# A NOTE ON LARGE SUMS OF DIVISOR-BOUNDED MULTIPLICATIVE FUNCTIONS 

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#### Abstract

Given a multiplicative function $f$, we let $S(x, f)=\sum_{n \leq x} f(n)$ be the associated partial sum. In this note, we show that lower bounds on partial sums of divisor-bounded functions result in lower bounds on the partial sums associated to their products. More precisely, we let $f_{j}, j=1,2$ be such that $\left|f_{j}(n)\right| \leq \tau(n)^{\kappa}$ for some $\kappa \in \mathbb{N}$, and assume their partial sums satisfy $\left|S\left(x_{j}, f_{j}\right)\right| \geq \eta x_{j}\left(\log x_{j}\right)^{2^{\kappa}-1}$ for some $x_{1}, x_{2} \gg 1$ and $\eta>\max _{j}\left\{\left(\log x_{j}\right)^{-1 / 100}\right\}$. We then show that there exists $x \geq \min \left\{x_{1}, x_{2}\right\}^{\xi^{2}}$ such that $\left|S\left(x, f_{1} f_{2}\right)\right| \geq \xi x(\log x)^{2^{2 \kappa}-1}$, where $\xi=C \eta^{1+2^{\kappa+3}}$ for some absolute constant $C>0$.


## 1. Introduction

A central theme in analytic number theory is studying mean values of multiplicative functions. One enduring quest in this area, which has garnered extensive research over the past century and continues to attract significant interest to date, revolves around understanding the asymptotic behaviour of character sums. Let $\chi$ be a primitive Dirichlet character modulo $q$ and let $S(x, \chi)=\sum_{n \leq x} \chi(n)$ be the associated character sum. Trivially, we have the bound

$$
|S(x, \chi)| \leq \min (x, q)
$$

An important result proven independently by Pólya and Vinogradov [1, pages 135-137] asserts that

$$
\max _{x}|S(x, \chi)| \ll \sqrt{q} \log q,
$$

which was further improved to

$$
\max _{x}|S(x, \chi)| \ll \sqrt{q} \log \log q,
$$

by assuming the Generalized Riemann Hypothesis (GRH) for Dirichlet $L$-functions thanks to the work of Montgomery and Vaughan [15]. It follows that

$$
\begin{equation*}
S(x, \chi)=o(x) \tag{1.1}
\end{equation*}
$$

when $x \geq q^{\frac{1}{2}+\epsilon}$. In 1957, Burgess established (1.1) in the range $x>q^{\frac{1}{4}+o(1)}$ for any quadratic character $\chi$ when $q$ is prime. This was generalized by Burgess himself to any non-principal character provided that $q$ is cubefree, and in the smaller range $x>q^{\frac{3}{8}+o(1)}$ otherwise. Burgess's range has

[^0]not been substantially improved over the last five decades, although it is widely believed that (1.1) should hold in the wider range $x>_{\epsilon} q^{\epsilon}$. In 2001, Granville and Soundararajan [4] proved that, for a primitive character $\chi \bmod q$, (1.1) holds in the range $\log x / \log \log q \rightarrow \infty$ assuming the GRH for $L(s, \chi)$. In [6], they also showed that this asymptotic holds under the weaker assumption that " $100 \%$ " of the zeros of $L(s, f)$ up to height $\frac{1}{4}$ lie on the critical line. This work of Granville and Soundararajan is of particular interest to us because it makes strong connections between character sums and zeros of the associated $L$-functions using general results from multiplicative number theory pertaining to mean values of 1-bounded multiplicative functions in the pretentious framework. In [6], the authors record various other interesting observations on large character sums applying variants of Halasz's theorem and Lipschitz estimate as developed in [5]. For example, they show that if $\chi_{1}$ and $\chi_{2}$ have large character sums then so does their product $\chi_{1} \chi_{2}$. This result follows as a direct application of [6. Theorem 6.2] which asserts that if the partial sums of two completely multiplicative 1-bounded functions are large, then the partial sum of their product is large as well. In this note, we extend this theorem to multiplicative functions valued outside of the unit disk, but whose magnitude is bounded by a power of the divisor function.
Theorem 1.1. Let $\kappa$ be a positive integer. Let $f_{1}$ and $f_{2}$ be multiplicative functions with $\left|f_{j}(n)\right| \leq$ $\tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Suppose that, for $x_{1}, x_{2} \gg 1$, there exists $\eta>\max _{j}\left\{\left(\log x_{j}\right)^{-1 / 100}\right\}$ such that
\[

$$
\begin{equation*}
\left|\sum_{n \leq x_{j}} f_{j}(n)\right| \geq \eta x_{j}\left(\log x_{j}\right)^{2^{\kappa}-1} \tag{1.2}
\end{equation*}
$$

\]

Then, with $\xi=C \eta^{1+2^{\kappa+3}}$ for some absolute constant $C>0$, there exists $x \geq \min \left\{x_{1}, x_{2}\right\}^{\xi^{2}}$ such that

$$
\left|\sum_{n \leq x} f_{1}(n) f_{2}(n)\right| \geq \xi x(\log x)^{2^{2 \kappa}-1}
$$

Theorem 1.1 is proved following the general structure of the argument of [6, Theorem 6.1], but we extend it to divisor-bounded multiplicative functions. The technical results employed in the proofs of the various theorems in [6] have their roots in the work of Halász [7, 8] and the subsequent works of Montgomery [14], Tenenbaum [16], Granville and Soundararajan [5], among other mathematicians, on mean values of multiplicative functions that take their values in the complex unit disc. These results have been extended to a broad class of divisor-bounded multiplicative functions by Granville, Harper and Soundararajan in [3], Mangerel in [11, 12], and Matthiesen in [13]. Our work in the present paper and our previous work 2 hinges upon such generalizations and is motivated by trying to study a $\mathrm{GL}_{2}$-analogue of (1.1). In fact, for a primitive cusp form $g(z)=\sum_{n \geq 1} \lambda_{g}(n) n^{\frac{k-1}{2}} e^{2 \pi i n z}$ in $S_{k}(1)$, Lamzouri [9, Corollary 1.2] proved that

$$
\begin{equation*}
\sum_{n \leq x} \lambda_{g}(n)=o(x \log x) \tag{1.3}
\end{equation*}
$$

holds in the range $\log x / \log \log k \rightarrow \infty$ assuming the GRH for $L(s, g)$. He also proved unconditionally that this range in $x$ is best possible [9, Corollary 1.4]. In [2], we extend the methods of [6] to show that (1.3) holds in the range $x \geq k^{\epsilon}$ under a weaker assumption than GRH. In connection with the main result of this paper, we derive the following immediate consequence for large sums of Fourier coefficients of primitive cusp forms, which can be viewed as a GL G $_{2}$-analogue of [6, Corollary 1.7].

