

Murnaghan-Type Representations for the Positive Elliptic Hall Algebra

Milo Bechtloff Weising

May 3, 2024

Abstract

We construct a new family of graded representations \widetilde{W}_λ for the positive elliptic Hall algebra \mathcal{E}^+ indexed by Young diagrams λ which generalize the standard \mathcal{E}^+ action on symmetric functions. These representations have homogeneous bases of eigenvectors for the action of the Macdonald element $P_{0,1} \in \mathcal{E}^+$ with distinct $\mathbb{Q}(q, t)$ -rational spectrum generalizing the symmetric Macdonald functions. The analysis of the structure of these representations exhibits interesting combinatorics arising from the stable limits of periodic standard Young tableaux. We find an explicit combinatorial rule for the action of the multiplication operators $e_r[X]^\bullet$ generalizing the Pieri rule for symmetric Macdonald functions. We will also naturally obtain a family of interesting (q, t) product-series identities which come from keeping track of certain combinatorial statistics associated to periodic standard Young tableaux.

Contents

1	Introduction	2
1.1	Overview	3
1.2	Acknowledgements	3
2	Definitions and Notations	3
2.1	Some Combinatorics	3
2.2	Finite Hecke Algebra	9
2.3	Affine Hecke Algebra	10
2.4	Positive Double Affine Hecke Algebra	13
2.5	Elliptic Hall Algebra	15
3	DAHA Modules from Young Diagrams	16
3.1	The \mathcal{D}_n -module V_λ	16
3.2	Connecting Maps Between $V_{\lambda^{(n)}}$	23
4	Positive EHA Representations from Young Diagrams	27
4.1	The $\mathcal{D}_n^{\text{spb}}$ -modules $W_{\lambda^{(n)}}$	27
4.2	Stable Limit of the $W_{\lambda^{(n)}}$	31
4.3	Positive Elliptic Hall Algebra Action on \widetilde{W}_λ	33
5	Pieri Rule for Generalized Macdonald Functions	35
5.1	Pieri Rule Preliminaries	35
5.2	Pieri Rule	39
5.3	Non-vanishing for e_1 Pieri Coefficients	41
6	Family of (q, t) Product-Sum Identities	44

1 Introduction

The space of symmetric functions, Λ , is a central object in algebraic combinatorics deeply connecting the fields of representation theory, geometry, and combinatorics. In his influential paper [Mac88], Macdonald introduced a special basis $P_\lambda[X; q, t]$ for Λ over $\mathbb{Q}(q, t)$ simultaneously generalizing many other important and well-studied symmetric function bases like the Schur functions $s_\lambda[X]$. These symmetric functions $P_\lambda[X; q, t]$, called the symmetric Macdonald functions, exhibit many striking combinatorial properties and can be defined as the eigenvectors of a certain operator $\Delta : \Lambda \rightarrow \Lambda$ called the Macdonald operator constructed using polynomial difference operators. It was discovered through the works of Bergeron, Garsia, Haiman, Tesler, and many others [Hai01] [BG99] [Ber+99] that variants of the symmetric Macdonald functions called the modified Macdonald functions $\tilde{H}_\lambda[X; q, t]$ have deep ties to the geometry of the Hilbert schemes $\text{Hilb}_n(\mathbb{C}^2)$. On the side of representation theory, it was shown first in full generality by Cherednik [Che03] that one can recover the symmetric Macdonald functions by considering the representation theory of certain algebras called the spherical double affine Hecke algebras (DAHAs) in type GL_n .

The positive elliptic Hall algebra (EHA), \mathcal{E}^+ , was introduced by Burban and Schiffmann [BS12] as the positive subalgebra of the Hall algebra of the category of coherent sheaves on an elliptic curve over a finite field. This algebra has connections to many areas of mathematics including, most importantly for the present paper, to Macdonald theory. In [SV13], Schiffmann and Vasserot realize \mathcal{E}^+ as a stable limit of the positive spherical DAHAs in type GL_n . They show further that there is a natural action of \mathcal{E}^+ on Λ aligning with the spherical DAHA representations originally considered by Cherednik. In particular, the action of $P_{0,1} \in \mathcal{E}^+$ gives the Macdonald operator Δ . The action of \mathcal{E}^+ on Λ can be realized as the action of certain generalized convolution operators on the torus equivariant K -theory of the schemes $\text{Hilb}_n(\mathbb{C}^2)$.

Dunkl and Luque in [DL11] introduced symmetric and non-symmetric vector-valued (vv.) Macdonald polynomials. The term vector-valued here refers to polynomial-like objects of the form $\sum_\alpha c_\alpha X^\alpha \otimes v_\alpha$ for some scalars c_α , monomials X^α , and vectors v_α lying in some $\mathbb{Q}(q, t)$ -vector space. The non-symmetric vv. Macdonald polynomials are distinguished bases for certain DAHA representations built from the irreducible representations of the finite Hecke algebras in type A. These DAHA representations are indexed by Young diagrams and exhibit interesting combinatorial properties relating to periodic Young tableaux. The symmetric vv. Macdonald polynomials are distinguished bases for the spherical (i.e. Hecke-invariant) subspaces of these DAHA representations. Naturally, the spherical DAHA acts on this spherical subspace with the special element $\xi_1 + \dots + \xi_n$ of spherical DAHA acting diagonally on the symmetric vv. Macdonald polynomials.

Dunkl and Luque in [DL11] (and in later work of Colmenarejo, Dunkl, and Luque [CDL22] and Dunkl [Dun19]) only consider the finite rank non-symmetric and symmetric vv. Macdonald polynomials. It is natural to ask if there is an infinite-rank stable-limit construction using the symmetric vv. Macdonald polynomials to give generalized symmetric Macdonald functions and an associated representation of the positive elliptic Hall algebra \mathcal{E}^+ . In this paper, we will describe such a construction (Thm. 4.13). We will obtain a new family of graded \mathcal{E}^+ -representations \tilde{W}_λ indexed by Young diagrams λ and a natural generalization of the symmetric Macdonald functions \mathfrak{P}_T indexed by certain labellings of infinite Young diagrams built as limits of the symmetric vv. Macdonald polynomials. For combinatorial reasons there is essentially a unique natural way to obtain this construction. For any λ we will consider the increasing chains of Young diagrams $\lambda^{(n)} = (n - |\lambda|, \lambda)$ for $n \geq |\lambda| + \lambda_1$ to build the representations \tilde{W}_λ . These special sequences of Young diagrams are central to Murnaghan's theorem [Mur38] regarding the reduced Kronecker coefficients. As such we refer to the \mathcal{E}^+ -representations \tilde{W}_λ as Murnaghan-type. For $\lambda = \emptyset$ we recover the \mathcal{E}^+ action on Λ and the symmetric Macdonald functions $P_\mu[X; q, t]$. We will obtain a Pieri rule for the action of the multiplication operators e_i^\bullet on the generalized symmetric Macdonald function basis \mathfrak{P}_T . After studying the particular case of the e_1 -Pieri coefficients we will show that the modules \tilde{W}_λ are cyclic generated by their unique elements of minimal degree $\mathfrak{P}_{T_\lambda^{\min}}$. Lastly, we will show that these Murnaghan-type representations \tilde{W}_λ are mutually non-isomorphic.

The existence of these representations of the elliptic Hall algebra raises many questions about possible new relations between Macdonald theory and geometry. Other authors have constructed families of \mathcal{E}^+ -representations [Fei+11] [Fei+12]. Although there should exist a relationship between the Murnaghan-type representations \tilde{W}_λ and those of other authors, the construction in this paper appears to be distinct

from prior \mathcal{E}^+ -module constructions.

For technical reasons (regarding the misalignment of the spectrum of the Cherednik operators ξ_i) we will need to reprove many of the results of Dunkl and Luque in [DL11] using a re-oriented version of the Cherednik operators θ_i . Since the elements θ_i are not uniformly conjugate to the ξ_i on the vector-valued polynomial spaces V_λ , we are not immediately able to use the results of Dunkl and Luque. This alternative choice of conventions greatly assists during the construction of the generalized Macdonald functions \mathfrak{P}_T . The θ_i satisfy additional stability properties which the ξ_i fail to satisfy. The combinatorics underpinning the non-symmetric v.v. Macdonald polynomials originally defined by Dunkl and Luque is also nearly identical but with reversed orientation to the conventions appearing in this paper.

1.1 Overview

Here we will give a brief overview of this paper. First, in Section 2 we will review relevant definitions and notations as well as recall the stable-limit spherical DAHA construction of Schiffmann-Vasserot. In Section 3 we will reprove many of the results of Dunkl-Luque but for the re-oriented Cherednik operators including describing the non-symmetric v.v. Macdonald polynomials F_τ and their associated Knop-Sahi relations (Prop. 3.2). We define (Def. 3.9) the DAHA modules $V_{\lambda^{(n)}}$ and connecting maps $\Phi_\lambda^{(n)} : V_{\lambda^{(n+1)}} \rightarrow V_{\lambda^{(n)}}$ which will be used in the stable-limit process. Next in Section 4, we describe the spherical subspaces $W_\lambda^{(n)}$ of Hecke invariants of $V_\lambda^{(n)}$ and the symmetric v.v. Macdonald polynomials P_T including an explicit expansion of the P_T into the F_τ (Cor. 4.5). We will use the connecting maps to define the stable-limit spaces \widetilde{W}_λ and show in Thm.4.13 that they possess a graded action of \mathcal{E}^+ having a distinguished basis of generalized symmetric Macdonald functions \mathfrak{P}_T . In Section 5 we will obtain a Pieri formula (Cor. 5.9) for the action of e_r^\bullet on the generalized Macdonald functions \mathfrak{P} . Lastly in Section 6, we will look at an interesting family of (q, t) product-series identities (Thm. 6.12) which follow naturally from the algebra/combinatorics in the prior sections of the paper.

1.2 Acknowledgements

The author would like to thank their advisor Monica Vazirani for her consistent guidance. The author would also like to thank Erik Carlsson, Daniel Orr, and Eugene Gorsky for helpful conversations about the elliptic Hall algebra and the geometry of Hilbert schemes. The author was supported during this work by the 2023 UC Davis Dean's Summer Research Fellowship.

2 Definitions and Notations

2.1 Some Combinatorics

We start with a description of many of the combinatorial objects which we will need for the remainder of this paper.

Definition 2.1. A *partition* is a (possibly empty) sequence of weakly decreasing positive integers. Denote by \mathbb{Y} the set of all partitions. Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ we set $\ell(\lambda) := r$ and $|\lambda| := \lambda_1 + \dots + \lambda_r$. For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Y}$ and $n \geq n_\lambda := |\lambda| + \lambda_1$ we set $\lambda^{(n)} := (n - |\lambda|, \lambda_1, \dots, \lambda_r)$. We will identify partitions as defined above with *Young diagrams* of the corresponding shape in English notation i.e. justified up and to the left.

Fix a partition λ with $|\lambda| = n$. We will require each of the following combinatorial constructions for types of labelling of the Young diagram λ . If a diagram λ appears as the domain of a labelling function then we are referring to the set of boxes of λ as the domain.

- A non-negative *reverse Young tableau* $\text{RYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \rightarrow \mathbb{Z}_{\geq 0}$ which is weakly decreasing along rows and columns.
- A non-negative *reverse semi-standard Young tableau* $\text{RSSYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \rightarrow \mathbb{Z}_{\geq 0}$ which is weakly decreasing along rows and strictly decreasing along columns.
- A *standard Young tableau* $\text{SYT}(\lambda)$ is a labelling $\tau : \lambda \rightarrow \{1, \dots, n\}$ which is strictly increasing along rows and columns.

- A non-negative *periodic standard Young tableau* $\text{PSYT}_{\geq 0}(\lambda)$ is a labelling $\tau : \lambda \rightarrow \{jq^b : 1 \leq j \leq n, b \geq 0\}$ in which each $1 \leq j \leq n$ occurs in exactly one box of λ and where the labelling is strictly increasing along rows and columns. Here we order the formal products jq^m by $jq^m < kq^\ell$ if $m > \ell$ or in the case that $m = \ell$ we have $j < k$. Note that $\text{SYT}(\lambda) \subset \text{PSYT}_{\geq 0}(\lambda)$.

Define $\tau_\lambda^{rs}, \tau_\lambda^{cs} \in \text{SYT}(\lambda)$ to be the row-standard and column-standard labellings of λ respectively.

Example 1.

$17q^7$	$15q^5$	$16q^5$	$11q^3$	$7q^1$	$2q^0$
$14q^6$	$12q^4$	$13q^4$	$9q^2$	$8q^0$	
$10q^2$	$4q^1$	$5q^1$	$6q^1$		
$3q^1$	$1q^0$				

 $\in \text{PSYT}_{\geq 0}(6, 5, 4, 2)$

Definition 2.2. Given a box, \square , in a Young diagram λ we define the **content** of \square as $c(\square) := a - b$ where $\square = (a, b)$ as drawn in the $\mathbb{N} \times \mathbb{N}$ grid (English notation). Let $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ and $1 \leq i \leq n$. Whenever $\tau(\square) = iq^b$ for some box $\square \in \lambda$ we will write

- $c_\tau(i) := c(\square)$
- $w_\tau(i) := b$.

Set $w_\tau := (w_\tau(1), \dots, w_\tau(n)) \in \mathbb{Z}_{\geq 0}^n$. Let $1 \leq j \leq n - 1$ and suppose that for some boxes $\square_1, \square_2 \in \lambda$ that $\tau(\square_1) = jq^m$ and $\tau(\square_2) = (j + 1)q^\ell$. Let τ' be the labelling defined by $\tau'(\square_1) = (j + 1)q^m$, $\tau'(\square_2) = jq^\ell$, and $\tau'(\square) = \tau(\square)$ for $\square \in \lambda \setminus \{\square_1, \square_2\}$. If $\tau' \in \text{PSYT}_{\geq 0}(\lambda)$ then we write $s_j(\tau) := \tau'$. Let $\Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda)$ be the labelling defined by whenever $\tau(\square) = kq^a$ then either $\Psi(\tau)(\square) = (k - 1)q^a$ when $k \geq 2$ or $\Psi(\tau)(\square) = nq^{a+1}$ when $k = 1$.

We give the set $\text{PSYT}_{\geq 0}(\lambda)$ a partial order \geq defined by the following cover relations.

- For all $\tau \in \text{PSYT}_{\geq 0}(\lambda)$, $\Psi(\tau) > \tau$.
- If $w_\tau(i) < w_\tau(i + 1)$ then $s_i(\tau) > \tau$.
- If $w_\tau(i) = w_\tau(i + 1)$ and $c_\tau(i) - c_\tau(i + 1) > 1$ then $s_i(\tau) > \tau$.

Define the map $\mathfrak{p}_\lambda : \text{PSYT}_{\geq 0}(\lambda) \rightarrow \text{RYT}_{\geq 0}(\lambda)$ by $\mathfrak{p}_\lambda(\tau)(\square) = b$ whenever $\tau(\square) = iq^b$. We will write $\text{PSYT}_{\geq 0}(\lambda; T)$ for the set of all $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T \in \text{RYT}_{\geq 0}(\lambda)$.

