x Q_\beta \overline{Q_\gamma} Q_p, \tag{2.57}$$

$$S_4 := \sum_{p,q,\alpha,\beta,\gamma=1}^n S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p Q_\alpha \partial_x \overline{Q_\beta} \overline{Q_\gamma} \overline{Q_q}, \tag{2.58}$$

and

$$\begin{aligned}
S_{j,r}^{\nabla R} & := a \sum_{p,q=1}^n \partial_x (S_{p,q,r}^j) \left(\partial_x^2 Q_p \overline{Q_q} + Q_p \overline{\partial_x^2 Q_q} \right) \\
& - a \sum_{p,q=1}^n \partial_x (S_{p,q,r}^j) \partial_x Q_p \overline{\partial_x Q_q} + \lambda \sum_{p,q=1}^n \partial_x (S_{p,q,r}^j) Q_p \overline{Q_q}. \tag{2.59}
\end{aligned}$$

Here, it follows that

$$|a_j^r + \sqrt{-1}b_j^r| = |h(\nabla_t e_r, e_j) + \sqrt{-1}h(\nabla_t e_r, J_u e_j)| \leq 2|\nabla_t e_r|_h,$$

where $|\cdot|_h = \sqrt{h(\cdot, \cdot)}$. Moreover, $\nabla_t e_r = O(|u_t|_h)$ holds, since N is compact. In addition, $|u_t(t, x)|_h = |(aJ_u \nabla_x^3 u_x + \dots)(t, x)|_h \rightarrow 0$ as $x \rightarrow -\infty$ for each $t \in (-T, T)$, since $u_x(t, \cdot)$ is in the Schwartz class. Combining them, we see

$$\lim_{x \rightarrow -\infty} (a_j^r + \sqrt{-1}b_j^r)(t, x) = 0 \quad (t \in (-T, T)). \tag{2.60}$$

Integrating both sides of (2.53) with respect to x , and using (2.60), we obtain

$$\sqrt{-1}(a_j^r + \sqrt{-1}b_j^r)(t, x)$$

$$\begin{aligned}
&= a \sum_{p,q=1}^n S_{p,q,r}^j \partial_x^2 Q_p \overline{Q_q} + a \sum_{p,q=1}^n S_{p,q,r}^j Q_p \overline{\partial_x^2 Q_q} - a \sum_{p,q=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} \\
&\quad + \lambda \sum_{p,q=1}^n S_{p,q,r}^j Q_p \overline{Q_q} - \int_{-\infty}^x S_{j,r}^R(t,y) dy - \int_{-\infty}^x S_{j,r}^{\nabla R}(t,y) dy. \tag{2.61}
\end{aligned}$$

By substitution of (2.46) and (2.61) into (2.49) multiplied by $\sqrt{-1}$, we deduce

$$\begin{aligned}
&\sqrt{-1} \partial_t Q_j + (a \partial_x^4 + \lambda \partial_x^2) Q_j \\
&= -(b+2c) \sum_{p,q,r=1}^n \partial_x \{ S_{p,q,r}^j \partial_x Q_p \overline{Q_q} Q_r \} + b \sum_{p,q,r=1}^n \partial_x \{ S_{p,q,r}^j Q_p \overline{\partial_x Q_q} Q_r \} \\
&\quad - a \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x^2 Q_p \overline{Q_q} Q_r - a \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x^2 Q_q} Q_r \\
&\quad + a \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} Q_r - \lambda \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{Q_q} Q_r \\
&\quad + \sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^R(t,y) dy \right) Q_r + \sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^{\nabla R}(t,y) dy \right) Q_r \\
&= (-a-b-2c) \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x^2 Q_p \overline{Q_q} Q_r \\
&\quad + (a-b-2c+b) \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} Q_r \\
&\quad + (-b-2c) \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{Q_q} \partial_x Q_r + (-a+b) \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x^2 Q_q} Q_r \\
&\quad + b \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x Q_q} \partial_x Q_r + (-b-2c) \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) \partial_x Q_p \overline{Q_q} Q_r \\
&\quad + b \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) Q_p \overline{\partial_x Q_q} Q_r - \lambda \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{Q_q} Q_r \\
&\quad + \sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^R(t,y) dy \right) Q_r + \sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^{\nabla R}(t,y) dy \right) Q_r.
\end{aligned}$$

Moreover, replacing indexes $(p, q, r) \rightarrow (r, q, p)$ in the summation and using (2.35) shows

$$\sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x Q_q} \partial_x Q_r = \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} Q_r.$$

Using this, we have

$$\sqrt{-1} \partial_t Q_j + (a \partial_x^4 + \lambda \partial_x^2) Q_j$$

$$\begin{aligned}
&= d_1 \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x^2 Q_p \overline{Q_q} Q_r + d_2 \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x^2 Q_q} Q_r + d_3 \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} Q_r \\
&+ d_4 \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{Q_q} \partial_x Q_r + (-b - 2c) \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) \partial_x Q_p \overline{Q_q} Q_r \\
&+ b \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) Q_p \overline{\partial_x Q_q} Q_r - \lambda \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{Q_q} Q_r \\
&+ \sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^R(t, y) dy \right) Q_r + \sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^{\nabla R}(t, y) dy \right) Q_r, \tag{2.62}
\end{aligned}$$

where d_1, d_2, d_3, d_4 are the same constants as those in the statement of Theorem 1.1.

Furthermore, we compute the last two terms of the right hand side of (2.62). First, recalling (2.54) with (2.55)-(2.58), we see $S_{j,r}^R$ is equal to $f_{j,r}^1(Q, \partial_x Q)$ in (1.11), and hence

$$\sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^R(t, y) dy \right) Q_r = \sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^1(Q, \partial_x Q)(t, y) dy \right) Q_r. \tag{2.63}$$

Second, it follows from (2.59)

$$\begin{aligned}
\int_{-\infty}^x S_{j,r}^{\nabla R}(t, y) dy &= a \int_{-\infty}^x \sum_{p,q=1}^n \left(\partial_x (S_{p,q,r}^j) (\partial_x^2 Q_p \overline{Q_q} + Q_p \overline{\partial_x^2 Q_q}) \right) (t, y) dy \\
&- a \int_{-\infty}^x \sum_{p,q=1}^n \left(\partial_x (S_{p,q,r}^j) \partial_x Q_p \overline{\partial_x Q_q} \right) (t, y) dy \\
&+ \lambda \int_{-\infty}^x \sum_{p,q=1}^n \left(\partial_x (S_{p,q,r}^j) Q_p \overline{Q_q} \right) (t, y) dy. \tag{2.64}
\end{aligned}$$

For the first term of the right hand side, we rewrite as

$$\begin{aligned}
\partial_x (S_{p,q,r}^j) (\partial_x^2 Q_p \overline{Q_q} + Q_p \overline{\partial_x^2 Q_q}) &= \partial_x \left\{ \partial_x (S_{p,q,r}^j) (\partial_x Q_p \overline{Q_q} + Q_p \overline{\partial_x Q_q}) \right\} \\
&- \partial_x^2 (S_{p,q,r}^j) (\partial_x Q_p \overline{Q_q} + Q_p \overline{\partial_x Q_q}) \\
&- 2 \partial_x (S_{p,q,r}^j) \partial_x Q_p \overline{\partial_x Q_q}. \tag{2.65}
\end{aligned}$$

Here, we see there exists a positive constant $C(N)$ depending only on N such that

$$|\partial_x (S_{p,q,r}^j)| \leq C(N) |Q|, \quad |\partial_x^2 (S_{p,q,r}^j)| \leq C(N) (|\partial_x Q| + |Q|),$$

since $S_{p,q,r}^j(t, x)$ depends on $u(t, x) \in N$ and N is compact. (This can be also proved by taking partial derivatives of the right hand of (2.8) with respect to x .) This ensures $\lim_{x \rightarrow -\infty} \partial_x (S_{p,q,r}^j) (\partial_x Q_p \overline{Q_q} + Q_p \overline{\partial_x Q_q})(t, x) = 0$, since $Q(t, \cdot) : \mathbb{R} \rightarrow \mathbb{C}^n$ is in the Schwartz class. Noting this and substituting (2.65) into (2.64) leads to

$$\begin{aligned}
\sum_{r=1}^n \left(\int_{-\infty}^x S_{j,r}^{\nabla R}(t, y) dy \right) Q_r &= a \sum_{p,q,r=1}^n \partial_x (S_{p,q,r}^j) (\partial_x Q_p \overline{Q_q} Q_r + Q_p \overline{\partial_x Q_q} Q_r) \\
&+ \sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^2(Q, \partial_x Q)(t, y) dy \right) Q_r, \tag{2.66}
\end{aligned}$$

where $f_{j,r}^2(Q, \partial_x Q)$ is given by (1.12). Substituting (2.63) and (2.66) into (2.62), we derive the desired expression (1.10) with (1.11) and (1.12), which completes the proof of Theorem 1.1. \square

3. EXAMPLES (1) AND (2)

In this section, taking two examples of (N, J, h) , we formulate (1.10) for Q in Theorem 1.1 more explicitly.

3.1. Example (1). Let (N, J, h) be a compact Riemann surface. Since $n = 1$ in this setting, the orthonormal frame introduced by (2.1) in Section 2.1 is $\{e_1, J_u e_1\}$, and thus only $S_{1,1,1}^1$ is required to compute (1.10). We see

$$S_{1,1,1}^1 = \frac{\kappa(u)}{2}, \quad (3.1)$$

where $(\kappa(u))(t, x) := \kappa(u(t, x))$ denotes the Gaussian curvature at $u(t, x) \in N$ which is known to be characterized by

$$\kappa(u) = h(R(e_1, J_u e_1)J_u e_1, e_1). \quad (3.2)$$

To see this, recall that (2.8) for $p, q, r, j = 1$ yields

$$\begin{aligned} 2S_{1,1,1}^1 &= h(R(e_1, e_1)e_1, e_1) + \sqrt{-1}h(R(e_1, e_1)e_1, J_u e_1) \\ &\quad + \sqrt{-1}\{h(R(e_1, J_u e_1)e_1, e_1) + \sqrt{-1}h(R(e_1, J_u e_1)e_1, J_u e_1)\}. \end{aligned}$$

Moreover, by (i) and (iii) in Proposition 2.1,

$$\begin{aligned} h(R(e_1, e_1)e_1, e_1) &= \sqrt{-1}h(R(e_1, e_1)e_1, J_u e_1) = h(R(e_1, J_u e_1)e_1, e_1) = 0, \\ h(R(e_1, J_u e_1)e_1, J_u e_1) &= -h(R(e_1, J_u e_1)J_u e_1, e_1) = -\kappa(u), \end{aligned}$$

which shows (3.1).

Next, we compute (1.11) and (1.12). We write $Q = Q_1$ for simplicity. As for (1.11) in this setting, it follows that $f_{1,1}^1(Q, \partial_x Q) = -(b + 2c)(S_1 + S_2) + b(S_3 + S_4)$ where

$$\begin{aligned} S_1 &= S_{1,1,1}^1 \overline{S_{1,1,1}^1} \partial_x Q Q \overline{Q} \overline{Q} = \frac{(\kappa(u))^2}{4} \partial_x \overline{Q} Q |Q|^2, \\ S_2 &= S_{1,1,1}^1 S_{1,1,1}^1 \partial_x Q \overline{Q} Q \overline{Q} = \frac{(\kappa(u))^2}{4} \partial_x Q \overline{Q} |Q|^2, \\ S_3 &= S_{1,1,1}^1 \overline{S_{1,1,1}^1} \overline{Q} \partial_x Q \overline{Q} Q = \frac{(\kappa(u))^2}{4} \partial_x Q \overline{Q} |Q|^2, \\ S_4 &= S_{1,1,1}^1 S_{1,1,1}^1 Q \overline{\partial_x Q} Q \overline{Q} = \frac{(\kappa(u))^2}{4} \overline{\partial_x Q} Q |Q|^2. \end{aligned}$$

Therefore

$$S_1 + S_2 = S_3 + S_4 = \frac{(\kappa(u))^2}{4} \partial_x (|Q|^2) |Q|^2 = \frac{(\kappa(u))^2}{8} \partial_x (|Q|^4).$$

This yields $f_{1,1}^1(Q, \partial_x Q) = -\frac{c}{4}(\kappa(u))^2 \partial_x (|Q|^4)$, and thus

$$\sum_{r=1}^1 \left(\int_{-\infty}^x f_{1,r}^1(Q, \partial_x Q)(t, y) dy \right) Q_r$$

$$\begin{aligned}
&= -\frac{c}{4} \left(\int_{-\infty}^x ((\kappa(u))^2 \partial_x (|Q|^4)) (t, y) dy \right) Q \\
&= -\frac{c}{4} (\kappa(u))^2 |Q|^4 Q + \frac{c}{2} \left(\int_{-\infty}^x (\kappa(u)(\kappa(u))_x |Q|^4) (t, y) dy \right) Q. \tag{3.3}
\end{aligned}$$