Corollary 1.2. Let $g \in S_{k}(r)$ and $h \in S_{\ell}(q)$ be primitive cusp forms whose Fourier coefficients are denoted by $\lambda_{g}(n)$ and $\lambda_{h}(n)$, respecctively. Suppose that

$$
\left|\sum_{n \leq x_{1}} \lambda_{g}(n)\right| \geq \eta x_{1} \log x_{1} \quad \text { and } \quad\left|\sum_{n \leq x_{2}} \lambda_{h}(n)\right| \geq \eta x_{2} \log x_{2} \text {, }
$$

for some $x_{1}, x_{2} \gg 1$ and $\eta>\max _{j}\left\{\left(\log x_{j}\right)^{-1 / 100}\right\}$. Then there exists $x \geq\left(\min \left(x_{1}, x_{2}\right)\right)^{\xi^{2}}$ such that

$$
\left|\sum_{n \leq x_{1}} \lambda_{g}(n) \lambda_{h}(n)\right| \geq \xi x(\log x)^{3}
$$

where $\xi=C \eta^{17}$ for some absolute positive constant $C$.
The paper is organized as follows. In Section 2, we record versions of Halász's Theorem and Lipschitz Formula that are suitable for our setup. We also prove some consequences of these results that are crucial for the proof of our main theorem. In Section 3, we introduce a distance function for divisor-bounded multiplicative functions, for which we prove a triangle inequality-type result. We also derive a lower bound on sums of multiplicative functions in terms of the distance function. These results form the main ingredients of the proof of Theorem 1.1, which appears in Section 4.

Conventions and Notation. In this work, we adopt the following conventions and notation. Given two functions $f(x)$ and $g(x)$ we write $f(x) \ll g(x), g(x) \gg f(x)$ or $f(x)=O(g(x))$ to mean there exists some positive constant $M$ such that $|f(x)| \leq M|g(x)|$ for $x$ large enough. The notation $f(x) \asymp g(x)$ is used when both estimates $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold simultaneously. We write $f(x)=o(g(x))$ when $g(x) \neq 0$ for sufficiently large $x$ and $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=0$. The letter $p$ will be exclusively used to represent a prime number.

## 2. Halász's Theorem and Consequences

Let $f(n)$ be a multiplicative function satisfying

$$
\begin{equation*}
|f(n)| \leq \tau(n)^{\kappa} \tag{2.1}
\end{equation*}
$$

for all $n \in \mathbb{N}$, and let $L(s, f)$ be its associated Dirichlet $L$-series given by

$$
L(s, f)=\sum_{n \geq 1} \frac{f(n)}{n^{s}}, \quad \operatorname{Re}(s)>1
$$

We begin with presenting slightly modified versions of Halász's Theorem and its corollary, following the ideas of [3, Theorem 1.1, Corollary 1.2].

Theorem 2.1. Let $f$ be a multiplicative function such than $|f(n)| \leq \tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Let $x$ be sufficiently large. Then for any $1 \ll T \leq x^{\frac{9}{10}}$, we have

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll \frac{1}{\log x} \int_{\frac{1}{\log x}}^{1}\left(\max _{|t| \leq T}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|\right) \frac{d \sigma}{\sigma}+O\left(\frac{(\log T)^{2^{\kappa}}}{\log x}+\frac{(\log x)^{2^{\kappa}}}{T}\right)
$$

Proof. The result follows from [3, Eqs. 2.1, 2.2, \& 2.9] with $y=T^{2}$.
To further proceed, let $1 \leq T_{0} \ll(\log x)^{2^{\kappa}}$, and let $M_{f}\left(x, T_{0}\right)$ be a real number satisfying

$$
\begin{equation*}
\max _{|t| \leq T_{0}}\left|\frac{L\left(1+\frac{1}{\log x}+i t, f\right)}{1+\frac{1}{\log x}+i t}\right|=e^{-M_{f}\left(x, T_{0}\right)}(\log x)^{2^{\kappa}} . \tag{2.2}
\end{equation*}
$$

Then, we obtain the following corollary.
Corollary 2.2. Suppose $f$ is a multiplicative function such that $|f(n)| \leq \tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Let $(\log x)^{2^{\kappa-1}} \ll T_{0} \ll(\log x)^{2^{k}}$ for some sufficiently large $x$, and let $M_{f}\left(x, T_{0}\right)$ be the quantity defined in (2.2). Then, we have

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll\left(M_{f}\left(x, T_{0}\right)+1\right) e^{-M_{f}\left(x, T_{0}\right)}(\log x)^{2^{\kappa}-1}+\frac{(\log x)^{2^{\kappa}-1}}{T_{0}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}
$$

Proof. We apply Theorem [2.1] with $T=T_{0} \log x$ to get

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll \frac{1}{\log x} \int_{\frac{1}{\log x}}^{1}\left(\max _{|t| \leq T}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|\right) \frac{d \sigma}{\sigma}+O\left(\frac{(\log x)^{2^{\kappa}-1}}{T_{0}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}\right) .
$$

Noting that $|f(n)| \leq \tau(n)^{\kappa}=\tau_{2}(n)^{\kappa} \leq \tau_{2^{\kappa}}(n)$, where $\tau_{\ell}$ is the $\ell$-fold divisor function (i.e., $\tau_{\ell}(n)$ is the coefficient of $n^{-s}$ in the Dirichlet series representation of $\left.\zeta(s)^{\ell}\right)$, and that $\sigma \leq 1$, we have

$$
L(1+\sigma+i t, f) \ll \zeta(1+\sigma)^{2^{\kappa}} \ll\left(\frac{1}{\sigma}\right)^{2^{\kappa}}
$$

and therefore,

$$
\max _{|t| \leq T}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|=\max _{|t| \leq T_{0}}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|+O\left(\frac{1}{\sigma^{2^{\kappa}} T_{0}}\right) .
$$

It follows that

$$
\frac{1}{\log x} \int_{\frac{1}{\log x}}^{1}\left(\max _{|t| \leq T}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|\right) \frac{d \sigma}{\sigma}=\frac{1}{\log x} \int_{\frac{1}{\log x}}^{1}\left(\max _{|t| \leq T_{0}}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|\right) \frac{d \sigma}{\sigma}+O\left(\frac{\left(\log x 2^{2^{\kappa}-1}\right.}{T_{0}}\right),
$$

and so

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll \frac{1}{\log x} \int_{\frac{1}{\log x}}^{1}\left(\max _{|t| \leq T_{0}}\left|\frac{L(1+\sigma+i t, f)}{1+\sigma+i t}\right|\right) \frac{d \sigma}{\sigma}+O\left(\frac{(\log x)^{2^{\kappa}-1}}{T_{0}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}\right) .
$$

Using the maximum modulus principle on the region $\left\{u+i v: 1+\frac{1}{\log x} \leq u \leq 2,|v| \leq T_{0}\right\}$, the rest of the proof follows by applying the same argument employed in the proof of [3, Corollary 1.2] with very minor changes. We remark here that the lower bound $T_{0} \gg(\log x)^{2^{\kappa-1}}$ is imposed so that we could use [3, Lemma 2.7] directly. In practice, we will apply this result with $T_{0} \asymp(\log x)^{2^{\kappa}}$.

