

An axiomatisation of two dimensional irreflexive Minkowski Spacetime $(\mathbb{R}^2, <)$

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Abstract

We define temporal axioms that are sound and complete for the temporal validities over $(\mathbb{R}^2, <)$.

Introduction The topic should need little motivation. If we consider events taking place at spacetime points, in the real world, we find that communications between them form a non-linear order, where one spacetime point might be neither in the accessible future nor past of another spacetime point, since there is a maximum speed for all signals. Here we assume that the set of all spacetime points forms a Euclidean metric space with one temporal dimension and some spatial dimensions, and that signals may travel at up to the speed of light. If we measure distances in light-seconds, the speed of light is one. Our problem is to find temporal axioms that are sound and complete over Kripke frames $(\mathbb{R}^n, <)$, where $<$ is defined by

$$(\bar{x}, t) < (\bar{x}', t') \iff (t < t' \wedge |\bar{x}' - \bar{x}| \leq t' - t)$$

It turns out that the temporal logic of such frames depends on the number $n - 1$ of spatial dimensions, indeed it is known that there are modal formulas valid over $(\mathbb{R}^2, <)$ but not over $(\mathbb{R}^3, <)$, for example axiom V below. One critical difference between $(\mathbb{R}^2, <)$ and higher dimensional frames is that (\mathbb{R}^2, \leq) forms a complete distributive lattice, whereas in higher dimensions two spatially displaced spacetime points do not have a supremum (the upper bounds are bound below not by a point but by a hyperbola in $(\mathbb{R}^3, <)$), nor do they have an infimum.

The modal logic of the reflexive closure \leq of $<$ over \mathbb{R}^n was shown to be S4.2 [Gol80] and the modal logic of the slower than light irreflexive frame $(\mathbb{R}^n, <)$ is OL.2 [SS03] for any $n \geq 2$, however axioms for the temporal logic of any of these frames is not known.

Here we focus on the propositional temporal logic of $(\mathbb{R}^2, <)$ where there is only one space dimension. Temporal formulas are built from propositions using connectives \neg, \vee and **F**, **P** (sometime in the future, sometime in the past) and standard abbreviations \wedge, \rightarrow and **G**, **H** (always in the future, always in the

past), formulas are evaluated in the normal way. Throughout this paper, a finite set of propositions *Props* will be known, and all temporal formulas will involve only propositions from *Props*.

We provide sound and complete axioms for the temporal validities over this frame, see theorems 6 and 9 below. It is known that the temporal validities of $(\mathbb{R}^2, <)$ are decidable, a PSPACE algorithm was provided in [HR18, theorem 5.2], but axioms were not given, indeed [HR18, Problem (1)] asks for sound and complete temporal axioms over (\mathbb{R}^2, \leq) and sound and complete axioms over $(\mathbb{R}^2, <)$.

First we recall a sound and complete set of axioms for $(\mathbb{R}, <)$ (see [BG85]). The proof rules are modus ponens, generalisation, $\vdash \phi \Rightarrow \vdash \mathbf{G}\phi$, $\vdash \phi \Rightarrow \vdash \mathbf{H}\phi$, and substitution (letters p, q, r may be replaced by arbitrary temporal formulas).

1. K , axioms for propositional logic plus $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$
2. Temporal $p \rightarrow \mathbf{G}Pp$ and $p \rightarrow \mathbf{H}Fp$
3. Serial, transitive, dense $\mathbf{F}\top \wedge \mathbf{P}\top$, $\mathbf{G}p \leftrightarrow \mathbf{G}Gp$
4. Weak linear

$$Lin(q, r) = \left(\begin{array}{c} (\mathbf{F}q \wedge \mathbf{F}r) \rightarrow \mathbf{F}((q \wedge \mathbf{F}r) \vee (q \wedge r) \vee (r \wedge \mathbf{F}q)) \\ \wedge \\ (\mathbf{P}q \wedge \mathbf{P}r) \rightarrow \mathbf{P}((q \wedge \mathbf{P}r) \vee (q \wedge r) \vee (r \wedge \mathbf{P}q)) \end{array} \right)$$

5. Dedekind complete

$$Ddk(q) = \left(\begin{array}{c} (\mathbf{F}\neg q \wedge \mathbf{F}Gq) \rightarrow \mathbf{F}(\mathbf{G}q \wedge \mathbf{H}F\neg q) \\ \wedge \\ (\mathbf{P}\neg q \wedge \mathbf{P}Hq) \rightarrow \mathbf{P}(\mathbf{H}q \wedge \mathbf{G}P\neg q) \end{array} \right)$$

Write $\vdash \psi$ if the temporal formula ψ can be proved using rules and axioms above.

It is easy to check the soundness of these axioms over $(\mathbb{R}, <)$. Completeness is proved by *filtration* and we outline the method as a guide to the proof of theorem 9 below. Recommended textbooks for the basic material (and much more) include [BRV01, GHR94, vB83], in particular see [BRV01, definition 2.16, theorem 2.20] for the next definition and proposition.

DEFINITION 1 *A binary relation $R \subseteq X \times Y$ is a bisimulation between $(X, <)$ and $(Y, <')$ if*

- $((x_1 < x_2) \wedge (x_1, y_1) \in R) \Rightarrow \exists y_2 (y_1 <' y_2 \wedge (x_2, y_2) \in R)$
- $((y_1 <' y_2) \wedge (x_1, y_1) \in R) \Rightarrow \exists x_2 (x_1 < x_2 \wedge (x_2, y_2) \in R)$.

If a bisimulation happens to be a function $: X \rightarrow Y$ we may call it a p -morphism.

So a p -morphism $f : (X, <) \rightarrow (Y, <')$ is an order-preserving function such that

- If $f(x_1) <' y_2$ then $\exists x_2(x_1 < x_2 \wedge f(x_2) = y_2)$,
- If $y_1 <' f(x_2)$ then $\exists x_1(x_1 < x_2 \wedge f(x_1) = y_1)$.

PROPOSITION 2 *Let $R \subseteq X \times Y$ be a bisimulation between $(X, <)$ and $(Y, <')$. Let $v : Props \rightarrow \wp(X)$, $v' : Props \rightarrow \wp(Y)$ be propositional valuations such that whenever $(x, y) \in R$ and $p \in Props$, we have $x \in v(p) \iff y \in v'(p)$. For all temporal formulas ψ , all $(x, y) \in R$*

$$(X, <), x \models_v \psi \iff (Y, <'), y \models_{v'} \psi.$$

The following definition is equivalent to [BRV01, definition 4.34], but here restricted to the finite language $Cl(\phi)$.

DEFINITION 3 . *Let $Cl(\phi)$ be the set of subformulas and negated subformulas of ϕ , let M be the set of maximal consistent subsets of $Cl(\phi)$, define an order $<$ over M by*

$$(m < n) \iff \bigwedge_{p \in Cl(\phi)} (p \in m \rightarrow \mathbf{P}p \in n). \quad (1)$$

or equivalently, by the temporal axiom, $p \in n \rightarrow \mathbf{F}p \in m$. The directed graph $\mathcal{M} = (M, <)$ is called the canonical frame. The map $c : Props \rightarrow \wp(M)$ defined by $c(p) = \{m \in M : p \in m\}$ is called the canonical valuation.

Below, we allow letters p, q to range over closure formulas. For each $p \in Cl(\phi)$, $m \in \mathcal{M}$ we have $p \in m \iff (\mathcal{M}, m \models_c p)$ (see [BRV01, lemma 4.21]) but note that arbitrary temporal formulas may be evaluated at (\mathcal{M}, m) with canonical valuation c , moreover

$$(\mathcal{M}, m) \models_c \psi \iff m \vdash \psi. \quad (2)$$

A proof of completeness of the axioms above for the logic of $(\mathbb{R}, <)$ was given in [Bur84, section 2.7].

An MCS $m \in M$ is *irreflexive* if there is $p \in Cl(\phi)$ where $\{\mathbf{G}p, \neg p\} \subseteq m$, else it is *reflexive* (for all $p \in Cl(\phi)$, $\mathbf{G}p \in m$ implies $p \in m$). Define an equivalence relation \equiv over M by, $m \equiv n \iff ((m < n \wedge n < m) \vee (m = n))$. Write $[m]$ for the \equiv -equivalence class of $m \in M$. For reflexive $m \in M$, the equivalence class $[m] = \{n \in M : m \equiv n\}$ is called the *cluster* of m , we write *Clusters* for the set of all clusters of reflexive MCSs. Write I for the set of singleton irreflexive MCSs, but may treat such singleton as its irreflexive member. So $Clusters \cup I$ is the set of all equivalence classes of M . If $m \equiv m'$ then $\mathbf{F}p \in m \iff \mathbf{F}p \in m'$, $\mathbf{P}p \in m \iff \mathbf{P}p \in m'$. We may write $m \leq m'$ for $m < m' \vee m = m'$. Since equivalent MCSs agree on all temporally bound formulas, $(Clusters \cup I, <)$ also inherits a well-defined order, transitive and dense by axiom (3) (although finite), reflexive over *Clusters* and irreflexive over I , and antisymmetric. We write $\mathbf{M} = (Clusters \cup I, <)$ for the quotient \mathcal{M}/\equiv , distinguished from \mathcal{M} by a change of font.

For $x, y \in Clusters \cup I$ we write $Suc(x, y)$ when $x < y$ and there is no $z \in (Clusters \cup I) \setminus \{x, y\}$ where $x < z < y$. For $x \in Clusters \cup I$, let $Successors(x) = \{y \in Clusters \cup I : Suc(x, y)\}$, the set of immediate successors of x . By density, an irreflexive is never the successor of an irreflexive. We write $m \models_c n$ as a shorthand for $m \models_c \bigwedge n$ where $m, n \in \mathcal{M}$ (so this holds iff $m = n$), and we write $m \models_c d$ for $m \models_c \bigvee_{n \in d} n$ where d is a cluster.