Example 2. Ψ

$1q^7$	$3q^5$	$5q^5$	$8q^2$	$12q^1$	$17q^0$
$2q^6$	$4q^5$	$6q^5$	$14q^0$	$16q^0$	
$7q^2$	$10q^1$	$11q^1$	$15q^0$		
$9q^1$	$13q^0$				

 $=$

$17q^8$	$2q^5$	$4q^5$	$7q^2$	$11q^1$	$16q^0$
$1q^6$	$3q^5$	$5q^5$	$13q^0$	$15q^0$	
$6q^2$	$9q^1$	$10q^1$	$14q^0$		
$8q^1$	$12q^0$				

We will frequently require the basic lemma regarding the ordering \leq on $\text{PSYT}_{\geq 0}(\lambda)$.

Lemma 2.3. Let $\lambda \in \mathbb{Y}$ and $T \in \text{RYT}_{\geq 0}(\lambda)$. There are unique $\min(T), \text{top}(T) \in \text{PSYT}_{\geq 0}(\lambda; T)$ such that for all $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T$, $\min(T) \leq \tau \leq \text{top}(T)$.

Proof. We can explicitly construct the elements $\text{top}(T), \min(T)$ directly. Define $\text{top}(T)$ by first filling in the boxes $\square \in \lambda$ of λ with the values $q^{T(\square)}$. Now we label these boxes with the values $\{1, \dots, n\}$ by first decomposing λ into skew diagrams where T is constant on each sub-diagram. This gives us an increasing chain of Young diagrams $\lambda^{(1)} \subset \dots \subset \lambda^{(r)} = \lambda$. Next we fill each diagram $\lambda^{(i)}$ with the values $\{|\lambda^{(1)}| + \dots + |\lambda^{(i-1)}| + 1, \dots, |\lambda^{(1)}| + \dots + |\lambda^{(i)}|\}$ in column-standard order. This gives a value iq^a in each box of λ .

For $\min(T)$, we proceed similarly by first filling in the boxes $\square \in \lambda$ of λ with the values $q^{T(\square)}$. Then we decompose λ into the same skew diagrams as before. Now we fill each diagram $\lambda^{(i)}$ with the values $\{n - (|\lambda^{(1)}| + \dots + |\lambda^{(i-1)}|), \dots, n - (|\lambda^{(1)}| + \dots + |\lambda^{(i)}|)\}$ in row-standard order. This gives a value iq^a in each box of λ . □

Example 3. Given $T =$

7	5	5	2	1	0
6	5	5	0	0	
2	1	1	0		
1	0				

 $\in \text{RYT}_{\geq 0}(6, 5, 4, 2)$ we have that

$$\min(T) =$$

$17q^7$	$12q^5$	$13q^5$	$10q^2$	$6q^1$	$1q^0$
$16q^6$	$14q^5$	$15q^5$	$2q^0$	$3q^0$	
$11q^2$	$7q^1$	$8q^1$	$4q^0$		
$9q^1$	$5q^0$				

 $\text{and top}(T) =$

$1q^7$	$3q^5$	$5q^5$	$8q^2$	$12q^1$	$17q^0$
$2q^6$	$4q^5$	$6q^5$	$14q^0$	$16q^0$	
$7q^2$	$10q^1$	$11q^1$	$15q^0$		
$9q^1$	$13q^0$				

Definition 2.4. Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \text{RYT}_{\geq 0}(\lambda)$. Define $\nu(T) \in \mathbb{Z}_{\geq 0}^n$ to be the vector formed by listing the values of T in decreasing order i.e. $\nu(T) = \text{sort}(w_\tau)$ for any $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$. Define $S(T) \in \text{SYT}(\lambda)$ by ordering the boxes of λ according to $\square_1 \leq \square_2$ if and only if

- $T(\square_1) > T(\square_2)$ or
- $T(\square_1) = T(\square_2)$ and \square_1 comes before \square_2 in the column-standard labelling of λ .

We will often write as a shorthand $\square_1 <_T \square_2$ whenever $S(T)(\square_1) < S(T)(\square_2)$. Define the statistic $b_T \in \mathbb{Z}_{\geq 0}$ by

$$b_T := \sum_{i=1}^n \nu(T)_i (c_{S(T)}(i) + i - 1).$$

Lastly, define the composition $\mu(T)$ of n as follows. Decompose λ into horizontal strips h_1, \dots, h_m where T is constant on each strip. We order these strips so that the $\min(T)$ labels in h_i are strictly less than those in h_{i+1} for all i . Note that, unless $T \in \text{RSSYT}_{\geq 0}(\lambda)$, we may have horizontal strips with the same T -value touching in adjacent rows. We see that each of these horizontal strips h_i has some labels $a_i, \dots, a_i + r_i$. Then $\mu(T)$ is given as (r_1, \dots, r_m) .

Remark 1. For every $T \in \text{RYT}_{\geq 0}(\lambda)$ we can recover T from the pair $(S(T), \nu(T))$ by labelling λ with the entries of $\nu(T)$ following the order of $S(T)$. Further, the standard Young tableau $S(T)$ is the largest such tableau following the partial order defined in Definition 2.2.

Below is an example calculation of the various data which we associate to $T \in \text{RYT}_{\geq 0}(\lambda)$.

Example 4. For $T \in \text{RYT}_{\geq 0}(6, 5, 4, 2)$ as in Example 3 we have that

$$S(T) =$$

1	3	5	8	12	17
2	4	6	14	16	
7	10	11	15		
9	13				

 $\in \text{SYT}(6, 5, 4, 2),$

$$\begin{aligned} \nu(T) &= (7, 6, 5, 5, 5, 5, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0) \in \mathbb{Z}_{\geq 0}^{17}, \\ b_T &= 0 + 0 + 15 + 15 + 30 + 30 + 8 + 20 + 5 + 8 + 10 + 15 + 0 + 0 + 0 + 0 + 0 = 156, \\ \text{and } \mu(T) &= (1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 2, 1, 1). \end{aligned}$$

The next definition will be crucial for many of the results in this paper.

Definition 2.5. Let $\lambda \in \mathbb{Y}$, with $|\lambda| = n$ and $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ with $T = \mathfrak{p}_\lambda(\tau)$. An ordered pair of boxes $(\square_1, \square_2) \in \lambda \times \lambda$ is called an *inversion pair* of τ if $S(T)(\square_1) < S(T)(\square_2)$ and $i > j$ where $\tau(\square_1) = iq^a$, $\tau(\square_2) = jq^b$ for some $a, b \geq 0$. The set of all inversion pairs of τ will be denoted by $\text{Inv}(\tau)$. We will use the shorthand $I(T)$ for the set $\text{Inv}(\min(T))$.

$17q^7$	$12q^5$	$13q^5$	$10q^2$	$6q^1$	$1q^0$
$16q^6$	$14q^5$	$15q^5$	$2q^0$	$3q^0$	
$11q^2$	$7q^1$	$8q^1$	$4q^0$		
$9q^1$	$5q^0$				

Example 5. In the labelling

we have that the pairs $(17q^7, 12q^5)$,

$(14q^5, 13q^5)$, and $(5q^0, 4q^0)$ are all inversions. Here we have referred to boxes according to their labels.

In the following definition our conventions for the Bruhat ordering differ from many other authors. These conventions are use to help properly state some triangularity properties later in the paper. However, one may obtain the below definition from the more standard conventions in [HHL08] by reversing the order of the entries of each vector $(a_1, \dots, a_n) \rightarrow (a_n, \dots, a_1)$ and rewriting their Bruhat ordering from this reversed perspective.

Definition 2.6. Define the Bruhat ordering \preceq on $\mathbb{Z}_{\geq 0}^n$ using the following cover relations for $\lambda \in \mathbb{Z}_{\geq 0}^n$:

- if $i < j$ with $\lambda_i < \lambda_j$ then $\lambda \prec (i, j)\lambda$
- if $i < j$ with $\lambda_i + 1 < \lambda_j$ then $\lambda \succ \lambda + e_i - e_j$.

Here e_i denotes the i -th standard basis vector of \mathbb{Z}^n and $(i, j) \in \mathfrak{S}_n$ denotes the simple transposition swapping i and j . For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ we define $\tilde{\gamma}(\alpha) := (\alpha_2, \dots, \alpha_n, \alpha_1 + 1)$. We will write $\text{sort}(\alpha)$ for the vector formed by listing the entries of α in weakly decreasing order and $\text{revsort}(\alpha)$ for the vector formed by listing the entries of α in weakly increasing order. We define $\text{Stab}(\alpha)$ to be the corresponding stabilizer subgroup of \mathfrak{S}_n for α i.e. the set of all $\sigma \in \mathfrak{S}_n$ with $\sigma(\alpha) = \alpha$.

We require the following simple lemma regarding the interplay between the map $\tilde{\gamma}$ on $\mathbb{Z}_{\geq 0}^n$ and the ordering \prec .

Lemma 2.7. If $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$ satisfy $\alpha \prec \beta$ then $\tilde{\gamma}(\alpha) \prec \tilde{\gamma}(\beta)$.

Proof. We will show that if $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$ and β covers α with respect to the Bruhat order then $\tilde{\gamma}(\alpha) \prec \tilde{\gamma}(\beta)$. We will proceed in cases. Let $\lambda \in \mathbb{Z}_{\geq 0}^n$.

First, suppose $1 < i < j$ and $\lambda_i < \lambda_j$. Then

$$\tilde{\gamma}(\lambda) \prec (i-1, j-1)\tilde{\gamma}(\lambda) = \tilde{\gamma}((i, j)\lambda).$$

Now suppose $1 < j$ and $\lambda_1 < \lambda_j$. Then

$$\tilde{\gamma}((1, j)\lambda) \succ \tilde{\gamma}((1, j)\lambda) + e_j - e_n \succeq (j, n)(\tilde{\gamma}((1, j)\lambda) + e_j - e_n) = \tilde{\gamma}(\lambda).$$

If now we have that $1 < i < j$ and $\lambda_i < \lambda_j - 1$ then

$$\tilde{\gamma}(\lambda) \succ \tilde{\gamma}(\lambda) + e_{i-1} - e_{j-1} = \tilde{\gamma}(\lambda + e_i - e_j).$$

Lastly, consider the case when $1 < j$ and $\lambda_1 < \lambda_j - 1$. If $\lambda_1 + 2 = \lambda_j$ then

$$\tilde{\gamma}(\lambda) \succ (j-1, n)\tilde{\gamma}(\lambda) = \tilde{\gamma}(\lambda + e_1 - e_j).$$

Instead if $\lambda_1 < \lambda_j - 2$ then

$$\tilde{\gamma}(\lambda) \succ (j-1, n)\tilde{\gamma}(\lambda) \succ (j-1, n)\tilde{\gamma}(\lambda) + e_{j-1} - e_n = \tilde{\gamma}(\lambda + e_1 - e_j).$$

□

Here we review some necessary details about the extended affine symmetric groups.

Definition 2.8. Define $\widehat{\mathfrak{S}}_n$ to be the extended affine symmetric group given by

$$\widehat{\mathfrak{S}}_n := \mathfrak{S}_n \ltimes \mathbb{Z}^n$$

where \mathfrak{S}_n acts on \mathbb{Z}^n by coordinate permutations. Denote by t_1, \dots, t_n the standard generators of $\mathbb{Z}^n \subset \widehat{\mathfrak{S}}_n$. Further, we define the special element $\tilde{\gamma}_n \in \widehat{\mathfrak{S}}_n$ given by

$$\tilde{\gamma}_n := t_n s_{n-1} \dots s_1.$$

For any $\beta \in \mathbb{Z}^n$ we will write

$$t_\beta := t_1^{\beta_1} \cdots t_n^{\beta_n}.$$

Define the positive submonoid of $\widehat{\mathfrak{S}}_n, \widehat{\mathfrak{S}}_n^+$, as the monoid generated by $\{s_1, \dots, s_{n-1}, \tilde{\gamma}_n\}$ (i.e. no $\tilde{\gamma}_n^{-1}$ s).

The length $\ell(\sigma)$ of $\sigma \in \widehat{\mathfrak{S}}_n$ is the minimal number of s_i required to express σ in terms of the generators $\{s_1, \dots, s_{n-1}, \tilde{\gamma}_n\}$. We denote by $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ the set of minimal length left coset representatives of $\widehat{\mathfrak{S}}_n$ with respect to the subgroup \mathfrak{S}_n . We will denote the set of positive minimal length coset representatives of $\widehat{\mathfrak{S}}_n$ with respect to the subgroup \mathfrak{S}_n by $(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n)^+ := (\widehat{\mathfrak{S}}_n/\mathfrak{S}_n) \cap \widehat{\mathfrak{S}}_n^+$. If $\mu = (\mu_1, \dots, \mu_r)$ is a composition of $n = \mu_1 + \dots + \mu_r$ then we will define the Young subgroup \mathfrak{S}_μ of \mathfrak{S}_n corresponding to μ as $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_r} \subset \mathfrak{S}_n$. We will write $\widehat{\mathfrak{S}}_n/\mathfrak{S}_\mu$ for the set of minimal length left coset representatives for $\widehat{\mathfrak{S}}_n$ with respect to the subgroup \mathfrak{S}_μ .

For $\beta \in \mathbb{Z}^n$ define $\sigma_\beta \in \widehat{\mathfrak{S}}_n$ by

$$\sigma_\beta := \sigma t_{\text{sort}(\beta)}$$

where σ is the unique minimal length coset representative in $\widehat{\mathfrak{S}}_n/\mathfrak{S}_{\text{Stab}(\text{sort}(\beta))}$ such that $\sigma(\text{sort}(\beta)) = \beta$.

The next two lemmas are standard in the theory of (extended) affine permutations and we leave them to the reader to verify.

Lemma 2.9. We have that

$$\widehat{\mathfrak{S}}_n/\mathfrak{S}_n = \{\sigma_\beta | \beta \in \mathbb{Z}^n\}$$

and

$$(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n)^+ = \{\sigma_\beta | \beta \in \mathbb{Z}_{\geq 0}^n\}.$$

Lemma 2.10. For all $\alpha \in \mathbb{Z}_{\geq 0}^n$ we have the following:

- If α is weakly decreasing then $\sigma_\alpha = t_\alpha$.
- If $s_i(\alpha) \succ \alpha$ then $\sigma_{s_i(\alpha)} = s_i \sigma_\alpha$.
- If $s_i(\alpha) = \alpha$ then $s_i \sigma_\alpha = \sigma_\alpha s_{\sigma^{-1}(i)}$ where σ is the minimal length permutation with $\sigma(\text{sort}(\alpha)) = \alpha$.
- $\sigma_{\tilde{\gamma}_n(\alpha)} = \tilde{\gamma}_n(\sigma_\alpha)$.

Recall that in Definition 2.2 we only defined $s_i(\tau)$ for $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ in the situation where swapping the i and $i+1$ labels in the boxes of τ resulted in an element of $\text{PSYT}_{\geq 0}(\lambda)$. We now generalize this notion to elements of $\widehat{\mathfrak{S}}_n^+$.