Let $\phi_{f}\left(x, T_{0}\right)$ be a real number in the range $|t| \leq T_{0}$ for which the function

$$
t \mapsto\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right|
$$

attains its maximum, and let $N_{f}\left(x, T_{0}\right)$ be a real number satisfying

$$
\begin{equation*}
\left|L\left(1+\frac{1}{\log x}+i \phi_{f}\left(x, T_{0}\right), f\right)\right|=e^{-N_{f}\left(x, T_{0}\right)}(\log x)^{2^{\kappa}} \tag{2.3}
\end{equation*}
$$

As mentioned in the proof of Corollary 2.2, we will mostly be interested in the case $T_{0} \asymp(\log x)^{2^{\kappa}}$. We also note that

$$
e^{-M_{f}\left(x, T_{0}\right)}(\log x)^{2^{\kappa}}=\max _{|t| \leq T_{0}}\left|\frac{L\left(1+\frac{1}{\log x}+i t, f\right)}{1+\frac{1}{\log x}+i t}\right| \leq \max _{|t| \leq T_{0}}\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right|=e^{-N_{f}\left(x, T_{0}\right)}(\log x)^{2^{\kappa}} .
$$

It then follows that $M_{f}\left(x, T_{0}\right) \geq N_{f}\left(x, T_{0}\right)$, and so $\left(M_{f}\left(x, T_{0}\right)+1\right) e^{-M_{f}\left(x, T_{0}\right)} \leq\left(N_{f}\left(x, T_{0}\right)+1\right) e^{-N_{f}\left(x, T_{0}\right)}$. Thus, we arrive at the following result.

Corollary 2.3. Suppose $f$ is a multiplicative function such that $|f(n)| \leq \tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Let $(\log x)^{2^{\kappa-1}} \ll T_{0} \ll(\log x)^{2^{\kappa}}$ for some sufficiently large $x$, and let $N_{f}\left(x, T_{0}\right)$ be the quantity defined in (2.3). Then, we have

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll\left(N_{f}\left(x, T_{0}\right)+1\right) e^{-N_{f}\left(x, T_{0}\right)}(\log x)^{2^{\kappa}-1}+\frac{(\log x)^{2^{\kappa}-1}}{T_{0}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}
$$

We now generalize [5. Theorem 2a], which will become useful in the proof of Proposition [2.7 at the end of this section.

Proposition 2.4. Suppose $f$ is a multiplicative function such that $|f(n)| \leq \tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Let $\phi=\phi_{f}\left(x,(\log x)^{2^{\kappa}}\right)$ as in (2.3) and assume that $|\phi| \gg(\log x)^{2^{\kappa-1}}$. Then

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll \frac{(\log x)^{2^{\kappa}-1}}{|\phi|-2}+\frac{(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)+1}}{(\log x)^{1-\frac{2^{\kappa+1}}{\pi}}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}
$$

Proof. We apply Corollary [2.3 with $T_{0}=|\phi|-2$ to get

$$
\begin{gathered}
\frac{1}{x} \sum_{n \leq x} f(n) \ll\left(N_{f}(x,|\phi|-2)+1\right) e^{-N_{f}(x,|\phi|-2)}(\log x)^{2^{\kappa}-1}+\frac{(\log x)^{2^{\kappa}-1}}{|\phi|-2}+\frac{(\log \log x)^{2^{\kappa}}}{\log x} \\
\ll \frac{1}{\log x} \max _{|t| \leq|\phi|-2}\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right| \log \left(\frac{\left(\log x 2^{2^{\kappa}}\right.}{\max _{|t| \leq|\phi|-2}\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right|}\right) \\
+\frac{(\log x)^{2^{\kappa}-1}}{|\phi|-2}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}
\end{gathered}
$$

By [3, Lemma 4.1], we have

$$
\begin{aligned}
L\left(1+\frac{1}{\log x}+i t, f\right) & \ll(\log x)^{\frac{2^{\kappa+1}}{\pi}}\left(\frac{\log x}{1+|t-\phi| \log x}+(\log \log x)^{2}\right)^{2^{\kappa\left(1-\frac{2}{\pi}\right)}} \\
& \ll(\log x)^{\frac{2^{\kappa+1}}{\pi}}(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)},
\end{aligned}
$$

for all $|t| \leq|\phi|-2$. It follows that

$$
\begin{aligned}
\frac{1}{x} \sum_{n \leq x} f(n) & \ll \frac{1}{\log x}(\log x)^{\frac{2^{\kappa+1}}{\pi}}(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)} \log \left(\frac{(\log x)^{2^{\kappa}}}{\max _{|t| \leq|\phi|-2}\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right|}\right) \\
& \quad+\frac{(\log x)^{2^{\kappa}-1}}{|\phi|-2}+\frac{(\log \log x)^{2^{\kappa}}}{\log x} \\
\ll & \frac{(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)+1}}{(\log x)^{1-\frac{2^{\kappa \kappa 1}}{\pi}}}+\frac{(\log x)^{2^{\kappa}-1}}{|\phi|-2}+\frac{(\log \log x)^{2^{\kappa}}}{\log x},
\end{aligned}
$$

as required.
Next, we require the following version of Lipschitz Theorem.
Theorem 2.5 (Lipschitz Theorem). Suppose $f$ is a multiplicative function such that $|f(n)| \leq \tau(n)^{\kappa}$. Let $\phi=\phi_{f}\left(x,(\log x)^{2^{\kappa}}\right)$, as given in (2.3). Then for all $1 \leq w \leq x^{1 / 3}$ we have

$$
\begin{aligned}
&\left|\frac{1}{x} \sum_{n \leq x} f(n) n^{-i \phi}-\frac{1}{x / w} \sum_{n \leq x / w} f(n) n^{-i \phi}\right| \\
&<\log \left(\frac{\log x}{\log e w}\right)\left(\frac{\log w+(\log \log x)^{2}}{\log x}\right)^{\min \left\{2^{\kappa}\left(1-\frac{2}{\pi}\right), 1\right\}}(\log x)^{2^{\kappa}-1}
\end{aligned}
$$

Proof. The proof is very similar to that of [3, Theorem 1.5], and the reader is referred to their detailed exposition.