As for (1.12), it follows from the definition and (3.1)

$$\begin{aligned}
&f_{1,1}^2(Q, \partial_x Q) \\
&= -a \partial_x^2 (S_{1,1,1}^1) (\partial_x Q \overline{Q} + Q \overline{\partial_x Q}) - 3a \partial_x (S_{1,1,1}^1) \partial_x Q \overline{\partial_x Q} + \lambda \partial_x (S_{1,1,1}^1) Q \overline{Q} \\
&= -\frac{a}{2} (\kappa(u))_{xx} (\partial_x Q \overline{Q} + Q \overline{\partial_x Q}) - \frac{3a}{2} (\kappa(u))_x |\partial_x Q|^2 + \frac{\lambda}{2} (\kappa(u))_x |Q|^2. \tag{3.4}
\end{aligned}$$

Substituting (3.1)-(3.4) into (1.10), we obtain

$$\begin{aligned}
&\sqrt{-1} \partial_t Q + (a \partial_x^4 + \lambda \partial_x^2) Q \\
&= \frac{d_1}{2} \kappa(u) \partial_x^2 Q |Q|^2 + \frac{d_2}{2} \kappa(u) \overline{\partial_x^2 Q} Q^2 + \frac{d_3}{2} \kappa(u) |\partial_x Q|^2 Q + \frac{d_4}{2} \kappa(u) (\partial_x Q)^2 \overline{Q} \\
&\quad + \frac{d_5}{2} (\kappa(u))_x \partial_x Q |Q|^2 + \frac{d_6}{2} (\kappa(u))_x Q^2 \overline{\partial_x Q} - \frac{\lambda}{2} \kappa(u) |Q|^2 Q - \frac{c}{4} \kappa^2(u) |Q|^4 Q \\
&\quad + \frac{1}{2} \left(\int_{-\infty}^x \mathcal{W}_1(Q, \partial_x Q)(t, y) dy \right) Q, \tag{3.5}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{W}_1(Q, \partial_x Q) &= -a (\kappa(u))_{xx} (\partial_x Q \overline{Q} + Q \overline{\partial_x Q}) - 3a (\kappa(u))_x |\partial_x Q|^2 \\
&\quad + c \kappa(u) (\kappa(u))_x |Q|^4 + \lambda (\kappa(u))_x |Q|^2.
\end{aligned}$$

Remark 3.1. If the Gaussian curvature of (N, J, h) is constant, then the nonlocal term in (3.5) vanishes, and it is easy to check that (3.5) under the setting (1.6) actually coincides with (1.9) which is transformed from (1.5) in [12]. This is natural because our orthonormal frame for $n = 1$ is essentially the same as that used in [12] and because the constancy of the sectional curvature on (N, J, h) ensures $\nabla R = 0$. In contrast, without the constancy of the curvature, (3.5) under the setting (1.6) includes the nonlocal term and does not coincide with (1.9), even though we rewrite the nonlocal term by using the fundamental theorem of calculus. This is not strange, because the definitions of (1.1) and (1.5) for curve flows are originally not the same unless $\nabla R = 0$.

3.2. Example (2). Let (N, J, h) be a compact Kähler manifold of complex dimension n with constant holomorphic sectional curvature K . It is known that

$$\begin{aligned}
R(U, V)W &= \frac{K}{4} \left\{ h(V, W)U - h(U, W)V + h(U, J_u W)J_u V \right. \\
&\quad \left. - h(V, J_u W)J_u U + 2h(U, J_u V)J_u W \right\} \tag{3.6}
\end{aligned}$$

for any $U, V, W \in \Gamma(u^{-1}TN)$, and $\nabla R = 0$ holds. In particular,

$$\begin{aligned}
R(e_p, e_q)e_r &= \frac{K}{4} (\delta_{qr} e_p - \delta_{pr} e_q), \\
R(e_p, J_u e_q)e_r &= -\frac{K}{4} (\delta_{pr} J_u e_q + \delta_{qr} J_u e_p + 2\delta_{pq} J_u e_r)
\end{aligned}$$

hold for all $p, q, r \in \{1, \dots, n\}$. Applying them to (2.4)-(2.6), we see

$$\begin{aligned} (\operatorname{Re}[R_{p,q,r}^{A,j}] =)h(R(e_p, e_q)r_r, e_j) &= \frac{K}{4}(\delta_{qr}\delta_{pj} - \delta_{pr}\delta_{qj}), \\ (\operatorname{Im}[R_{p,q,r}^{A,j}] =)h(R(e_p, e_q)r_r, J_u e_j) &= 0, \\ (\operatorname{Re}[R_{p,q,r}^{B,j}] =)h(R(e_p, J_u e_q)r_r, e_j) &= 0, \\ (\operatorname{Im}[R_{p,q,r}^{B,j}] =)h(R(e_p, J_u e_q)r_r, J_u e_j) &= -\frac{K}{4}(\delta_{pr}\delta_{qj} + \delta_{qr}\delta_{pj} + 2\delta_{pq}\delta_{rj}) \end{aligned}$$

for all $p, q, r, j \in \{1, \dots, n\}$. Substituting them into (2.8), we obtain

$$\begin{aligned} S_{p,q,r}^j &= \frac{K}{8}(\delta_{qr}\delta_{pj} - \delta_{pr}\delta_{qj}) + \frac{K}{8}(\delta_{pr}\delta_{qj} + \delta_{qr}\delta_{pj} + 2\delta_{pq}\delta_{rj}) \\ &= \frac{K}{4}(\delta_{qr}\delta_{pj} + \delta_{pq}\delta_{rj}) \quad (\in \mathbb{R}) \quad (\forall p, q, r, j \in \{1, \dots, n\}). \end{aligned} \quad (3.7)$$

From this, we also see $\partial_x(S_{p,q,r}^j) \equiv 0$. This does not conflict with Proposition 2.3.

We use (3.7) to compute the right hand side of (1.10) with (1.11) and (1.12). It follows that

$$\begin{aligned} \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x^2 Q_p \overline{Q_q} Q_r &= \frac{K}{4} \sum_{p,q,r=1}^n (\delta_{qr}\delta_{pj} + \delta_{pq}\delta_{rj}) \partial_x^2 Q_p \overline{Q_q} Q_r \\ &= \frac{K}{4} \sum_{q=1}^n \sum_{r=1}^n \delta_{qr} \partial_x^2 Q_j \overline{Q_q} Q_r + \frac{K}{4} \sum_{p=1}^n \sum_{q=1}^n \delta_{pq} \partial_x^2 Q_p \overline{Q_q} Q_j \\ &= \frac{K}{4} |Q|^2 \partial_x^2 Q_j + \frac{K}{4} \sum_{p=1}^n \partial_x^2 Q_p \overline{Q_p} Q_j, \\ \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{\partial_x Q_q} Q_r &= \frac{K}{4} \sum_{p,q,r=1}^n (\delta_{qr}\delta_{pj} + \delta_{pq}\delta_{rj}) \partial_x Q_p \overline{\partial_x Q_q} Q_r \\ &= \frac{K}{4} \sum_{q=1}^n \overline{\partial_x Q_q} Q_q \partial_x Q_j + \frac{K}{4} |\partial_x Q|^2 Q_j, \\ \sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x Q_p \overline{Q_q} \partial_x Q_r &= \frac{K}{4} \sum_{p,q,r=1}^n (\delta_{qr}\delta_{pj} + \delta_{pq}\delta_{rj}) \partial_x Q_p \overline{Q_q} \partial_x Q_r \\ &= \frac{K}{2} \sum_{q=1}^n \partial_x Q_q \overline{Q_q} \partial_x Q_j, \\ \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x^2 Q_q} Q_r &= \frac{K}{2} \sum_{q=1}^n \overline{\partial_x^2 Q_q} Q_q Q_j, \\ \sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{Q_q} Q_r &= \frac{K}{2} |Q|^2 Q_j. \end{aligned}$$

Since $\partial_x(S_{p,q,r}^j) = 0$ for all $p, q, r, j \in \{1, \dots, n\}$, it is immediate to see

$$\sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^2(Q, \partial_x Q)(t, y) dy \right) Q_r = 0.$$

On the other hand, by a lengthy computation, we can show

$$\begin{aligned} & \sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^1(Q, \partial_x Q)(t, y) dy \right) Q_r \\ &= -\frac{(b+4c)K^2}{16} |Q|^4 Q_j + \frac{bK^2}{8} \sum_{r=1}^n \left(\int_{-\infty}^x (Q_j \overline{Q_r} \partial_x (|Q|^2))(t, y) dy \right) Q_r. \end{aligned} \quad (3.8)$$

We demonstrate the computation here. Recall that $f_{j,r}^1(Q, \partial_x Q) = -(b+2c)(S_1 + S_2) + b(S_3 + S_4)$, where S_1, \dots, S_4 are given by (2.55)-(2.58). Obtaining the exact expressions of $S_{p,q,r}^j S_{\alpha,\beta,\gamma}^q$ and $S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p$ is sufficient to compute S_1, \dots, S_4 , since $S_{p,q,r}^j = S_{p,q,r}^j$ holds for any p, q, r, j in this example. (We need to compute them separately, since $S_{p,q,r}^j \neq S_{q,p,r}^j$.) The result of computation is as follows:

$$\begin{aligned} S_{p,q,r}^j S_{\alpha,\beta,\gamma}^q &= \frac{K^2}{16} (\delta_{qr} \delta_{pj} + \delta_{pq} \delta_{rj}) (\delta_{\beta\gamma} \delta_{\alpha q} + \delta_{\alpha\beta} \delta_{\gamma q}) \\ &= \frac{K^2}{16} (\delta_{qr} \delta_{pj} \delta_{\beta\gamma} \delta_{\alpha q} + \delta_{qr} \delta_{pj} \delta_{\alpha\beta} \delta_{\gamma q} + \delta_{pq} \delta_{rj} \delta_{\beta\gamma} \delta_{\alpha q} + \delta_{pq} \delta_{rj} \delta_{\alpha\beta} \delta_{\gamma q}), \\ S_{p,q,r}^j S_{\alpha,\beta,\gamma}^p &= \frac{K^2}{16} (\delta_{qr} \delta_{pj} + \delta_{pq} \delta_{rj}) (\delta_{\beta\gamma} \delta_{\alpha p} + \delta_{\alpha\beta} \delta_{\gamma p}) \\ &= \frac{K^2}{16} (\delta_{qr} \delta_{pj} \delta_{\beta\gamma} \delta_{\alpha p} + \delta_{qr} \delta_{pj} \delta_{\alpha\beta} \delta_{\gamma p} + \delta_{pq} \delta_{rj} \delta_{\beta\gamma} \delta_{\alpha p} + \delta_{pq} \delta_{rj} \delta_{\alpha\beta} \delta_{\gamma p}). \end{aligned}$$

Hence,

$$\begin{aligned} S_1 &= \frac{K^2}{16} \left(\sum_{\beta=1}^n \overline{\partial_x Q_r} Q_\beta \overline{Q_\beta} Q_j + \sum_{\beta=1}^n \overline{\partial_x Q_\beta} Q_\beta \overline{Q_r} Q_j \right. \\ &\quad \left. + \delta_{jr} \sum_{\alpha=1}^n \sum_{\beta=1}^n \overline{\partial_x Q_\alpha} Q_\beta \overline{Q_\beta} Q_\alpha + \delta_{jr} \sum_{\beta=1}^n \sum_{\gamma=1}^n \overline{\partial_x Q_\beta} Q_\beta \overline{Q_\gamma} Q_\gamma \right) \\ &= \frac{K^2}{16} \left(\overline{\partial_x Q_r} Q_j |Q|^2 + \overline{Q_r} Q_j \sum_{\beta=1}^n \overline{\partial_x Q_\beta} Q_\beta + 2\delta_{jr} \sum_{\alpha=1}^n \overline{\partial_x Q_\alpha} Q_\alpha |Q|^2 \right), \end{aligned} \quad (3.9)$$

$$\begin{aligned} S_3 &= \frac{K^2}{16} \left(\sum_{\beta=1}^n \overline{Q_r} \partial_x Q_\beta \overline{Q_\beta} Q_j + \sum_{\beta=1}^n \overline{Q_\beta} \partial_x Q_\beta \overline{Q_r} Q_j \right. \\ &\quad \left. + \delta_{jr} \sum_{\alpha=1}^n \sum_{\beta=1}^n \overline{Q_\alpha} \partial_x Q_\beta \overline{Q_\beta} Q_\alpha + \delta_{jr} \sum_{\beta=1}^n \sum_{\gamma=1}^n \overline{Q_\beta} \partial_x Q_\beta \overline{Q_\gamma} Q_\gamma \right) \\ &= \frac{K^2}{16} \left(2\overline{Q_r} Q_j \sum_{\beta=1}^n \partial_x Q_\beta \overline{Q_\beta} + 2\delta_{jr} \sum_{\alpha=1}^n \partial_x Q_\alpha \overline{Q_\alpha} |Q|^2 \right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} S_2 &= \frac{K^2}{16} \left(\sum_{\beta=1}^n \partial_x Q_j \overline{Q_\beta} Q_\beta \overline{Q_r} + \sum_{\beta=1}^n \partial_x Q_\beta \overline{Q_\beta} Q_j \overline{Q_r} \right. \\ &\quad \left. + \delta_{jr} \sum_{\alpha=1}^n \sum_{\beta=1}^n \partial_x Q_\alpha \overline{Q_\beta} Q_\beta \overline{Q_\alpha} + \delta_{jr} \sum_{\beta=1}^n \sum_{\gamma=1}^n \partial_x Q_\beta \overline{Q_\beta} Q_\gamma \overline{Q_\gamma} \right) \end{aligned}$$

$$= \frac{K^2}{16} \left(\partial_x Q_j \overline{Q_r} |Q|^2 + Q_j \overline{Q_r} \sum_{\beta=1}^n \partial_x Q_\beta \overline{Q_\beta} + 2\delta_{jr} \sum_{\alpha=1}^n \partial_x Q_\alpha \overline{Q_\alpha} |Q|^2 \right), \quad (3.11)$$

$$\begin{aligned} S_4 &= \frac{K^2}{16} \left(\sum_{\beta=1}^n Q_j \overline{\partial_x Q_\beta} Q_\beta \overline{Q_r} + \sum_{\beta=1}^n Q_\beta \overline{\partial_x Q_\beta} Q_j \overline{Q_r} \right. \\ &\quad \left. + \delta_{jr} \sum_{\alpha=1}^n \sum_{\beta=1}^n Q_\alpha \overline{\partial_x Q_\beta} Q_\beta \overline{Q_\alpha} + \delta_{jr} \sum_{\beta=1}^n \sum_{\gamma=1}^n Q_\beta \overline{\partial_x Q_\beta} Q_\gamma \overline{Q_\gamma} \right) \\ &= \frac{K^2}{16} \left(2\overline{Q_r} Q_j \sum_{\beta=1}^n \overline{\partial_x Q_\beta} Q_\beta + 2\delta_{jr} \sum_{\alpha=1}^n \overline{\partial_x Q_\alpha} Q_\alpha |Q|^2 \right). \end{aligned} \quad (3.12)$$