PROPOSITION 4 *Axioms (1)–(5) are complete over $(\mathbb{R}, <)$.*

PROOF:

Given a consistent temporal formula ϕ we are required to find a propositional valuation $v : Props \rightarrow \wp(\mathbb{R})$ such that $(\mathbb{R}, <), x \models_v \phi$, for some $x \in \mathbb{R}$. It suffices to construct a p -morphism $f' : (\mathbb{R}, <) \rightarrow \mathcal{M}$. The construction has two stages: first we define a p -morphism $f : (\mathbb{R}, <) \rightarrow \mathbf{M}$; then we ‘bulldoze’ f to obtain a p -morphism $f' : (\mathbb{R}, <) \rightarrow \mathcal{M}$.

By the future weak linearity axiom (4), each element of $Clusters \cup I$ has at most one successor and by the dual axiom it has at most one predecessor. Hence \mathbf{M} is a spatial union of linear orders, i.e. there is no ordering between elements of distinct linear orders in \mathbf{M} . By density (3), one irreflexive is never the successor of another. By Dedekind completeness (5), one cluster is never the successor of another. Hence \mathbf{M} is a disjoint union of *traces*, that is, linearly ordered alternating sequences of clusters and irreflexives. By seriality and its dual each trace starts and ends with a cluster.

Since ϕ is consistent, there is $m \in s \in \mathbf{M}$ (some s) where $\phi \in m$. Let $\tau = (c_0, m_0, c_1, m_1, \dots, m_{k-1}, c_k)$ be a trace containing such an s .

Define $f : \mathbb{R} \rightarrow \mathbf{M}$ as follows.

$$f(x) = \begin{cases} m_i & x = i \in \mathbb{N}, i \leq k-1 \\ c_0 & x < 0 \\ c_k & x > k-1 \\ c_{i+1} & i < x < i+1 \end{cases}$$

It is clear that f , so defined, is order-preserving. Also, if $a, b \in Clusters \cup I$, $Suc(a, b)$ and $f(x) = a$ then there is $y > x$ where $f(y) = b$, and similar for predecessors. Hence f is a p -morphism from $(\mathbb{R}, <)$ to \mathbf{M} . The range of f is the trace τ . But, since two equivalent MCSs need not agree on all propositions, there is no way of defining a valuation over \mathbf{M} in a truth-preserving way.

To fix that, we *bulldoze* each cluster to get a p -morphism $f' : \mathbb{R} \rightarrow \mathcal{M}$, as follows. Observe that $f^{-1}(c)$ is an open interval, including neither a maximal nor a minimal point, for each cluster c . By lemma 8 below, we can define $f' : \mathbb{R} \rightarrow \mathcal{M}$ such that for all $x \in \mathbb{R}$ we have $f'(x) \in f(x)$, and whenever $f(x) = c$ and $m \in c$ there

are $y < x < y'$ such that $f'(y) = f'(y') = m$. By this property, $f' : \mathbb{R} \rightarrow \mathcal{M}$ is a p -morphism.

Finally, define a propositional valuation $v : Props \rightarrow \wp(\mathbb{R})$ by letting $v(p) = \{x \in \mathbb{R} : p \in f'(x)\}$. By proposition 2, since f' is a p -morphism, for all $p \in Cl(\phi)$ and $x \in \mathbb{R}$, we have

$$(\mathbb{R}, <), x \models_v p \iff p \in f'(x)$$

Hence there is $x \in \mathbb{R}$ where $(\mathbb{R}, <), x \models_v \phi$, as required.

□

Over $(\mathbb{R}^2, <)$ the first few axioms remain sound, but the linearity axiom fails (since $(\mathbb{R}^2, <)$ is non-linear) and the Dedekind completeness axiom also fails [for a counter-model, let $l \subseteq \mathbb{R}^2$ be any lightline and let $v(q)$ be the set of all points strictly above l , then $\mathbf{G}q$ holds strictly above l but not elsewhere, the premise $\mathbf{F}\neg q \wedge \mathbf{F}\mathbf{G}q$ holds at all points on or below l , but $(\mathbf{G}q \wedge \mathbf{H}\mathbf{F}\neg q)$ holds nowhere, so the consequent fails.] To axiomatise the temporal logic of $(\mathbb{R}^2, <)$ we must drop the linearity and Dedekind completeness axioms (3), (4) and replace them by weaker axioms (see axioms (IX), (XII)–(XIV) below). The transitivity axiom is included (since it remains valid) but is strengthened to axiom (V).

Notation and Definitions For convenience, we may adopt a change of bases, so axes are lightlines and

$$(x, y) < (x', y') \iff (x \leq x' \wedge y \leq y' \wedge (x, y) \neq (x', y')).$$

However, in the figures, lightlines are drawn at 45° , so time goes up the page and space goes across.

- For $x, y \in \mathbb{R}^2$ let $x \wedge y$, $x \vee y$ denote the infimum and supremum of $\{x, y\}$, let x^\uparrow , x^\downarrow denote $\{y \in \mathbb{R}^2 : x \leq y\}$, $\{y \in \mathbb{R}^2 : x \geq y\}$ respectively.
- An interval is a non-empty, convex set of reals. A product of intervals $I \times J$ is called a *rectangle*, which may be a point, a segment, or proper. A proper rectangle includes a corner iff it includes both bounding edge segments incident with the corner. The left and right corners of any rectangle are called spatial corners. A rectangle includes both spatial corners iff it includes all four corners. Two disjoint rectangles are neighbours if they share a spatial corner.
- For $x, y \in \mathbb{R}^2$ we write $x \sim y$ if $x \not\leq y$ and $y \not\leq x$.
- Let $v : prop \rightarrow \wp(\mathbb{R}^2)$ be a propositional valuation. For a temporal formula ϕ , write $v(\phi)$ for $\{x \in \mathbb{R}^2 : (\mathbb{R}^2, <), x \models_v \phi\}$. We will freely interchange the equivalent statements $x \in v(p)$ and $x \models_v p$.

- Given temporal formulas ϕ, ψ we may define the *relativization* $\psi \upharpoonright \phi$ of ψ to ϕ

$$\begin{aligned} p \upharpoonright \phi &= p \wedge \phi & (\neg\psi) \upharpoonright \phi &= \phi \wedge \neg(\psi \upharpoonright \phi) \\ (\psi_1 \vee \psi_2) \upharpoonright \phi &= (\psi_1) \upharpoonright \phi \vee (\psi_2) \upharpoonright \phi \\ (\mathbf{F}\psi) \upharpoonright \phi &= \phi \wedge \mathbf{F}(\psi \upharpoonright \phi) & (\mathbf{P}\psi) \upharpoonright \phi &= \phi \wedge \mathbf{P}(\psi \upharpoonright \phi) \end{aligned}$$

We will write down several temporal axioms then prove their soundness and completeness for the logic of $(\mathbb{R}^2, <)$. The axioms are a bit complicated, but they would look far worse without the introduction of some notational abbreviations, which we'll discuss shortly.

Notation	Definition
$\Box(p), \Diamond(p)$	$\mathbf{GH}p, \mathbf{FP}p$
$\mathbf{F}^0(p), \mathbf{P}^0(p)$	$p \vee \mathbf{F}p, p \vee \mathbf{P}p$
$\uparrow(p, q), \downarrow(p, q)$	$\mathbf{F}p \wedge \mathbf{F}q \wedge \mathbf{G}\neg(\mathbf{F}p \wedge \mathbf{F}q), \mathbf{P}p \wedge \mathbf{P}q \wedge \mathbf{H}\neg(\mathbf{P}p \wedge \mathbf{P}q)$
$\#p$	$\neg p \wedge \mathbf{G}\neg p \wedge \mathbf{H}\neg p$
$p \sim q$	$\Box(p \rightarrow \#q)$
$Pt(p)$	$p \wedge \mathbf{G}\neg p \wedge \mathbf{GP}\#p \wedge \mathbf{HF}\#p$
$After(p, q)$	$\Box((p \rightarrow (\mathbf{F}q \wedge \mathbf{H}\neg q)) \wedge (q \rightarrow \mathbf{P}p) \wedge ((\mathbf{P}p \wedge \mathbf{F}q) \rightarrow (p \vee q)))$

A set $S \subseteq \mathbb{R}^2$ is *spatial* if $x \neq y \in S \Rightarrow x \sim y$. For $S, T \subseteq \mathbb{R}^2$ we write $S \sim T$ when $\forall x \in S, t \in T (x \sim t)$. A set of (not necessarily spatial) sets $S_i \subseteq \mathbb{R}^2$ for $i \in I$, is spatial if $i \neq j \in I \Rightarrow S_i \sim S_j$. A set $S \subseteq \mathbb{R}^2$ is *timelike* if $s, t \in S \Rightarrow (s \leq t \vee t \leq s)$, a maximal timelike set is called a *timeline*, so all lightlines are timelines (but not conversely).

The axiom of 2-density $(\mathbf{F}p \wedge \mathbf{F}q) \rightarrow \mathbf{F}(\mathbf{F}p \wedge \mathbf{F}q)$ is not valid over $(\mathbb{R}^2, <)$, indeed it fails at a point where $\uparrow(p, q)$ holds, and this happens when p and q hold at points in the future at exactly the speed of light, but not in the same direction, and neither p nor q hold in the future at less than the speed of light, nor on the other lightline.