Corollary 2.6. Let $f$ and $\phi$ be as in Theorem 2.5. Then

$$
\frac{1}{x} \sum_{n \leq x} f(n)=\frac{x^{i \phi}}{1+i \phi} \cdot \frac{1}{x} \sum_{n \leq x} f(n) n^{-i \phi}+O\left(E_{\kappa}(x)\right)
$$

where

$$
E_{\kappa}(x)=\left\{\begin{array}{l}
(\log x)^{-1+4 / \pi}(\log \log x)^{5-8 / \pi} \quad \text { if } \kappa=1  \tag{2.4}\\
(\log x)^{2^{\kappa}-2}(\log \log x)^{3} \quad \text { if } \kappa>1 .
\end{array}\right.
$$

Proof. We will show the equivalent statement that

$$
\frac{1}{x} \sum_{n \leq x} f(n) n^{-i \phi}=\frac{1+i \phi}{x^{1+i \phi}} \sum_{n \leq x} f(n)+O\left(|\phi| E_{\kappa}(x)\right) .
$$

By partial summation, we have

$$
\frac{1}{x} \sum_{n \leq x} f(n) n^{-i \phi}=\frac{1}{x} \int_{1}^{x} u^{-i \phi} d\left\{\sum_{n \leq x} f(n)\right\}=\frac{1}{x^{1+i \phi}} \sum_{n \leq x} f(n)+\frac{i \phi}{x} \int_{1}^{x} \frac{1}{u^{1+i \phi}} \sum_{n \leq u} f(n) d u
$$

We split the integral into two pieces as follows:

$$
\int_{1}^{x} \frac{1}{u^{1+i \phi}} \sum_{n \leq u} f(n) d u=\int_{1}^{x /(\log x)^{2}} \frac{1}{u^{1+i \phi}} \sum_{n \leq u} f(n) d u+\int_{x /(\log x)^{2}}^{x} \frac{1}{u^{1+i \phi}} \sum_{n \leq u} f(n) d u
$$

For the first integral, we apply the trivial bound

$$
\begin{equation*}
\sum_{n \leq x}|f(n)| \leq \sum_{n \leq x} \tau_{2^{\kappa}}(n) \ll x(\log x)^{2^{\kappa}-1} \tag{2.5}
\end{equation*}
$$

(see [10] for example) to get

$$
\begin{aligned}
\frac{i \phi}{x} \int_{1}^{x /(\log x)^{2}} \frac{1}{u^{1+i \phi}} \sum_{n \leq u} f(n) d u & \ll \frac{|\phi|}{x} \int_{1}^{x /(\log x)^{2}} \frac{1}{u} \sum_{n \leq u}|f(n)| d u \\
& \leq \frac{|\phi|}{x} \int_{1}^{x /(\log x)^{2}}(\log u)^{2^{\kappa}-1} d u \ll|\phi| E_{\kappa}(x)
\end{aligned}
$$

Since $w=x / u$ is in the range of Theorem [2.5, the second integral is equal to

$$
\begin{aligned}
& \frac{i \phi}{x} \int_{x /(\log x)^{2}}^{x}\left(\frac{1}{x^{1+i \phi}} \sum_{n \leq x} f(n)+O\left(\log \left(\frac{\log x}{\log e w}\right)\left(\frac{\log w+(\log \log x)^{2}}{\log x}\right)^{\min \left\{2^{\kappa}\left(1-\frac{2}{\pi}\right), 1\right\}}(\log x)^{2^{\kappa}-1}\right)\right) d u \\
= & \frac{i \phi}{x}\left(\frac{1}{x^{1+i \phi}} \sum_{n \leq x} f(n)\right) \int_{x /(\log x)^{2}}^{x} d u+\frac{i \phi}{x} \cdot O\left(\log \log x\left(\frac{(\log \log x)^{2}}{\log x}\right)^{\min \left\{2^{\kappa}\left(1-\frac{2}{\pi}\right), 1\right\}}(\log x)^{2^{\kappa}-1}\right) \int_{x /(\log x)^{2}}^{x} d u \\
= & \frac{i \phi}{x^{1+i \phi}} \sum_{n \leq x} f(n)+O\left(|\phi| E_{\kappa}(x)\right) .
\end{aligned}
$$

Combining the two integrals gives the desired result.
We now utilize Proposition 2.4 (when $|\phi|$ is large) and Corollary 2.6 (when $|\phi|$ is small) to get the proposition that will be used in our argument.
Proposition 2.7. Let $f$ be a multiplicative function such that $|f(n)| \leq \tau(n)^{\kappa}$, and let $N=N_{f}\left(x,(\log x)^{2^{\kappa}}\right)$ and $\phi=\phi_{f}\left(x,(\log x)^{2^{\kappa}}\right)$ for sufficiently large $x$. Then,

$$
\frac{1}{x} \sum_{n \leq x} f(n) \ll(\log x)^{2^{\kappa}-1}\left(\frac{(N+1) e^{-N}}{1+|\phi|}+\frac{E_{\kappa}(x)}{(\log x)^{2^{\kappa}-1}}\right),
$$

where $E_{\kappa}(x)$ is given in (2.4).

Proof. Suppose that $|\phi| \geq \frac{1}{2}(\log x)^{2^{\kappa}}$. By Proposition [2.4, we have
$\frac{1}{x} \sum_{n \leq x} f(n) \ll \frac{(\log x)^{2^{\kappa}-1}}{(\log x)^{2^{\kappa}}}+\frac{(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)+1}}{(\log x)^{1-\frac{2^{\kappa+1}}{\pi}}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x} \ll \frac{(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)+1}}{(\log x)^{1-\frac{2^{\kappa+1}}{\pi}}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}$.
It follows that

$$
\frac{1}{x} \sum_{n \leq x} f(n)-(\log x)^{2^{\kappa}-1} \frac{(N+1) e^{-N}}{1+|\phi|} \ll \frac{(\log \log x)^{2^{\kappa+1}\left(1-\frac{2}{\pi}\right)+1}}{(\log x)^{1-\frac{2^{\kappa+1}}{\pi}}}+\frac{(\log \log x)^{2^{\kappa}}}{\log x} \ll E_{\kappa}(x) .
$$

Hence, we may assume that $|\phi| \leq \frac{1}{2}(\log x)^{2^{\kappa}}$. Applying Corollary 2.3 to the function $g(n)=f(n) n^{-i \phi}$ with $T_{0}=\frac{1}{2}(\log x)^{2^{\kappa}}$ gives

$$
\begin{align*}
\frac{1}{x} \sum_{n \leq x} f(n) n^{-i \phi}=\frac{1}{x} \sum_{n \leq x} g(n) & \ll\left(N_{g}\left(x, \frac{(\log x)^{2^{\kappa}}}{2}\right)+1\right) e^{-N_{g}\left(x, \frac{1}{2}(\log x)^{2^{\kappa}}\right)}(\log x)^{2^{\kappa}-1}+\frac{(\log \log x)^{2^{\kappa}}}{\log x} \\
& =(N+1) e^{-N}(\log x)^{2^{\kappa}-1}+\frac{(\log \log x)^{2^{\kappa}}}{\log x} . \tag{2.6}
\end{align*}
$$