Combining them, we have

$$\begin{aligned} S_1 + S_2 &= \frac{K^2}{16} \left\{ \partial_x (Q_j \overline{Q_r}) |Q|^2 + Q_j \overline{Q_r} \partial_x (|Q|^2) + 2\delta_{jr} |Q|^2 \partial_x (|Q|^2) \right\} \\ &= \frac{K^2}{16} \left\{ \partial_x (Q_j \overline{Q_r} |Q|^2) + \delta_{jr} \partial_x (|Q|^4) \right\} \\ &= \partial_x \left\{ \frac{K^2}{16} (Q_j \overline{Q_r} |Q|^2 + \delta_{jr} |Q|^4) \right\}, \\ S_3 + S_4 &= \frac{K^2}{16} \left(2Q_j \overline{Q_r} \partial_x (|Q|^2) + 2\delta_{jr} |Q|^2 \partial_x (|Q|^2) \right) \\ &= \frac{K^2}{16} \left(2Q_j \overline{Q_r} \partial_x (|Q|^2) + \delta_{jr} \partial_x (|Q|^4) \right) \\ &= \partial_x \left\{ \frac{K^2}{16} \delta_{jr} |Q|^4 \right\} + \frac{K^2}{8} Q_j \overline{Q_r} \partial_x (|Q|^2). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{r=1}^n \left(\int_{-\infty}^x (S_1 + S_2)(t, y) dy \right) Q_r &= \frac{K^2}{16} \sum_{r=1}^n (Q_j \overline{Q_r} |Q|^2 + \delta_{jr} |Q|^4) Q_r \\ &= \frac{K^2}{8} |Q|^4 Q_j, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \sum_{r=1}^n \left(\int_{-\infty}^x (S_3 + S_4)(t, y) dy \right) Q_r &= \frac{K^2}{16} \sum_{r=1}^n \delta_{jr} |Q|^4 Q_r + \frac{K^2}{8} \sum_{r=1}^n \left(\int_{-\infty}^x (Q_j \overline{Q_r} \partial_x (|Q|^2)) (t, y) dy \right) Q_r \\ &= \frac{K^2}{16} |Q|^4 Q_j + \frac{K^2}{8} \sum_{r=1}^n \left(\int_{-\infty}^x (Q_j \overline{Q_r} \partial_x (|Q|^2)) (t, y) dy \right) Q_r. \end{aligned} \quad (3.14)$$

Combining (3.13) and (3.14), we get the desired (3.8).

Finally, substituting the result of computation into (1.10), we arrived at

$$\sqrt{-1} \partial_t Q_j + (a \partial_x^4 + \lambda \partial_x^2) Q_j$$

$$\begin{aligned}
&= \frac{K}{4}d_1 \left(|Q|^2 \partial_x^2 Q_j + \sum_{r=1}^n \partial_x^2 Q_r \overline{Q_r} Q_j \right) + \frac{K}{2}d_2 \sum_{r=1}^n \overline{\partial_x^2 Q_r} Q_r Q_j \\
&+ \frac{K}{4}d_3 \left(\sum_{r=1}^n \overline{\partial_x Q_r} Q_r \partial_x Q_j + |\partial_x Q|^2 Q_j \right) + \frac{K}{2}d_4 \sum_{r=1}^n \partial_x Q_r \overline{Q_r} \partial_x Q_j - \frac{K}{2}\lambda |Q|^2 Q_j \\
&- \frac{(b+4c)K^2}{16} |Q|^4 Q_j + \frac{bK^2}{8} \sum_{r=1}^n \left(\int_{-\infty}^x (Q_j \overline{Q_r} \partial_x (|Q|^2))(t, y) dy \right) Q_r. \tag{3.15}
\end{aligned}$$

Remark 3.2. If $n = 1$, then the final nonlocal term of the right hand of (3.15) simply becomes a local one, in that

$$\begin{aligned}
\left(\int_{-\infty}^x (Q_1 \overline{Q_1} \partial_x (|Q_1|^2))(t, y) dy \right) Q_1 &= \frac{1}{2} \left(\int_{-\infty}^x (\partial_x (|Q_1|^4))(t, y) dy \right) Q_1 \\
&= \frac{1}{2} |Q_1|^4 Q_1.
\end{aligned}$$

In this case, the derived equation (3.15) for $Q = Q_1$ turns out to coincide with (3.5) where $\kappa(u) \equiv K$. This does not conflict with the fact that the holomorphic sectional curvature for the Riemann surface coincides with the Gaussian curvature.

4. EXAMPLE (3)

We investigate the case (N, J, h) is a compact complex Grassmannian as a Hermitian symmetric space. Looking at many famous literature (e.g., [3], [19], [21], [27], [34]), there are some models of complex Grassmannians. To avoid the confusion, following [4, 10] mainly, we start from stating the setting we use in this paper.

4.1. Setting of complex Grassmannians. Fix integers n_0, k_0 with $1 \leq k_0 < n_0$, and set $m_0 = n_0 - k_0$. Let N be the complex Grassmannian G_{n_0, k_0} defined as the set of all k_0 -dimensional linear subspaces through the origin of the complex Euclidean space \mathbb{C}^{n_0} . With a slight abuse of notation, this can be identified with the set of Hermitian rank- k_0 projectors:

$$G_{n_0, k_0} = \{A \in H(n_0) \mid A^2 = A \text{ and } \text{rank } A = k_0\}, \tag{4.1}$$

where $H(n_0) = \{A \in \mathcal{M}_{n_0 \times n_0} \mid A^* = A\}$ being the set of Hermitian matrices. ($\mathcal{M}_{n_0 \times n_0}$ denotes the space of $n_0 \times n_0$ complex-matrices and $A^* = \overline{A}^t$ denotes the conjugate transpose of A .)

Set $U(n_0) = \{B \in \mathcal{M}_{n_0 \times n_0} \mid B^* B = B B^* = I\}$ to denote the unitary group of degree n_0 . (In this section, the identity matrix of size n_0 is denoted by I and the identity matrices of size $k \in \{1, \dots, n_0 - 1\}$ are by I_k .) Then $U(n_0)$ is a compact Lie group and the Lie algebra consists of the set of skew-Hermitian matrices:

$$\mathfrak{u}(n_0) := T_I U(n_0) = \{\Omega \in \mathcal{M}_{n_0 \times n_0} \mid \Omega^* = -\Omega\}.$$

Let us define an isometric group action of $U(n_0)$ on $H(n_0)$ by

$$\Phi : U(n_0) \times H(n_0) \rightarrow H(n_0), \quad (B, H) \mapsto B H B^*,$$

and take

$$A_0 := \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} \in G_{n_0, k_0}$$

as the origin of G_{n_0, k_0} . (Here and hereafter, all matrices in $\mathcal{M}_{n_0 \times n_0}$ are written as a block form where the submatrix in the upper left corner is of order $k_0 \times k_0$, and the four zero-matrices as the submatrix are simply denoted by 0.) The same argument as that in [4, Section 2.1] shows $G_{n_0, k_0} = \Phi(U(n_0), A_0)$ being the orbit of A_0 under Φ . Moreover, Φ is a transitive action of $U(n_0)$ on G_{n_0, k_0} and the isotropy group at A_0 is

$$\left\{ \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \middle| K_1 \in U(k_0), K_2 \in U(m_0) \right\} \cong U(k_0) \times U(m_0).$$

In fact, G_{n_0, k_0} is diffeomorphic to $U(n_0)/(U(k_0) \times U(m_0))$ via the canonical map $G_{n_0, k_0} \ni \Phi(H, A_0) \mapsto H(U(k_0) \times U(m_0)) \in U(n_0)/(U(k_0) \times U(m_0))$ and is an embedded submanifold of $H(n_0)$ satisfying

$$\dim_{\mathbb{R}} G_{n_0, k_0} = \dim_{\mathbb{R}} U(n_0) - \dim_{\mathbb{R}} (U(k_0) \times U(m_0)) = 2k_0m_0$$

which implies $n = \dim_{\mathbb{C}} G_{n_0, k_0} = k_0m_0$. In addition, the involution which is given by

$$\sigma : U(n_0) \rightarrow U(n_0), \quad B \mapsto \begin{pmatrix} I_{k_0} & 0 \\ 0 & -I_{m_0} \end{pmatrix} B \begin{pmatrix} I_{k_0} & 0 \\ 0 & -I_{m_0} \end{pmatrix}^{-1}$$

makes $U(n_0)/(U(k_0) \times U(m_0))$ symmetric.

Next, set $\pi = \Phi(\cdot, A_0)$, that is,

$$\pi : U(n_0) \rightarrow G_{n_0, k_0}, \quad B \mapsto BA_0B^*.$$

For any $A \in G_{n_0, k_0}$, there exists $B \in U(n_0)$ such that $A = \pi(B)$ and the tangent space of G_{n_0, k_0} at $A \in G_{n_0, k_0}$ can be expressed by

$$T_A G_{n_0, k_0} = \left\{ B \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} B^* \middle| V \in \mathcal{M}_{k_0 \times m_0} \right\}. \quad (4.2)$$

This follows from the same argument as that in [4, Section 2.1]. To see this more concretely, note that the tangent space of $U(n_0)$ at an arbitrary $B \in U(n_0)$ is given by the left translation of $\mathfrak{u}(n_0)$,

$$T_B U(n_0) = \{B\Omega \in \mathcal{M}_{n_0 \times n_0} \mid \Omega \in \mathfrak{u}(n_0)\}. \quad (4.3)$$

It turns out that the differential $(d\pi)_B : T_B U(n_0) \rightarrow T_{\pi(B)} G_{n_0, k_0}$ at $B \in U(n_0)$ is given by

$$(d\pi)_B \left(B \begin{pmatrix} \omega_{11} & -\omega_{12} \\ (\omega_{12})^* & \omega_{22} \end{pmatrix} \right) = B \begin{pmatrix} 0 & \omega_{12} \\ (\omega_{12})^* & 0 \end{pmatrix} B^* \quad (4.4)$$

for all $\begin{pmatrix} \omega_{11} & -\omega_{12} \\ (\omega_{12})^* & \omega_{22} \end{pmatrix} \in \mathfrak{u}(n_0)$. Since π is submersion, (4.2) is obtained.