If $x \models_v Pt(p)$ then p holds at x but fails in the strict future and past of x , moreover there are points $y \sim x$ arbitrarily close to x on both sides, such that $y \models_v \#p$. If $\models_v After(p, q)$ then $v(p) \leq v(q)$ and no points are strictly between. The following equivalences follow from K_t

$$\#p \equiv \neg(p \vee \mathbf{F}p \vee \mathbf{P}p) \quad \#(p \vee q) \equiv (\#p \wedge \#q) \quad \#p \equiv \#\#\#p.$$

and

$$\Box(p \rightarrow q) \Rightarrow \Box(\#q \rightarrow \#p).$$

Let $R \subseteq \mathbb{R}^2$ be a rectangle. Then

$$v(p) \subseteq R \Rightarrow v(\#\#p) \subseteq R. \quad (3)$$

Rules and axioms For proof rules we use modus ponens, generalisation and substitution. Axioms for $(\mathbb{R}^2, <)$:

I. K , axioms for propositional logic plus $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$

II. Temporal $p \rightarrow \mathbf{G}\mathbf{P}p$ and $p \rightarrow \mathbf{H}\mathbf{F}p$

III. transitive, dense, serial $\mathbf{G}p \leftrightarrow \mathbf{G}\mathbf{G}p$, $\mathbf{F}T$

IV. Between from below implies between from above

$$((p_0 \sim p_1 \sim p_2 \sim p_0) \wedge \Box((\mathbf{F}p_0 \wedge \mathbf{F}p_2) \rightarrow \mathbf{F}p_1)) \rightarrow \Box((\mathbf{P}p_0 \wedge \mathbf{P}p_2) \rightarrow \mathbf{P}p_1)$$

V. ‘3 – 2’ density

$$(\bigwedge_{i < 3} \mathbf{F}p_i) \rightarrow \bigvee_{j \neq k < 3} \mathbf{F}(\mathbf{F}p_j \wedge \mathbf{F}p_k)$$

VI. Weak confluence

$$(\mathbf{P}(\mathbf{F}q \wedge \mathbf{F}r) \leftrightarrow \mathbf{F}(\mathbf{P}q \wedge \mathbf{P}r)) \upharpoonright (\#p)$$

VII. Weak confluence and downward seriality

$$((\mathbf{P}(\mathbf{F}q \wedge \mathbf{F}r) \leftrightarrow \mathbf{F}(\mathbf{P}q \wedge \mathbf{P}r)) \wedge \mathbf{P}\top) \upharpoonright (\mathbf{P} \uparrow (p, p') \wedge \mathbf{P}p \wedge \mathbf{P}p')$$

VIII. Branching points are isolated.

$$\uparrow (p, \#p) \rightarrow (\mathbf{G}\mathbf{P}\# \uparrow (p, \#p) \wedge \mathbf{H}\mathbf{F}\# \uparrow (p, \#p))$$

IX. Branching points determine linear, Dedekind complete lightlines

$$(\text{Lin}(q, r) \wedge \text{Ddk}(q)) \upharpoonright (\mathbf{P} \uparrow (p, \#p) \wedge \#p)$$

X. Branching points determine grids

$$\uparrow (p \wedge \mathbf{H}\neg p, q \wedge \mathbf{H}\neg q) \rightarrow \mathbf{F} \downarrow (p \wedge \mathbf{H}\neg p, q \wedge \mathbf{H}\neg q)$$

XI. Closed cells include left and right corners

$$[(\mathbf{H}\perp \wedge \uparrow (q, \#q) \wedge \mathbf{F}(\mathbf{G}\perp \wedge \downarrow (r, \#r))) \rightarrow (\mathbf{F}(\#q \wedge (\#r \vee \#\#r)) \wedge \mathbf{F}(\#\#q \wedge (\#\#r \vee \#r)))] \upharpoonright (\#p)$$

XII. Densely shuffled rectangles (with at least three) are closed

$$\left(\begin{array}{c} (p \sim q) \wedge \mathbf{F}p \wedge \mathbf{F}q \\ \wedge \\ \mathbf{G}((\mathbf{F}p \wedge \mathbf{F}q) \rightarrow \mathbf{F}\#(p \vee q)) \\ \wedge \\ \mathbf{G}(\#(p \vee q) \rightarrow (\mathbf{F}p \leftrightarrow \mathbf{F}q)) \end{array} \right) \rightarrow \mathbf{F}(\#(p \vee q) \wedge \mathbf{H}\neg \#(p \vee q))$$

XIII. Spatial Cauchy sequences converge

$$(\mathbf{F}p \wedge \mathbf{F}\#p \wedge \mathbf{G}(\#p \rightarrow \mathbf{P}^0(\#p \wedge \mathbf{H}\neg\#p))) \rightarrow \mathbf{F}(Pt(p) \vee Pt(\#p))$$

XIV. Neighbouring open rectangles cover common corner

$$\left(\begin{array}{c} \mathbf{F}p \wedge \mathbf{F}\#p \\ \wedge \\ \mathbf{G}(\#p \rightarrow (\#\#p'_0 \wedge \#\#p'_1)) \\ \wedge \\ \mathbf{G}(\#\#p \rightarrow (\#\#p_0 \wedge \#\#p_1)) \end{array} \right) \rightarrow \mathbf{F} \left(\begin{array}{c} \uparrow(p, \#p) \vee \downarrow(p, \#p) \\ \vee \\ (\mathbf{G}(\bigwedge_{i<2} \mathbf{P}p_i \wedge \mathbf{H}(\bigwedge_{i<2} \mathbf{F}p_i)) \uparrow(\#\#p)) \\ \vee \\ (\mathbf{G}(\bigwedge_{i<2} \mathbf{P}p_i \wedge \mathbf{H}(\bigwedge_{i<2} \mathbf{F}p_i)) \uparrow(\#p)) \end{array} \right)$$

XV. Four adjacent rectangles cover corners

$$\left(\begin{array}{c} \mathbf{F}p \wedge \mathbf{F}(\#p \wedge \mathbf{F}q \wedge \mathbf{F}\#q) \\ \wedge \\ \neg(\mathbf{F}q \wedge \mathbf{F}\#q \wedge \mathbf{P}p \wedge \mathbf{P}\#p) \\ \wedge \\ \mathbf{G}(\#p \rightarrow (\#\#p'_0 \wedge \#\#p'_1)) \\ \wedge \\ \mathbf{G}(\#\#p \rightarrow (\#\#p_0 \wedge \#\#p_1)) \end{array} \right) \rightarrow \mathbf{F} \left(\begin{array}{c} Pt(p) \vee Pt(\#p) \vee Pt(q) \vee Pt(\#q) \\ \vee \\ \uparrow(p, \#p) \vee \downarrow(p, \#p) \\ \vee \\ \left[\begin{array}{c} (\mathbf{G} \bigwedge_{i<2} \mathbf{P}p_i \wedge \mathbf{H} \bigwedge_{i<2} \mathbf{F}p_i) \uparrow(\#\#p) \\ \wedge \\ (\mathbf{G} \bigwedge_{i<2} \mathbf{P}p'_i \wedge \mathbf{H} \bigwedge_{i<2} \mathbf{F}p'_i) \uparrow(\#p) \end{array} \right] \end{array} \right)$$

plus duals. We will prove the soundness of these axioms shortly, but first a preliminary lemma.

LEMMA 5 (Generalised Trichotomy) *A consequence of K_t plus transitivity is the validity of*

$$(\mathbf{F}p \wedge \mathbf{F}\#p) \vee (\mathbf{P}p \wedge \mathbf{P}\#p) \vee \#\#p \vee \#p,$$

and the four disjuncts are pairwise inconsistent.

PROOF:

$(\#p \vee \neg\#p) \wedge (\#\#p \vee \neg\#\#p)$ is a propositional tautology, equivalent to

$$(\#p \wedge \#\#p) \vee (\#p \wedge \neg\#\#p) \vee (\neg\#p \wedge \#\#p) \vee (\neg\#p \wedge \neg\#\#p) \quad (4)$$

The first three disjuncts of (4) are equivalent to \perp , $\#p$, $\#\#p$, respectively, using $\neg(p \wedge \#p)$ and $p \rightarrow \#\#p$. The last disjunct, $(\neg\#p) \wedge (\neg\#\#p)$, is equivalent to

$$(p \vee \mathbf{F}p \vee \mathbf{P}p) \wedge (\#p \vee \mathbf{F}\#p \vee \mathbf{P}\#p).$$

The term $(p \vee \mathbf{F}p \vee \mathbf{P}p) \wedge \#p$ is inconsistent, the term $p \wedge (\#p \vee \mathbf{F}\#p \vee \mathbf{P}\#p)$ is also inconsistent, using $p \rightarrow \#\#p$. Terms $\mathbf{F}p \wedge$

$\mathbf{P}\#p$, $\mathbf{P}p \wedge \mathbf{F}\#p$ are inconsistent by propositional logic, transitivity and the definition of $\#$. So the last disjunct of (4) is equivalent to

$$(\mathbf{F}p \wedge \mathbf{F}\#p) \vee (\mathbf{P}p \wedge \mathbf{P}\#p)$$

and it follows that the tautology (4) is equivalent to $\#p \vee \#\#p \vee (\mathbf{F}p \wedge \mathbf{F}\#p) \vee (\mathbf{P}p \wedge \mathbf{P}\#p)$, as required.

Inconsistency of any pair of disjuncts follows from $\neg(p \wedge \#p)$ and the definition of $\#$.

□

Soundness

THEOREM 6 (Soundness) *Axioms (I)–(XV), above, are valid over $(\mathbb{R}^2, <)$.*

PROOF:

Let $v : Prop \rightarrow \wp(\mathbb{R}^2)$ be an arbitrary propositional valuation. For brevity, we write $x \models_v \psi$ instead of $(\mathbb{R}^2, <), x \models_v \psi$, and $\models_v \psi$ instead of $(\mathbb{R}^2, <) \models_v \psi$. We check each axiom.