The last line follows from the fact that $N_{g}\left(x, \frac{1}{2}(\log x)^{2^{\kappa}}\right)=N_{f}\left(x,(\log x)^{2^{\kappa}}\right)=N$. Indeed, this can be observed as following: Since $L(s, g)=L(s+i \phi, f)$, we have

$$
\begin{aligned}
\max _{|t|<\frac{1}{2}(\log x)^{2^{\kappa}}}\left|L\left(1+\frac{1}{\log x}+i t, g\right)\right| & =\max _{|t|<\frac{1}{2}(\log x)^{2^{\kappa}}}\left|L\left(1+\frac{1}{\log x}+i(t+\phi), f\right)\right| \\
& =\max _{|t|<(\log x)^{2^{2}}}\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right| .
\end{aligned}
$$

Applying Corollary 2.6 and (2.6), we get

$$
\begin{aligned}
\frac{1}{x} \sum_{n \leq x} f(n) & \ll\left|\frac{x^{i \phi}}{1+i \phi}\right|\left((N+1) e^{-N}(\log x)^{2^{\kappa}-1}+\frac{(\log \log x)^{2^{\kappa}}}{\log x}\right)+O\left(E_{\kappa}(x)\right) \\
& \ll \frac{(N+1) e^{-N}}{1+|\phi|}(\log x)^{2^{\kappa}-1}+O\left(\frac{(\log \log x)^{2^{\kappa}}}{\log x}\right)+O\left(E_{\kappa}(x)\right)
\end{aligned}
$$

The first error term is subsumed into the second one since it is smaller.
Before concluding this section, we establish the following lemma, which will play a pivotal role in Section 4 ,
Lemma 2.8. Let $f$ be a multiplicative function such that $|f(n)| \leq \tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Let $N=$ $N_{f}\left(x,(\log x)^{2^{\kappa}}\right)$ and $\phi=\phi_{f}\left(x,(\log x)^{2^{\kappa}}\right)$ as in (2.3). Suppose that there exists $x \gg 1$ and $\eta>$ $(\log x)^{-1 / 100}$ such that

$$
\begin{equation*}
\left|\sum_{n \leq x} f(n)\right| \geq \eta x(\log x)^{2^{\kappa}-1} \tag{2.7}
\end{equation*}
$$

Then,

$$
|\phi| \ll \frac{1}{\eta} \quad \text { and } \quad N \leq 2 \log \left(\frac{1}{\eta}\right)+O(1) .
$$

Proof. By Proposition [2.7, we know that

$$
\left|\sum_{n \leq x} f(n)\right| \ll x(\log x)^{2^{\kappa}-1}\left(\frac{(N+1) e^{-N}}{1+|\phi|}+\frac{E_{\kappa}(x)}{(\log x)^{2^{\kappa}-1}}\right) .
$$

Therefore, together with our assumption (2.7), we have

$$
\begin{equation*}
\eta \leq \frac{C(N+1) e^{-N}}{1+|\phi|}+\frac{C E_{\kappa}(x)}{(\log x)^{2^{\kappa}-1}}, \tag{2.8}
\end{equation*}
$$

for some absolute constant $C>0$. Writing the second term on the right-hand side as $A_{\kappa}(x)$, the inequality (2.8) can be written as

$$
(1+|\phi|)\left(\eta-A_{\kappa}(x)\right) \leq C(N+1) e^{-N} .
$$

Since $\eta \geq 2 A_{\kappa}(x)$ for sufficiently large $x$, we see that

$$
\begin{equation*}
1+|\phi| \leq \frac{C(N+1) e^{-N}}{\eta-A_{\kappa}(x)} \leq \frac{2 C(N+1) e^{-N}}{\eta} \tag{2.9}
\end{equation*}
$$

In particular, $|\phi| \ll \frac{1}{\eta}$ and $\eta \ll e^{-\frac{N}{2}}$. It follows that $N \leq 2 \log \left(\frac{1}{\eta}\right)+O(1)$ as desired.

## 3. Technical Results

While the notion of a distance between multiplicative functions makes most sense in the context of functions taking values in the complex unit disc, we adapt the standard definition (see for example [6. Section 2]) and set the following notation for a distance function associated with multiplicative functions $f$ satisfying (2.1):

$$
\begin{equation*}
\mathbb{D}^{2}\left(f, n^{i t} ; x\right)=\sum_{p \leq x} \frac{2^{\kappa}-\operatorname{Re}\left(f(p) p^{-i t}\right)}{p} . \tag{3.1}
\end{equation*}
$$

The distance function $\mathbb{D}^{2}\left(f, n^{i t} ; x\right)$ is related to the Dirichlet series $L(s, f)$ via

$$
\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right| \asymp(\log x)^{2^{\kappa}} \exp \left(-\mathbb{D}^{2}\left(f, n^{i t} ; x\right)\right)
$$

which can be observed in the following way. We have

$$
\begin{align*}
\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right| & =\exp \left(\sum_{p \leq x} \frac{\operatorname{Re}\left(f(p) p^{-i t}\right)}{p}+O(1)\right)  \tag{3.2}\\
& =\exp \left(\sum_{p \leq x} \frac{2^{\kappa}}{p}-\sum_{p \leq x} \frac{2^{\kappa}-\operatorname{Re}\left(f(p) p^{-i t}\right)}{p}+O(1)\right) \\
& \asymp(\log x)^{2^{\kappa}} \exp \left(-\sum_{p \leq x} \frac{2^{\kappa}-\operatorname{Re}\left(f(p) p^{-i t}\right)}{p}\right) \\
& =(\log x)^{2^{\kappa}} \exp \left(-\mathbb{D}^{2}\left(f, n^{i t} ; x\right)\right)
\end{align*}
$$

where the first equality follows from [11, Lemma 2.2.15].
We note that this distance function is related to $N=N_{f}\left(x,(\log x)^{2^{\kappa}}\right)$, defined in (2.3), in the following way:

$$
\begin{equation*}
e^{-N}=\frac{1}{(\log x)^{2^{\kappa}}}\left|L\left(1+\frac{1}{\log x}+i \phi, f\right)\right| \asymp \exp \left(-\mathbb{D}^{2}\left(f, n^{i \phi} ; x\right)\right) . \tag{3.3}
\end{equation*}
$$

It is known that the standard distance function between 1-bounded multiplicative functions satisfies a triangle inequality. While such a result does not hold in our setting, we next establish a triangle-type inequality satisfied by the function $\mathbb{D}\left(f, n^{i t} ; x\right)$ we introduced in (3.1).

Lemma 3.1. For $j=1,2$, let $f_{j}$ be a multiplicative function such that $\left|f_{j}(n)\right| \leq \tau(n)^{\kappa}$, and let $t_{j}$ be a real number. We have

$$
\mathbb{D}\left(f_{1}, n^{i t_{1}} ; x_{1}\right)+\mathbb{D}\left(f_{2}, n^{i t_{2}} ; x_{2}\right) \geq \frac{1}{\sqrt{2^{\kappa}}} \mathbb{D}\left(f_{1} f_{2}, n^{i\left(t_{1}+t_{2}\right)} ; \min \left\{x_{1}, x_{2}\right\}\right),
$$

where the distance function on the right-hand side is defined with respect to the condition $\left|f_{1}(n) f_{2}(n)\right| \leq$ $\tau(n)^{2 \kappa}$.