The complex structure J_A at the point $A = \pi(B) \in G_{n_0, k_0}$ is given by

$$J_A : T_A G_{n_0, k_0} \rightarrow T_A G_{n_0, k_0}, \quad B \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} B^* \mapsto B \begin{pmatrix} 0 & \sqrt{-1}V \\ (\sqrt{-1}V)^* & 0 \end{pmatrix} B^*. \quad (4.5)$$

The Riemannian metric on G_{n_0, k_0} is taken to be $U(n_0)$ -invariant by the following standard manner: We take an Ad-invariant metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{u}(n_0)$ which is defined by

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2} \text{tr} (\Omega_1 (\Omega_2)^*), \quad (4.6)$$

and define $\langle \cdot, \cdot \rangle_B$ for each $B \in U(n_0)$ by

$$\langle B\Omega_1, B\Omega_2 \rangle_B = \langle \Omega_1, \Omega_2 \rangle \quad (4.7)$$

for any $B\Omega_1, B\Omega_2 \in T_B U(n_0)$, which gives a bi-invariant Riemannian metric on $U(n_0)$. Let $A = \pi(B) \in G_{n_0, k_0}$ and let $\Delta_i \in T_A G_{n_0, k_0}$ ($i = 1, 2$). By (4.2) and (4.4), there exist $\Omega_i \in \mathfrak{u}(n_0)$ ($i = 1, 2$) such that

$$\Delta_i = (d\pi)_B(B\Omega_i), \quad \Omega_i = \begin{pmatrix} 0 & -\omega_i \\ (\omega_i)^* & 0 \end{pmatrix}, \quad \omega_i \in \mathcal{M}_{k_0 \times m_0} \quad (i = 1, 2).$$

We define $h_A(\Delta_1, \Delta_2)$ by

$$h_A(\Delta_1, \Delta_2) = \langle B\Omega_1, B\Omega_2 \rangle_B \left(= \frac{1}{2} \operatorname{tr}(\Omega_1(\Omega_2)^*) \right).$$

Then, $h = \{h_A\}$ is a $U(n_0)$ -invariant Riemannian metric on G_{n_0, k_0} . Furthermore, by the fundamental properties of the trace for complex-component matrices,

$$h_A(\Delta_1, \Delta_2) = \frac{1}{2} \operatorname{tr} \left(\begin{pmatrix} 0 & -\omega_1 \\ (\omega_1)^* & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_2 \\ (\omega_2)^* & 0 \end{pmatrix}^* \right) = \operatorname{Re} [\operatorname{tr}(\omega_1(\omega_2)^*)]. \quad (4.8)$$

Remark 4.1. The metric h is the same as that used in, e.g., [21, 27, 34]. It is also the same as that in [13, 36, 11] up to a constant multiple. To investigate the expression of (1.1), we do not need to be aware of the difference of the constant multiple, because the Levi-Civita connection and R as a $(1, 3)$ -tensor used to formulate (1.1) are invariant under the homothetic change $h \rightarrow ch$ (c is a positive constant). In other words, although the holomorphic sectional curvature is multiplied by $1/c$, the derived final expression (1.10) is not changed.

Remark 4.2. For each $B \in U(n_0)$, $T_B U(n_0)$ can be decomposed into the kernel of the differential $(d\pi)_B$ and the orthogonal complement with respect to $\langle \cdot, \cdot \rangle_B$:

$$T_B U(n_0) = \mathfrak{k}_B + \mathfrak{m}_B, \quad (4.9)$$

where

$$\mathfrak{k}_B := \operatorname{Ker}((d\pi)_B) = \left\{ B \begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix} \middle| \omega_{11} \in \mathfrak{u}(k_0), \omega_{22} \in \mathfrak{u}(m_0) \right\}, \quad (4.10)$$

$$\mathfrak{m}_B := (\mathfrak{k}_B)^\perp = \left\{ B \begin{pmatrix} 0 & \omega_{12} \\ -(\omega_{12})^* & 0 \end{pmatrix} \middle| \omega_{12} \in \mathcal{M}_{k_0 \times m_0} \right\}. \quad (4.11)$$

Comparing (4.2) and (4.11), we see that the tangent space $T_A G_{n_0, k_0}$ at $A = \pi(B)$ can be identified with \mathfrak{m}_B by the map $((d\pi)_B)|_{\mathfrak{m}_B}: \mathfrak{m}_B \rightarrow T_A G_{n_0, k_0}$. In addition, a direct computation using (4.10) and (4.11) easily shows that $\mathfrak{u}(n_0)$ is a symmetric Lie algebra, that is,

$$[\mathfrak{k}_I, \mathfrak{k}_I] \subset \mathfrak{k}_I, \quad [\mathfrak{k}_I, \mathfrak{m}_I] \subset \mathfrak{m}_I, \quad [\mathfrak{m}_I, \mathfrak{m}_I] \subset \mathfrak{k}_I.$$

This does not conflict with the fact that $U(n_0)/(U(k_0) \times U(m_0))$ is a symmetric space with involution σ (see, e.g., [3, Proposition 6.4]).

Remark 4.3. Some other equivalent expressions of the tangent space are known. For example, as is used in [10], the following implicit expression also holds:

$$T_A G_{n_0, k_0} = \{H \in H(n_0) | HA + AH = H\}. \quad (4.12)$$

Indeed, by definition of $H(n_0)$ and (4.1), the right hand side of (4.12) turns out to coincide with that of (4.2).

The Riemann curvature tensor R at $A \in G_{n_0, k_0}$ is given by the following:

$$(R(X, Y)Z)(A) = [[X, Y], Z] \quad (X, Y, Z \in T_A G_{n_0, k_0}), \quad (4.13)$$

where $[\cdot, \cdot]$ is the bracket of the matrices defined by $[X_1, X_2] = X_1 X_2 - X_2 X_1$. The above expression is derived in [10] by using (4.12). (As is commented in [10], the expression of R differs by a sign from the familiar one (e.g., [3, 14, 19, 20, 21, 34]), since our tangent vectors are Hermitian rather than skew-Hermitian. See Remark 4.4 also.) Let $A = \pi(B) \in G_{n_0, k_0}$, and let $X, Y, Z, W \in T_A G_{n_0, k_0}$ be expressed by

$$X = B \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} B^*, Y = B \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} B^*, Z = B \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} B^*, W = B \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} B^*, \quad (4.14)$$

where $x, y, z, w \in \mathcal{M}_{k_0 \times m_0}$. (The notation x is not a variable of functions only here.) Then, the substitution of them into (4.13) shows

$$(R(X, Y)Z)(A) = B \left[\left[\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} \right], \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \right] B^* = B S B^*, \quad (4.15)$$

where $S \in T_{A_0} G_{n_0, k_0}$ is determined by

$$S = \begin{pmatrix} 0 & s \\ s^* & 0 \end{pmatrix}, \quad s = xy^*z - yx^*z - zx^*y + zy^*x \in \mathcal{M}_{k_0 \times m_0}. \quad (4.16)$$

This combined with (4.8) gives

$$\begin{aligned} h_A(R(X, Y)Z, W) &= \operatorname{Re}(\operatorname{tr}(sw^*)) \\ &= \operatorname{Re}(\operatorname{tr}(xy^*zw^* - yx^*zw^* - zx^*yw^* + zy^*xw^*)). \end{aligned} \quad (4.17)$$

Remark 4.4. The Riemann curvature tensor at A_0 can be rewritten via the identification

$$((d\pi)_I)|_{\mathfrak{m}_I}: \mathfrak{m}_I \rightarrow T_{A_0} G_{n_0, k_0}, \quad \begin{pmatrix} 0 & -\omega \\ \omega^* & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \omega \\ \omega^* & 0 \end{pmatrix}.$$

To see this, let $\iota: T_{A_0} G_{n_0, k_0} \rightarrow \mathfrak{m}_I$ be the inverse of $((d\pi)_I)|_{\mathfrak{m}_I}$. Then, for any

$$X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \in T_{A_0} G_{n_0, k_0},$$

the following relation holds:

$$(R(X, Y)Z)(A_0) = -((d\pi)_I)|_{\mathfrak{m}_I} [[\iota(X), \iota(Y)], \iota(Z)]. \quad (4.18)$$

This does not conflict with [3, 14, 19, 20, 21, 34] where R is expressed by “ $R(X, Y)Z = -[[X, Y], Z]$ ” in the context of the right hand side of (4.18) via the above identification.

4.2. Computation of (1.10). Let $N = G_{n_0, k_0}$ as above. We compute (1.10) in Theorem 1.1. Recall that $n = k_0 m_0$ is the complex dimension of G_{n_0, k_0} . For any $j \in \{1, \dots, n\}$, there exists a unique pair of integers $j_1 \in \{1, \dots, k_0\}$ and $j_2 \in \{1, \dots, m_0\}$ such that $j = (j_2 - 1)k_0 + j_1$. In what follows, we denote it by $j = (j_1, j_2)$.

Since $u \in C_{u^\infty}((-T, T) \times \mathbb{R}; N)$ in Theorem 1.1, there exist $B^\infty \in U(n_0)$ and $B = B(t, x) : (-T, T) \times \mathbb{R} \rightarrow U(n_0)$ such that $u^\infty = B^\infty A_0 (B^\infty)^*$ and $u(t, x) = B(t, x) A_0 (B(t, x))^*$. We take $e_j^\infty \in T_{u^\infty} N$ for each $j = (j_1, j_2) \in \{1, \dots, n\}$ to satisfy

$$e_j^\infty = B_\infty \begin{pmatrix} 0 & E_j \\ (E_j)^* & 0 \end{pmatrix} (B_\infty)^*, \quad (4.19)$$

where $E_j = E_{(j_1, j_2)} \in \mathcal{M}_{k_0 \times m_0}$ denotes a constant matrix with entry 1 where the j_1 -th row and the j_2 -th column meet, and all other entries being 0. It is easy to see $E_{(j_1, j_2)}(E_{(\ell_1, \ell_2)})^* = \delta_{j_2 \ell_2} E_{j_1, \ell_1}^{(k_0)}$ where $E_{j_1, \ell_1}^{(k_0)} \in \mathcal{M}_{k_0 \times k_0}$ denotes a square matrix with entry 1 where the j_1 -th row and the ℓ_1 -th column meet, and all other entries being 0. This combined with (4.8) shows

$$h_{u^\infty}(e_j^\infty, e_\ell^\infty) = \operatorname{Re}[\operatorname{tr}(E_j(E_\ell)^*)] = \delta_{j_2 \ell_2} \delta_{j_1 \ell_1} = \delta_{j\ell} \quad (\forall j, \ell \in \{1, \dots, n\}).$$

Therefore, $\{e_j^\infty, J e_j^\infty\}_{j=1}^n$ is actually an orthonormal basis for $T_{u^\infty} N$. Let $\{e_j, J e_j\}_{j=1}^n$ be the associated orthonormal frame for $u^{-1} T N$ that satisfies (2.1)-(2.2). The parallelity of R and J with respect to ∇ shows that $e_j, J u e_j, R(e_p, e_q) e_r$, and $R(e_p, J u e_q) e_r$ are respectively the parallel displacement of $e_j^\infty, J u e_j^\infty, R(e_p^\infty, e_q^\infty) e_r^\infty$, and $R(e_p^\infty, J u e_q^\infty) e_r^\infty$. In addition, h is invariant under the parallel displacement. Therefore, the expression (2.8) reduces to

$$\begin{aligned} S_{p,q,r}^j &= \frac{1}{2} \{h(R(e_p^\infty, e_q^\infty) e_r^\infty, e_j^\infty) + \sqrt{-1} h(R(e_p^\infty, e_q^\infty) e_r^\infty, J u e_j^\infty)\} \\ &\quad + \frac{\sqrt{-1}}{2} \{h(R(e_p^\infty, J u e_q^\infty) e_r^\infty, e_j^\infty) + \sqrt{-1} h(R(e_p^\infty, J u e_q^\infty) e_r^\infty, J u e_j^\infty)\}. \end{aligned}$$

Here, we apply (4.17) for $A = u^\infty$ to deduce

$$h(R(e_p^\infty, e_q^\infty) e_r^\infty, e_j^\infty) = \operatorname{Re}(\operatorname{tr}(\Xi_1)),$$

where

$$\Xi_1 = E_p(E_q)^* E_r(E_j)^* - E_q(E_p)^* E_r(E_j)^* - E_r(E_p)^* E_q(E_j)^* + E_r(E_q)^* E_p(E_j)^*.$$

Moreover, noting

$$J u e_j^\infty = B_\infty \begin{pmatrix} 0 & \sqrt{-1} E_j \\ (\sqrt{-1} E_j)^* & 0 \end{pmatrix} (B_\infty)^*,$$

we repeat the above computation replacing E_j with $\sqrt{-1} E_j$, which provides

$$h(R(e_p^\infty, e_q^\infty) e_r^\infty, J u e_j^\infty) = \operatorname{Re}(-\sqrt{-1} \operatorname{tr}(\Xi_1)) = \operatorname{Im}(\operatorname{tr}(\Xi_1)).$$

In the same way as above, we deduce

$$h(R(e_p^\infty, J u e_q^\infty) e_r^\infty, e_j^\infty) = \operatorname{Im}(\operatorname{tr}(\Xi_2)),$$

where

$$\Xi_2 = E_p(E_q)^* E_r(E_j)^* + E_q(E_p)^* E_r(E_j)^* + E_r(E_p)^* E_q(E_j)^* + E_r(E_q)^* E_p(E_j)^*,$$

and

$$h(R(e_p^\infty, J u e_q^\infty) e_r^\infty, J u e_j^\infty) = \operatorname{Im}(-\sqrt{-1} \operatorname{tr}(\Xi_2)) = -\operatorname{Re}(\operatorname{tr}(\Xi_2)).$$

Combining them, we obtain

$$S_{p,q,r}^j = \frac{1}{2} (\operatorname{tr}(\Xi_1) + \operatorname{tr}(\Xi_2)) = \operatorname{tr}(E_p(E_q)^* E_r(E_j)^* + E_r(E_q)^* E_p(E_j)^*). \quad (4.20)$$