- (I)–(III) These axioms are valid since $(\mathbb{R}^2, <)$ is transitive, dense, serial.
- (IV) Assume the premises, and suppose also that $x \models_v (\mathbf{P}p_0 \wedge \mathbf{P}p_2)$ for some $x \in \mathbb{R}^2$, we must show that $x \models_v \mathbf{P}p_1$. To prove this, since $x \models_v \mathbf{P}p_0 \wedge \mathbf{P}p_2$ there are $y_0, y_1 < x$ where $y_i \models_v p_i$, and by a premise $y_i \sim y_j$ for $i \neq j < 3$. Then $y_0 \wedge y_2 \models_v \mathbf{F}p_0 \wedge \mathbf{F}p_2$, so $y_0 \wedge y_2 \models_v \mathbf{F}p_1$, by the other premise. Hence there is $z > y_0 \wedge y_2$ where $z \sim y_0$, $z \sim y_2$ and $z \models_v p_1$, it follows that z is in the interior of the rectangle with corners $y_0 \wedge y_2$, y_0 , y_2 , $y_0 \vee y_2$, so $z < y_0 \vee y_2 \leq x$, proving $x \models_v \mathbf{P}p_1$, as required.
- (V) Assume the premise holds at $x \in \mathbb{R}^2$ under v , so there are $x_i > x$ where $x_i \in v(p_i)$, for $i < 3$. There are just two future lightlines through x , so it is impossible that each of the three x_i belong to different lightlines through x , so one of the lightlines l includes neither x_j nor x_k for some $j \neq k < 3$. It follows that x_j and x_k are above a point $y > x$ on the other lightline, strictly above x . So, $y \models_v \mathbf{F}p_j \wedge \mathbf{F}p_k$, hence $x \models_v \mathbf{F}(\mathbf{F}p_j \wedge \mathbf{F}p_k)$.
- (VI) Note in passing that the unrestricted axiom, requiring weak confluence over the whole frame, is the special case where p is falsity. Observe that

$$\begin{aligned} v(\#p) &= \bigcap_{x \in v(p)} \{y \in v(p) : y \sim x\} \\ &= \bigcap_{x \in v(p)} (L(x) \cup R(x)) \\ &= \bigcup_{S \subseteq v(p)} \left(\bigcap_{x \in S} L(x) \cap \bigcap_{x \in v(p) \setminus S} R(x) \right) \end{aligned}$$

where $L(x) \sim R(x)$ are the open rectangles strictly to the left or right of x respectively, by infinite distribution. Hence $v(\#p)$ is a spatial union of rectangles. A maximal rectangle R of $v(\#p)$ satisfies $R \subseteq v(\#p)$ and is not properly contained in any rectangle in $v(\#p)$. By Zorn's lemma, $v(\#p)$ is a spatial union of maximal rectangles. Suppose $x \models_v (\mathbf{F}q \wedge \mathbf{F}r) \uparrow (\#p)$. There must be $y, z > x \in v(\#p)$ in the same maximal rectangle as x , where $y \models_v q \uparrow (\#p)$ and $z \models_v r \uparrow (\#p)$. By rectangularity, the supremum w of y and z is in the same rectangle, and $w \models_v (\mathbf{P}q \wedge \mathbf{P}r) \uparrow (\#p)$, so $x \models_v \mathbf{F}(\mathbf{P}q \wedge \mathbf{P}r) \uparrow (\#p)$, as required.

(VII) Similarly, observe that $v(\mathbf{P} \uparrow (p, p') \wedge \mathbf{P}p \wedge \mathbf{P}p')$ is rectangular, hence weakly confluent, also it has no minimal element so $\mathbf{P}\top$ holds over the rectangle.

So $v(\#\#p)$ is also a spatial union of maximal rectangles. Before continuing with our soundness proof, we have a few comments about these rectangles.

The maximal rectangles in $v(\#\#p)$ may be computed from $v(p)$ in two steps: first extend $v(p)$ to $v(p \vee (\mathbf{F}p \wedge \mathbf{P}p)) \subseteq v(\#\#p)$; secondly each connected subset $S \subseteq v(p \vee (\mathbf{F}p \wedge \mathbf{P}p))$ generates a maximal rectangle $\{x \in v(\#\#p) : \text{every timeline through } x \text{ meets } S\}$. Each maximal rectangle of $v(\#\#p)$ includes points in $v(p)$, by (3). A spatial corner x of a maximal rectangle in $v(\#\#p)$ satisfies

$$x \models_v p \vee (\mathbf{F}p \wedge \mathbf{G}p). \quad (5)$$

Similarly $v(\#p)$ is a spatial union of rectangles, spatial with $v(\#\#p)$. Thus $v(\#p) \cup v(\#\#p) = \bigcup_{\lambda \in \Lambda} R_\lambda$ for some index set Λ , where

- $R_\lambda \subseteq v(\#p)$ or $R_\lambda \subseteq v(\#\#p)$ is a maximal rectangle (for all $\lambda \in \Lambda$),
- $R_\lambda \sim R_\mu$ (for all $\lambda \neq \mu \in \Lambda$),
- $v(\#(\bigcup_{\lambda \in \Lambda} R_\lambda)) = \emptyset$ and
- If rectangle $R \subseteq v(\#p) \cup v(\#\#p)$ then either $R \subseteq v(\#p)$ or $R \subseteq v(\#\#p)$.

In view of the last point, a maximal rectangle in $v(\#p)$ is also maximal in $v(\#p) \cup v(\#\#p)$, and vice versa. Recall that two rectangles are neighbours if they share a spatial corner. So R_λ neighbours R_μ iff $\lambda \neq \mu$ and there is no $\rho \in \Lambda$ where R_ρ is spatially between R_λ and R_μ . Each rectangle may be open, partially open, or closed, and has up to four corners $\{t, b, l, r\}$, not necessarily distinct and not necessarily in the rectangle. If all corners are identical R_λ is a point $\{t\}$, else if $b = l$ and $t = r$, or $b = r$ and $t = l$ we have a light segment (open, closed or semi), else all are distinct and we have a proper rectangle (open, closed or partial).

If $p \sim q$ and no maximal rectangle in $v(\#p)$ neighbours a maximal rectangle in $v(\#q)$ then

$$v(\#\#(p \vee q)) = v(\#\#p) \cup v(\#\#q). \quad (6)$$

If $\models_v \text{After}(\#p, \#q)$ then for each maximal rectangle $R \subseteq v(\#p)$ there is a maximal rectangle $S \subseteq v(\#q)$ *adjacent* to R . Adjacent rectangles are disjoint but matching, there are three types of adjacencies: (i) two proper rectangles are adjacent if they share a common bounding edge segment, included in one rectangle but not the other, (ii) an edge segment is adjacent to a proper rectangle if the edge segment is a matching excluded boundary edge of the proper rectangle, or (iii) a singleton rectangle is adjacent to an edge segment if it is an excluded bound, of course these adjacencies may be reversed.

Apologies for the digression, we return to the proof of theorem 6.

(VIII) Suppose $x \models_v \uparrow(p, \#p)$. From the definition of $\uparrow(p, \#p)$ we know that $x^\uparrow \cap v(\uparrow(p, \#p)) = x^\downarrow \cap v(\uparrow(p, \#p)) = \{x\}$. And there are $y, z > x$ on the two lightlines through x , where $y \models_v p$, $z \models_v \#p$. Then $y^\downarrow \cap v(\uparrow(p, \#p)) = z^\downarrow \cap v(\uparrow(p, \#p)) = \emptyset$. By (3), the maximal rectangle $R(x) \subseteq v(\#\# \uparrow(p, \#p))$ is a single point $\{x\} \subseteq v(\#\# \uparrow(p, \#p))$. By maximality of $\{x\}$ there are points arbitrarily close to x on both sides in $v(\# \uparrow(p, \#p))$. Hence, $x \models_v \mathbf{GP} \# \uparrow(p, \#p) \wedge \mathbf{HF} \# \uparrow(p, \#p)$, as required.

(IX) Consider $v(\#p \wedge \mathbf{P} \uparrow(p, \#p))$, see figure 1. Let $x \in v(\#p \wedge \mathbf{P} \uparrow(p, \#p)) \subseteq v(\#p)$, let $R(x)$ be the maximal rectangle of $v(\#p)$ including x . There is $y < x$ where $y \models_v \uparrow(p, \#p)$. So p holds at some point on one of the future halfflightlines l_1 through y and $\#p$ holds at some point on the other future halfflightline l_2 through y , and neither p nor $\#p$ holds in the slower than light future of y , nor on the other halfflightline. Hence $v(\#p \wedge \mathbf{P} \uparrow(p, \#p)) \cap R(x) = l_2 \cap R(x)$, which is a linear and Dedekind complete ordered set, so the restricted axioms are true.

(X) Assume the premise holds at x , so $x \models_v \uparrow(p \wedge \mathbf{H}\neg p, q \wedge \mathbf{H}\neg q)$. There are y, z on the two lightlines through x where $y \models_v p \wedge \mathbf{H}\neg p$, $z \models_v q \wedge \mathbf{H}\neg q$. Then $(y \vee z) \models_v \downarrow(p \wedge \mathbf{H}\neg p, q \wedge \mathbf{H}\neg q)$, so the conclusion of the axiom holds at x .

(XI) The axiom is relativized to $\#p$. The premise $\mathbf{H}\perp$ can only hold at a minimal point of $v(\#p)$, at the (included) bottom corner of a maximal rectangle $R \subseteq v(\#p)$. The premise $\uparrow(q, \#q)$ implies that one lower edge of R is covered by $v(\#q)$ and the other is covered by $v(\#\#q)$. The premise $\mathbf{F}(\downarrow(r, \#r) \wedge \mathbf{G}\perp) \uparrow(\#p)$ implies that $\downarrow(r, \#r) \wedge \#p$ holds at the top corner of R , one upper edge is covered by $v(\#r)$ and the other is covered by

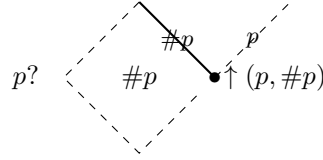


Figure 1: Axiom (IX). $v(\mathbf{P} \uparrow (p, \#p)) \cap v(\#p)$ is shown as a thick line, linear and Dedekind complete. [If there are any points in $v(p)$ where $p?$ is indicated then max rectangle in $v(\#p)$ is bounded on the left, otherwise not.]

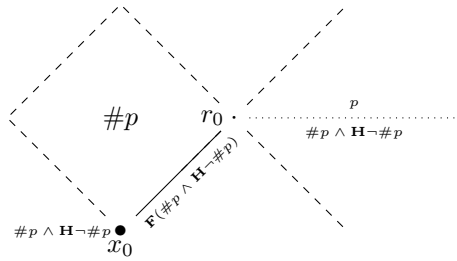


Figure 2: Points witnessing p and points witnessing $\#p \wedge \mathbf{H} \neg \#p$ arbitrarily close to the right corner of $R(x_0) \subseteq v(\#p)$

$v(\#\#r)$. Either the left or the right corner of R belongs to $v(\#p \wedge \#q \wedge (\#r \vee \#\#r))$, and the other corner belongs to $v(\#p \wedge \#\#q \wedge (\#\#r \vee \#r))$.

(XII) Assume the premise holds at x , so the three sets $v(p), v(q), v(\#(p \vee q))$ are non-empty and dense within each other, i.e. spatially between any points in two distinct sets there is a point in the third set (all restricted to points above x).