Proof. For simplicity, let us denote $\mathbb{D}_{j}=\mathbb{D}\left(f_{j}, n^{i t_{j}} ; x_{j}\right)$ for $j=1,2, x=\min \left\{x_{1}, x_{2}\right\}$, and $\mathbb{D}=$ $\mathbb{D}\left(f_{1} f_{2}, n^{i\left(t_{1}+t_{2}\right)} ; x\right)$. Then, we have

$$
\begin{aligned}
\left(\mathbb{D}_{1}+\mathbb{D}_{2}\right)^{2}= & \sum_{p \leq x_{1}} \frac{2^{\kappa}-\operatorname{Re}\left(f_{1}(p) p^{-i t_{1}}\right)}{p}+\sum_{p \leq x_{2}} \frac{2^{\kappa}-\operatorname{Re}\left(f_{2}(p) p^{-i t_{2}}\right)}{p} \\
& +2\left(\sum_{p \leq x_{1}} \frac{2^{\kappa}-\operatorname{Re}\left(f_{1}(p) p^{-i t_{1}}\right)}{p}\right)^{1 / 2}\left(\sum_{p \leq x_{2}} \frac{2^{\kappa}-\operatorname{Re}\left(f_{2}(p) p^{-i t_{2}}\right)}{p}\right)^{1 / 2} \\
\geq & \sum_{p \leq x} \frac{2^{\kappa+1}-\operatorname{Re}\left(f_{1}(p) p^{-i t_{1}}\right)-\operatorname{Re}\left(f_{2}(p) p^{-i t_{2}}\right)}{p} \\
& +2\left(\sum_{p \leq x} \frac{2^{\kappa}-\operatorname{Re}\left(f_{1}(p) p^{\left.-i t_{1}\right)}\right.}{p}\right)^{1 / 2}\left(\sum_{p \leq x} \frac{2^{\kappa}-\operatorname{Re}\left(f_{2}(p) p^{-i t_{2}}\right)}{p}\right)^{1 / 2} \\
\geq & \sum_{p \leq x} \frac{2^{\kappa}}{p}\left[2-\operatorname{Re}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right)-\operatorname{Re}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right)\right] \\
& \quad+\sum_{p \leq x} \frac{2^{\kappa}}{p} \sqrt{2\left(1-\operatorname{Re}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right)\right)} \sqrt{2\left(1-\operatorname{Re}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right)\right)} \\
\geq & \sum_{p \leq x} \frac{2^{\kappa}}{p}\left[2-\operatorname{Re}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right)-\operatorname{Re}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right)+\left|\operatorname{Im}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right) \operatorname{Im}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right)\right|\right]
\end{aligned}
$$

where the last inequality follows from $\left|\frac{f_{j}(p)}{2^{k}} p^{-i t_{j}}\right| \leq 1$ for $j=1,2$. This fact also gives us

$$
\begin{aligned}
& 2-\operatorname{Re}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right)-\operatorname{Re}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right)+\left|\operatorname{Im}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right) \operatorname{Im}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i \phi_{2}}\right)\right| \\
& \geq 1-\operatorname{Re}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right) \operatorname{Re}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right)+\operatorname{Im}\left(\frac{f_{1}(p)}{2^{\kappa}} p^{-i t_{1}}\right) \operatorname{Im}\left(\frac{f_{2}(p)}{2^{\kappa}} p^{-i t_{2}}\right) \\
&\left.=1-\operatorname{Re}\left(\frac{\left.f_{1}(p) f_{2}(p)\right)}{2^{2 \kappa}} p^{-i\left(t_{1}+t_{2}\right.}\right)\right) .
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\left(\mathbb{D}_{1}+\mathbb{D}_{2}\right)^{2} & \left.\geq \sum_{p \leq x} \frac{2^{\kappa}}{p}\left(1-\operatorname{Re}\left(\frac{\left.f_{1}(p) f_{2}(p)\right)}{2^{2 \kappa}} p^{-i\left(\phi_{1}+\phi_{2}\right.}\right)\right)\right) \\
& =\frac{1}{2^{\kappa}} \sum_{p \leq x} \frac{2^{2 \kappa}-\operatorname{Re}\left(f_{1}(p) f_{2}(p) p^{-i\left(\phi_{1}+\phi_{2}\right)}\right)}{p}=\frac{1}{2^{\kappa}} \mathbb{D}^{2},
\end{aligned}
$$

as desired.

To conclude this section, we exhibit lower bounds of sums of multiplicative functions in terms of the distance function. More precisely, we show that if the distance $\mathbb{D}\left(f, n^{i \psi} ; x\right)$ is small then the partial sums of $f$ get large in suitable ranges.

Proposition 3.2. Let $f$ be a multiplicative function satisfying $|f(n)| \leq \tau(n)^{\kappa}$ for all $n \in \mathbb{N}$. Let $\psi=\psi_{f}\left(x,(\log x)^{2^{k}}\right)$ be a number in the range $|t| \leq(\log x)^{2^{\kappa}}$ where the maximum in (2.2) is attained. Let $x \gg 1$ and set $\lambda=\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)+\log (1+|\psi|)+c$ where $c$ is a suitably large constant. Then, there exists $y \in\left[x^{\gamma}, x\right]$, with $\gamma=1 /\left(\lambda e^{\lambda}\right)$, such that

$$
\begin{equation*}
\left|\sum_{n \leq y} f(n)\right| \gg \frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right)}{1+|\psi|} y(\log y)^{2^{\kappa}-1} \tag{3.4}
\end{equation*}
$$

Proof. The strategy of the proof is to produce contradictory upper and lower bounds on the quantity

$$
\left|L\left(1+\frac{2 \lambda}{\log x}+i \psi, f\right)\right|
$$

under the assumption that (3.4) fails.