Definition 2.11. Suppose $z_r \cdots z_1$ is a reduced word in $\widehat{\mathfrak{S}}_n^+$ written in the generators $z_i \in \{s_1, \dots, s_{n-1}, \tilde{\gamma}_n\}$. We define inductively on $r \geq 1$ if $z_{r-1} \cdots z_1(\tau) \in \text{PSYT}_{\geq 0}(\lambda)$ the element $z_r \cdots z_1(\tau)$ of $\text{PSYT}_{\geq 0}(\lambda)$ as either

- $\Psi(z_{r-1} \cdots z_1(\tau))$ if $z_r = \tilde{\gamma}_n$
- $s_i(z_{r-1} \cdots z_1(\tau))$ if $z_r = s_i$ and swapping the i and $i+1$ labels in the boxes of $z_{r-1} \cdots z_1(\tau)$ results in an element of $\text{PSYT}_{\geq 0}(\lambda)$.

Otherwise we will leave this symbol undefined. This definition is only dependent on the element $z_r \cdots z_1$ of $\widehat{\mathfrak{S}}_n^+$ in that if $z_r \cdots z_1 = z'_r \cdots z'_1$ is another reduced word then $z_r \cdots z_1(\tau)$ is defined if and only if $z'_r \cdots z'_1(\tau)$ is defined. Thus we will write $\sigma(\tau) = z_r \cdots z_1(\tau)$ unambiguously in this situation if $\sigma = z_r \cdots z_1$.

We will need the following result later in the paper.

Lemma 2.12. For $T \in \text{RYT}_{\geq 0}(\lambda)$ we have that

$$\text{top}(T) = \zeta_1^{\nu(T)_1 - \nu(T)_2} \cdots \zeta_n^{\nu(T)_n} (S(T))$$

where for all $1 \leq i \leq n$

$$\zeta_i := (s_i \cdots s_{n-1} \Psi)^i.$$

Proof. One may check by direct computation that if $T \in \text{RYT}_{\geq 0}(\lambda)$ and $1 \leq i \leq n$ then $\zeta_i(\text{top}(T))$ is well defined according to Definition 2.11 and in particular, $\zeta_i(\text{top}(T)) = \text{top}(T')$ where $T'(\square) = T(\square) + 1$ for $S(T)(\square) \leq i$ and $T'(\square) = T(\square)$ otherwise. Note that $S(T) = S(T')$ so applying ζ_i does not change the underlying diagram ordering corresponding to the labelling T . Thus given any $T \in \text{RYT}_{\geq 0}(\lambda)$ by applying each ζ_i one at a time we see that $\zeta_1^{\nu(T)_1 - \nu(T)_2} \cdots \zeta_n^{\nu(T)_n} (S(T))$ must equal $\text{top}(T)$. \square

We will need to identify an explicit bijection between $\text{PSYT}_{\geq 0}(\lambda)$ and $(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n)^+ \times \text{SYT}(\lambda)$. We already have a map $\text{PSYT}_{\geq 0, \geq 0}(\lambda) \rightarrow (\widehat{\mathfrak{S}}_n/\mathfrak{S}_n)^+$ given by $\tau \rightarrow \sigma_{w_\tau}$. This is not bijective so we will use elements of $\text{SYT}(\lambda)$ to refine this map to yield a bijection. We now identify the correct choice of $\text{SYT}(\lambda)$ for a given $\tau \in \text{PSYT}_{\geq 0, \geq 0}(\lambda)$.

Definition 2.13. For $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ we define $S(\tau) \in \text{SYT}(\lambda)$ by the following recursion:

- $S(\text{top}(T)) := S(T)$ as defined in Definition 2.4
- If $w_\tau(i) < w_\tau(i+1)$ then $S(s_i(\tau)) = S(\tau)$.
- $S(\Psi(\tau)) = S(\tau)$
- If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then $S(s_i(\tau)) = s_j S(\tau)$ where $j = \sigma^{-1}(i)$ and σ is the minimal length permutation with $\sigma(\text{sort}(w_\tau)) = w_\tau$.

Proposition 2.14. For $\tau \in \text{PSYT}_{\geq 0}(\lambda)$

$$\tau = \sigma_{w_\tau}(S(\tau)).$$

Proof. Using Lemma 2.10 and Lemma 2.12 we see that for all $T \in \text{RYT}_{\geq 0}(\lambda)$

$$\sigma_{w_{\text{top}(T)}}(S(\mathfrak{p}_\lambda(\text{top}(T)))) = \sigma_{\nu(T)}(S(T)) = t_{\nu(T)}(S(T)) = \zeta_1^{\nu(T)1 - \nu(T)2} \dots \zeta_n^{\nu(T)n}(S(T)) = \text{top}(T).$$

Let $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ and suppose for sake of induction that $\tau = \sigma_{w_\tau}(S(\tau))$. Now let $s_i(\tau) < \tau$. If $w_\tau(i) > w_\tau(i+1)$ then $S(s_i(\tau)) = S(\tau)$ and $\sigma_{w_{s_i(\tau)}} = s_i \sigma_{w_\tau}$ so that

$$\sigma_{w_{s_i(\tau)}}(S(s_i(\tau))) = s_i \sigma_{w_\tau}(S(\tau)) = s_i(\tau).$$

In the case instead that $w_\tau(i) = w_\tau(i+1)$ with $c_\tau(i+1) - c_\tau(i) > 1$ then $S(s_i(\tau)) = s_j(S(\tau))$ and $\sigma_{w_{s_i(\tau)}} = \sigma_{w_\tau}$ where $j = \sigma^{-1}(i)$ and σ is the minimal length permutation with $\sigma(\text{sort}(w_\tau)) = w_\tau$. Then

$$\begin{aligned} & \sigma_{w_{s_i(\tau)}}(S(s_i(\tau))) \\ &= \sigma_{w_\tau}(s_j S(\tau)) \\ &= (\sigma_{w_\tau} s_j)(S(\tau)) \\ &= (s_i \sigma_{w_\tau})(S(\tau)) \\ &= s_i(\tau). \end{aligned}$$

□

We may now obtain the desired bijection.

Proposition 2.15. The map $\Xi_\lambda : \text{PSYT}_{\geq 0}(\lambda) \rightarrow (\widehat{\mathfrak{S}}_n/\mathfrak{S}_n)^+ \times \text{SYT}(\lambda)$ given by

$$\Xi_\lambda(\tau) := (\sigma_{w_\tau}, S(\tau))$$

is a bijection.

Proof. It is immediate from Proposition 2.14 that Ξ_λ is injective. But it is straightforward to check inductively that given any $\sigma \in (\widehat{\mathfrak{S}}_n/\mathfrak{S}_n)^+$ and $S \in \text{SYT}(\lambda)$, $\sigma(S)$ is a well defined element of $\text{PSYT}_{\geq 0}(\lambda)$ in the sense of Definition 2.11. This shows that Ξ_λ is also surjective and thus bijective. □

2.2 Finite Hecke Algebra

Here we detail the conventions for the finite Hecke algebras in type A_{n-1} occurring in this paper.

Definition 2.16. Define the finite Hecke algebra \mathcal{H}_n to be the $\mathbb{Q}(q, t)$ -algebra generated by T_1, \dots, T_{n-1} subject to the relations

- $(T_i - 1)(T_i + t) = 0$ for $1 \leq i \leq n - 1$
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i \leq n - 2$
- $T_i T_j = T_j T_i$ for $|i - j| > 1$.

We define the special elements $\bar{\theta}_1, \dots, \bar{\theta}_n \in \mathcal{H}_n$ by $\bar{\theta}_1 := 1$ and $\bar{\theta}_{i+1} := tT_i^{-1}\bar{\theta}_i T_i^{-1}$ for $1 \leq i \leq n - 1$. Further, define $\bar{\varphi}_1, \dots, \bar{\varphi}_{n-1}$ by $\bar{\varphi}_i := (tT_i^{-1})\bar{\theta}_i - \bar{\theta}_i(tT_i^{-1})$. For a permutation $\sigma \in \mathfrak{S}_n$ and a reduced expression $\sigma = s_{i_1} \cdots s_{i_r}$ we write $T_\sigma := T_{i_1} \cdots T_{i_r}$.

Note that the definition of \mathcal{H}_n in [DL11] differs from our definition given above. To translate between our differing conventions we may identify as follows:

- $s \rightarrow t$ (parameters)
- $T_i \rightarrow tT_i^{-1}$ (Hecke elements).

Remark 2. There are natural algebra inclusions $\mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ given by $T_i \rightarrow T_i$ for $1 \leq i \leq n - 1$. Under this embedding $\bar{\theta}_i \rightarrow \bar{\theta}_i$ for $1 \leq i \leq n$ so we can naturally identify the copies of $\bar{\theta}_i$ in both \mathcal{H}_n and \mathcal{H}_{n+1} .

We require the following list of relations.

Proposition 2.17. The following relations hold:

- $\bar{\theta}_i = t^{i-1}T_{i-1}^{-1} \cdots T_1^{-1}T_1^{-1} \cdots T_{i-1}^{-1}$ for $1 \leq i \leq n$
- $\bar{\theta}_i \bar{\theta}_j = \bar{\theta}_j \bar{\theta}_i$ for $1 \leq i, j \leq n$
- $T_i \bar{\theta}_j = \bar{\theta}_j T_i$ for $j \notin \{i, i + 1\}$
- $\bar{\varphi}_i = tT_i^{-1}(\bar{\theta}_i - \bar{\theta}_{i+1}) + (t - 1)\bar{\theta}_{i+1}$ for $1 \leq i \leq n - 1$
- $\bar{\varphi}_i \bar{\varphi}_{i+1} \bar{\varphi}_i = \bar{\varphi}_{i+1} \bar{\varphi}_i \bar{\varphi}_{i+1}$ for $1 \leq i \leq n - 1$
- $\bar{\varphi}_i \bar{\varphi}_j = \bar{\varphi}_j \bar{\varphi}_i$ for $|i - j| > 1$
- $\bar{\varphi}_i \bar{\theta}_j = \bar{\theta}_{s_i(j)} \bar{\varphi}_i$ for $1 \leq i \leq n - 1$ and $1 \leq j \leq n$
- $\bar{\varphi}_i^2 = (t\bar{\theta}_i - \bar{\theta}_{i+1})(t\bar{\theta}_{i+1} - \bar{\theta}_i)$.

Proof. This result follows directly from using the map ρ_n defined in Definition 2.23 and Proposition 2.22 which will be independently proven later. \square

The following definition gives a description of the irreducible representations of \mathcal{H}_n . There are many equivalent methods for defining these representations but we choose to specify eigenvectors for the Jucys-Murphy elements $\bar{\theta}_i$ directly as we will require these eigenvectors throughout this paper.

Definition 2.18. Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$. Define S_λ to be the \mathcal{H}_n -module spanned by e_τ for $\tau \in \text{SYT}(\lambda)$ defined by the following relations:

- $\bar{\theta}_i(e_\tau) = t^{c_\tau(i)} e_\tau$
- If $s_i(\tau) > \tau$ then $\bar{\varphi}_i(e_\tau) = (t^{c_\tau(i)} - t^{c_\tau(i+1)})e_{s_i(\tau)}$.
- If the labels $i, i + 1$ are in the same row in τ then $T_i(e_\tau) = e_\tau$.
- If the labels $i, i + 1$ are in the same column in τ then $T_i(e_\tau) = -te_\tau$.

Using the relations from Proposition 2.17 we can show the following more explicit form for the action of the T_i on the $\text{SYT}(\lambda)$ basis:

- If $s_i(\tau) > \tau$ then

$$T_i(e_\tau) = e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} e_\tau.$$

- If $s_i(\tau) < \tau$ then

$$T_i(e_\tau) = -\frac{(t^{c_\tau(i+1)+1} - t^{c_\tau(i)})(t^{c_\tau(i)+1} - t^{c_\tau(i+1)})}{(t^{c_\tau(i+1)} - t^{c_\tau(i)})^2} e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} e_\tau.$$

Proposition 2.19. Definition 2.18 is well-posed i.e. the action of the operators T_i on S_λ define an irreducible \mathcal{H}_n -module.

Proof. As this construction is standard we will only give an outline. It follows from standard theory for the finite Hecke algebra \mathcal{H}_n (analogous to that of the symmetric group \mathfrak{S}_n in characteristic 0) that there exists an irreducible representation of \mathcal{H}_n , S_λ , corresponding to the partition λ with a basis of weight vectors for the Jucys-Murphy elements $\bar{\theta}_i$, v_τ say, indexed by $\tau \in \text{SYT}(\lambda)$. Further, the weights are given by $\bar{\theta}_i(v_\tau) = t^{c_\tau(i)}v_\tau$. As these weights are all distinct it follows that this basis is unique up to re-normalization by nonzero scalars. The presentation given in Definition 2.18 fixes a specific normalization given by choosing first $e_{\tau_{\lambda^s}} = v_{\tau_{\lambda^s}}$ and then building the full basis e_τ using the intertwiner $\bar{\varphi}_i$ relations in Proposition 2.17 with the choice that whenever $s_i(\tau) > \tau$ we have that $\bar{\varphi}_i(e_\tau) = (t^{c_\tau(i)} - t^{c_\tau(i+1)})e_{s_i(\tau)}$. Up to an initial arbitrary choice for the scalar multiple of $e_{\tau_{\lambda^s}}$, this uniquely determines the rest of the coefficients of the e_τ . \square

Remark 3. The set $\{\lambda \in \mathbb{Y} : |\lambda| = n\}$ gives a full set of irreducible \mathcal{H}_n -modules up to isomorphism. Note that for $\tau, \tau' \in \text{SYT}(\lambda)$, the $\bar{\theta}$ -weights of $e_\tau = e_{\tau'}$ are equal if and only if $\tau = \tau'$.

Lemma 2.20. Let $\lambda \in \mathbb{Y}$ and $n \geq n_\lambda$. Let \square_0 be the unique square in the skew-diagram $\lambda^{(n+1)}/\lambda^{(n)}$. Consider the map $q_\lambda^{(n)} : \lambda^{(n+1)} \rightarrow \lambda^{(n)}$ given for $\tau \in \text{SYT}(\lambda^{(n+1)})$ as

$$q_\lambda^{(n)}(e_\tau) := \begin{cases} e_{\tau|_{\lambda^{(n)}}} & \tau(\square_0) = n+1 \\ 0 & \tau(\square_0) \neq n+1. \end{cases}$$

Then $q_\lambda^{(n)}$ is a \mathcal{H}_n -module map.

Proof. Let $\tau \in \text{SYT}(\lambda^{(n+1)})$. First, assume that $\tau(\square_0) \neq n+1$ so that $q_\lambda^{(n)}(e_\tau) = 0$. Then for $1 \leq i \leq n-1$, from the relations in Definition 2.18, we see that $T_i(e_\tau)$ is either a scalar multiple of e_τ or a linear combination of e_τ and $e_{s_i(\tau)}$. In either case $q_\lambda^{(n)}(T_i(e_\tau)) = 0 = T_i q_\lambda^{(n)}(e_\tau)$. Now assume $\tau(\square_0) = n+1$. We will be more detailed about this case as we will need to be careful about the combinatorics regarding the coefficients of expanding $T_i(e_\tau)$ into the $\text{SYT}(\lambda^{(n)})$ basis. For $1 \leq i \leq n-1$ we have the cases

- $T_i(e_\tau) = e_\tau$ if $i, i+1$ are in the same row of τ
- $T_i(e_\tau) = -te_\tau$ if $i, i+1$ are in the same column of τ
- $T_i(e_\tau) = e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}e_\tau$ if $s_i(\tau) > \tau$
- $T_i(e_\tau) = -\frac{(t^{c_\tau(i+1)+1} - t^{c_\tau(i)})(t^{c_\tau(i)+1} - t^{c_\tau(i+1)})}{(t^{c_\tau(i+1)} - t^{c_\tau(i)})^2}e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}e_\tau$ if $s_i(\tau) < \tau$.