Let $y > x$, $y \models_v \#(p \vee q)$ and let $R(y) \subseteq v(\#(p \vee q))$ be the maximal rectangle containing y . We claim that $R(y)$ includes its right corner r . To prove this, suppose instead that $r \notin R(y)$. By maximality of $R(y)$ we know $r \in v(\#\#(p \vee q))$. By the premise $\mathbf{G}((\mathbf{F}p \wedge \mathbf{F}q) \rightarrow \mathbf{F}\#(p \vee q))$, a maximal rectangle in $v(\#\#p)$ can never neighbour a maximal rectangle in $v(\#\#q)$, so by (6), $r \in v(\#\#p)$ or $r \in v(\#\#q)$, without loss assume the former. Since r is the included left corner of a maximal rectangle in $v(\#\#p)$, either $r \in v(p)$ or $r \in v(\mathbf{F}p \wedge \mathbf{P}p)$. But then there are points $z < r$ below $R(y)$ on a lightline through r where $z \models_v \mathbf{F}(\#(p \vee q)) \wedge \mathbf{F}p \wedge \mathbf{G}\neg q$, contradicting the premise. This proves the claim: $R(y)$ includes r , similarly it includes its left corner too, hence it contains its bottom corner b . So $b \in R(y)$ and the consequent to the axiom is witnessed at b .

(XIII) Assume the premise holds at x and recall that $v(\#p) \cup v(\#\#p)$

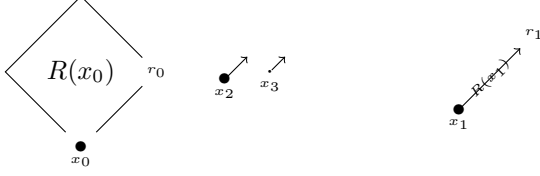


Figure 3: The start of sequence x_0, x_1, \dots where no rectangle right of $R(x_0)$ includes any point on leftgoing lightline.

is covered spatially by maximal rectangles. We claim that there is a singleton maximal rectangle $\{y\} \subseteq v(\#p) \cup v(\#\#p)$. We prove the claim shortly, but observe that at such a singleton $\{y\}$ we have $y \models_v Pt(p) \vee Pt(\#p)$, as required by the axiom.

For the claim, suppose for contradiction that none of the maximal rectangles in $(v(\#p) \cup v(\#\#p)) \cap x^\uparrow$ are points

$$x \models_v \mathbf{G} \neg (Pt(p) \vee Pt(\#p)). \quad (7)$$

By the premise each maximal rectangle of $R \subseteq v(\#p)$ includes its bottom corner, but also at least one of its spatial corners l, r is distinct from b . If $b \neq r$ we say that R points right, if $b \neq l$ then we say that R points left. Since R is not a point, it points left or right but we should not assume it points both left and right, as proper rectangles do.

We define a spatial sequence $x_0, x_1, \dots \subseteq v(\#p \wedge \mathbf{H} \neg \#p)$ as follows. Let $x_0 \in v(\#p \wedge \mathbf{H} \neg \#p)$ be arbitrary. Let $R(x_0) \subseteq v(\#p)$ be the maximal rectangle including x_0 , with corners x_0, t_0, l_0, r_0 . Without loss, suppose $R(x_0)$ points right. The whole sequence after x_0 will be to the right and within one of r_0 (if $R(x_0)$ does not point right, it points left, and the whole sequence is to the left of l_0 within one). There are points $z < y$ where $z \in v(\mathbf{F}p \wedge \mathbf{F}\#p)$, outside but arbitrarily close to $R(x_0)$, spatial with x_0 , see figure 2. So by the premise, $z \models_v (\mathbf{F}(\#p \wedge \mathbf{H} \neg \#p))$. It follows that

$$\forall \delta > 0 \exists w (w \models_v (\#p \wedge \mathbf{H} \neg \#p), |w - r_0| < \delta) \quad (8)$$

where $|w - r_0|$ denotes the Euclidean distance from w to r_0 .

If there is a point $w \models_v \#p \wedge \mathbf{H} \neg \#p$ within one of r_0 such that the maximal rectangle $R(w) \subseteq v(\#p)$ is left pointing then let x_1 be such a point. The sequence continues x_2, \dots between x_0 and x_1 where x_2 is nearer the left corner l_1 of $R(x_1)$ than to r_0 , as before. On the other hand, if for every $w \in v(\#p \wedge \mathbf{H} \neg \#p)$ within one of r_0 , the rectangle $R(w)$ is not left pointing, then let x_1 be any such w , so $R(w)$ is right pointing, illustrated in

figure 3. By (8) there is $w \in v(\#p \wedge \mathbf{H}\neg\#p)$ between x_0 and x_1 , not necessarily nearer l_1 than r_0 and we let x_2 be any such point. By current assumptions, $R(x_2)$ is right pointing, so we may continue with x_3 between the right corner r_2 of x_2 and the left corner l_1 of $R(x_1)$, and nearer r_2 than l_1 . We continue the sequence in this way so that x_{i+2} is between x_i and x_{i+1} , and the gap between $R(x_{i+2})$ and $R(x_{i+3})$ is at most half the gap between $R(x_i)$ and $R(x_{i+1})$, for $i \geq 0$.

Clearly, this defines a spatial Cauchy sequence, let x_∞ be the limit of the sequence. The sequence x_0, x_2, x_4, \dots converges to x_∞ from one side and the sequence x_1, x_3, \dots converges to x_∞ from the other side. Since $x_\infty \sim R(x_i)$ (all i) we have $x_\infty \in v(\#p) \cup v(\#\#p)$ and the maximal rectangle of $v(\#p) \cup v(\#\#p)$ including x_∞ is the singleton $\{x_\infty\} \subseteq v(\#p) \cup v(\#\#p)$. This contradicts our assumption (7), proves the claim and, as noted above, this proves that the consequent to the axiom holds at x , as required.

(XIV) Assume the premises. Let y be a point where the boundaries of $v(\mathbf{F}p)$ and $v(\mathbf{F}\#p)$ meet. By lemma 5 either $y \in v(\mathbf{F}p \wedge \mathbf{F}\#p)$, $y \in v(\mathbf{P}p \wedge \mathbf{P}\#p)$, $y \in v(\#p)$ or $y \in v(\#\#p)$. In the first two cases we get $y \models_v \uparrow(p, \#p)$ and $y \models_v \downarrow(p, \#p)$, respectively. For the third case $y \models_v \#p$, consider $R(y) \subseteq v(\#p)$ the maximal rectangle including y . Since the boundaries meet at y , it must be an included spatial corner of $R(y) \subseteq v(\#p)$. By the premise $\mathbf{G}(\#p \rightarrow \#\#p'_i)$ we have $y \in v(\#\#p'_i)$, for $i < 2$. By (5) it follows that $y \in v(p'_i)$ or $y \in v(\mathbf{F}p'_i \wedge \mathbf{P}p'_i)$. Either way, it follows that $y \models_v (\bigwedge_{i<2} \mathbf{G}p'_i \wedge \mathbf{H}p'_i) \uparrow R(y)$. Similarly, for the fourth case, $y \models_v \#\#p$ we get $y \models_v (\mathbf{G} \bigwedge_{i<2} \mathbf{P}p_i \wedge \mathbf{H} \bigwedge_{i<2} \mathbf{F}p_i) \uparrow (\#\#p)$, so the axiom holds.

(XV) Assume the premise. Consider a maximal rectangle $R \subseteq v(\#p)$. By the first premise, it is below a maximal rectangle S (say) in $v(\#q)$ and by the second premise it is adjacent to S . Similarly, R is adjacent below a maximal rectangle T in $v(\#\#q)$, S is adjacent above a maximal rectangle $U \subseteq v(\#\#p)$ and T is adjacent above a rectangle $V \subseteq v(\#\#p)$. All of that is to show that R is bound on both sides and similarly, all the maximal rectangles in either $v(\#p)$, $v(\#\#p)$, $v(\#q)$ or $v(\#\#q)$ are bound on both sides, so all four corners of these rectangles exist. If a maximal rectangle in $v(\#p) \cup v(\#\#p)$ includes both its left and right corners, it must also include its bottom corner b , so either $b \models_v (\#p \wedge \mathbf{H}\neg\#p)$ or $b \models_v (\#\#p \wedge \mathbf{H}\neg\#\#p)$. Either way, by axiom XIII, we conclude $Pt(\#p) \vee Pt(\#\#p)$, giving the first line of disjuncts in the consequent. Similarly, if a maximal rectangle in $v(\#q)$ or in $v(\#\#q)$ includes both its left and right corners, a disjunct in the first line of the consequent must hold.

So suppose all maximal rectangles omit either their left or right corner. Say $R \subseteq v(\#p)$ omits a corner γ , either left or right. By lemma 5 γ belongs to $v(\mathbf{F}p \wedge \mathbf{F}\#p), v(\mathbf{P}p \wedge \mathbf{P}\#p), v(\#p)$ or $v(\#\#p)$. If $\gamma \in v(\mathbf{P}\#p \wedge \mathbf{P}\#\#p)$ then $\gamma \models_v \downarrow (\#p, \#\#p)$, similarly $\gamma \in v(\mathbf{F}p \wedge \mathbf{F}\#p)$ implies $\gamma \models_v \uparrow (p, \#p)$, and a disjunct in the second line of the consequent holds. We can't have $\gamma \in v(\#p)$ since we are assuming that the maximal $R \subseteq v(\#p)$ omits γ . Hence $\gamma \in v(\#\#p)$ is a corner of a neighbouring rectangle $U \subseteq v(\#\#p)$. As in the previous case this implies that $\gamma \models_v (\mathbf{G}(\mathbf{P}p_0 \wedge \mathbf{P}p_1) \wedge \mathbf{H}(\mathbf{F}p_0 \wedge \mathbf{F}p_1)) \uparrow (\#\#p)$. But also, the neighbour $U \subseteq v(\#\#p)$ omits its other corner which must be the corner of a maximal rectangle of $v(\#p)$, where $(\mathbf{G}(\mathbf{P}p'_0 \wedge \mathbf{P}p'_1) \wedge \mathbf{H}(\mathbf{F}p'_0 \wedge \mathbf{F}p'_1)) \uparrow (\#p)$ holds. Hence, if the first two lines of disjuncts all fail then both conjuncts in the final disjunct must hold.