Furthermore, set $p = (p_1, p_2)$, $q = (q_1, q_2)$, and $r = (r_1, r_2)$ where $p_1, q_1, r_1 \in \{1, \dots, k_0\}$ and $p_2, q_2, r_2 \in \{1, \dots, m_0\}$. A simple computation yields

$$E_p(E_q)^* E_r(E_j)^* = \delta_{p_2 q_2} E_{p_1, q_1}^{(k_0)} \delta_{r_2 j_2} E_{r_1, j_1}^{(k_0)} = \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} E_{p_1, j_1}^{(k_0)}, \quad (4.21)$$

and thus

$$\operatorname{tr}(E_p(E_q)^* E_r(E_j)^*) = \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1}.$$

Hence, for any $p = (p_1, p_2), q = (q_1, q_2), r = (r_1, r_2), j = (j_1, j_2) \in \{1, \dots, n\}$,

$$S_{p,q,r}^j = \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} + \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1}. \quad (4.22)$$

Based on (4.22), we proceed the computation of (1.10). First, we compute the form

$$\sum_{p,q,r=1}^n S_{p,q,r}^j X_p \overline{Y_q} Z_r = \sum_{p_1, q_1, r_1=1}^{k_0} \sum_{p_2, q_2, r_2=1}^{m_0} S_{(p_1, p_2), (q_1, q_2), (r_1, r_2)}^{(j_1, j_2)} X_{(p_1, p_2)} \overline{Y_{(q_1, q_2)}} Z_{(r_1, r_2)},$$

where X_p, Y_q, Z_r for $p, q, r \in \{1, \dots, n\}$ denote complex-valued functions of (t, x) . The right hand side of above is divided by the sum of L_1 and L_2 :

$$\begin{aligned} L_1 &= \sum_{p_1, q_1, r_1=1}^{k_0} \sum_{p_2, q_2, r_2=1}^{m_0} \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} X_{(p_1, p_2)} \overline{Y_{(q_1, q_2)}} Z_{(r_1, r_2)}, \\ L_2 &= \sum_{p_1, q_1, r_1=1}^{k_0} \sum_{p_2, q_2, r_2=1}^{m_0} \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} X_{(p_1, p_2)} \overline{Y_{(q_1, q_2)}} Z_{(r_1, r_2)}. \end{aligned} \quad (4.23)$$

By a simple computation,

$$\begin{aligned} L_1 &= \sum_{q_1, r_1=1}^{k_0} \sum_{p_2, q_2=1}^{m_0} \delta_{p_2 q_2} \delta_{q_1 r_1} X_{(j_1, p_2)} \overline{Y_{(q_1, q_2)}} Z_{(r_1, j_2)} \\ &= \sum_{r_1=1}^{k_0} \sum_{p_2=1}^{m_0} X_{(j_1, p_2)} \overline{Y_{(r_1, p_2)}} Z_{(r_1, j_2)}, \\ L_2 &= \sum_{p_1, q_1=1}^{k_0} \sum_{q_2, r_2=1}^{m_0} \delta_{r_2 q_2} \delta_{q_1 p_1} X_{(p_1, j_2)} \overline{Y_{(q_1, q_2)}} Z_{(j_1, r_2)} \\ &= \sum_{p_1=1}^{k_0} \sum_{r_2=1}^{m_0} Z_{(j_1, r_2)} \overline{Y_{(p_1, r_2)}} X_{(p_1, j_2)}. \end{aligned}$$

Combining them, we obtain

$$\begin{aligned} &\sum_{p,q,r=1}^n S_{p,q,r}^j X_p \overline{Y_q} Z_r \\ &= \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} X_{(j_1, s_1)} \overline{Y_{(s_2, s_1)}} Z_{(s_2, j_2)} + \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Z_{(j_1, s_1)} \overline{Y_{(s_2, s_1)}} X_{(s_2, j_2)}. \end{aligned} \quad (4.24)$$

Applying (4.24), we have

$$\begin{aligned} &\sum_{p,q,r=1}^n S_{p,q,r}^j \partial_x^2 Q_p \overline{Q_q} Q_r \\ &= \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} \partial_x^2 Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, j_2)} + \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} \partial_x^2 Q_{(s_2, j_2)}, \\ &\sum_{p,q,r=1}^n S_{p,q,r}^j Q_p \overline{\partial_x^2 Q_q} Q_r \end{aligned} \quad (4.25)$$

$$\begin{aligned}
&= \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{\partial_x^2 Q_{(s_2, s_1)}} Q_{(s_2, j_2)} + \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{\partial_x^2 Q_{(s_2, s_1)}} Q_{(s_2, j_2)} \\
&= 2 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{\partial_x^2 Q_{(s_2, s_1)}} Q_{(s_2, j_2)}, \tag{4.26}
\end{aligned}$$

$$\begin{aligned}
&\sum_{p, q, r=1}^n S_{p, q, r}^j \partial_x Q_p \overline{\partial_x Q_q} Q_r \\
&= \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} \partial_x Q_{(j_1, s_1)} \overline{\partial_x Q_{(s_2, s_1)}} Q_{(s_2, j_2)} + \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{\partial_x Q_{(s_2, s_1)}} \partial_x Q_{(s_2, j_2)}, \tag{4.27}
\end{aligned}$$

$$\sum_{p, q, r=1}^n S_{p, q, r}^j \partial_x Q_p \overline{Q_q} \partial_x Q_r = 2 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} \partial_x Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} \partial_x Q_{(s_2, j_2)}, \tag{4.28}$$

$$\sum_{p, q, r=1}^n S_{p, q, r}^j Q_p \overline{Q_q} Q_r = 2 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, j_2)}. \tag{4.29}$$

Next, we compute

$$\sum_{r=1}^n \left(\int_{-\infty}^x f_{j, r}^1(Q, \partial_x Q)(t, y) dy \right) Q_r,$$

where $f_{j, r}^1(Q, \partial_x Q) = -(b + 2c)(S_1 + S_2) + b(S_3 + S_4)$ and S_1, \dots, S_4 are defined by (2.55)-(2.58). By (4.22), we see

$$\begin{aligned}
S_{p, q, r}^j \overline{S_{\alpha, \beta, \gamma}^q} &= S_{p, q, r}^j S_{\alpha, \beta, \gamma}^q = S_{(p_1, p_2), (q_1, q_2), (r_1, r_2)}^{(j_1, j_2)} S_{(\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2)}^{(q_1, q_2)} \\
&= \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 q_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 q_1} + \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 q_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 q_1} \\
&\quad + \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 q_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 q_1} + \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 q_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 q_1}.
\end{aligned}$$

Let $X = \partial_x Q$ and $Y = Z = W = Q$. It follows that

$$S_1 = \sum_{p, q, \alpha, \beta, \gamma=1}^n S_{p, q, r}^j \overline{S_{\alpha, \beta, \gamma}^q} \overline{X_\alpha} Y_\beta \overline{Z_\gamma} W_p =: L_3 + L_4 + L_5 + L_6, \tag{4.30}$$

$$S_2 = \sum_{p, q, \alpha, \beta, \gamma=1}^n S_{p, q, r}^j S_{\alpha, \beta, \gamma}^p X_\alpha \overline{Y_\beta} Z_\gamma \overline{W_q} =: L_7 + L_8 + L_9 + L_{10}, \tag{4.31}$$

where

$$\begin{aligned}
L_3 &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 q_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 q_1} \overline{X_\alpha} Y_\beta \overline{Z_\gamma} W_p, \\
L_4 &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 q_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 q_1} \overline{X_\alpha} Y_\beta \overline{Z_\gamma} W_p, \\
L_5 &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 q_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 q_1} \overline{X_\alpha} Y_\beta \overline{Z_\gamma} W_p,
\end{aligned}$$

$$L_6 = \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 q_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 q_1} \overline{X_\alpha Y_\beta Z_\gamma W_p},$$

and

$$\begin{aligned} L_7 &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 p_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 p_1} X_\alpha \overline{Y_\beta Z_\gamma W_q}, \\ L_8 &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{p_2 q_2} \delta_{r_2 j_2} \delta_{q_1 r_1} \delta_{p_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 p_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 p_1} X_\alpha \overline{Y_\beta Z_\gamma W_q}, \\ L_9 &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 p_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 p_1} X_\alpha \overline{Y_\beta Z_\gamma W_q}, \\ L_{10} &= \sum_{\#1}^{k_0} \sum_{\#2}^{m_0} \delta_{r_2 q_2} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{r_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 p_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 p_1} X_\alpha \overline{Y_\beta Z_\gamma W_q}, \end{aligned}$$

and we use the notation $\sum_{\#1}^{k_0} := \sum_{p_1, q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0}$ and $\sum_{\#2}^{m_0} := \sum_{p_2, q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0}$. Note that $r = (r_1, r_2)$

and $j = (j_1, j_2)$ are fixed here. By a simple but a bit careful computation, we deduce

$$\begin{aligned} L_3 &= \delta_{r_2 j_2} \sum_{q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\ &\quad \times \delta_{p_2 q_2} \delta_{q_1 r_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 q_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 q_1} \overline{X_{(\alpha_1, \alpha_2)} Y_{(\beta_1, \beta_2)} Z_{(\gamma_1, \gamma_2)} W_{(j_1, p_2)}} \\ &= \delta_{r_2 j_2} \sum_{q_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, q_2, \alpha_2, \beta_2=1}^{m_0} \delta_{p_2 q_2} \delta_{q_1 r_1} \delta_{\alpha_2 \beta_2} \delta_{\beta_1 \gamma_1} \overline{X_{(q_1, \alpha_2)} Y_{(\beta_1, \beta_2)} Z_{(\gamma_1, q_2)} W_{(j_1, p_2)}} \\ &= \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{X_{(r_1, \beta_2)} Y_{(\beta_1, \beta_2)} Z_{(\beta_1, q_2)} W_{(j_1, q_2)}}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} L_4 &= \delta_{r_2 j_2} \sum_{q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\ &\quad \times \delta_{p_2 q_2} \delta_{q_1 r_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 q_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 q_1} \overline{X_{(\alpha_1, \alpha_2)} Y_{(\beta_1, \beta_2)} Z_{(\gamma_1, \gamma_2)} W_{(j_1, p_2)}} \\ &= \delta_{r_2 j_2} \sum_{q_1, \alpha_1, \beta_1=1}^{k_0} \sum_{p_2, q_2, \beta_2, \gamma_2=1}^{m_0} \delta_{p_2 q_2} \delta_{q_1 r_1} \delta_{\gamma_2 \beta_2} \delta_{\beta_1 \alpha_1} \overline{X_{(\alpha_1, q_2)} Y_{(\beta_1, \beta_2)} Z_{(q_1, \gamma_2)} W_{(j_1, p_2)}} \\ &= \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{X_{(\beta_1, q_2)} Y_{(\beta_1, \beta_2)} Z_{(r_1, \beta_2)} W_{(j_1, q_2)}}, \end{aligned} \tag{4.33}$$

$$L_5 = \delta_{r_1 j_1} \sum_{p_1, q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0}$$

$$\begin{aligned}
& \times \delta_{r_2 q_2} \delta_{q_1 p_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 q_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 q_1} \overline{X_{(\alpha_1, \alpha_2)}} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, \gamma_2)}} W_{(p_1, j_2)} \\
& = \delta_{r_1 j_1} \sum_{p_1, q_1, \beta_1, \gamma_1=1}^{k_0} \sum_{q_2, \alpha_2, \beta_2=1}^{m_0} \delta_{r_2 q_2} \delta_{q_1 p_1} \delta_{\alpha_2 \beta_2} \delta_{\beta_1 \gamma_1} \overline{X_{(q_1, \alpha_2)}} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, q_2)}} W_{(p_1, j_2)} \\
& = \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{X_{(q_1, \beta_2)}} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\beta_1, r_2)}} W_{(q_1, j_2)}, \tag{4.34}
\end{aligned}$$

$$\begin{aligned}
L_6 & = \delta_{r_1 j_1} \sum_{p_1, q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\
& \quad \times \delta_{r_2 q_2} \delta_{q_1 p_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 q_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 q_1} \overline{X_{(\alpha_1, \alpha_2)}} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, \gamma_2)}} W_{(p_1, j_2)} \\
& = \delta_{r_1 j_1} \sum_{p_1, q_1, \alpha_1, \beta_1=1}^{k_0} \sum_{q_2, \beta_2, \gamma_2=1}^{m_0} \delta_{r_2 q_2} \delta_{q_1 p_1} \delta_{\gamma_2 \beta_2} \delta_{\beta_1 \alpha_1} \overline{X_{(\alpha_1, q_2)}} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(q_1, \gamma_2)}} W_{(p_1, j_2)} \\
& = \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{X_{(\beta_1, r_2)}} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(q_1, \beta_2)}} W_{(q_1, j_2)}. \tag{4.35}
\end{aligned}$$