In any of these cases since $\tau(\square_0) = n+1$ and $1 \leq i \leq n-1$, we have that $s_i(\tau)(\square_0) = n+1$ as well. Further, the placement of the boxes labelled $i, i+1$ in the labellings $\tau, s_i(\tau)$ is unchanged when restricted to $\lambda^{(n)}$ i.e. in the labellings $\tau|_{\lambda^{(n)}}, s_i(\tau)|_{\lambda^{(n)}}$. Let $\tau' := \tau|_{\lambda^{(n)}}$. Therefore we have the cases:

- $q_\lambda^{(n)}(T_i(e_\tau)) = e_{\tau|_{\lambda^{(n)}}} = T_i q_\lambda^{(n)}(e_\tau)$ if $i, i+1$ are in the same row of τ
- $q_\lambda^{(n)}(T_i(e_\tau)) = -te_{\tau|_{\lambda^{(n)}}} = T_i q_\lambda^{(n)}(e_\tau)$ if $i, i+1$ are in the same column of τ
- $q_\lambda^{(n)}(T_i(e_\tau)) = e_{s_i(\tau')} + \frac{(1-t)t^{c_{\tau'}(i)}}{t^{c_{\tau'}(i)} - t^{c_{\tau'}(i+1)}}e_{\tau'} = T_i q_\lambda^{(n)}(e_\tau)$ if $s_i(\tau) > \tau$
- $q_\lambda^{(n)}(T_i(e_\tau)) = -\frac{(t^{c_{\tau'}(i+1)+1} - t^{c_{\tau'}(i)})(t^{c_{\tau'}(i)+1} - t^{c_{\tau'}(i+1)})}{(t^{c_{\tau'}(i+1)} - t^{c_{\tau'}(i)})^2}e_{s_i(\tau')} + \frac{(1-t)t^{c_{\tau'}(i)}}{t^{c_{\tau'}(i)} - t^{c_{\tau'}(i+1)}}e_{\tau'} = T_i q_\lambda^{(n)}(e_\tau)$ if $\tau > s_i(\tau)$.

Thus in all cases we have that $q_\lambda^{(n)}(T_i(e_\tau)) = T_i q_\lambda^{(n)}(e_\tau)$. Hence, $q_\lambda^{(n)}$ is a \mathcal{H}_n -module map. \square

2.3 Affine Hecke Algebra

We will be interested in the presentation of the AHAs of type GL_n which follows.

Definition 2.21. Define the affine Hecke algebra \mathcal{A}_n to be the $\mathbb{Q}(q, t)$ -algebra generated by T_1, \dots, T_{n-1} and $\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}$ subject to the relations

- T_1, \dots, T_{n-1} generate \mathcal{H}_n

- $\theta_i \theta_j = \theta_j \theta_i$ for all $1 \leq i, j \leq n$
- $\theta_{i+1} = tT_i^{-1}\theta_i T_i^{-1}$ for $1 \leq i \leq n-1$
- $T_i \theta_j = \theta_j T_i$ for $j \notin \{i, i+1\}$

Define the special elements π_n and $\varphi_1, \dots, \varphi_{n-1}$ of \mathcal{A}_n by

- $\pi_n := t^{n-1}\theta_1 T_1^{-1} \cdots T_{n-1}^{-1}$
- $\varphi_i := (tT_i^{-1})\theta_i - \theta_i(tT_i^{-1})$.

We will denote by $\theta^{(n)}$ the commutative subalgebra of \mathcal{A}_n generated by $\theta_n^{(1)}, \dots, \theta_n^{(n)}$.

It is important to note that when converting between the AHA conventions in this paper and those in Dunkl-Luque [DL11] the standard Cherednik elements ξ_i of Dunkl-Luque do **not** align with the θ_i above. In particular, after the appropriate translation into our conventions we have that ξ_i are given by $\xi_i = t^{n-i+1}T_{i-1} \cdots T_1 \pi_n T_{n-1}^{-1} \cdots T_i^{-1}$ as opposed to $\theta_i = t^{-(n-i)}T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1} \cdots T_i$. The distinction between the standard Cherednik elements ξ_i and the reversed orientation Cherednik elements θ_i will be important in this paper since the latter will yield operators with additional stability properties which the ξ_i do not satisfy.

Remark 4. We will use the notation $\theta_i^{(n)}$ and $\theta_i^{(m)}$ to differentiate between the copies of θ_i in \mathcal{A}_n and \mathcal{A}_m for $n \neq m$.

The following proposition is standard in AHA theory and will be required at many points throughout this paper. We include the proofs of these relations for completeness and to emphasize that we may use intertwiner theory for AHA with the θ_i elements instead of the standard ξ_i with only slight differences.

Proposition 2.22. The following relations hold:

- $\varphi_i = tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1} = (\theta_{i+1} - \theta_i)tT_i^{-1} + (1-t)\theta_{i+1}$ for $1 \leq i \leq n-1$
- $\varphi_i \theta_j = \theta_{s_i(j)} \varphi_i$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$
- $\varphi_i^2 = (t\theta_i - \theta_{i+1})(t\theta_{i+1} - \theta_i)$
- $\varphi_i \varphi_{i+1} \varphi_i = \varphi_{i+1} \varphi_i \varphi_{i+1}$ for $1 \leq i \leq n-2$
- $\varphi_i \varphi_j = \varphi_j \varphi_i$ for $|i-j| > 1$.

Proof. The proofs of the correctness of the above relations are standard but we include them for completeness. We will proceed by proving each of these relations in the order in which they appear above.

Let $1 \leq i \leq n-1$. Then

$$\begin{aligned} \varphi_i &= tT_i^{-1}\theta_i - \theta_i(tT_i^{-1}) \\ &= tT_i^{-1}\theta_i - T_i\theta_{i+1} \\ &= tT_i^{-1}\theta_i - (tT_i^{-1} + 1 - t)\theta_{i+1} \\ &= tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}. \end{aligned}$$

By a similar calculation we also get

$$\varphi_i = (\theta_{i+1} - \theta_i)tT_i^{-1} + (1-t)\theta_{i+1}.$$

This can also be written as

$$\varphi_i = (\theta_{i+1} - \theta_i)T_i + (1-t)\theta_i$$

which we will need later in this proof.

Now we see

$$\begin{aligned}
\varphi_i \theta_i &= tT_i^{-1}(\theta_i - \theta_{i+1})\theta_i + (t-1)\theta_{i+1}\theta_i \\
&= tT_i^{-1}\theta_i(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}\theta_i \\
&= \theta_{i+1}T_i(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}\theta_i \\
&= \theta_{i+1}(T_i(\theta_i - \theta_{i+1}) + (t-1)\theta_i) \\
&= \theta_{i+1}((tT_i^{-1} + 1 - t)(\theta_i - \theta_{i+1}) + (t-1)\theta_i) \\
&= \theta_{i+1}(tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}) \\
&= \theta_{i+1}\varphi_i
\end{aligned}$$

and

$$\begin{aligned}
\varphi_i \theta_{i+1} &= tT_i^{-1}(\theta_i - \theta_{i+1})\theta_{i+1} + (t-1)\theta_{i+1}^2 \\
&= (T_i + t - 1)\theta_{i+1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}^2 \\
&= T_i\theta_{i+1}(\theta_i - \theta_{i+1}) + (t-1)(\theta_{i+1}(\theta_i - \theta_{i+1}) + \theta_{i+1}^2) \\
&= t\theta_i T_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_i\theta_{i+1} \\
&= \theta_i(tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}) \\
&= \theta_i\varphi_i.
\end{aligned}$$

For any $j \notin \{i, i+1\}$ it follows since θ_j commutes with both θ_i and T_i that

$$\varphi_i \theta_j = \theta_j \varphi_i.$$

Thus for any $1 \leq j \leq n$

$$\varphi_i \theta_j = \theta_{s_i(j)} \varphi_i.$$

Now we have that

$$\begin{aligned}
\varphi_i^2 &= (tT_i^{-1}\theta_i - \theta_i tT_i^{-1})^2 \\
&= t^2 T_i^{-1}\theta_i T_i^{-1}\theta_i - t^2 T_i^{-1}\theta_i^2 T_i^{-1} - t^2 \theta_i T_i^{-2}\theta_i + t^2 \theta_i T_i^{-1}\theta_i T_i^{-1} \\
&= t\theta_{i+1}\theta_i - t\theta_{i+1}T_i\theta_i T_i^{-1} - t\theta_i(1 + (t-1)T_i^{-1})\theta_i + t\theta_i\theta_{i+1} \\
&= 2t\theta_i\theta_{i+1} - t\theta_{i+1}(tT_i^{-1} + 1 - t)\theta_i T_i^{-1} - t\theta_i^2 + t(1-t)\theta_i T_i^{-1}\theta_i \\
&= 2t\theta_i\theta_{i+1} - t^2\theta_{i+1}T_i^{-1}\theta_i T_i^{-1} + t(t-1)\theta_i\theta_{i+1}T_i^{-1} - t\theta_i^2 + (1-t)\theta_i\theta_{i+1}T_i \\
&= 2t\theta_i\theta_{i+1} - t\theta_{i+1}^2 - t\theta_i^2 + (1-t)\theta_i\theta_{i+1}(T_i - tT_i^{-1}) \\
&= 2t\theta_i\theta_{i+1} - t\theta_{i+1}^2 - t\theta_i^2 + (1-t)^2\theta_i\theta_{i+1} \\
&= (1+t^2)\theta_i\theta_{i+1} - t\theta_{i+1}^2 - t\theta_i^2 \\
&= (t\theta_i - \theta_{i+1})(t\theta_{i+1} - \theta_i).
\end{aligned}$$

Now suppose $1 \leq i \leq n-2$. By expanding each of the φ_j from right to left using $\varphi_j = (\theta_{j+1} - \theta_j)T_j + (1-t)\theta_j$ and repeatedly applying the relation $\varphi_j\theta_k = \theta_{s_j(k)}\varphi_j$ we find

$$\begin{aligned}
\varphi_i \varphi_{i+1} \varphi_i &= (\theta_{i+2} - \theta_{i+1})(\theta_{i+2} - \theta_i)(\theta_{i+1} - \theta_i)T_i T_{i+1} T_i + (1-t)\theta_i(\theta_{i+2} - \theta_{i+1})(\theta_{i+2} - \theta_i)T_{i+1} T_i \\
&\quad + (1-t)\theta_{i+1}(\theta_{i+2} - \theta_i)(\theta_{i+1} - \theta_i)T_i T_{i+1} + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2} - \theta_i)T_i \\
&\quad + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2} - \theta_i)T_{i+1} \\
&\quad + (t(1-t)\theta_i(\theta_{i+2} - \theta_{i+1})(\theta_{i+1} - \theta_i) + (1-t)^3\theta_i^2\theta_{i+1}).
\end{aligned}$$

Using the same method we also see that

$$\begin{aligned}
\varphi_{i+1}\varphi_i\varphi_{i+1} &= (\theta_{i+1} - \theta_i)(\theta_{i+2} - \theta_i)(\theta_{i+2} - \theta_{i+1})T_{i+1}T_iT_{i+1} + (1-t)\theta_{i+1}(\theta_{i+1} - \theta_i)(\theta_{i+2} - \theta_i)T_iT_{i+1} \\
&\quad + (1-t)\theta_i(\theta_{i+2} - \theta_i)(\theta_{i+2} - \theta_{i+1})T_{i+1}T_i + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2} - \theta_i)T_{i+1} \\
&\quad + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2} - \theta_i)T_i \\
&\quad + (t(1-t)\theta_i(\theta_{i+1} - \theta_i)(\theta_{i+2} - \theta_{i+1}) + (1-t)^3\theta_i^2\theta_{i+1}).
\end{aligned}$$

From here we may use the braid relation $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ and some rearrangement of terms to see $\varphi_i\varphi_{i+1}\varphi_i = \varphi_{i+1}\varphi_i\varphi_{i+1}$.

Lastly, consider $|i - j| > 1$. Since $T_iT_j = T_jT_i$, $T_i\theta_j = \theta_jT_i$, and $\theta_i\theta_j = \theta_j\theta_i$ we readily find that $\varphi_i\varphi_j = \varphi_j\varphi_i$. \square

In this paper we will be interested in AHA modules which are *pulled back* from irreducible finite Hecke representations. To do this we need to define algebra surjections $\mathcal{A}_n \rightarrow \mathcal{H}_n$. There are many such choices for these maps but we choose the maps ρ_n defined below carefully so that the AHA modules we consider in this paper satisfy nontrivial stability conditions.

Definition 2.23. Define the $\mathbb{Q}(q, t)$ -algebra homomorphism $\rho_n : \mathcal{A}_n \rightarrow \mathcal{H}_n$ by

- $\rho_n(T_i) = T_i$ for $1 \leq i \leq n - 1$
- $\rho_n(\theta_1) = 1$.

For a \mathcal{H}_n -module V we will denote by $\rho_n^*(V)$ the \mathcal{A}_n -module with action defined for $v \in V$ and $X \in \mathcal{A}_n$ by $X(v) := \rho_n(X)(v)$.

Remark 5. Note that $\rho_n(\pi_n) = t^{n-1}T_1^{-1} \cdots T_{n-1}^{-1}$ and for all $1 \leq i \leq n$, $\rho_n(\theta_i) = \bar{\theta}_i$. Further, for $\lambda \in \mathbb{Y}$ with $|\lambda| = n$, $\rho_n^*(\lambda)$ is an irreducible \mathcal{A}_n -module with a basis of θ -weight vectors $\{e_\tau\}_{\tau \in \text{SYT}(\lambda)}$ with distinct weights.

2.4 Positive Double Affine Hecke Algebra

We will use the following presentation for the positive DAHAs in type GL_n .

Definition 2.24. Define the positive double affine Hecke algebra \mathcal{D}_n to be the $\mathbb{Q}(q, t)$ -algebra generated by T_1, \dots, T_{n-1} , $\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}$, and X_1, \dots, X_n subject to the relations

- T_1, \dots, T_{n-1} and $\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}$ satisfy the relations of \mathcal{A}_n in Definition 2.21
- $X_iX_j = X_jX_i$ for $1 \leq i, j \leq n$
- $X_{i+1} = tT_i^{-1}X_iT_i^{-1}$ for $1 \leq i \leq n - 1$
- $T_iX_j = X_jT_i$ for $1 \leq i \leq n - 1$ and $1 \leq j \leq n$ with $j \notin \{i, i + 1\}$
- $\pi_n X_i \pi_n^{-1} = X_{i+1}$ for $1 \leq i \leq n - 1$
- $\pi_n X_n \pi_n^{-1} = qX_1$.