□

Completeness To prove completeness of the axioms we use a filtration method, along the lines of the proof of proposition 4. The terms and definitions of the proof of proposition 4 are unchanged in particular definitions 1, 3, equation (1) : $Cl(\phi), \mathcal{M}, <, \equiv, Clusters, I, \models_c, \mathbf{M}, Suc$ etc. The only change is that the axioms used to define consistency have changed considerably. The task is to construct a p -morphism $f : (\mathbb{R}^2, <) \rightarrow \mathcal{M}$.

By density, its still the case that one irreflexive cannot succeed another, but without linearity a node can have more than one successor and more than one predecessor, and without Dedekind completeness, one cluster may succeed another. To partially make up for this, by weak confluence and finiteness, we know that \mathbf{M} is a spatial union of *interval frames* $\mathbf{M}(u, v) = \{s \in Clusters \cup I : u \leq s \leq t\}$, where $u \leq v \in \mathbf{M}$. By seriality, these interval frames have maximal and minimal clusters.

For $S, T \subseteq \mathcal{M}$ we write $S \sim T$ when for all $s \in S, t \in T$ we have $s \sim t$. For any $S \subseteq \mathcal{M}$ we write $\#(S)$ for $\{m \in \mathcal{M} : m \sim S\}$. We say that S is *spatial* if $s \neq t \in S \rightarrow s \sim t$. Every singleton set is spatial. A set $\{S_i : i < k\}$ of (not necessarily spatial) subsets of \mathcal{M} is spatial if $i \neq j < k \rightarrow S_i \sim S_j$. For temporal formula p we write $\mathcal{M}(p)$ for the restriction of the frame to $\{f \in \mathcal{M} : f \models_c p\}$.

LEMMA 7 (Non-lattice) *Suppose $\models_c Suc(s_0, t_0) \wedge Suc(s_0, t_1) \wedge Suc(s_1, t_0) \wedge Suc(s_1, t_1)$ and $s_0 \sim s_1, t_0 \sim t_1$, for some $s_0, s_1, t_0, t_1 \in \mathbf{M}$. Also suppose that s_0 is irreflexive. Then s_1, t_0, t_1 are clusters and*

$$s_0 \equiv \uparrow (t_0, t_1), s_1 \equiv (\mathbf{F}t_0 \wedge \mathbf{F}t_1 \wedge \mathbf{G}\neg \uparrow (t_0, t_1)) \equiv \#s_0$$

PROOF:

Suppose s_0 is irreflexive. Since $s_0 < t_0, t_1$ we have $s_0 \models_c \mathbf{F}t_0 \wedge \mathbf{F}t_1$. Since s_0 is irreflexive, $s_0 \models_c \mathbf{G}\neg s_0$, hence $s_0 \models_c \mathbf{G}\neg(\mathbf{F}t_0 \wedge \mathbf{F}t_1)$.

$\mathbf{F}t_1$), so $s_0 \models_c \uparrow (t_0, t_1)$. Since they succeed an irreflexive and by density, t_0, t_1 are clusters. And $s_1 \sim \uparrow (t_0, t_1)$, $s_1 < t_0, t_1$ implies $s_1 \models_c \#s_0 \wedge \mathbf{F}t_0 \wedge \mathbf{F}t_1$. Finally, s_1 cannot be irreflexive, else $s_1 \models_c \uparrow (t_0, t_1) \wedge \# \uparrow (t_0, t_1)$, an impossibility. \square

LEMMA 8 *If $f : (\mathbb{R}^2, <) \rightarrow \mathbf{M}$ is a p -morphism where for each cluster c the set $f^{-1}(c)$ has no maximal or minimal points, then there is a p -morphism $f' : (\mathbb{R}^2, <) \rightarrow \mathcal{M}$.*

PROOF:

Define f' over $f^{-1}(c)$ by mapping a countable sequences of arbitrarily high and low points to m for each $m \in c$ (such sequences exist since the reals are Archimedean) and mapping all remaining points of $f^{-1}(c)$ to arbitrary MCSs in c , and repeat this for all clusters c . Let $f'(m) = f(m)$ for irreflexives $m \in \mathcal{M}$. Then f' is easily seen to be a p -morphism. \square

For f' constructed as in the proof and $c \in \text{Clusters}$, we may say that f' maps *densely* to cluster c .

A map $f : \mathbb{R}^2 \rightarrow \mathcal{M}$ determines a propositional valuation v_f over \mathbb{R}^2 by $v_f(p) = \{x \in \mathbb{R}^2 : p \in f(x)\}$. If $f : (\mathbb{R}^2, <) \rightarrow \mathcal{M}$ is a p -morphism then by proposition 2,

$$(\mathbb{R}^2, <), x \models_{v_f} \theta \iff \mathcal{M}, f(x) \models_c \theta \quad (9)$$

for all $x \in \mathbb{R}^2$ and all temporal formulas θ .

THEOREM 9 *Axioms (I)–(XV) are complete over $(\mathbb{R}^2, <)$.*

PROOF:

We aim to show that there is a p -morphism from $(\mathbb{R}^2, <)$ to \mathcal{M} . To get there, we prove for any $u \leq v \in \mathcal{M}$ that there is a p -morphism $f : R \rightarrow \mathcal{M}(u, v)$ for some rectangle $R \subseteq \mathbb{R}^2$. [Clearly, if R, R' are rectangles that include corresponding corners with the same equalities between them, then there is an invertible order preserving bijection $(R, <) \rightarrow (R', <)$.] The rectangle R constructed will include its top corner iff v is irreflexive, and include its bottom corner iff u is irreflexive. The proof is by induction on the size of $\mathbf{M}(u, v)$, i.e. the number of irreflexives and clusters of $\mathcal{M}(u, v)$. In the base case we have $u = v$, either a cluster or irreflexive. In the former case let R be any proper rectangle excluding its top and bottom corners and map densely from R to the MCSs in the single cluster, see lemma 8. In the latter case, let $R = \{x\}$ be a single point and map x to the single irreflexive, to obtain a p -morphism. So suppose $u \neq v$.

Let $\text{Fat}(u, v) = \{f \in \mathcal{M}(u, v) : \mathcal{M}(u, v) \models_c \neg \#f\}$, so fat elements are ordered with everything in the frame, note that $u, v \in \text{Fat}(u, v)$, and $\text{Fat}(u, v)$ is linearly ordered by $<$. For $f \in \mathcal{F}(u, v)$ we have $f \in \text{Fat}(u, v) \iff (\mathcal{F}(\#f) \cap \mathcal{F}(u, v) = \emptyset)$.

An irreflexive m cannot have more than two successors, by axiom (V). If m has two successors c, d then $m \models_c \uparrow(c, d)$ and by axiom (VIII) it is not fat, similarly if m has two predecessors it is not fat. Hence if $m \in \text{Fat}(u, v)$, $u < m < v$ is irreflexive, it has a unique successor and a unique predecessor in $\mathcal{M}(u, v)$ and these are both fat, by uniqueness. If there is a cluster $c \in \text{Fat}(u, v) \setminus \{u, v\}$, then $\mathcal{M}(u, c)$ and $\mathcal{M}(c, v)$ have p -morphisms, inductively. A p -morphism for $\mathcal{M}(u, v)$ may be obtained from a p -morphism $f : R \rightarrow \mathcal{M}(c, v)$ by picking $x \in f^{-1}(c)$, in the interior, and overwriting the interior of $R \cap x^\downarrow$ using a scaled copy of a p -morphism for $\mathcal{M}(u, c)$. So now we assume no clusters of $\mathcal{M}(u, v) \setminus \{u, v\}$ are fat.

Suppose u has an irreflexive successor m and v is a successor of m , so u, v are both clusters by the density axiom. Consider $\mathcal{M}(\#m)$. If m is fat then it is the only successor of u , $\mathcal{M}(u, v) \cap \mathcal{M}(\#m) = \emptyset$ and we define $f : \mathbb{R}^2 \rightarrow \mathcal{M}(u, v)$ by mapping all points on some extensive spatial line to m , points above the line map densely to v and points below map densely to u . Now we can assume that $\mathcal{M}(u, v) \setminus \{u, v\}$ has no fat elements at all.

We also consider the case where $\models_c \text{Suc}(u, m) \wedge \text{Suc}(m, v)$ for some $m \in I$ but m is not fat. By weak confluence (axiom VI), $\mathcal{M}(\#m)$ is a spatial union of interval frames $\mathcal{M}(\lambda^b, \lambda^t)$ where $\lambda^b \leq \lambda^t \in \mathcal{M}(\#m)$ for $\lambda \in \Lambda$, some non-empty index set. For now, suppose $|\Lambda| \geq 2$. By axiom XIII (and dual), each λ^b, λ^t is irreflexive. Define $f : R \rightarrow \mathcal{M}(u, v)$ by the following *shuffle* procedure (see [HR18, definition 3.3]). Initialise a queue with the single rectangle R , whose width is positive. Choose a maximal interval $\mathcal{M}(\lambda^b, \lambda^t) \subseteq \mathcal{M}(\#m')$. If this interval is a point, $\lambda^b = \lambda^t \in I$, then let $f(x) = \lambda^b$, and enqueue the open quadrant spatially left of x and the open quadrant spatially right of x , each rectangle is half the width of R . Otherwise, let S be the closed rectangle with centre x , one third the size of R in both dimensions. Use a p -morphism for $\mathcal{M}(\lambda^b, \lambda^t)$ to define f over S . Enqueue the open quadrant spatially left of S in R and the open quadrant spatially right of S in R , each of them is one third the width of R . Repeat this process countably often, using a fair schedule when picking rectangles from the queue and interval frames. Let f be the map defined in the limit of this procedure. Extend f to the whole of R by mapping all undefined points below the rectangles densely to u , mapping all undefined points above the rectangles densely to v , and mapping all points in the Cantor set of points spatial to all the rectangles, to m' . This defines a surjective map $f : R \rightarrow \mathcal{M}(u, v)$.

It is clear that f , so defined, is order preserving, moreover as the rectangles are all closed and all rectangles and all $\lambda \in \Lambda$ are chosen, f is a p -morphism.