$$\begin{aligned}
L_7 & = \delta_{r_2 j_2} \sum_{p_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\
& \quad \times \delta_{p_2 q_2} \delta_{p_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 p_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 p_1} X_{(\alpha_1, \alpha_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, \gamma_2)}} \overline{W_{(r_1, q_2)}} \\
& = \delta_{r_2 j_2} \sum_{p_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, q_2, \alpha_2, \beta_2=1}^{m_0} \delta_{p_2 q_2} \delta_{p_1 j_1} \delta_{\alpha_2 \beta_2} \delta_{\beta_1 \gamma_1} X_{(p_1, \alpha_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, p_2)}} \overline{W_{(r_1, q_2)}} \\
& = \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} X_{(j_1, \beta_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\beta_1, q_2)}} \overline{W_{(r_1, q_2)}}, \tag{4.36}
\end{aligned}$$

$$\begin{aligned}
L_8 & = \delta_{r_2 j_2} \sum_{p_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, q_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\
& \quad \times \delta_{p_2 q_2} \delta_{p_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 p_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 p_1} X_{(\alpha_1, \alpha_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, \gamma_2)}} \overline{W_{(r_1, q_2)}} \\
& = \delta_{r_2 j_2} \sum_{p_1, \alpha_1, \beta_1=1}^{k_0} \sum_{p_2, q_2, \beta_2, \gamma_2=1}^{m_0} \delta_{p_2 q_2} \delta_{p_1 j_1} \delta_{\gamma_2 \beta_2} \delta_{\beta_1 \alpha_1} X_{(\alpha_1, p_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(p_1, \gamma_2)}} \overline{W_{(r_1, q_2)}} \\
& = \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} X_{(\beta_1, q_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(j_1, \beta_2)}} \overline{W_{(r_1, q_2)}}, \tag{4.37}
\end{aligned}$$

$$\begin{aligned}
L_9 & = \delta_{r_1 j_1} \sum_{p_1, q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\
& \quad \times \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{\alpha_2 \beta_2} \delta_{\gamma_2 p_2} \delta_{\beta_1 \gamma_1} \delta_{\alpha_1 p_1} X_{(\alpha_1, \alpha_2)} \overline{Y_{(\beta_1, \beta_2)}} \overline{Z_{(\gamma_1, \gamma_2)}} \overline{W_{(q_1, r_2)}}
\end{aligned}$$

$$\begin{aligned}
&= \delta_{r_1 j_1} \sum_{p_1, q_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, \alpha_2, \beta_2=1}^{m_0} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{\alpha_2 \beta_2} \delta_{\beta_1 \gamma_1} X_{(p_1, \alpha_2)} \overline{Y_{(\beta_1, \beta_2)}} Z_{(\gamma_1, p_2)} \overline{W_{(q_1, r_2)}} \\
&= \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} X_{(q_1, \beta_2)} \overline{Y_{(\beta_1, \beta_2)}} Z_{(\beta_1, j_2)} \overline{W_{(q_1, r_2)}}, \tag{4.38}
\end{aligned}$$

$$\begin{aligned}
L_{10} &= \delta_{r_1 j_1} \sum_{p_1, q_1, \alpha_1, \beta_1, \gamma_1=1}^{k_0} \sum_{p_2, \alpha_2, \beta_2, \gamma_2=1}^{m_0} \\
&\quad \times \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{\gamma_2 \beta_2} \delta_{\alpha_2 p_2} \delta_{\beta_1 \alpha_1} \delta_{\gamma_1 p_1} X_{(\alpha_1, \alpha_2)} \overline{Y_{(\beta_1, \beta_2)}} Z_{(\gamma_1, \gamma_2)} \overline{W_{(q_1, r_2)}} \\
&= \delta_{r_1 j_1} \sum_{p_1, q_1, \alpha_1, \beta_1=1}^{k_0} \sum_{p_2, \beta_2, \gamma_2=1}^{m_0} \delta_{p_2 j_2} \delta_{q_1 p_1} \delta_{\gamma_2 \beta_2} \delta_{\beta_1 \alpha_1} X_{(\alpha_1, p_2)} \overline{Y_{(\beta_1, \beta_2)}} Z_{(p_1, \gamma_2)} \overline{W_{(q_1, r_2)}} \\
&= \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} X_{(\beta_1, j_2)} \overline{Y_{(\beta_1, \beta_2)}} Z_{(q_1, \beta_2)} \overline{W_{(q_1, r_2)}}. \tag{4.39}
\end{aligned}$$

Summing (4.32), (4.33), (4.36) and (4.37), and substituting $X = \partial_x Q$, $Y = Z = W = Q$, we deduce

$$\begin{aligned}
L_3 + L_4 + L_7 + L_8 &= \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{\partial_x Q_{(r_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \\
&\quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{\partial_x Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(r_1, \beta_2)}} Q_{(j_1, q_2)} \\
&\quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \partial_x Q_{(j_1, \beta_2)} \overline{Q_{(\beta_1, \beta_2)}} Q_{(\beta_1, q_2)} \overline{Q_{(r_1, q_2)}} \\
&\quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \partial_x Q_{(\beta_1, q_2)} \overline{Q_{(\beta_1, \beta_2)}} Q_{(j_1, \beta_2)} \overline{Q_{(r_1, q_2)}} \\
&= \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{\partial_x Q_{(r_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \\
&\quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{\partial_x Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(r_1, \beta_2)}} Q_{(j_1, q_2)} \\
&\quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \partial_x Q_{(j_1, q_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(r_1, \beta_2)}} \\
&\quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \partial_x Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \overline{Q_{(r_1, \beta_2)}} \\
&= \partial_x \left\{ \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{Q_{(r_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \right\}. \tag{4.40}
\end{aligned}$$

In the same way as above, summing (4.34), (4.35), (4.38), and (4.39), and substituting $X = \partial_x Q$, $Y = Z = W = Q$, we deduce

$$\begin{aligned}
L_5 + L_6 + L_9 + L_{10} &= \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{\partial_x Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} \\
&\quad + \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{\partial_x Q_{(\beta_1, r_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(q_1, \beta_2)}} Q_{(q_1, j_2)} \\
&\quad + \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \partial_x Q_{(q_1, \beta_2)} \overline{Q_{(\beta_1, \beta_2)}} Q_{(\beta_1, j_2)} \overline{Q_{(q_1, r_2)}} \\
&\quad + \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \partial_x Q_{(\beta_1, j_2)} \overline{Q_{(\beta_1, \beta_2)}} Q_{(q_1, \beta_2)} \overline{Q_{(q_1, r_2)}} \\
&= \partial_x \left\{ \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} \right\}. \tag{4.41}
\end{aligned}$$

From (4.40) and (4.41), we get

$$\begin{aligned}
S_1 + S_2 &= \partial_x \left\{ \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} \right\} \\
&\quad + \partial_x \left\{ \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{Q_{(r_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \right\}. \tag{4.42}
\end{aligned}$$

Using this and replacing indexes, we deduce

$$\begin{aligned}
&\sum_{r=1}^n \left(\int_{-\infty}^x (S_1 + S_2)(t, y) dy \right) Q_r \\
&= \sum_{r_1=1}^{k_0} \sum_{r_2=1}^{m_0} \left(\delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} \right) Q_{(r_1, r_2)} \\
&\quad + \sum_{r_1=1}^{k_0} \sum_{r_2=1}^{m_0} \left(\delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{Q_{(r_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \right) Q_{(r_1, r_2)} \\
&= \sum_{q_1, \beta_1=1}^{k_0} \sum_{r_2, \beta_2=1}^{m_0} \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} Q_{(j_1, r_2)} \\
&\quad + \sum_{r_1, \beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{Q_{(r_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} Q_{(r_1, j_2)} \\
&= \sum_{q_1, \beta_1=1}^{k_0} \sum_{r_2, \beta_2=1}^{m_0} Q_{(j_1, r_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(q_1, \beta_2)}} Q_{(q_1, j_2)}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{r_1, \beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} Q_{(j_1, q_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(r_1, \beta_2)}} Q_{(r_1, j_2)} \\
& = 2 \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \overline{Q_{(s_4, s_3)}} Q_{(s_4, j_2)}. \tag{4.43}
\end{aligned}$$

Although the expression of $S_3 + S_4$ can be obtained in the same way as above, $S_3 + S_4$ is not expressed as an image of ∂_x . Indeed, by applying (4.32)-(4.39) for $X = Z = W = Q$ and $Y = \partial_x Q$, we see $S_3 + S_4 = (L_3 + L_4 + L_7 + L_8) + (L_5 + L_6 + L_9 + L_{10})$ where

$$\begin{aligned}
& L_3 + L_4 + L_7 + L_8 \\
& = \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{Q_{(r_1, \beta_2)}} \partial_x Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \\
& \quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \overline{Q_{(\beta_1, q_2)}} \partial_x Q_{(\beta_1, \beta_2)} \overline{Q_{(r_1, \beta_2)}} Q_{(j_1, q_2)} \\
& \quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} Q_{(j_1, q_2)} \overline{\partial_x Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \overline{Q_{(r_1, \beta_2)}} \\
& \quad + \delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} Q_{(\beta_1, \beta_2)} \overline{\partial_x Q_{(\beta_1, q_2)}} Q_{(j_1, q_2)} \overline{Q_{(r_1, \beta_2)}} \\
& = 2\delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} Q_{(j_1, q_2)} \partial_x \{ \overline{Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(r_1, \beta_2)}},
\end{aligned}$$

$$\begin{aligned}
& L_5 + L_6 + L_9 + L_{10} \\
& = \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{Q_{(q_1, \beta_2)}} \partial_x Q_{(\beta_1, \beta_2)} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} \\
& \quad + \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \overline{Q_{(\beta_1, r_2)}} \partial_x Q_{(\beta_1, \beta_2)} \overline{Q_{(q_1, \beta_2)}} Q_{(q_1, j_2)} \\
& \quad + \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} Q_{(q_1, \beta_2)} \overline{\partial_x Q_{(\beta_1, \beta_2)}} Q_{(\beta_1, j_2)} \overline{Q_{(q_1, r_2)}} \\
& \quad + \delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} Q_{(\beta_1, j_2)} \overline{\partial_x Q_{(\beta_1, \beta_2)}} Q_{(q_1, \beta_2)} \overline{Q_{(q_1, r_2)}} \\
& = 2\delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \partial_x \{ \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
S_3 + S_4 &= 2\delta_{r_1 j_1} \sum_{q_1, \beta_1=1}^{k_0} \sum_{\beta_2=1}^{m_0} \partial_x \{ \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} \\
&\quad + 2\delta_{r_2 j_2} \sum_{\beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} Q_{(j_1, q_2)} \partial_x \{ \overline{Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(r_1, \beta_2)}}. \tag{4.44}
\end{aligned}$$

Using this and replacing indexes, we obtain

$$\begin{aligned}
&\sum_{r=1}^n \left(\int_{-\infty}^x (S_3 + S_4)(t, y) dy \right) Q_r \\
&= 2 \sum_{r_1, q_1, \beta_1=1}^{k_0} \sum_{r_2, \beta_2=1}^{m_0} \delta_{r_1 j_1} \left(\int_{-\infty}^x \partial_x \{ \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} dy \right) Q_{(r_1, r_2)} \\
&\quad + 2 \sum_{r_1, \beta_1=1}^{k_0} \sum_{r_2, q_2, \beta_2=1}^{m_0} \delta_{r_2 j_2} \left(\int_{-\infty}^x Q_{(j_1, q_2)} \partial_x \{ \overline{Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(r_1, \beta_2)}} dy \right) Q_{(r_1, r_2)} \\
&= 2 \sum_{q_1, \beta_1=1}^{k_0} \sum_{r_2, \beta_2=1}^{m_0} Q_{(j_1, r_2)} \left(\int_{-\infty}^x \partial_x \{ \overline{Q_{(q_1, \beta_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(\beta_1, r_2)}} Q_{(q_1, j_2)} dy \right) \\
&\quad + 2 \sum_{r_1, \beta_1=1}^{k_0} \sum_{q_2, \beta_2=1}^{m_0} \left(\int_{-\infty}^x Q_{(j_1, q_2)} \partial_x \{ \overline{Q_{(\beta_1, q_2)}} Q_{(\beta_1, \beta_2)} \} \overline{Q_{(r_1, \beta_2)}} dy \right) Q_{(r_1, j_2)} \\
&= 2 \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_3)} \left(\int_{-\infty}^x \overline{Q_{(s_4, s_3)}} \partial_x \{ Q_{(s_4, s_1)} \overline{Q_{(s_2, s_1)}} \} Q_{(s_2, j_2)} dy \right) \\
&\quad + 2 \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} \left(\int_{-\infty}^x Q_{(j_1, s_1)} \partial_x \{ \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \} \overline{Q_{(s_4, s_3)}} dy \right) Q_{(s_4, j_2)}. \tag{4.45}
\end{aligned}$$