Define the special element $\gamma_n := X_n T_{n-1} \cdots T_1$.

The element γ_n satisfies some nice intertwining relations which we will need later.

Lemma 2.25. The following hold:

- $\theta_i \gamma_n = \gamma_n \theta_{i+1}$ for $1 \leq i \leq n - 1$
- $\theta_n \gamma_n = \gamma_n q \theta_1$.

Proof. Let $1 \leq i \leq n-1$. We find that

$$\begin{aligned}
\theta_i \gamma_n &= t^{-(n-i)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1} \cdots T_i X_n T_{n-1} \cdots T_1 \\
&= t^{-(n-i)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1} X_n T_{n-2} \cdots T_i T_{n-1} \cdots T_1 \\
&= t^{-(n-i)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_n t X_{n-1} T_{n-1}^{-1} T_{n-2} \cdots T_i T_{n-1} \cdots T_1 \\
&= t^{-(n-(i+1))} T_{i-1}^{-1} \cdots T_1^{-1} X_n \pi_n T_{n-1}^{-1} T_{n-2} \cdots T_i T_{n-1} \cdots T_1 \\
&= t^{-(n-(i+1))} X_n T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1}^{-1} T_{n-2} \cdots T_i (T_{n-1} \cdots T_1).
\end{aligned}$$

From the braid relations we see that for all $1 \leq j \leq n-2$

$$T_j(T_{n-1} \cdots T_1) = (T_{n-1} \cdots T_1)T_{j+1}$$

and hence

$$\begin{aligned}
&t^{-(n-(i+1))} X_n T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1}^{-1} T_{n-2} \cdots T_i (T_{n-1} \cdots T_1) \\
&= t^{-(n-(i+1))} X_n T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1}^{-1} (T_{n-1} \cdots T_1) T_{n-1} \cdots T_{i+1} \\
&= t^{-(n-(i+1))} X_n T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-2} \cdots T_1 T_{n-1} \cdots T_{i+1} \\
&= t^{-(n-(i+1))} X_n T_{i-1}^{-1} \cdots T_1^{-1} T_{n-1} \cdots T_2 \pi_n T_{n-1} \cdots T_{i+1} \\
&= t^{-(n-(i+1))} X_n T_{i-1}^{-1} \cdots T_1^{-1} T_{n-1} \cdots T_2 T_1 T_1^{-1} \pi_n T_{n-1} \cdots T_{i+1} \\
&= t^{-(n-(i+1))} X_n T_{n-1} \cdots T_1 T_i^{-1} \cdots T_2^{-1} T_1^{-1} \pi_n T_{n-1} \cdots T_{i+1} \\
&= (X_n T_{n-1} \cdots T_1) (t^{-(n-(i+1))} T_i^{-1} \cdots T_1^{-1} \pi_n T_{n-1} \cdots T_{i+1}) \\
&= \gamma_n \theta_{i+1}.
\end{aligned}$$

Now we consider the last case:

$$\begin{aligned}
\theta_n \gamma_n &= T_{n-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1} \cdots T_1 \\
&= T_{n-1}^{-1} \cdots T_1^{-1} q X_1 \pi_n T_{n-1} \cdots T_1 \\
&= t^{-(n-1)} X_n T_{n-1} \cdots T_1 q \pi_n T_{n-1} \cdots T_1 \\
&= (X_n T_{n-1} \cdots T_1) (q t^{-(n-1)} \pi_n T_{n-1} \cdots T_1) \\
&= \gamma_n q \theta_1.
\end{aligned}$$

□

Recall the definition of the intertwiner elements φ_i in Definition 2.21. As is standard in DAHA theory we will use the elements $\{\varphi_1, \dots, \varphi_{n-1}, \gamma_n\}$ to define intertwiner operators corresponding to elements of $\widehat{\mathfrak{S}}_n^+$.

Definition 2.26. For any $\sigma \in \widehat{\mathfrak{S}}_n^+$ with $\sigma = (s_{i_1} \cdots s_{i_{j_1}}) \tilde{\gamma}_n \cdots \tilde{\gamma}_n (s_{i_{j_1+\dots+j_{r-1}+1}} \cdots s_{i_{j_1+\dots+j_r}})$ written minimally in terms of the generators $\{s_1, \dots, s_{n-1}, \tilde{\gamma}\}$ define $\varphi_\sigma \in \mathcal{D}_n$ by

$$\varphi_\sigma := (\varphi_{i_1} \cdots \varphi_{i_{j_1}}) \gamma_n \cdots \gamma_n (\varphi_{i_{j_1+\dots+j_{r-1}+1}} \cdots \varphi_{i_{j_1+\dots+j_r}}) \in \mathcal{D}_n.$$

In particular, we have that $\varphi_{s_i} = \varphi_i$ and $\varphi_{\tilde{\gamma}_n} = \gamma_n$.

The main utility of considering the intertwiner operators φ_σ comes from the next lemma.

Lemma 2.27. If v is a $\theta^{(n)}$ -weight vector in some \mathcal{D}_n -module with weight $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\sigma \in \mathfrak{S}_n$ with $\varphi_\sigma(v) \neq 0$ then $\varphi_\sigma(v)$ is a $\theta^{(n)}$ -weight vector with weight α^σ given by the following recursion:

- $\alpha^{s_i} = (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n)$
- $\alpha^{\tilde{\gamma}_n} = (\alpha_2, \dots, \alpha_n, q\alpha_1)$

- $(\alpha^{\sigma_2})^{\sigma_1} = \alpha^{\sigma_1\sigma_2}$.

Proof. This result follows easily by using induction on $\widehat{\mathfrak{S}}_n^+$ using the relations in Proposition 2.22 and Lemma 2.25. We leave the details to the reader. \square

We will be primarily interested in modules for the *spherical* subalgebra of \mathcal{D}_n .

Definition 2.28. Let $\epsilon^{(n)} \in \mathcal{H}_n$ denote the (normalized) trivial idempotent given by

$$\epsilon^{(n)} := \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} t^{\binom{n}{2} - \ell(\sigma)} T_\sigma$$

where $[n]_t! := \prod_{i=1}^n \left(\frac{1-t^i}{1-t}\right)$. The positive spherical double affine Hecke algebra $\mathcal{D}_n^{\text{sph}}$ is the non-unital subalgebra of \mathcal{D}_n given by $\mathcal{D}_n^{\text{sph}} := \epsilon^{(n)} \mathcal{D}_n \epsilon^{(n)}$.

The element $\epsilon^{(n)} := \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} t^{\binom{n}{2} - \ell(\sigma)} T_\sigma \in \mathcal{H}_n$ is uniquely determined by the following properties:

- $\epsilon^{(n)} \neq 0$ (non-zero)
- $(\epsilon^{(n)})^2 = \epsilon^{(n)}$ (idempotent)
- $\epsilon^{(n)} T_i = T_i \epsilon^{(n)}$ for all $1 \leq i \leq n-1$ (central)
- $T_i \epsilon^{(n)} = \epsilon^{(n)}$ (trivial-like).

We will use without proof that $\epsilon^{(n)}$ as defined in Definition 2.28 satisfies these properties but it is straightforward to check this using the defining relations of \mathcal{H}_n . Since $(\epsilon^{(n)})^2 = \epsilon^{(n)}$ we see that $\mathcal{D}_n^{\text{sph}}$ is a unital algebra with unit $\epsilon^{(n)}$. The algebra $\mathcal{D}_n^{\text{sph}}$ contains both of the subalgebras $\mathbb{Q}(q, t)[X_1, \dots, X_n]^{\mathfrak{S}_n} \epsilon^{(n)}$ and $\mathbb{Q}(q, t)[\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}]^{\mathfrak{S}_n} \epsilon^{(n)}$.

We may use $\epsilon^{(n)}$ to generate modules for the spherical DAHA. Given any \mathcal{D}_n -module V the space $\epsilon^{(n)}(V)$ is naturally a $\mathcal{D}_n^{\text{sph}}$ -module. In the standard picture of Cherednik theory the standard polynomial representation of \mathcal{D}_n on $\mathbb{Q}(q, t)[x_1, \dots, x_n]$ is symmetrized using $\epsilon^{(n)}$ to yield the standard symmetric polynomial representation of $\mathcal{D}_n^{\text{sph}}$ on $\mathbb{Q}(q, t)[x_1, \dots, x_n]^{\mathfrak{S}_n}$.

Remark 6. We will use without proof the standard result that \mathcal{D}_n is a *free* right \mathcal{A}_n module with basis $\{X^\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^n}$. This follows from the standard PBW-type result for DAHA. Importantly, for our purposes, this implies that for any \mathcal{A}_n -module V with $\mathbb{Q}(q, t)$ -basis $\{v_i\}_{i \in I}$, the induced module

$$\text{Ind}_{\mathcal{A}_n}^{\mathcal{D}_n} V := \mathcal{D}_n \otimes_{\mathcal{A}_n} V$$

has $\mathbb{Q}(q, t)$ -basis $\{X^\alpha \otimes v_i \mid \alpha \in \mathbb{Z}_{\geq 0}^n, i \in I\}$.

2.5 Elliptic Hall Algebra

Here we give a brief description of the positive elliptic Hall algebra which will be a sufficient introduction for the purposes of this paper. For a more complete description of the elliptic Hall algebra we direct the reader to the original paper of Burban-Schiffmann [BS12] introducing EHA and the subsequent paper of Schiffmann-Vasserot [SV13] which develops more of the relations between EHA and Macdonald theory.

Definition 2.29. For $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ define the special elements $P_{0,\ell}^{(n)}, P_{r,0}^{(n)} \in \mathcal{D}_n^{\text{sph}}$ by

- $P_{0,\ell}^{(n)} = \epsilon^{(n)} \left(\sum_{i=1}^n \theta_i^\ell \right) \epsilon^{(n)}$
- $P_{r,0}^{(n)} = q^r \epsilon^{(n)} \left(\sum_{i=1}^n X_i^r \right) \epsilon^{(n)}$.

Theorem 2.30. [SV13] The elements $P_{0,\ell}^{(n)}, P_{r,0}^{(n)}$ for $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ generate $\mathcal{D}_n^{\text{sph}}$ as a $\mathbb{Q}(q, t)$ -algebra. There is a unique $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ grading on $\mathcal{D}_n^{\text{sph}}$ determined by

- $\deg(P_{0,\ell}^{(n)}) = (0, \ell)$
- $\deg(P_{r,0}^{(n)}) = (r, 0)$.

There is a graded algebra surjection $\mathcal{D}_{n+1}^{\text{sph}} \rightarrow \mathcal{D}_n^{\text{sph}}$ determined for $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ by $P_{0,\ell}^{(n+1)} \rightarrow P_{0,\ell}^{(n)}$ and $P_{r,0}^{(n+1)} \rightarrow P_{r,0}^{(n)}$.

The existence of the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ -graded algebra surjections $\mathcal{D}_{n+1}^{\text{sph}} \rightarrow \mathcal{D}_n^{\text{sph}}$ allows for the following definition.

Definition 2.31. [SV13] The *positive elliptic Hall algebra* \mathcal{E}^+ is the stable limit of the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ -graded algebras $\mathcal{D}_n^{\text{sph}}$ with respect to the maps $\mathcal{D}_{n+1}^{\text{sph}} \rightarrow \mathcal{D}_n^{\text{sph}}$. For $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ define the special elements of \mathcal{E}^+ , $P_{0,\ell} := \lim_n P_{0,\ell}^{(n)}$ and $P_{r,0} := \lim_n P_{r,0}^{(n)}$.

The positive elliptic Hall algebra contains elements $P_{a,b}$ for $(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \setminus \{(0,0)\}$ which may be defined using repeated commutators of the elements $P_{0,\ell}, P_{r,0}$. For example, $P_{1,1} = [P_{0,1}, P_{1,0}]$. We will not require an explicit description of these elements for the purposes of this paper. Further, we will not require knowledge of the full elliptic Hall algebra \mathcal{E} which is obtained as the Drinfeld double of \mathcal{E}^+ with respect to a certain Hopf pairing. In the standard Macdonald theory picture, we can realize the action of the full EHA on the ring of symmetric functions Λ using multiplication operators p_r^\bullet , skewing operators p_r^\perp , and Macdonald operators $p_\ell[\Delta]$ roughly corresponding to the elements $P_{r,0}, P_{-r,0}, P_{0,\ell}$ respectively.

Remark 7. We will be considering the $\mathbb{Z}_{\geq 0}$ -grading on \mathcal{E}^+ obtained by the specialization $(a,b) \rightarrow a$ i.e. for $r > 0$ and $\ell \in \mathbb{Z} \setminus \{0\}$

- $\deg(P_{0,\ell}) = 0$
- $\deg(P_{r,0}) = r$.

When we refer to a \mathcal{E}^+ -module V as *graded* we are referring to the $\mathbb{Z}_{\geq 0}$ -grading on \mathcal{E}^+ .

3 DAHA Modules from Young Diagrams

3.1 The \mathcal{D}_n -module V_λ

We begin by defining a collection of DAHA modules indexed by Young diagrams $\lambda \in \mathbb{Y}$. These modules are the same as those appearing in [DL11] but we take the approach of using induction from \mathcal{A}_n to \mathcal{D}_n for their definition.

Definition 3.1. Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$. Define the \mathcal{D}_n -module V_λ to be the induced module $V_\lambda := \text{Ind}_{\mathcal{A}_n}^{\mathcal{D}_n} \rho_n^*(S_\lambda)$.

The modules V_λ naturally have the basis given by $X^\alpha \otimes e_\tau$ where X^α is a monomial and $\tau \in \text{SYT}(\lambda)$. We will refer to this as the standard basis of V_λ .

Using the theory of intertwiners for DAHA and some combinatorics we are able to show the following structural results. The F_τ appearing below are the version of the non-symmetric vv. Macdonald polynomials from [DL11] following our conventions. These do **not** align with the vv. Macdonald polynomials of [DL11].

Proposition 3.2. There exists a basis of V_λ consisting of $\theta^{(n)}$ -weight vectors $\{F_\tau : \tau \in \text{PSYT}_{\geq 0}(\lambda)\}$ with distinct $\theta^{(n)}$ -weights such that the following hold:

- $\theta_i^{(n)}(F_\tau) = q^{w_\tau(i)} t^{c_\tau(i)} F_\tau$
- If $\tau \in \text{SYT}(\lambda)$ then $F_\tau = 1 \otimes e_\tau$.
- If $s_i(\tau) > \tau$ then

$$\left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right) (F_\tau) = F_{s_i(\tau)}.$$

- $F_{\Psi(\tau)} = q^{w_1(\tau)} X_n \pi_n^{-1}(F_\tau)$.