For the lower bound, we observe that

$$
\begin{align*}
\left|L\left(1+\frac{2 \lambda}{\log x}+i \psi, f\right)\right| & =\exp \left(\sum_{p \leq x^{\frac{1}{2 \lambda}}} \frac{\operatorname{Re}\left(f(p) p^{-i \psi}\right)}{p}+O(1)\right) \\
& \geq \exp \left(\sum_{p \leq x^{\frac{1}{2 \lambda}}} \frac{\operatorname{Re}\left(f(p) p^{-i \psi}\right)}{p}-\sum_{x^{\frac{1}{2 \lambda}}<p \leq x} \frac{2^{\kappa}-\operatorname{Re}\left(f(p) p^{-i \psi}\right)}{p}+O(1)\right) \\
& \geq \exp \left(\sum_{p \leq x} \frac{\operatorname{Re}\left(f(p) p^{-i \psi}\right)}{p}-\sum_{x^{\frac{1}{2 \lambda}} \leq p \leq x} \frac{2^{\kappa}}{p}+O(1)\right)  \tag{3.5}\\
& =(2 \lambda)^{-2^{\kappa}} \exp \left(\sum_{p \leq x} \frac{\operatorname{Re}\left(f(p) p^{-i \psi}\right)}{p}+O(1)\right) \\
& =(2 \lambda)^{-2^{\kappa}}(\log x)^{2^{\kappa}} \exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right) .
\end{align*}
$$

We now move to the upper bound. Set $\delta:=\frac{2 \lambda}{\log x}$. From the Mellin transform representation of Dirichlet series, we get

$$
\begin{equation*}
|L(1+\delta+i \psi, f)| \leq|1+\delta+i \psi|\left(\int_{1}^{\infty} \frac{1}{y^{2+\delta}}\left|\sum_{n \leq y} f(n)\right| d y\right) \tag{3.6}
\end{equation*}
$$

Applying (2.5), the integral on the right-hand side of (3.6) is bounded by a constant multiple of

$$
\int_{1}^{\infty} y^{-1-\delta}(\log y)^{2^{\kappa}-1} d y
$$

Observe that

$$
\int y^{-1-\delta}(\log y)^{2^{\kappa}-1} d y=-\frac{P(\delta \log y)}{\delta^{2^{\kappa}} y^{\delta}}
$$

where

$$
P(X):=\left(2^{\kappa}-1\right)!\sum_{k=0}^{2^{\kappa}-1} \frac{X^{k}}{k!}
$$

For contradiction, we now assume that the upper bound

$$
\begin{equation*}
\left|\sum_{n \leq y} f(n)\right| \leq e^{-\lambda} y(\log y)^{2^{\kappa}} \tag{3.7}
\end{equation*}
$$

holds for all $y \in\left[x^{\gamma}, x\right]$, and split the integral into three parts, so that

$$
\int_{1}^{\infty} y^{-1-\delta}(\log y)^{2^{\kappa}-1} d y=\left[-\frac{P(\delta \log y)}{\delta^{2^{\kappa}} y^{\delta}}\right]_{1}^{x^{\gamma}}+\left[-e^{-\lambda} \frac{P(\delta \log y)}{\delta^{2^{\kappa}} y^{\delta}}\right]_{x^{\gamma}}^{x}+\left[-\frac{P(\delta \log y)}{\delta^{2^{\kappa}} y^{\delta}}\right]_{x}^{\infty}
$$

Noting that $P(0)=\left(2^{\kappa}-1\right)!$ and recalling that $\delta:=\frac{2 \lambda}{\log x}$, we get

$$
\begin{aligned}
\int_{1}^{\infty} y^{-1-\delta}(\log y)^{2^{\kappa}-1} d y & =\frac{\left(2^{\kappa}-1\right)!}{\delta^{2^{\kappa}}}-\frac{P(\delta \gamma \log x)}{\delta^{2^{\kappa}} x^{\delta \gamma}}+e^{-\lambda} \frac{P(\delta \gamma \log x)}{\delta^{2^{\kappa}} x^{\delta \gamma}}-e^{-\lambda} \frac{P(\delta \log x)}{\delta^{2^{\kappa}} x^{\delta}}+\frac{P(\delta \log x)}{\delta^{2^{\kappa}} x^{\delta}} \\
= & \frac{\left(2^{\kappa}-1\right)!(\log x)^{2^{\kappa}}}{(2 \lambda)^{2^{\kappa}}}-\frac{(\log x)^{2^{\kappa}} P(2 \gamma \lambda)}{\left(2 \lambda 2^{2^{\kappa}} e^{2 \lambda \gamma}\right.}+e^{-\lambda} \frac{(\log x)^{2^{\kappa}} P(2 \gamma \lambda)}{(2 \lambda)^{2^{\kappa}} e^{2 \lambda \gamma}} \\
& -e^{-\lambda} \frac{(\log x)^{2^{\kappa}} P(2 \lambda)}{(2 \lambda)^{2^{\kappa}} e^{2 \lambda}}+\frac{(\log x)^{2^{\kappa}} P(2 \lambda)}{(2 \lambda)^{2^{\kappa}} e^{2 \lambda}} \\
= & \frac{e^{-\lambda}\left(\log x 2^{2^{\kappa}}\right.}{(2 \lambda)^{2^{\kappa}}}\left(\left(2^{\kappa}-1\right)!e^{\lambda}-\frac{e^{\lambda} P(2 \gamma \lambda)}{e^{2 \lambda \gamma}}+\frac{P(2 \gamma \lambda)}{e^{2 \lambda \gamma}}-\frac{P(2 \lambda)}{e^{2 \lambda}}+\frac{e^{\lambda} P(2 \lambda)}{e^{2 \lambda}}\right)
\end{aligned}
$$

We want to verify that the inside of the parenthesis is bounded as $\lambda \rightarrow \infty$. Recall that $\gamma=\frac{1}{\lambda e^{\lambda}}$ and $\kappa \geq 1$, so that

$$
\frac{P(2 \gamma \lambda)}{e^{2 \lambda \gamma}}-\frac{P(2 \lambda)}{e^{2 \lambda}}+\frac{e^{\lambda} P(2 \lambda)}{e^{2 \lambda}} \ll 1
$$

For the first two terms in the parenthesis, let $Q(X):=P(X)-\left(2^{\kappa}-1\right)!$. We have

$$
\left(2^{\kappa}-1\right)!e^{\lambda}-\frac{e^{\lambda} P(2 \gamma \lambda)}{e^{2 \lambda \gamma}}=\left(2^{\kappa}-1\right)!e^{\lambda}\left(1-e^{-2 \lambda \gamma}\right)+\frac{e^{\lambda} Q\left(2 e^{-\lambda}\right)}{e^{2 e^{-\lambda}}}=\left(2^{\kappa}-1\right)!e^{\lambda}\left(1-e^{-2 \lambda \gamma}\right)+O(1)
$$

since $Q(X)$ is a polynomial with vanishing constant term. It remains to bound $\left(2^{\kappa}-1\right)!e^{\lambda}\left(1-e^{-2 \lambda \gamma}\right)$. The Taylor expansion of $e^{X}$ for $X=o(1)$ gives

$$
\left(2^{\kappa}-1\right)!e^{\lambda}\left(1-e^{-2 \lambda \gamma}\right)=\left(2^{\kappa}-1\right)!e^{\lambda}\left(2 \lambda \gamma+O\left((2 \lambda \gamma)^{2}\right)\right)=\left(2^{\kappa}-1\right)!e^{\lambda}\left(2 e^{-\lambda}+O\left(e^{-2 \lambda}\right)\right)=O(1)
$$