Combining (4.43) and (4.45), we derive

$$\begin{aligned}
&\sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^1(Q, \partial_x Q)(t, y) dy \right) Q_r \\
&= -2(b + 2c) \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \overline{Q_{(s_4, s_3)}} Q_{(s_4, j_2)} \\
&\quad + 2b \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_3)} \left(\int_{-\infty}^x \overline{Q_{(s_4, s_3)}} \partial_x \{ Q_{(s_4, s_1)} \overline{Q_{(s_2, s_1)}} \} Q_{(s_2, j_2)} dy \right) \\
&\quad + 2b \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} \left(\int_{-\infty}^x Q_{(j_1, s_1)} \partial_x \{ \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \} \overline{Q_{(s_4, s_3)}} dy \right) Q_{(s_4, j_2)}. \tag{4.46}
\end{aligned}$$

On the other hand, since G_{n_0, k_0} in this setting is Hermitian symmetric ($\nabla R = 0$), we see $f_{j,r}^2(Q, \partial_x Q) = 0$ for any $j, r \in \{1, \dots, n\}$ and hence

$$\sum_{r=1}^n \left(\int_{-\infty}^x f_{j,r}^2(Q, \partial_x Q)(t, y) dy \right) Q_r = 0.$$

Finally, substituting (4.25)-(4.29) and (4.46) into (1.10), we obtain

$$\begin{aligned} & \sqrt{-1} \partial_t Q_j + (a \partial_x^4 + \lambda \partial_x^2) Q_j \\ &= d_1 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} \partial_x^2 Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, j_2)} + d_1 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} \partial_x^2 Q_{(s_2, j_2)} \\ &+ 2d_2 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{\partial_x^2 Q_{(s_2, s_1)}} Q_{(s_2, j_2)} \\ &+ d_3 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} \partial_x Q_{(j_1, s_1)} \overline{\partial_x Q_{(s_2, s_1)}} Q_{(s_2, j_2)} + d_3 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{\partial_x Q_{(s_2, s_1)}} \partial_x Q_{(s_2, j_2)} \\ &+ 2d_4 \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} \partial_x Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} \partial_x Q_{(s_2, j_2)} - 2\lambda \sum_{s_2=1}^{k_0} \sum_{s_1=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, j_2)} \\ &+ (-2b - 4c) \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \overline{Q_{(s_4, s_3)}} Q_{(s_4, j_2)} \\ &+ 2b \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_3)} \left(\int_{-\infty}^x \overline{Q_{(s_4, s_3)}} \partial_x \{ Q_{(s_4, s_1)} \overline{Q_{(s_2, s_1)}} \} Q_{(s_2, j_2)} dy \right) \\ &+ 2b \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} \left(\int_{-\infty}^x Q_{(j_1, s_1)} \partial_x \{ \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \} \overline{Q_{(s_4, s_3)}} dy \right) Q_{(s_4, j_2)}. \end{aligned} \quad (4.47)$$

Proof of Corollary 1.2. Under the setting (1.6), it follows that $d_1 = -2\beta + 16\gamma$, $d_2 = 8\gamma$, $d_3 = 2\beta + 32\gamma$, $d_4 = -\beta + 16\gamma$, $-2b - 4c = -2\beta + 32\gamma$, $2b = 2\beta + 16\gamma$, and thus the system of (4.47) for Q_1, \dots, Q_n is rewritten as

$$\begin{aligned} \sqrt{-1} q_t &= -\beta q_{xxxx} + \alpha q_{xx} + (-2\beta + 16\gamma)(q_{xx} q^* q + q q^* q_{xx}) + 16\gamma q q_{xx}^* q \\ &+ (2\beta + 32\gamma)(q_x q_x^* q + q q_x^* q_x) + (-2\beta + 32\gamma) q_x q^* q_x + 2\alpha q q^* q \\ &+ (-2\beta + 32\gamma) q q^* q q^* q \\ &+ (2\beta + 16\gamma) \left\{ q \left(\int_{-\infty}^x q^* (q q^*)_s q ds \right) + \left(\int_{-\infty}^x q (q^* q)_s q^* ds \right) q \right\} \end{aligned} \quad (4.48)$$

for $q = (Q_{(j_1, j_2)})$ being an $\mathcal{M}_{k_0 \times m_0}$ -valued function whose (j_1, j_2) -component is $Q_{(j_1, j_2)} = Q_{(j_2-1)k_0 + j_1}$. Furthermore, (4.48) is also formulated as follows:

$$\begin{aligned} \sqrt{-1} q_t &= \alpha \left\{ q_{xx} + 2q q^* q \right\} - \beta \left\{ q_{xxxx} + 4q_{xx} q^* q + 2q q_{xx}^* q + 4q q^* q_{xx} \right. \\ &\quad \left. + 2q_x q_x^* q + 6q_x q^* q_x + 2q q_x^* q_x + 6q q^* q q^* q \right\} \end{aligned}$$

$$\begin{aligned}
& + 2(\beta + 8\gamma) \left\{ (qq^*q)_{xx} + 2qq^*qq^*q \right. \\
& \left. + q \left(\int_{-\infty}^x q^*(qq^*)_s q ds \right) + \left(\int_{-\infty}^x q(q^*q)_s q^* ds \right) q \right\}. \tag{4.49}
\end{aligned}$$

In addition, let $A(t) : (-T, T) \rightarrow \mathfrak{u}(m_0)$ and $B(t) : (-T, T) \rightarrow \mathfrak{u}(k_0)$ be defined by

$$\begin{aligned}
A(t) &= 2(\beta + 8\gamma)\sqrt{-1} \left(\int_{-\infty}^0 q^*(qq^*)_s q ds \right), \\
B(t) &= 2(\beta + 8\gamma)\sqrt{-1} \left(\int_{-\infty}^0 q(q^*q)_s q^* ds \right).
\end{aligned}$$

Noting $(A(t))^* = -A(t)$ and $(B(t))^* = -B(t)$, we see there exist $y = y(t) : (-T, T) \rightarrow U(m_0)$ and $z = z(t) : (-T, T) \rightarrow U(k_0)$ such that

$$\frac{dy}{dt} = A(t)y, \quad y(0) = I_{m_0}, \quad \frac{dz}{dt} = zB(t), \quad z(0) = I_{k_0}.$$

It is easy to find that (4.49) is transformed to (1.8) by $q(t, x) \mapsto z(t)q(t, x)y(t)$. We omit the detail. \square

Remark 4.5. It is known that $G_{n+1,1}$ where $k_0 = 1$, $m_0 = n_0 - k_0 = n$ is identified with the complex projective space $P_n(\mathbb{C})$ with the Fubini-Study metric, and is a complex Kähler manifold of complex dimension n with constant holomorphic sectional curvature $K = 4$ in our setting of h . Hence, (4.22) and (4.47) should coincide with (3.7) and (3.15) with $K = 4$ respectively. This may not be obvious immediately from the expressions of (4.47) and (3.15), but actually holds. See Appendix for the reason.

4.3. Relationship with the method in [11]. Let $N = G_{n_0, k_0}$ be as above. Corollary 1.2 reveals that the system of (1.10) for Q_1, \dots, Q_n with (1.6) is essentially the same as (1.8) derived in [11]. However, one may want a more theoretical reason why they coincide with each other, since our method using the parallel orthonormal frame and that used to derive (1.8) in [11] are seemingly different. Hence, we here try to make a more convincing explanation of the reason by comparing the two methods.

To begin with, we review the outline of the method in [11] briefly: The authors in [11] started from identifying G_{n_0, k_0} with $\{E^{-1}\sigma_3 E \mid E \in U(n_0)\}$ which is the adjoint orbit embedded in $\mathfrak{u}(n_0)$ at $\sigma_3 = \frac{\sqrt{-1}}{2} \begin{pmatrix} I_{k_0} & 0 \\ 0 & -I_{m_0} \end{pmatrix} \in \mathfrak{u}(n_0)$. We can see that the identification is verified by the one-to-one corresponding $\Psi : G_{n_0, k_0} \rightarrow \{E^{-1}\sigma_3 E \mid E \in U(n_0)\}$ such that

$$\Psi(BA_0B^*) = B\sigma_3B^* \tag{4.50}$$

for $A = BA_0B^* \in G_{n_0, k_0}$ with $B \in U(n_0)$. They next expressed the solution to the generalized bi-Schrödinger flow equation by $\varphi(t, x) = (E(t, x))^{-1}\sigma_3 E(t, x)$. Here, $E = E(t, x) : (-T, T) \times \mathbb{R} \rightarrow U(n_0)$ and satisfies $E_x = PE$ for some $P = P(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathfrak{m}_I$, where \mathfrak{m}_I is defined by (4.11). Based on the setting, they showed that the generalized bi-Schrödinger flow equation for φ is equivalent to the fourth-order matrix-nonlinear Schrödinger-like equation for $P = E_x E^{-1} = E_x E^*$ up to a gauge transformation. Since P takes values in \mathfrak{m}_I ,

$$P = \begin{pmatrix} 0 & \mathbf{q} \\ -\mathbf{q}^* & 0 \end{pmatrix}, \tag{4.51}$$

for some $\mathbf{q} = \mathbf{q}(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathcal{M}_{k_0 \times m_0}$. (Equations satisfied by P and \mathbf{q} are respectively given by (42) and (62) in [11], and (62) is just (1.8) for q up to a gauge transformation.) Their proof is based on the geometric concept of PDEs with given (non-zero) curvature representation. See [11, Theorem 3] for more detail on their proof.

Remark 4.6. The embedding of G_{n_0, k_0} in $u(n_0)$ was adopted also by Terng and Uhlenbeck [39] to show the equivalence of the Schrödinger flow equation for maps with values in G_{n_0, k_0} and the matrix-nonlinear Schrödinger equation. In fact, from their results in [39], it turns out that the above $E = E(t, x)$ exists uniquely under some assumptions on the map. For example, assume that $\varphi = \varphi(t, x) : (-T, T) \times \mathbb{R} \rightarrow \{E^{-1}\sigma_3 E \mid E \in U(n_0)\}$ is a smooth map such that $\lim_{x \rightarrow -\infty} \varphi(t, x) = \sigma_3$ and $\varphi_x(t, \cdot)$ is in the Schwartz class for any $t \in (-T, T)$. (The assumption on φ is equivalently to $u \in C_{A_0}((-T, T) \times \mathbb{R}; G_{n_0, k_0})$ for $u = \Psi^{-1}(\varphi) : (-T, T) \times \mathbb{R} \rightarrow G_{n_0, k_0}$.) Then Corollary 3.3 in [39] shows that there exists a unique $E = E(t, x) : (-T, T) \times \mathbb{R} \rightarrow U(n_0)$ such that $\varphi(t, x) = (E(t, x))^{-1}\sigma_3 E(t, x)$, $\lim_{x \rightarrow -\infty} E(t, x) = I$, and EE_x^* takes values in \mathfrak{m}_I .

Next, we observe our method to derive (4.48): Let $u \in C_{u^\infty}((-T, T) \times \mathbb{R}; G_{n_0, k_0})$ be a solution to (1.1) with (1.6) where $u^\infty = B^\infty A_0 (B^\infty)^*$ and $B^\infty \in U(n_0)$. We can assume $B^\infty = I$ without loss of generality, by retaking a G_{n_0, k_0} -valued map $(B^\infty)^* u(t, x) B^\infty$ as $u(t, x)$. Let $\{e_j, J e_j\}_{j=1}^n$ be the orthonormal frame for $u^{-1} T G_{n_0, k_0}$ constructed in Section 2.1, and let $Q_j : (-T, T) \times \mathbb{R} \rightarrow \mathbb{C}$ for $j \in \{1, \dots, n (= k_0 m_0)\}$ be the functions defined by (2.41) in Section 2.2. We continue to denote $j = (j_1, j_2)$ for $j \in \{1, \dots, n (= k_0 m_0)\}$ if there exist $j_1 \in \{1, \dots, k_0\}$ and $j_2 \in \{1, \dots, m_0\}$ such that $j = (j_2 - 1)k_0 + j_1$. As used in the proof of Corollary 1.2, let $(Q_{(j_1, j_2)})$ denote the $\mathcal{M}_{k_0 \times m_0}$ -valued function whose (j_1, j_2) -components are $Q_{(j_1, j_2)}$.

The aim of this subsection is to verify the following:

Proposition 4.7. *Under the assumption as above for $u \in C_{A_0}((-T, T) \times \mathbb{R}; G_{n_0, k_0})$, the relation*

$$P = \begin{pmatrix} 0 & (Q_{(j_1, j_2)}) \\ - (Q_{(j_1, j_2)})^* & 0 \end{pmatrix}, \quad (4.52)$$

holds, where P is given by (4.51).

This shows that the two $\mathcal{M}_{k_0 \times m_0}$ -valued functions \mathbf{q} obtained in [11] and $(Q_{(j_1, j_2)})$ constructed by our method (and thus the equations satisfied by them) essentially coincide with each other. Therefore, we can be convinced that Corollary 1.2 holds without doing the computation in Section 4.2.

The key of the proof is that both $\{e_j, J e_j\}_{j=1}^n$ and $(Q_{(j_1, j_2)})$ can be expressed explicitly with the aid of the co-diagonal lifting (or the horizontal lifting) of $u(t, \cdot) : \mathbb{R} \rightarrow G_{n_0, k_0}$.