To cover the case where $\Lambda = \{\lambda\}$ is a singleton, we slightly extend the shuffle procedure above to cover this case too. When choosing a

rectangle S co-central with R and one third the size, S must include its supremum iff λ^t is irreflexive, and include its infimum iff λ^b is irreflexive. If S omits its bottom corner b then it also omits either the left corner l and the edge (b, l) , or the corner r and the edge (b, r) . Then we use a p -morphism to $\mathcal{M}(\lambda^b, \lambda^t)$ suitably scaled, over S . As before we repeat countably often, fill in remainder with u below, v above and m' between, but when checking the p -morphism condition, we must also consider a point y where $f(y) = u$, on the lower boundary of some rectangle S that omits its bottom corner b . Since $u \models_c \mathbf{F}m'$ we need to find $z > y$, $f(z) = m'$. If S omits its left corner l then l will never be covered by a rectangle in the shuffle process, hence $f(l) = m'$, we may take $z = l$ and we have $y \models_c \mathbf{F}m'$. Similarly if $y \in (b, r)$ then $f(y) \models_c \mathbf{F}m'$. If $y = b$ then S omits l (and (b, l)) or r (and (b, r)), either way $f(y) \models \mathbf{F}m'$, as before. Dually, predecessors of v are witnessed, so f is a p -morphism.

So we may now assume that in $\mathcal{M}(u, v)$,

$$\models_c \neg \bigvee_{m \in I} (Suc(u, m) \wedge Suc(m, v)) \quad (10)$$

Consider $Successors(u)$, the set of successors of u . Since $u < v$, u has at least one successor. If $Successors(u) = \{v\}$ then an open p -morphism over $(0, 2) \times (0, 1)$ is easily obtained by mapping densely to u over $(0, 1] \times (0, 1)$ and mapping densely to v over $(1, 2) \times (0, 1)$, noting that the common boundary segment $\{1\} \times (0, 1)$ is open in the second coordinate.

Since u and v are the only fat elements, we may now assume that u has at least two successors. But u cannot have three or more successors without contradicting (10). To see why, suppose u has three or more successors, say $Successors(u) = \{s_i : i < k\}$, where $k \geq 3$ and $s_i \in Clusters \cup I$. For each $i < k$ let $s_i^+ \geq s_i$ be maximal in $\mathcal{M}(\#\#s_i)$, so $\mathcal{M}(\#\#s_i) = \mathcal{M}(s_i, s_i^+)$. Note that $\models_c Suc(u, s_i) \wedge Suc(s_i^+, v)$. By axiom (V), u is a cluster. The premise of axiom (XII) holds with $s_0, s_1, \#(s_0 \vee s_1)$ for $\#(p \vee q), p, q$, respectively. Hence there is an irreflexive MCS m where $m \models_c s_0 \wedge \mathbf{H} \neg s_0$. But then, by axiom XIII, there is an irreflexive m' where $m' \models_c Pt(s_0) \vee Pt(\#s_0)$. If $m' \models_c Pt(s_0)$ then $s_0 = s_0^+$ so $\models_c Suc(u, m') \wedge Suc(m', v)$, contrary to (10), but similarly if $m' \models_c Pt(\#s_0)$ then there is $0 < i < k$ where $m' \models_c (Pt(s_i))$ and as before (10) is contradicted.

So the remaining case is where u has exactly two successors. Here we break into subcases, according to whether u is irreflexive or a cluster.

Suppose u is irreflexive with two successors c, c' . Rename $m_{00} = u$. By axiom (IX), $\mathcal{M}(\mathbf{P}m_{00} \wedge \#c') = (c_0, m_{10}, c_1, \dots, c-1, m_{k-1,0})$ or $\mathcal{M}(\mathbf{P}m_{00} \wedge \#c') = (c_0, m_{10}, c_1, \dots, c_{k-1}, m_{k-1,0}, c_k)$, for some $k \geq 1$, some some clusters c_i and irreflexives m_i . Similarly $\mathcal{M}(\mathbf{P}m_{00} \wedge$

The diagram illustrates the lattice $\mathcal{M}(A_2)$. It consists of several levels of nodes connected by directed edges. The bottom-most node is m_{00} , which has two outgoing edges labeled c_0 and c'_0 leading to m_{10} and m_{01} respectively. From m_{10} and m_{01} , edges lead to m_{20} and m_{11} , and then to m_{21} . There are also edges from m_{21} to m_{12} and v . The top-most node is v .

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$\#c) = (c'_0, m_{01}, c'_1, \dots, m_{0,k'-1})$ or $\mathcal{M}(\mathbf{P}m_{00} \wedge \#c) = (c'_0, m_{01}, c'_1, \dots, m_{0,k'-1}, c_{k'})$.

The former choice applies in each case when v is irreflexive, and the latter when v is a cluster. This is illustrated in figure 4 where v is a cluster, $k = 2$, $k' = 1$. For each irreflexive $m_{i,0}$ in the first trace and $m_{0,j}$ in the second trace, there is an irreflexive $m_{i,j}$ where $m_{i,j} \models_{c\downarrow} (m_{i,0}, m_{0,j})$, by axiom (X). When v is irreflexive we have $v = m_{k-1,k'-1}$. For $i < k - 1$, $j < k - 1'$ there is an interval frame $\mathcal{M}(m_{i,j}, m_{i+1,j+1})$. If v is a cluster then there is a top external interval frame $\mathcal{M}(m_{k-1,k'-1}, v)$. Both traces end in clusters, $c_{k-1}, c'_{k'-1}$, there are also 'external' interval frames $\mathcal{M}(m_{k-1,j}, c_{k-1,j+1})$ for $j < k - 1'$ and $\mathcal{M}(m_{j,k'-1}, c'_{j+1})$ for $j < k - 1$, shown in figure 4.

If $k > 0$ or $k' > 0$, each interval frame has a p -morphism, inductively. A p -morphism for $\mathcal{M}(u, v)$ may be obtained from these p -morphisms by joining them together in a grid.

Otherwise, $k = k' = 0$. We are currently assuming that u has two successors c, c' and u is irreflexive, v might or might not be. Then $u \models_c \uparrow (c, c')$. By axiom (VII), the weak confluent law and downward serial law $\mathbf{P}\top$ holds over the restriction to $\mathbf{P} \uparrow (c, c') \wedge \mathbf{P}c \wedge \mathbf{P}c'$, so there is a single cluster e successor to both c and to c' . Inductively, there is a p -morphism $f : R \rightarrow \mathcal{M}(e, v)$, where R is open when v is a cluster, includes top corner only when v is irreflexive. Extend f by including the lower boundary of R in the domain, mapping the bottom corner to u , points on one open lightline segment through the bottom corner map densely to c , points on the other open lightline segment map densely to c' , and in the case where $v \models_{c\downarrow} (d, d')$ is irreflexive extend to left and right corners of R an MCS where $\#c \wedge (\#d \vee \#d')$ or $(\#c' \wedge (\#d' \vee \#d))$ holds, respectively, such MCSs exist by axiom XI. This completes the case where u is irreflexive and has two successors. The case where v is irreflexive with two predecessors is similar.

So now suppose $u < v$ are clusters, $\text{Successors}(u) = \{a, a'\}$ and v has two predecessors $b \sim b'$. Since there are only two successors, $\mathcal{M}(\#a)$ is weakly confluent and connected, hence actually confluent. Since $a \in \mathcal{M}(\#\#a)$ is a successor of u we know that a is the bottom element of $\mathcal{M}(\#\#a)$, let the top element be a^+ , so $\mathcal{M}(\#\#a) = \mathcal{M}(a, a^+)$, similarly $\mathcal{M}(\#a) = \mathcal{M}(a', a'^+)$ for some $a'^+ \geq a'$. Note that $\mathcal{M}(\#a) = \mathcal{M}(\#(a^+))$, $\mathcal{M}(\#(a')) = \mathcal{M}(\#(a'^+)) = \mathcal{M}(\#\#a)$.

In the first subcase we suppose $\models_c \text{Suc}(a^+, v) \wedge \text{Suc}(a'^+, v)$. There are essentially two ways this can happen, illustrated in figure 5. Suppose a or a' is irreflexive, by axiom XIII we have $\models_c \text{Pt}(a) \vee \text{Pt}(\#a)$ so there is an irreflexive successor m of u where v is a successor of m and $\text{Pt}(m)$ holds. For this case, a p -morphism $f : \mathbb{R}^2 \rightarrow \mathcal{M}(u, v)$ is defined as follows, illustrated in the first part of figure 5. Let $f(0, 0) = m$, let f map $(0, 0)^\uparrow \setminus \{(0, 0)\}$ densely to v , map $(0, 0)^\downarrow \setminus \{(0, 0)\}$ densely to u , use an open p -morphism for $\mathcal{M}(\#a)$ over the open quadrant $\{(x, y) : x < 0 < y\}$, and another

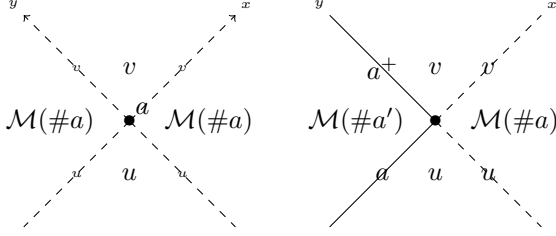


Figure 5: $\mathcal{M}(\#a) = \mathcal{M}(a', a'^+)$, $\mathcal{M}(\#a') = \mathcal{M}(a, a^+)$. Left, $\bullet \models_v Pt(a)$. Right, $\bullet \models_v (\mathbf{GPa}^+ \wedge \mathbf{HFa}) \upharpoonright (\#a')$.

copy of the p -morphism for $\mathcal{M}(\#a)$ over $\{(x, y) : y < 0 < x\}$.