Proof. Using Mackey Decomposition we find

$$\begin{aligned}
& gr. \operatorname{Res}_{\theta^{(n)}}^{\mathcal{D}_n}(V_\lambda) \\
&= gr. \operatorname{Res}_{\theta^{(n)}}^{\mathcal{D}_n} \operatorname{Ind}_{\mathcal{A}_n}^{\mathcal{D}_n} \rho_n^*(S_\lambda) \\
&= \bigoplus_{\sigma \in (\widehat{\mathfrak{S}}_n / \mathfrak{S}_n)^+} \left(\operatorname{Res}_{\theta^{(n)}}^{\mathcal{A}_n} \rho_n^*(S_\lambda) \right)^\sigma \\
&= \bigoplus_{\substack{\sigma \in (\widehat{\mathfrak{S}}_n / \mathfrak{S}_n)^+ \\ \tau \in \operatorname{SYT}(\lambda)}} \mathbb{Q}(q, t)(\varphi_\sigma \otimes e_\tau).
\end{aligned}$$

As a consequence we find that the set $\{\varphi_\sigma \otimes e_\tau\}_{(\sigma, \tau) \in (\widehat{\mathfrak{S}}_n / \mathfrak{S}_n)^+ \times \operatorname{SYT}(\lambda)}$ is a generalized $\theta^{(n)}$ -weight basis for V_λ . We now define

$$F_\tau := g_\tau \varphi_{\sigma_{w_\tau}} \otimes e_{S(\tau)}$$

of V_λ where the scalars g_τ are chosen uniquely to satisfy the conditions detailed in this proposition's statement. It is easy to check that since every $\tau \in \operatorname{PSYT}_{\geq 0}(\lambda)$ may be obtained by applying σ_{w_τ} to $S(\tau)$ the scalars g_τ are uniquely determined by setting $g_{\tau^s} = 1$. By Proposition 2.15 this assignment produces a basis for V_λ labelled by $\operatorname{PSYT}_{\geq 0}(\lambda)$. Further, by induction using Lemma 2.27 and Proposition 2.14 we see that no matter our choice of nonzero scalars g_τ each F_τ is a $\theta^{(n)}$ -weight vector with $\theta_i^{(n)}(F_\tau) = q^{w_\tau(i)} t^{c_\tau(i)} F_\tau$.

The only remaining step to justify is that if $\tau \in \operatorname{PSYT}_{\geq 0}(\lambda)$ then $\gamma_n(F_\tau)$ agrees with $X_n \pi_n^{-1}(F_\tau)$ up to some nonzero scalar. We see that

$$\begin{aligned}
& \gamma_n(F_\tau) \\
&= X_n T_{n-1} \cdots T_1(F_\tau) \\
&= X_n \pi_n^{-1} \pi_n T_{n-1} \cdots T_1(F_\tau) \\
&= t^{n-1} X_n \pi_n^{-1} \theta_1(F_\tau) \\
&= t^{n-1} q^{w_\tau(1)} t^{c_\tau(1)} X_n \pi_n^{-1}(F_\tau).
\end{aligned}$$

Therefore, there is no issue in defining the coefficient $g_{\Psi(\tau)}$ so that $F_{\Psi(\tau)} = q^{w_\tau(1)} X_n \pi_n^{-1}(F_\tau)$. \square

Example 6.

$$\begin{aligned}
F_{\begin{array}{|c|} \hline 1q2q \\ \hline 3 \\ \hline \end{array}} &= t^{-2} X_1 X_2 \otimes e_{\begin{array}{|c|} \hline 12 \\ \hline 3 \\ \hline \end{array}} + t^{-2} \left(\frac{1-t}{1-qt^2} \right) X_2 X_3 \otimes e_{\begin{array}{|c|} \hline 13 \\ \hline 2 \\ \hline \end{array}} \\
&+ \frac{t^{-2}}{1+t} \left(\frac{1-t}{1-qt^2} \right) X_2 X_3 \otimes e_{\begin{array}{|c|} \hline 12 \\ \hline 3 \\ \hline \end{array}} - t^{-3} \left(\frac{1-t}{1-qt^2} \right) X_1 X_3 \otimes e_{\begin{array}{|c|} \hline 13 \\ \hline 2 \\ \hline \end{array}} \\
&+ \frac{t^{-1}}{1+t} \left(\frac{1-t}{1-qt^2} \right) X_1 X_3 \otimes e_{\begin{array}{|c|} \hline 12 \\ \hline 3 \\ \hline \end{array}}
\end{aligned}$$

Remark 8. Note that from Proposition 3.2 we get that

$$\gamma_n(F_\tau) = t^{n-1+c_\tau(1)} F_{\Psi(\tau)}.$$

By induction we see that

$$\gamma_n^r(F_\tau) = t^{r(n-1)} t^{c_\tau(1)+\dots+c_\tau(r)} F_{\Psi^r(\tau)}.$$

Proposition 3.3. The \mathcal{D}_n -module V_λ has the following decomposition into \mathcal{A}_n -submodules:

$$\operatorname{Res}_{\mathcal{A}_n}^{\mathcal{D}_n} V_\lambda = \bigoplus_{T \in \operatorname{RYT}_{\geq 0}(\lambda)} U_T$$

where $U_T := \operatorname{span}_{\mathbb{Q}(q,t)} \{F_\tau \mid \tau \in \operatorname{PSYT}_{\geq 0}(\lambda; T)\}$. Further, each \mathcal{A}_n -module U_T is irreducible.

Proof. Let $T \in \text{RYT}_{\geq 0}(\lambda)$. Note that it follows immediately from Proposition 3.2 that each U_T is a \mathcal{A}_n -submodule of V_λ . Further, trivially $U_T \cap U_{T'} = \emptyset$ for $T \neq T'$ since the F_τ are a basis for V_λ and the sets $\text{PSYT}_{\geq 0}(\lambda; T)$ partition $\text{PSYT}_{\geq 0}(\lambda)$. Therefore,

$$\text{Res}_{\mathcal{A}_n}^{\mathcal{D}_n} V_\lambda = \bigoplus_{T \in \text{RYT}_{\geq 0}(\lambda)} U_T.$$

Now let $T \in \text{RYT}_{\geq 0}(\lambda)$. If $U \subset U_T$ is a nonzero \mathcal{A}_n -submodule then U must contain some $\theta^{(n)}$ weight vector as U_T is spanned by $\theta^{(n)}$ weight vectors. Thus there exists some $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $F_{\tau_0} \in U$. But then it follows readily from Proposition 3.2 that by using intertwiner operators φ_i given any $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ we may find $A \in \mathcal{A}_n$ such that $A(F_{\tau_0}) = F_\tau$. Therefore, $U = U_T$ and hence U_T is irreducible. \square

Remark 9. It follows by using Frobenius Reciprocity and Proposition 3.3 that in fact there are surjective \mathcal{A}_n module maps

$$\text{Ind}_{\mathcal{A}_{\mu(T)}}^{\mathcal{A}_n} \chi_T \rightarrow U_T$$

where χ_T is the 1-dimensional representation of $\mathcal{A}_{\mu(T)}$ determined by the $\theta^{(n)}$ -weight of $F_{\min(T)}$ and $T_i \rightarrow 1$ for relevant T_i . Thus each U_T is a quotient of an induced module from a parabolic subalgebra of \mathcal{A}_n . In the case of $T \in \text{RSSYT}_{\geq 0}(\lambda)$ this map is an isomorphism. We may witness the implied bijection between $\text{PSYT}_{\geq 0}(\lambda; T)$ and $\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$ combinatorially using the map $\sigma \rightarrow \sigma(\min(T))$ for $\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$. It is straightforward to check by decomposing λ into horizontal strip diagrams where T is constant along rows that this map is actually an isomorphism of posets.

The following lemma exhibits triangularity for the T_i^{-1} operators with respect to the reversed Bruhat order on $\mathbb{Z}_{\geq 0}^n$.

Lemma 3.4. For $1 \leq i \leq n-1$ and $a \geq 0$,

$$(tT_i^{-1})X_{i+1}^a = X_i^a(tT_i^{-1}) + (t-1)X_{i+1} \frac{X_i^a - X_{i+1}^a}{X_i - X_{i+1}}.$$

Further, every monomial occurring in the term $X_{i+1} \frac{X_i^a - X_{i+1}^a}{X_i - X_{i+1}}$ is strictly lower than X_i^a with respect to the Bruhat ordering \preceq . Consequently, it follows that for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ with $s_i(\alpha) \succeq \alpha$ the following expansion holds for some scalars c_β

$$(tT_i^{-1})X^\alpha = X^{s_i(\alpha)}(tT_i^{-1}) + \sum_{\beta \prec s_i(\alpha)} c_\beta X^\beta.$$

Proof. We start with

$$\begin{aligned} tT_i^{-1}X_{i+1}^a &= (T_i + t - 1)X_{i+1}^a \\ &= T_i X_{i+1}^a + (t-1)X_{i+1}^a \\ &= X_i^a T_i + (1-t)X_i \frac{X_{i+1}^a - X_i^a}{X_i - X_{i+1}} - (1-t)X_{i+1}^a \\ &= X_i^a(tT_i^{-1} + 1 - t) + (1-t)X_i \frac{X_{i+1}^a - X_i^a}{X_i - X_{i+1}} - (1-t)X_{i+1}^a \\ &= X_i^a tT_i^{-1} + (1-t)X_i^a - (1-t)X_{i+1}^a + (1-t)X_i \frac{X_{i+1}^a - X_i^a}{X_i - X_{i+1}} \\ &= X_i^a tT_i^{-1} + (t-1)X_{i+1} \frac{X_{i+1}^a - X_i^a}{X_{i+1} - X_i}. \end{aligned}$$

Further,

$$X_{i+1} \frac{X_{i+1}^a - X_i^a}{X_{i+1} - X_i} = X_{i+1}^a + X_{i+1}^{a-1} X_i + \dots + X_{i+1}^2 X_i^{a-2} + X_{i+1} X_i^{a-1}$$

so that

$$tT_i^{-1}X_{i+1}^\alpha = X_i^\alpha tT_i^{-1} + (t-1)(X_{i+1}^\alpha + X_{i+1}^{\alpha-1}X_i + \dots + X_{i+1}^2X_i^{\alpha-2} + X_{i+1}X_i^{\alpha-1}).$$

Now let $\alpha \in \mathbb{Z}_{\geq 0}^n$ with $s_i(\alpha) \succ \alpha$ i.e. $\alpha_i < \alpha_{i+1}$. Then

$$\begin{aligned} & tT_i^{-1}X^\alpha \\ &= tT_i^{-1}X_1^{\alpha_1} \cdots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i} X_{i+1}^{\alpha_{i+1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} \\ &= X_1^{\alpha_1} \cdots X_{i-1}^{\alpha_{i-1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} tT_i^{-1} X_i^{\alpha_i} X_{i+1}^{\alpha_{i+1}} \\ &= X_1^{\alpha_1} \cdots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i} X_{i+1}^{\alpha_{i+1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} tT_i^{-1} X_{i+1}^{\alpha_{i+1}-\alpha_i} \\ &= X_1^{\alpha_1} \cdots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i} X_{i+1}^{\alpha_{i+1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} \left(X_i^{\alpha_{i+1}-\alpha_i} tT_i^{-1} + (t-1)(X_{i+1}^{\alpha_{i+1}-\alpha_i} + X_{i+1}^{\alpha_{i+1}-\alpha_i-1} X_i + \dots + X_{i+1} X_i^{\alpha_{i+1}-\alpha_i-1}) \right) \\ &= X^{s_i(\alpha)} tT_i^{-1} + (t-1) \sum_{j=0}^{\alpha_{i+1}-\alpha_i-1} X^{\alpha+j(e_i-e_{i+1})}. \end{aligned}$$

Lastly, from Definition 2.6 it is clear that for all $0 \leq j \leq \alpha_{i+1} - \alpha_i - 1$, $s_i(\alpha) \succ \alpha + j(e_i - e_{i+1})$. \square

Corollary 3.5. For $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ each F_τ has a triangular monomial expansion with respect to the Bruhat order on $\mathbb{Z}_{\geq 0}^n$ of the form

$$F_\tau = X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_\lambda$ where $f(\tau) \in S_\lambda$ is given by the following recurrence relations:

- If $\tau \in \text{SYT}(\lambda)$ then $f(\tau) = e_\tau$.
- $f(\Psi(\tau)) = t^{-(n-1)} T_{n-1} \cdots T_1(f(\tau))$
- If $w_\tau(i) < w_\tau(i+1)$ then $f(s_i(\tau)) = tT_i^{-1}f(\tau)$.
- If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then

$$f(s_i(\tau)) = \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) (f(\tau)).$$

Proof. We will proceed by induction with respect to the partial ordering on $\text{PSYT}_{\geq 0}(\lambda)$ defined in Definition 2.2. We will at the same time verify the recurrence relations given for $f(\tau) \in S_\lambda$ given above.

From Proposition 3.2 we know that if $\tau \in \text{SYT}(\lambda)$ then $F_\tau = 1 \otimes e_\tau$. Hence, F_τ trivially has a triangular monomial expansion of the correct form in this case and that $f(\tau) = e_\tau$.

In what follows assume that for $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ we have that

$$F_\tau = X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_\lambda$.

First, we see that

$$\begin{aligned} F_{\Psi(\tau)} &= q^{w_1(\tau)} X_n \pi_n^{-1} (F_\tau) \\ &= q^{w_1(\tau)} X_n \pi_n^{-1} X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} q^{w_1(\tau)} X_n \pi_n^{-1} X^\beta \otimes v_\beta \\ &= q^{w_1(\tau)} q^{-w_1(\tau)} X^{\tilde{w}(\tau)} \pi_n^{-1} \otimes f(\tau) + \sum_{\beta \prec w_\tau} q^{w_1(\tau)} q^{-\beta_1} X^{\tilde{\gamma}(\beta)} \pi_n^{-1} \otimes v_\beta \\ &= X^{\tilde{w}(\tau)} \otimes \rho_n(\pi_n^{-1}) f(\tau) + \sum_{\beta \prec w_\tau} X^{\tilde{\gamma}(\beta)} \otimes q^{w_1(\tau)-\beta_1} \rho_n(\pi_n^{-1}) v_\beta \\ &= X^{\tilde{w}(\tau)} \otimes t^{-(n-1)} T_{n-1} \cdots T_1 f(\tau) + \sum_{\beta \prec w_\tau} X^{\tilde{\gamma}(\beta)} \otimes q^{w_1(\tau)-\beta_1} t^{-(n-1)} T_{n-1} \cdots T_1 v_\beta. \end{aligned}$$

From Lemma 2.7 we know that if $\beta \prec w_\tau$ then $\tilde{\gamma}(\beta) \prec \tilde{\gamma}(w_\tau)$. Therefore, we find that $F_{\Psi(\tau)}$ has the expansion

$$F_{\Psi(\tau)} = X^{\tilde{\gamma}(w_\tau)} \otimes t^{-(n-1)} T_{n-1} \cdots T_1 f(\tau) + \sum_{\beta \prec \tilde{\gamma}(\tau)} X^\beta \otimes v'_\beta$$

for some $v'_\beta \in S_\lambda$. From this we see that $f(\Psi(\tau)) = t^{-(n-1)} T_{n-1} \cdots T_1(f(\tau))$.