Proof of Proposition 4.7. Recall that $\lim_{x \rightarrow -\infty} u(t, x) = u^\infty = A_0$ here. By following the argument in [2], a map $C = C(t, \cdot) : \mathbb{R} \rightarrow U(n_0)$ is called a co-diagonal lifting of $u = u(t, \cdot) : \mathbb{R} \rightarrow G_{n_0, k_0}$ for each fixed $t \in (-T, T)$, if

$$C(t, x) A_0 (C(t, x))^* = u(t, x) \text{ and } \sqrt{-1} (C(t, x))^* C_x(t, x) \in T_{u(t, x)} G_{n_0, k_0} \quad (4.53)$$

hold for any $x \in \mathbb{R}$. We pick a co-diagonal lifting $C = C(t, x)$ of u satisfying $C(t, -\infty) := \lim_{x \rightarrow -\infty} C(t, x) = I$ for any t . By the argument to show Lemma 2.6 in [2], such a co-diagonal

lifting exists uniquely and is characterized as the unique solution to $C_x = [u_x, u]C$ satisfying $C(t, -\infty) = I$, where $[u_x, u] = u_x u - u u_x$.

We investigate the expression of C^*C_x . It is immediate to see $C^*C = I$ and thus $C^*C_x + (C_x)^*C = O$ holds, since C is $U(n_0)$ -valued. This implies $C^*C_x = -(C_x)^*C = -(C^*C_x)^*$ and hence C^*C_x is $\mathfrak{u}(n_0)$ -valued. Therefore, we can write

$$C^*C_x = \begin{pmatrix} C_{11} & C_{12} \\ -C_{12}^* & C_{22} \end{pmatrix}, \quad (4.54)$$

where $C_{11} = C_{11}(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathfrak{u}(k_0)$, $C_{22} = C_{22}(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathfrak{u}(m_0)$, and $C_{12} = C_{12}(t, x) : (-T, T) \times \mathbb{R} \rightarrow \mathcal{M}_{k_0 \times m_0}$. On the other hand, $u = CA_0C^*$ in (4.53) and $CC^* = I$ yields

$$u_x = C_x A_0 C^* + C A_0 (C_x)^* = C \{C^* C_x A_0 + A_0 (C_x)^* C\} C^*. \quad (4.55)$$

Using this, $C^*C = I$ and $A_0^2 = A_0$, we deduce

$$\begin{aligned} [u_x, u] &= C \{C^* C_x A_0 + A_0 (C_x)^* C\} C^* C A_0 C^* \\ &\quad - C A_0 C^* C \{C^* C_x A_0 + A_0 (C_x)^* C\} C^* \\ &= C \{C^* C_x A_0 + A_0 (C_x)^* C A_0 - A_0 C^* C_x A_0 - A_0 (C_x)^* C\} C^*. \end{aligned}$$

Substituting this into $C^*C_x = C^*[u_x, u]C$ which follows from $C_x = [u_x, u]C$, and using $CC^* = C^*C = I$ and $(C_x)^*C = -C^*C_x$ which follows from $C^*C = I$, we deduce

$$\begin{aligned} C^*C_x &= C^*C_x A_0 + A_0 (C_x)^* C A_0 - A_0 C^* C_x A_0 - A_0 (C_x)^* C \\ &= C^*C_x A_0 + A_0 C^* C_x - 2A_0 C^* C_x A_0. \end{aligned} \quad (4.56)$$

Therefore, substituting (4.54) into the right hand side of (4.56), we obtain

$$\begin{aligned} C^*C_x &= \begin{pmatrix} C_{11} & C_{12} \\ -C_{12}^* & C_{22} \end{pmatrix} \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ -C_{12}^* & C_{22} \end{pmatrix} \\ &\quad - 2 \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ -C_{12}^* & C_{22} \end{pmatrix} \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & C_{12} \\ -C_{12}^* & 0 \end{pmatrix}, \end{aligned} \quad (4.57)$$

which implies C^*C_x takes values in \mathfrak{m}_I . If we adopt the identification (4.50), then what we obtained here is interpreted as $\Psi(u)(t, x) = C(t, x)\sigma_3(C(t, x))^*$, where $C = C(t, x) : (-T, T) \times \mathbb{R} \rightarrow U(n_0)$, $C^*(t, -\infty) = I$, and C^*C_x takes values in \mathfrak{m}_I . Hence, the uniqueness result stated in Remark 4.6 verifies $C = E^{-1}(= E^*)$. From this, we see

$$P = E_x E^* = (C_x)^* C = -C^* C_x = - \begin{pmatrix} 0 & C_{12} \\ -C_{12}^* & 0 \end{pmatrix}. \quad (4.58)$$

We next investigate the expression of Q_j , which by (2.40)-(2.41) is represented as

$$Q_j = h(u_x, e_j) + \sqrt{-1}h(u_x, J_u e_j).$$

Set $j = (j_1, j_2) \in \{1, \dots, n\}$. Recall that e_j is constructed as the parallel transport of $e_j^\infty \in T_{u^\infty} G_{n_0, k_0}$ along $u(t, \cdot) : \mathbb{R} \rightarrow G_{n_0, k_0}$, and e_j^∞ is defined by (4.19). In the present setting, $B^\infty = I$ and $u^\infty = A_0$, and hence we can write

$$e_j^\infty = \begin{pmatrix} 0 & E_{(j_1, j_2)} \\ (E_{(j_1, j_2)})^* & 0 \end{pmatrix}.$$

Note that the parallel transport of tangent vectors on G_{n_0, k_0} along $u(t, \cdot) = C(t, \cdot)A_0(C(t, \cdot))^*$ can be represented by using the co-diagonal lifting $C(t, \cdot)$. Indeed, following the argument in [2], we see $e_j(t, \cdot) = C(t, \cdot)e_j^\infty(C(t, \cdot))^*$, that is,

$$e_j(t, x) = C(t, x) \begin{pmatrix} 0 & E^{(j_1, j_2)} \\ (E^{(j_1, j_2)})^* & 0 \end{pmatrix} (C(t, x))^*. \quad (4.59)$$

From the expression, it follows that

$$J_u e_j(t, x) = C(t, x) \begin{pmatrix} 0 & \sqrt{-1}E^{(j_1, j_2)} \\ (\sqrt{-1}E^{(j_1, j_2)})^* & 0 \end{pmatrix} (C(t, x))^*. \quad (4.60)$$

In addition, using $(C_x)^*C = -C^*C_x$ and substituting (4.57) into (4.55), we deduce

$$u_x = C \{C^*C_x A_0 - A_0 C^*C_x\} C^* = C \begin{pmatrix} 0 & -C_{12} \\ (-C_{12})^* & 0 \end{pmatrix} C^*. \quad (4.61)$$

Let us also recall (4.8) to see $h(\Delta_1, \Delta_2) = \text{Re} [\text{tr}(\omega_1(\omega_2)^*)]$ for $\Delta_k = C \begin{pmatrix} 0 & \omega_k \\ (\omega_k)^* & 0 \end{pmatrix} C^* \in T_u G_{n_0, k_0}$ ($k = 1, 2$). Then, using (4.59)-(4.61), we have

$$\begin{aligned} h(u_x, e_j) &= \text{Re} [\text{tr}((-C_{12})(E^{(j_1, j_2)})^*)], \\ h(u_x, J e_j) &= \text{Re} [\text{tr}((-C_{12})(\sqrt{-1}E^{(j_1, j_2)})^*)] = \text{Im} [\text{tr}((-C_{12})(E^{(j_1, j_2)})^*)]. \end{aligned}$$

This shows $Q_j(t, x) = \text{tr}((-C_{12}(t, x))(E^{(j_1, j_2)})^*)$. Since the right hand side equals to the (j_1, j_2) -component of $(-C_{12})(t, x) \in \mathcal{M}_{k_0 \times m_0}$, we see $(Q_{(j_1, j_2)}) = -C_{12}$, that is,

$$\begin{pmatrix} 0 & (Q_{(j_1, j_2)}) \\ -(Q_{(j_1, j_2)})^* & 0 \end{pmatrix} = - \begin{pmatrix} 0 & C_{12} \\ -C_{12}^* & 0 \end{pmatrix}.$$

Comparing this with (4.58), we derive the desired (4.52), which completes the proof. \square

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5. APPENDIX

Supplemental comments on Remark 4.5 are presented below.

Let $N = G_{n+1, 1}$ where $k_0 = 1$, $m_0 = n_0 - k_0 = n$. As commented in Remark 4.5, (4.22) and (4.47) turn out to be the same as (3.7) and (3.15) for $K = 4$ respectively. We here state the outline of how to check it. Since $p_1 = q_1 = r_1 = j_1 = 1$ here, it is immediate to see (4.22) becomes

$$S_{p, q, r}^j = \delta_{p_2 q_2} \delta_{r_2 j_2} + \delta_{r_2 q_2} \delta_{p_2 j_2} = \delta_{pq} \delta_{rj} + \delta_{rq} \delta_{pj},$$

the right hand side of which actually coincides with (3.7) for $K = 4$. Second, any index $j \in \{1, \dots, n\}$ is expressed as $j = (1, j_2)$ by a unique $j_2 \in \{1, \dots, n\}$, and (4.47) for $Q_{(1, j_2)}$ turns out to reduce to (3.15) for $Q_j = Q_{(1, j_2)}$. We omit the detail, except to show that the sum of the final three terms of the right hand side of (4.47) actually reduces to that of the final

two terms of the right hand side of (3.15), because the correspondence of the other terms is obvious. To show this, set

$$\begin{aligned} I_1 &= \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_1)} \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \overline{Q_{(s_4, s_3)}} Q_{(s_4, j_2)}, \\ I_2 &= \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} Q_{(j_1, s_3)} \left(\int_{-\infty}^x \overline{Q_{(s_4, s_3)}} \partial_x \{ Q_{(s_4, s_1)} \overline{Q_{(s_2, s_1)}} \} Q_{(s_2, j_2)} dy \right) \\ I_3 &= \sum_{s_2, s_4=1}^{k_0} \sum_{s_1, s_3=1}^{m_0} \left(\int_{-\infty}^x Q_{(j_1, s_1)} \partial_x \{ \overline{Q_{(s_2, s_1)}} Q_{(s_2, s_3)} \} \overline{Q_{(s_4, s_3)}} dy \right) Q_{(s_4, j_2)}. \end{aligned}$$

We compute $(-2b - 4c)I_1 + 2bI_2 + 2bI_3$ where $k_0 = 1$, $m_0 = n$ and $Q_j = Q_{(1, j_2)}$. A simple computation shows

$$\begin{aligned} I_1 &= \sum_{s_1, s_3=1}^n Q_{(1, s_1)} \overline{Q_{(1, s_1)}} Q_{(1, s_3)} \overline{Q_{(1, s_3)}} Q_{(1, j_2)} = \sum_{s_1=1}^n |Q_{(1, s_1)}|^2 \sum_{s_3=1}^n |Q_{(1, s_3)}|^2 Q_{(1, j_2)} \\ &= |Q|^4 Q_j, \\ I_2 &= \sum_{s_1, s_3=1}^n Q_{(1, s_3)} \left(\int_{-\infty}^x \overline{Q_{(1, s_3)}} \partial_x \{ Q_{(1, s_1)} \overline{Q_{(1, s_1)}} \} Q_{(1, j_2)} dy \right) \\ &= \sum_{s_3=1}^n Q_{(1, s_3)} \left(\int_{-\infty}^x \overline{Q_{(1, s_3)}} \partial_x (|Q|^2) Q_{(1, j_2)} dy \right) \\ &= \sum_{r=1}^n \left(\int_{-\infty}^x \overline{Q_r} Q_j \partial_x (|Q|^2) dy \right) Q_r. \end{aligned}$$

The fundamental theorem of calculus shows

$$\begin{aligned} I_3 &= \sum_{s_1, s_3=1}^n \left(\int_{-\infty}^x Q_{(1, s_1)} \partial_x \{ \overline{Q_{(1, s_1)}} Q_{(1, s_3)} \} \overline{Q_{(1, s_3)}} dy \right) Q_{(1, j_2)} \\ &= \sum_{s_1, s_3=1}^n Q_{(1, s_1)} \overline{Q_{(1, s_1)}} Q_{(1, s_3)} \overline{Q_{(1, s_3)}} Q_{(1, j_2)} \\ &\quad - \sum_{s_1, s_3=1}^n \left(\int_{-\infty}^x Q_{(1, s_3)} \partial_x \{ \overline{Q_{(1, s_3)}} Q_{(1, s_1)} \} \overline{Q_{(1, s_1)}} dy \right) Q_{(1, j_2)} \\ &= |Q|^4 Q_j - I_3, \end{aligned}$$

which implies $I_3 = \frac{1}{2}|Q|^4 Q_j$. Combining them, we see

$$\begin{aligned} &(-2b - 4c)I_1 + 2bI_2 + 2bI_3 \\ &= (-b - 4c)|Q|^4 Q_j + 2b \sum_{r=1}^n \left(\int_{-\infty}^x \overline{Q_r} Q_j \partial_x (|Q|^2) dy \right) Q_r, \end{aligned}$$

which actually equals to the sum of the final two terms of the right hand side of (3.15) for $K = 4$.

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