We conclude that

$$
\left|L\left(1+\frac{2 \lambda}{\log x}+i \psi, f\right)\right| \ll\left|1+\frac{2 \lambda}{\log x}+i \psi\right| \frac{e^{-\lambda}(\log x)^{2^{\kappa}}}{(2 \lambda)^{2^{\kappa}}} .
$$

From the definition of $\lambda$, we have

$$
\lambda=\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)+\log (1+|\psi|)+c=O(\log \log x),
$$

so that

$$
\begin{aligned}
\left|L\left(1+\frac{2 \lambda}{\log x}+i \psi, f\right)\right| & \ll|1+i \psi| \frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right) \exp (-\log |1+i \psi|-c)(\log x)^{2^{\kappa}}}{(2 \lambda)^{2^{\kappa}}} \\
& =\frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right) \exp (-c)(\log x)^{2^{\kappa}}}{(2 \lambda)^{2^{\kappa}}} .
\end{aligned}
$$

Comparing the upper bound from this last line with the conclusion of (3.5), we get

$$
\frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right)(\log x)^{2^{\kappa}}}{(2 \lambda)^{2^{\kappa}}} \ll \frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right) e^{-c}(\log x)^{2^{\kappa}}}{(2 \lambda)^{2^{\kappa}}}
$$

which can be made false for $c$ large enough. It follows that the assumption (3.7) is false, and that the result holds.

## 4. Proof of Theorem 1.1

Now, we will prove the main theorem. For $j=1,2$, we set $\phi_{j}=\phi_{f_{j}}\left(x_{j},\left(\log x_{j}\right)^{2^{\kappa}}\right)$ and $N_{j}=$ $N_{f_{j}}\left(x_{j},\left(\log x_{j}\right)^{2^{\kappa}}\right)$ as in (2.3). We also set $f=f_{1} f_{2}$ and $\phi=\phi_{f}\left(x,(\log x)^{2^{\kappa}}\right)$ with $x=\min \left\{x_{1}, x_{2}\right\}$. By Lemma 3.1, we have

$$
\begin{aligned}
\mathbb{D}^{2}\left(f_{1} f_{2}, n^{i\left(\phi_{1}+\phi_{2}\right)} ; x\right) & \leq 2^{\kappa}\left[\mathbb{D}\left(f_{1}, n^{i \phi_{1}} ; x_{1}\right)+\mathbb{D}\left(f_{2}, n^{i \phi_{2}} ; x_{2}\right)\right]^{2} \\
& \leq 2^{\kappa+2} \max _{j}\left\{\mathbb{D}^{2}\left(f_{j}, n^{i \phi_{j}} ; x_{j}\right)\right\} .
\end{aligned}
$$

Together with (3.3), it follows that

$$
\exp \left(-\mathbb{D}^{2}\left(f_{1} f_{2}, n^{i\left(\phi_{1}+\phi_{2}\right)} ; x\right)\right) \geq \min _{j}\left\{e^{-2^{\kappa+2} N_{j}}\right\}
$$

Applying Lemma 2.8 yields

$$
\begin{equation*}
\exp \left(-\mathbb{D}^{2}\left(f_{1} f_{2}, n^{i\left(\phi_{1}+\phi_{2}\right)} ; x\right)\right) \geq\left(\eta^{2}\right)^{2^{\kappa+2}}=\eta^{2^{\kappa+3}} \tag{4.1}
\end{equation*}
$$

On the other hand, by Proposition [3.2, there exists $y \in\left[x^{1 /\left(\lambda e^{\lambda}\right)}, x\right]$ such that

$$
\begin{align*}
\left|\sum_{n \leq y} f_{1}(n) f_{2}(n)\right| & \gg \frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)\right) y(\log y)^{2^{\kappa}-1}}{1+|\psi|} \\
& \gg \frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i\left(\phi_{1}+\phi_{2}\right)} ; x\right)\right) y(\log y)^{2^{\kappa}-1}}{1+\left|\phi_{1}+\phi_{2}\right|} \tag{4.2}
\end{align*}
$$

where we recall that $\lambda=\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)+\log (1+|\psi|)+c$ for some sufficiently large constant $c$. Observe that the last inequality in (4.2) follows from

$$
\begin{aligned}
\frac{(\log x)^{2^{\kappa}} \exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi_{f}} ; x\right)\right)}{\left|1+\frac{1}{\log x}+i \psi_{f}\right|} & \asymp \frac{\left|L\left(1+\frac{1}{\log x}+i \psi_{f}, f\right)\right|}{\left|1+\frac{1}{\log x}+i \psi_{f}\right|}=\max _{|t| \leq(\log x)^{2^{\kappa}}} \frac{\left|L\left(1+\frac{1}{\log x}+i t, f\right)\right|}{\left|1+\frac{1}{\log x}+i t\right|} \\
& \geq \frac{\left|L\left(1+\frac{1}{\log x}+i\left(\phi_{1}+\phi_{2}\right), f\right)\right|}{\left|1+\frac{1}{\log x}+i\left(\phi_{1}+\phi_{2}\right)\right|} \\
& \asymp \frac{(\log x)^{2^{\kappa}} \exp \left(-\mathbb{D}^{2}\left(f, n^{i\left(\phi_{1}+\phi_{2}\right)} ; x\right)\right)}{\left|1+\frac{1}{\log x}+i\left(\phi_{1}+\phi_{2}\right)\right|} .
\end{aligned}
$$

Now, putting (4.1) and (4.2) together gives

$$
\left|\sum_{n \leq y} f_{1}(n) f_{2}(n)\right| \gg \frac{\eta^{2^{\kappa+3}} y(\log y)^{2^{2 \kappa}-1}}{1+\left|\phi_{1}+\phi_{2}\right|}
$$

Finally, since $1+\left|\phi_{1}+\phi_{2}\right| \leq 1+\left|\phi_{1}\right|+\left|\phi_{2}\right| \ll 1 / \eta$ by Lemma 2.8, we conclude that

$$
\left|\sum_{n \leq y} f_{1}(n) f_{2}(n)\right| \gg \eta^{1+2^{\kappa+3}} y(\log y)^{2^{2 \kappa}-1}
$$

which completes the proof by observing that

$$
\begin{aligned}
\frac{1}{\lambda e^{\lambda}} & \geq e^{-2 \lambda} \\
& =\left(\exp \left(-\mathbb{D}^{2}\left(f, n^{i \psi} ; x\right)-\log (1+|\psi|)-c\right)\right)^{2} \\
& \gg\left(\frac{\exp \left(-\mathbb{D}^{2}\left(f, n^{i\left(\phi_{1}+\phi_{2}\right)} ; x\right)\right)}{\left|1+\frac{1}{\log x}+i\left(\phi_{1}+\phi_{2}\right)\right|}\right)^{2} \\
& \gg\left(\eta^{1+2^{\kappa+3}}\right)^{2}
\end{aligned}
$$

and setting $\xi=C \eta^{1+2^{\kappa+3}}$, for some absolute positive constant $C$.

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