Now suppose $s_i(\tau) > \tau$. From Proposition 3.2 we get

$$\begin{aligned} F_{s_i(\tau)} &= \left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right) (F_\tau) \\ &= \left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right) \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \right) \\ &= tT_i^{-1} \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \right) + \left(\frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right) \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \right). \end{aligned}$$

For any $\beta \prec w_\tau$ using Lemma 3.4 we find that

$$tT_i^{-1} X^\beta \otimes v_\beta = \sum_{\beta' \prec s_i(w_\tau)} X^{\beta'} \otimes u_{\beta', \beta}$$

for some $u_{\beta', \beta} \in S_\lambda$; that is to say, each of the monomials $X^{\beta'}$ that appears in the standard basis expansion of $tT_i^{-1} X^\beta \otimes v_\beta$ must have $\beta' \prec s_i(w_\tau)$.

Assume $w_\tau(i) < w_\tau(i+1)$. By Lemma 3.4 we see

$$\begin{aligned} (tT_i^{-1}) X^{w_\tau} \otimes f(\tau) &= X^{s_i(w_\tau)} (tT_i^{-1}) \otimes f(\tau) + \sum_{\beta \prec s_i(w_\tau)} c_\beta X^\beta \otimes f(\tau) \\ &= X^{s_i(w_\tau)} \otimes (tT_i^{-1}) f(\tau) + \sum_{\beta \prec s_i(w_\tau)} c_\beta X^\beta \otimes f(\tau). \end{aligned}$$

Therefore, $F_{s_i(\tau)}$ has the expansion

$$F_{s_i(\tau)} = X^{s_i(w_\tau)} \otimes tT_i^{-1} f(\tau) + \sum_{\beta \prec s_i(w_\tau)} X^\beta \otimes v'_\beta$$

where $v'_\beta \in S_\lambda$. Since $s_i(w_\tau) = w_{s_i(\tau)}$ we have

$$F_{s_i(\tau)} = X^{w_{s_i(\tau)}} \otimes tT_i^{-1} f(\tau) + \sum_{\beta \prec w_{s_i(\tau)}} X^\beta \otimes v'_\beta$$

and $f(s_i(\tau)) = tT_i^{-1} f(\tau)$.

Now assume instead that $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$. Then $T_i X^{w_\tau} = X^{w_\tau} T_i$ so

$$\begin{aligned}
F_{s_i(\tau)} &= \left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right) (F_\tau) \\
&= \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) (F_\tau) \\
&= \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \right) \\
&= \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) X^{w_\tau} \otimes f(\tau) + \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \\
&= X^{w_\tau} \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) \otimes f(\tau) + \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \\
&= X^{w_\tau} \otimes \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) f(\tau) + \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta.
\end{aligned}$$

Therefore, since $w_\tau = w_{s_i(\tau)}$ we find that

$$F_{s_i(\tau)} = X^{w_{s_i(\tau)}} \otimes \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) f(\tau) + \sum_{\beta \prec w_{s_i(\tau)}} X^\beta \otimes v'_\beta$$

for some $v'_\beta \in S_\lambda$ and

$$f(s_i(\tau)) = \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) (f(\tau)).$$

□

Using the ζ_i operators on $\text{PSYT}_{\geq 0}(\lambda)$ we may compute $f(\text{top}(T))$ explicitly.

Proposition 3.6. For $T \in \text{RYT}_{\geq 0}(\lambda)$ we have that

$$f(\text{top}(T)) = \mathcal{C}_1^{\nu(T)_1 - \nu(T)_2} \dots \mathcal{C}_n^{\nu(T)_n} (e_{S(T)})$$

where define for $1 \leq i \leq n$,

$$\mathcal{C}_i := \left((tT_i^{-1}) \dots (tT_{n-1}^{-1}) (t^{-(n-1)} T_{n-1} \dots T_1) \right)^i.$$

Proof. Using the recurrence relations in Corollary 3.5 for the elements $f(\tau)$ and Proposition 2.14 we see that for any $T \in \text{RSSYT}_{\geq 0}(\lambda)$ since

$$\text{top}(T) = \zeta_1^{\nu(T)_1 - \nu(T)_2} \dots \zeta_n^{\nu(T)_n} (S(T))$$

with each $\zeta_i := (s_i \dots s_{n-1} \Psi)^i$ then we have a similar expression for $f(\text{top}(T))$:

$$f(\text{top}(T)) = \mathcal{C}_1^{\nu(T)_1 - \nu(T)_2} \dots \mathcal{C}_n^{\nu(T)_n} (e_{S(T)})$$

where $\mathcal{C}_i := \left((tT_i^{-1}) \dots (tT_{n-1}^{-1}) (t^{-(n-1)} T_{n-1} \dots T_1) \right)^i$ is obtained by replacing each s_j and Ψ in ζ_i with tT_j^{-1} and $t^{-(n-1)} T_{n-1} \dots T_1$ respectively. Importantly, when we apply ζ_i to any element of the form $\text{top}(T')$ we never perform any swaps $s_j(\tau) > \tau$ such that $w_\tau(j) = w_\tau(j+1)$ and hence never require the more complicated recurrence relation:

$$f(s_j(\tau)) = \left(tT_j^{-1} + \frac{(t-1)t^{c_\tau(j+1)}}{t^{c_\tau(j)} - t^{c_\tau(j+1)}} \right) (f(\tau)).$$

□

The \mathcal{C}_i operators can be identified concretely using the $\overline{\theta}_j$ elements of the finite Hecke algebra.

Lemma 3.7. For all $1 \leq i \leq n$,

$$\mathcal{C}_i = A_i \cdots A_1$$

where $A_j := t^{-(j-1)}\overline{\theta}_j^{-1}$.

Proof. Let $0 \leq k \leq i-1$. We first show by induction that

$$(t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^{i-k-1}\overline{\theta}_{i-k}^{-1}) = (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-1}).$$

To start we see that for $k=0$ we have

$$t^{i-1}\overline{\theta}_i^{-1} = T_{i-1} \cdots T_1^2 \cdots T_{i-1} = (T_{i-1} \cdots T_1)^1 (T_1 \cdots T_{i-1}).$$

Now suppose that for $0 \leq k \leq i-2$ the formula above holds. Then

$$\begin{aligned} & (t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^{i-(k+1)}\overline{\theta}_{i-(k+1)}^{-1}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-1}) (t^{i-(k+1)}\overline{\theta}_{i-(k+1)}^{-1}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-1}) (T_{i-k-2} \cdots T_1^2 \cdots T_{i-k-2}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_2 \cdots T_{i-k}) (T_1 \cdots T_{i-k-1}) (T_{i-k-2} \cdots T_1) (T_1 \cdots T_{i-k-2}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_2 \cdots T_{i-k}) (T_{i-k-1} \cdots T_2) (T_1 \cdots T_{i-k-1}) (T_1 \cdots T_{i-k-2}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_2 \cdots T_{i-k}) (T_{i-k-1} \cdots T_1) (T_2 \cdots T_{i-k-1}) (T_1 \cdots T_{i-k-2}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_3 \cdots T_{i-k+1}) (T_2 \cdots T_{i-k}) (T_{i-k-1} \cdots T_1) (T_2 \cdots T_{i-k-1}) (T_1 \cdots T_{i-k-2}) \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{k+1} \cdots T_{i-1}) (T_k \cdots T_{i-2}) \cdots (T_3 \cdots T_{i-k+1}) (T_{i-k} \cdots T_1) (T_3 \cdots T_{i-k}) (T_2 \cdots T_{i-k-1}) (T_1 \cdots T_{i-k-2}) \\ &= \cdots \\ &= (T_{i-1} \cdots T_1)^{k+1} (T_{i-1} \cdots T_1) (T_{k+2} \cdots T_{i-1}) (T_{k+1} \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-2}) \\ &= (T_{i-1} \cdots T_1)^{k+2} (T_{k+2} \cdots T_{i-1}) (T_{k+1} \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-2}). \end{aligned}$$

By taking $k = i-1$ we find

$$(t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^0\overline{\theta}_1^{-1}) = (T_{i-1} \cdots T_1)^i.$$

Now we see that

$$\begin{aligned} \mathfrak{C}_i &= \left((tT_i^{-1}) \cdots (tT_{n-1}^{-1}) (t^{-(n-1)}T_{n-1} \cdots T_1) \right)^i \\ &= t^{-i(i-1)} (T_{i-1} \cdots T_1)^i \\ &= t^{-i(i-1)} (t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^0\overline{\theta}_1^{-1}) \\ &= t^{-2(i-1)-2(i-2)-\cdots-2(1)-2(0)} (t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^0\overline{\theta}_1^{-1}) \\ &= (t^{-(i-1)}\overline{\theta}_i^{-1}) \cdots (t^0\overline{\theta}_1^{-1}) \\ &= A_i \cdots A_1 \end{aligned}$$

where $A_j := t^{-(j-1)}\overline{\theta}_j^{-1}$. □

Putting the results of this section together gives the following:

Corollary 3.8. For $T \in \text{RYT}_{\geq 0}(\lambda)$, the triangular expansion of $F_{\text{top}(T)}$ has the form

$$F_{\text{top}(T)} = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)} + \sum_{\beta \prec \nu(T)} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_\lambda$.

Proof. First, notice that for $T \in \text{RYT}_{\geq 0}(\lambda)$ $w_{\text{top}(T)} = \nu(T)$. From Proposition 3.6 and Lemma 2.12

$$\begin{aligned}
f(\text{top}(T)) &= \mathfrak{C}_1^{\nu(T)_1 - \nu(T)_2} \dots \mathfrak{C}_n^{\nu(T)_n} (e_{S(T)}) \\
&= A_1^{\nu(T)_1 - \nu(T)_2} (A_1 A_2)^{\nu(T)_2 - \nu(T)_3} \dots (A_1 \dots A_n)^{\nu(T)_n} (e_{S(T)}) \\
&= A_1^{(\nu(T)_1 - \nu(T)_2) + \dots + (\nu(T)_{n-1} - \nu(T)_n) + \nu(T)_n} \dots A_{n-1}^{(\nu(T)_{n-1} - \nu(T)_n) + \nu(T)_n} A_n^{\nu(T)_n} (e_{S(T)}) \\
&= (\bar{\theta}_1^{-1})^{\nu(T)_1} \dots (\bar{\theta}_n^{-1})^{\nu(T)_n} (e_{S(T)}) \\
&= t^{-\nu(T)_1 (c_{S(T)}(1) - (1-1))} \dots t^{-\nu(T)_n (c_{S(T)}(n) - (n-1))} e_{S(T)} \\
&= t^{-\sum_{i=1}^n \nu(T)_i (c_{S(T)}(i) + i - 1)} e_{S(T)} \\
&= t^{-b_T} e_{S(T)}.
\end{aligned}$$

Therefore, the leading term of $F_{\text{top}(T)}$ is

$$X^{w_{\text{top}(T)}} \otimes f(\text{top}(T)) = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)}.$$

□

3.2 Connecting Maps Between $V_{\lambda^{(n)}}$

Definition 3.9. Let $\lambda \in \mathbb{Y}$. For $n \geq n_\lambda$ define $\Phi_\lambda^{(n)} : V_{\lambda^{(n+1)}} \rightarrow V_{\lambda^{(n)}}$ as the $\mathbb{Q}(q, t)$ -linear map given on any element $X^\alpha \otimes v \in V_{\lambda^{(n+1)}}$ by

$$\Phi_\lambda^{(n)}(X^\alpha \otimes v) = \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \dots X_n^{\alpha_n} \otimes \mathfrak{q}_\lambda^{(n)}(v).$$

Proposition 3.10. The map $\Phi_\lambda^{(n)}$ satisfies the following relations:

- $\Phi_\lambda^{(n)} T_i = T_i \Phi_\lambda^{(n)}$ for $1 \leq i \leq n-1$
- $\Phi_\lambda^{(n)} X_i = X_i \Phi_\lambda^{(n)}$ for $1 \leq i \leq n$
- $\Phi_\lambda^{(n)} X_{n+1} = 0$
- $\Phi_\lambda^{(n)} t^{-n} \pi_{n+1} T_n = t^{-(n-1)} \pi_n \Phi_\lambda^{(n)}$
- $\Phi_\lambda^{(n)} \theta_i^{(n+1)} = \theta_i^{(n)} \Phi_\lambda^{(n)}$ for $1 \leq i \leq n$
- $\Phi_\lambda^{(n)} (\theta_{n+1}^{(n+1)} - t^{n-|\lambda|}) = 0$.

Proof. From Lemma 2.20 and Definition 3.9 it follows immediately for all $1 \leq i \leq n-1$ and $1 \leq j \leq n$ that $\Phi_\lambda^{(n)} T_i = T_i \Phi_\lambda^{(n)}$, $\Phi_\lambda^{(n)} X_j = X_j \Phi_\lambda^{(n)}$, and $\Phi_\lambda^{(n)} X_{n+1} = 0$.

Let $X^\alpha \otimes v \in V_{\lambda^{(n+1)}}$. By direct calculation we find

$$\begin{aligned}
&\Phi_\lambda^{(n)} t^{-n} \pi_{n+1} T_n (X_1^{\alpha_1} \dots X_{n+1}^{\alpha_{n+1}} \otimes v) \\
&= \Phi_\lambda^{(n)} t^{-n} \pi_{n+1} X_1^{\alpha_1} \dots X_{n-1}^{\alpha_{n-1}} T_n (X_n^{\alpha_n} X_{n+1}^{\alpha_{n+1}} \otimes v) \\
&= \Phi_\lambda^{(n)} t^{-n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \pi_{n+1} T_n (X_n^{\alpha_n} X_{n+1}^{\alpha_{n+1}} \otimes v) \\
&= t^{-n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \Phi_\lambda^{(n)} \pi_{n+1} T_n (X_n^{\alpha_n} X_{n+1}^{\alpha_{n+1}} \otimes v) \\
&= t^{-n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \Phi_\lambda^{(n)} \pi_{n+1} \left(X_{n+1}^{\alpha_n} X_n^{\alpha_{n+1}} T_n \otimes v + (1-t) X_n \frac{X_n^{\alpha_n} X_{n+1}^{\alpha_{n+1}} - X_{n+1}^{\alpha_n} X_n^{\alpha_{n+1}}}{X_n - X_{n+1}} \otimes v \right) \\
&= t^{-n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \Phi_\lambda^{(n)} \left(q^{\alpha_n} X_1^{\alpha_n} X_{n+1}^{\alpha_{n+1}} \pi_{n+1} T_n \otimes v + (1-t) X_{n+1} \pi_{n+1} \frac{X_n^{\alpha_n} X_{n+1}^{\alpha_{n+1}} - X_{n+1}^{\alpha_n} X_n^{\alpha_{n+1}}}{X_n - X_{n+1}} \otimes v \right) \\
&= \mathbb{1}(\alpha_{n+1} = 0) t^{-n} q^{\alpha_n} X_1^{\alpha_n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \Phi_\lambda^{(n)} (1 \otimes \rho_{n+1}(\pi_{n+1} T_n) v) \\
&= \mathbb{1}(\alpha_{n+1} = 0) t^{-n} q^{\alpha_n} X_1^{\alpha_n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \Phi_\lambda^{(n)} (1 \otimes t^n T_1^{-1} \dots T_{n-1}^{-1} v) \\
&= \mathbb{1}(\alpha_{n+1} = 0) q^{\alpha_n} X_1^{\alpha_n} X_2^{\alpha_1} \dots X_n^{\alpha_{n-1}} \otimes T_1^{-1} \dots T_{n-1}^{-1} \mathfrak{q}_\lambda^{(n)}(v).
\end{aligned}$$

On the other hand we see

$$\begin{aligned}
& t^{-(n-1)} \pi_n \Phi_\lambda^{(n)} (X_1^\alpha \cdots X_{n+1}^{\alpha_{n+1}} \otimes v) \\
&= \mathbb{1}(\alpha_{n+1} = 0) t^{-(n-1)} \pi_n (X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes \mathfrak{q}_\lambda^{(n)}(v)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) t^{-(n-1)} q^{\alpha_n} X_1^{\alpha_n} X_2^{\alpha_1} \cdots X_n^{\alpha_{n-1}} \otimes \rho_n(\pi_n)(\mathfrak{q}_\lambda^{(n)}(v)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) q^{\alpha_n} X_1^{\alpha_n} X_2^{\alpha_1} \cdots X_n^{\alpha_{n-1}} \otimes T_1^{-1} \cdots T_{n-1}^{-1} \mathfrak{q}_\lambda^{(n)}(v).
\end{aligned}$$

Therefore, $\Phi_\lambda^{(n)} t^{-n} \pi_{n+1} T_n = t^{-(n-1)} \pi_n \Phi_\lambda^{(n)}$ as desired.