Otherwise, a, a' are both clusters and we may apply axiom XIV to obtain an MCS where $\uparrow(a, \#a) \vee \downarrow(a, \#a) \vee (\mathbf{G}(\mathbf{Pa}^+ \wedge \mathbf{Pa}'^+) \wedge \mathbf{H}(\mathbf{Fa} \wedge \mathbf{Fa}')) \upharpoonright (\#\#a) \vee (\mathbf{G}(\mathbf{Pa}^+ \wedge \mathbf{Pa}'^+) \wedge \mathbf{H}(\mathbf{Fa} \wedge \mathbf{Fa}')) \upharpoonright (\#a)$. The first disjunct contradicts our assumption $Suc(u, a)$, the second contradicts $v = a \vee a'$, so the third or fourth disjunct must hold. If the third disjunct $\models_c \mathbf{F}(\mathbf{G}(\mathbf{Pa}^+ \wedge \mathbf{Pa}'^+) \wedge \mathbf{H}(\mathbf{Fa} \wedge \mathbf{Fa}')) \upharpoonright (\#\#a)$ construct an open p -morphism for $\mathcal{M}(u, v)$, illustrated in the right of figure 5, by mapping $\{(x, y) : 0 < x, 0 \leq y\}$ densely to v , mapping $\{(x, y) : x \leq 0, y < 0\}$ densely to u , using an open p -morphism for $\mathcal{M}(\#a)$ over $\{(x, y) : y < 0 < x\}$ and using a semi-open p -morphism for $\mathcal{M}(\#a')$ over $\{(x, y) : x \leq 0 \leq y\}$, so the positive y -axis map sensely to $\#a'^+$, the negative x -axis maps densely to $(\#a')$, and $f(0, 0) \models_c (\mathbf{GPa}^+ \wedge \mathbf{HFa}^+) \upharpoonright (\#\#a)$. The case where the fourth disjunct $\mathbf{G}(\mathbf{Pa}^+ \wedge \mathbf{Pa}'^+) \wedge \mathbf{H}(\mathbf{Fa} \wedge \mathbf{Fa}')) \upharpoonright (\#a)$ holds is similar.

That leaves the case where $\{\#a^+, \#a'^+\} \cap \{b, b'\} = \emptyset$. Since only u, v are fat, $(\#a)^+$ has two successors $a_1 \sim a'_1$ and these are also successors of $(\#a')^+$. Letting $(a_0, a'_0) = (a, a')$, we get a chain

$$u < \{a_0, a'_0\} < \{a_1, a'_1\} < \dots < \{a_k, a'_k\} < v$$

for some $k \geq 1$, where

- $a_i \sim a'_i$ (all $i \leq k$),
- $\mathcal{M}(\#a_i) = \mathcal{M}(a'_i, (a'_i)^+)$, $\mathcal{M}(\#\#a_i) = \mathcal{M}(a_i, a_i^+)$ (some $a_i^+ \geq a_i$, $(a'_i)^+ \geq a'_i$),
- $Suc(\#a_i^+, \#a_{i+1})$, $Suc(\#a_i^+, \#a'_{i+1})$, $Suc(\#a'_i^+, \#a_{i+1})$ and $Suc(\#a'_i^+, \#a'_{i+1})$ (for $i < k$), and
- $Suc(u, a_0)$, $Suc(u, a'_0)$, $Suc(\#a_k^+, v)$, $Suc(\#a'_k^+, v)$.

The two ways this can happen are illustrated in figure 6, for $k = 2$.

We know that $a_0 \models_c \mathbf{Fa}_1 \wedge \mathbf{Fa}'_1$. If a_0 is irreflexive $a_0 \models_c \mathbf{G}\neg(\mathbf{Fa}_1 \wedge \mathbf{Fa}'_1)$ so $a_0 \models_c \uparrow(a_1, a'_1)$. Since $a'_0 \sim a_0$, $a'_0 \models_c \mathbf{Fa}_1 \wedge \mathbf{Fa}'_1$ we must have $a'_0 \models_c \mathbf{Fa}_1 \wedge \mathbf{Fa}'_1 \wedge \mathbf{G}\neg \uparrow(a_1, a'_1)$. Since $a_2 \models_c$

$\mathbf{P}a_1 \wedge \mathbf{P}a'_1$ and $a'_2 \models_c \mathbf{P}a_1 \wedge \mathbf{P}a'_1$ are successors of a_1, a'_1 , either $a_2 \models_{c\downarrow} (a_1, a'_1)$, $a'_2 \models_c \mathbf{P}a_1 \wedge \mathbf{P}a'_1 \wedge \mathbf{H}\neg \downarrow (a_1, a'_1)$, or $a'_2 \models_{c\downarrow} (a_1, a'_1)$, $a_2 \models_c \mathbf{P}a_1 \wedge \mathbf{P}a'_1 \wedge \mathbf{H}\neg \downarrow (a_1, a'_1)$. Without loss (or by renaming) suppose $a_2 \models_{c\downarrow} (a_1, a'_1)$. By repeating this renaming, we have $a_{2i} \models_{c\downarrow} (a_{2i-1}, a'_{2i-1}) \wedge \uparrow (a_{2i+1}, a'_{2i+1})$ whenever $0 < 2i < k$. A p -morphism for the case $k = 2$ is illustrated on the left in figure 6. A p -morphism f may be constructed where $f(i, j) = a_{i+j}$ and points spatially between (i, j) and $(i + 1, j - 1)$ use a p -morphism for $\mathcal{M}(\#a_{i+j})$, whenever $i + j$ is an even integer in $[0, k]$. Also when $i + j$ is even, the open segment $((i, j), (i + 1, j))$ maps densely to $(\#a_{i+j+1})^+$ and the segment $((i, j), (i, j + 1))$ maps densely to $(\#\#a_{i+j+1})^+$. Points above all these nodes, edges and rectangles map densely to v and points below map densely to u .

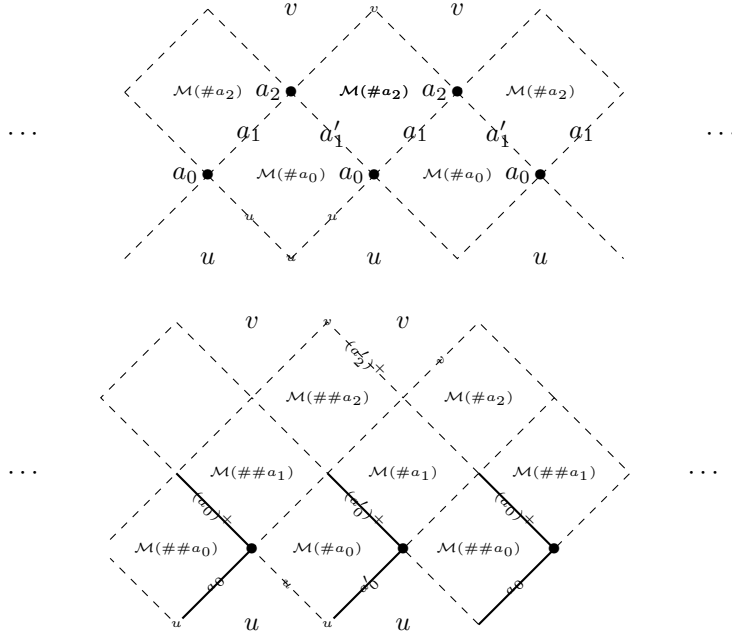


Figure 6: There is a chain $u < \{a_0, a'_0\} < \{a_1, a'_1\} < \dots < \{a_k, a'_k\} < v$, where $k \geq 1$ (here $k = 2$) for $i \leq k$ we have $a_i \sim a'_i$. On the left we have $\models_c Pt(a_i)$ when $i \in [0, k]$ is even (four corners exist and are equal), but no other rectangle includes a corner, $a_{2i} \models_{c\uparrow} (a_{2i+1}, a'_{2i+1})$ for $2i \in [0, k - 1]$ and $a_{2i} \models_{c\downarrow} (a_{2i-1}, a'_{2i-1})$ for $2i \in [1, k]$. On the right each rectangle includes its right corner only, illustrated in bold for $\mathcal{M}(\#a_0)$ and $\mathcal{M}(\#\#a_0)$, not indicated on other rectangles.

So finally suppose u has exactly two successors, both clusters. A p -morphism for this case with $k = 2$ is illustrated on the right of figure 6.

For $x \sim y \in \mathbb{R}^2$ recall that $x \vee y, x \wedge y$ denote the supremum and infimum of $\{x, y\}$. Let $x \sim y \in \mathbb{R}^2$. We write $[x, y]$ for the rectangle $\{z \in \mathbb{R}^2 : x \wedge y \leq z \leq x \vee y\}$, define a rectangle (x, y) by deleting both lightlines through x from $[x, y]$, define rectangles $[x, y), (x, y)$ similarly. A p -morphism f is constructed by using a p -morphism for $\mathcal{M}(\#a_{i+j})$ over $((i, j), (i+1, j-1))$ mapping open edge segment $((i, j), (i+1, j))$ densely to $(\#a_{i+j})$ and the segment $((i_1, j), (i+1, j+1))$ densely to $(\#a_{i+j})^+$, when $i+j$ is even and $\leq k$, noting by axiom (XV) that we can label the rightmost corner $(i+1, j-1)$ by an MCS where $\mathbf{GP}(\#a_{i+j})^+ \wedge \mathbf{HF}(\#a_{i+j})$ holds. Similarly, we use a p -morphism for $\mathcal{M}(\#a_{i+j})$ when $i+j$ is odd $\leq k$, see the second part of figure 6. Points above and below all these rectangles map densely to v, u respectively.

This completes the final case, where u has exactly two successors. Thus, $\mathcal{M}(u, v)$ has a p -morphism, for all $u \leq v \in \mathcal{M}$. Now let $u = b, v = t$ the bottom and top clusters and we have a p -morphism $f : \mathbb{R}^2 \rightarrow \mathcal{M}$. We may assume that $\phi \in f(x)$ for some $x \in \mathbb{R}^2$, then $(\mathbb{R}^2, <), x \models_v \phi$ under the valuation $v(p) = \{x \in \mathbb{R}^2 : p \in f(x)\}$. Thus, every consistent formula has a model, proving theorem 9.

□

Problems

- Eliminate any redundancies from axioms I–XV, and simplify them.
- Find sound and complete axioms for the temporal logics of the following frames: (\mathbb{R}^2, \leq) , $(\mathbb{R}^2, <)$, (\mathbb{R}^2, \preceq) where $<, \preceq$ denote slower than light accessibility.
- Find sound and complete axioms for the temporal logic of any higher dimensional frame, e.g. $(\mathbb{R}^n, <)$ for $n \geq 3$.
- Is there are temporal formula that is valid in $(\mathbb{R}^n, <)$ but not in $(\mathbb{R}^{n+1}, <)$, for any $n \geq 3$?

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