Now let $1 \leq i \leq n$. We see that

$$\begin{aligned}
\Phi_\lambda^{(n)} \theta_i^{(n+1)} &= \Phi_\lambda^{(n)} t^{-(n-i+1)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_{n+1} T_n \cdots T_i \\
&= t^{i-1} T_{i-1}^{-1} \cdots T_1^{-1} (\Phi_\lambda^{(n)} t^{-n} \pi_{n+1} T_n) T_{n-1} \cdots T_i \\
&= t^{i-1} T_{i-1}^{-1} \cdots T_1^{-1} (t^{-(n-1)} \pi_n \Phi_\lambda^{(n)}) T_{n-1} \cdots T_i \\
&= t^{-(n-i)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1} \cdots T_i \Phi_\lambda^{(n)} \\
&= \theta_i^{(n)} \Phi_\lambda^{(n)}.
\end{aligned}$$

Now let $\alpha \in \mathbb{Z}_{\geq 0}^{n+1}$ and $\tau \in \text{SYT}(\lambda^{(n+1)})$. We find

$$\begin{aligned}
& \Phi_\lambda^{(n)} \theta_{n+1}^{(n+1)} (X^\alpha \otimes e_\tau) \\
&= \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} \pi_{n+1} (X^\alpha \otimes e_\tau) \\
&= \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} q^{\alpha_{n+1}} X_1^{\alpha_{n+1}} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} \otimes \rho_{n+1}(\pi_{n+1}) e_\tau \\
&= q^{\alpha_{n+1}} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_1^{\alpha_{n+1}} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1} (e_\tau)).
\end{aligned}$$

Now if $\alpha_{n+1} > 0$ then this evaluates to 0 since

$$\Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_1 = t^{-n} \Phi_\lambda^{(n)} X_{n+1} T_n \cdots T_1 = 0.$$

Hence,

$$\Phi_\lambda^{(n)} \theta_{n+1}^{(n+1)} (X^\alpha \otimes e_\tau) = \mathbb{1}(\alpha_{n+1} = 0) \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1} (e_\tau)).$$

Now we by repeatedly applying Lemma 3.4 we see that as maps $V_{\lambda^{(n+1)}} \rightarrow V_{\lambda^{(n)}}$

$$\begin{aligned}
& \Phi_\lambda^{(n)} T_n^{-1} \cdots T_2^{-1} (T_1^{-1} X_2^{\alpha_1}) X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= \Phi_\lambda^{(n)} T_n^{-1} \cdots T_2^{-1} \left(X_1^{\alpha_1} T_1^{-1} + (1-t^{-1}) X_2 \frac{X_1^{\alpha_1} - X_2^{\alpha_1}}{X_1 - X_2} \right) X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} + (1-t^{-1}) \Phi_\lambda^{(n)} T_n^{-1} \cdots T_2^{-1} X_2 \frac{X_1^{\alpha_1} - X_2^{\alpha_1}}{X_1 - X_2} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} + (1-t^{-1}) t^{-(n-2)} \Phi_\lambda^{(n)} X_{n+1} T_{n-1} \cdots T_2 \frac{X_1^{\alpha_1} - X_2^{\alpha_1}}{X_1 - X_2} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} + 0 \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_3^{-1} (T_2^{-1} X_3^{\alpha_2}) T_1^{-1} X_4^{\alpha_3} \cdots X_{n+1}^{\alpha_n} \\
&= \dots \\
&= X_1^{\alpha_1} X_2^{\alpha_2} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_4^{\alpha_3} \cdots X_{n+1}^{\alpha_n} \\
&= \dots \\
&= X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1}.
\end{aligned}$$

As usual let \square_0 denote the unique square of the skew diagram $\lambda^{(n+1)}/\lambda^{(n)}$. Returning to our main calculation now shows

$$\begin{aligned}
& \Phi_\lambda^{(n)} \theta_{n+1}^{(n+1)} (X^\alpha \otimes e_\tau) \\
&= \mathbb{1}(\alpha_{n+1} = 0) \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} (1 \otimes t^n T_n^{-1} \cdots T_1^{-1} T_1^{-1} \cdots T_n^{-1} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} (1 \otimes \bar{\theta}_{n+1}^{(n+1)} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} (1 \otimes t^{c_\tau(n+1)} e_\tau) \\
&= t^{c_\tau(n+1)} \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes \mathfrak{q}_\lambda^{(n)} (e_\tau) \\
&= t^{c(\square_0)} \mathbb{1}(\alpha_{n+1} = 0) \mathbb{1}(\tau(\square_0) = n+1) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes e_{\tau|\lambda^{(n)}} \\
&= t^{n-|\lambda|} \mathbb{1}(\alpha_{n+1} = 0) \mathbb{1}(\tau(\square_0) = n+1) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes e_{\tau|\lambda^{(n)}} \\
&= \Phi_\lambda^{(n)} (t^{n-|\lambda|} X^\alpha \otimes e_\tau).
\end{aligned}$$

Therefore,

$$\Phi_\lambda^{(n)} (\theta_{n+1}^{(n+1)} - t^{n-|\lambda|}) = 0.$$

□

Corollary 3.11. Let $n \geq n_\lambda$ and $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$. For $\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)})$ we have

$$\Phi_\lambda^{(n)} (F_\tau) := \begin{cases} F_{\tau|\lambda^{(n)}} & \tau(\square_0) = n+1 \\ 0 & \tau(\square_0) \neq n+1. \end{cases}$$

Proof. We will first deal with the case when $\tau(\square_0) = n+1$. Let $T \in \text{RYT}_{\geq 0}(\lambda^{(n)})$ and let $T' \in \text{RYT}_{\geq 0}(\lambda^{(n+1)})$ with $T'(\square_0) = 0$ and $T'|_{\lambda^{(n)}} = T|_{\lambda^{(n)}}$. By looking at the eigenvalues of $\theta_1^{(n+1)}, \dots, \theta_n^{(n+1)}$ on $F_{\text{top}(T')}$ and the eigenvalues of $\theta_1^{(n)}, \dots, \theta_n^{(n)}$ on $F_{\text{top}(T)}$ we see that $\Phi_\lambda^{(n)} (F_{\text{top}(T')}) = \beta F_{\text{top}(T)}$ for some scalar β . We will now show that $\beta = 1$. From Corollary 3.8 we know that

$$F_{\text{top}(T')} = t^{-b_{T'}} X^{\nu(T')} \otimes e_{S(T')} + \sum_{\beta \prec \nu(T')} X^\beta \otimes v'_\beta$$

and

$$F_{\text{top}(T)} = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)} + \sum_{\beta \prec \nu(T)} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_{\lambda^{(n)}}$ and $v'_\beta \in S_{\lambda^{(n+1)}}$. Since $T'(\square_0) = 0$ and $T'|_{\lambda^{(n)}} = T|_{\lambda^{(n)}}$, it follows that $b_{T'} = b_T$, $\nu(T') = \nu(T) * 0$, and $\mathfrak{q}_\lambda^{(n)} (e_{S(T')}) = e_{S(T)}$. Therefore,

$$\Phi_\lambda^{(n)} (t^{-b_{T'}} X^{\nu(T')} \otimes e_{S(T')}) = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)}.$$

Now if $\beta \prec \nu(T')$ then $\Phi_\lambda^{(n)} (X^\beta \otimes v'_\beta) = \mathbb{1}(\beta_{n+1} = 0) X_1^{\beta_1} \cdots X_n^{\beta_n} \otimes \mathfrak{q}_\lambda^{(n)} (v'_\beta)$ cannot be of the form $X^{\nu(T)} \otimes w$ for any $w \in S_{\lambda^{(n)}}$. As such the coefficient of $X^{\nu(T)} \otimes e_{S(T)}$ in the standard basis expansion of $\Phi_\lambda^{(n)} (F_{\text{top}(T')})$ is t^{-b_T} . Since this agrees with the same coefficient in the expansion of $F_{\text{top}(T)}$ we know that $\beta = 1$ and thus $\Phi_\lambda^{(n)} (F_{\text{top}(T')}) = F_{\text{top}(T)}$.

Now consider any $\tau' \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)})$ with $\tau'(\square_0) = n+1$. Let $T' := \mathfrak{p}_{\lambda^{(n+1)}}(\tau) \in \text{RYT}_{\geq 0}(\lambda^{(n+1)})$. Then $T'(\square_0) = 0$ so if we set $T := T'|_{\lambda^{(n)}}$ we have that $\Phi_\lambda^{(n)} (F_{\text{top}(T')}) = F_{\text{top}(T)}$. Write $\tau := \tau'|_{\lambda^{(n)}}$. As seen before there exists a sequence $\tau < s_{i_1}(\tau) < \dots < s_{i_r}(\tau) = \text{top}(T)$. Since $\tau'(\square_0) = n+1$, we see that $\tau' < s_{i_1}(\tau') < \dots < s_{i_r}(\tau') = \text{top}(T')$ as well. For each $1 \leq j \leq r$ we will consider using the intertwiner operators from Proposition 3.2 to obtain $F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau)}$ from $F_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}$. We have that

$$\left(t T_{i_j}^{-1} + \frac{(t-1) q^{w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_{j+1})} t^{c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_{j+1})}}{q^{w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j)} t^{c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j)} - q^{w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_{j+1})} t^{c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_{j+1})}} \right) (F_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}) = F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau)}.$$

Now the same exact formula holds with τ replaced by τ' . Importantly, we have that $w_{s_{i_j-1} \dots s_{i_1}(\tau)}(i_j+1) = w_{s_{i_j-1} \dots s_{i_1}(\tau')}(i_j+1)$ and $c_{s_{i_j-1} \dots s_{i_1}(\tau)}(i_j+1) = c_{s_{i_j-1} \dots s_{i_1}(\tau')}(i_j+1)$. Therefore, we may write

$$D_j(F_{s_{i_j-1} \dots s_{i_1}(\tau)}) = F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau)}$$

and

$$D_j(F_{s_{i_j-1} \dots s_{i_1}(\tau')}) = F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau')}$$

for $D_j \in \mathcal{A}_n \subset \mathcal{A}_{(n,1)}$ of the form $D_j = T_{i_j} + \alpha_j$ where $\alpha_j \in \mathbb{Q}(q, t)$. Here we have used $tT_{i_j}^{-1} = T_{i_j} + t - 1$. By using the quadratic relation for T_{i_j} we may locally invert the operator D_j in the sense that there exists operators $C_j \in \mathcal{A}_n$ with

$$F_{s_{i_j-1} \dots s_{i_1}(\tau)} = C_j(F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau)})$$

and

$$F_{s_{i_j-1} \dots s_{i_1}(\tau')} = C_j(F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau')}).$$

Therefore, if we assume that $\Phi_\lambda^{(n)}(F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau')}) = F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau)}$ then

$$\begin{aligned} & \Phi_\lambda^{(n)}(F_{s_{i_j-1} \dots s_{i_1}(\tau')}) \\ &= \Phi_\lambda^{(n)}(C_j(F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau')})) \\ &= C_j \Phi_\lambda^{(n)}(F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau')}) \\ &= C_j F_{s_{i_j} s_{i_j-1} \dots s_{i_1}(\tau)} \\ &= F_{s_{i_j-1} \dots s_{i_1}(\tau)}. \end{aligned}$$

Thus by induction, since we know $\Phi_\lambda^{(n)}(F_{\text{top}(T')}) = F_{\text{top}(T)}$, it follows that $\Phi_\lambda^{(n)}(F_{\tau'}) = F_\tau$.

Lastly, we consider the case of $\tau(\square_0) \neq n+1$. Then $\tau(\square_0) = iq^a$ with either $i \neq n+1$ or $a \geq 0$. If $a > 0$ then $\tau = \Psi(\tau')$ for some τ' and thus from Proposition 3.2 we know X_{n+1} divides F_τ . Since $\Phi_\lambda^{(n)} X_{n+1} = 0$ it follows that $\Phi_\lambda^{(n)}(F_\tau) = 0$. Now suppose $a = 0$ and $i \neq n+1$. Notice for any $m \geq n_\lambda$ that the largest power of t occurring in the $\theta^{(m)}$ -weight of any $F_{\tau'}$ with $\tau' \in \text{PSYT}_{\geq 0}(\lambda^{(m)})$ is exactly $t^{m-|\lambda|-1}$. Since $i \neq n+1$ we know that if $\Phi_\lambda^{(n)}(F_\tau) = \beta F_{\tau'}$ for some nonzero scalar β and $\tau' \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)})$ then the maximal power of t occurring in the $\theta^{(n)}$ -weight of $F_{\tau'}$ is $t^{n-|\lambda|}$ coming from

$$\begin{aligned} & \theta_i(F_{\tau'}) \\ &= \theta_i(\Phi_\lambda^{(n)}(F_\tau)) \\ &= \Phi_\lambda^{(n)}(\theta_i(F_\tau)) \\ &= \Phi_\lambda^{(n)}(t^{c(\square_0)} F_\tau) \\ &= \Phi_\lambda^{(n)}(t^{n-|\lambda|} F_\tau) \\ &= t^{n-|\lambda|} F_{\tau'}. \end{aligned}$$

Thus $\Phi_\lambda^{(n)}(F_\tau)$ cannot be a $\theta^{(n)}$ -weight vector in $V_{\lambda^{(n)}}$ and so $\beta = 0$. □

The maps $\Phi_\lambda^{(n)}$ possess another important stability property.

Proposition 3.12. For all $\ell \in \mathbb{Z} \setminus \{0\}$ and $n \geq n_\lambda$,

$$\Phi_\lambda^{(n)} \left(\sum_{j=1}^{n+1} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)} \right) = \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \Phi_\lambda^{(n)}.$$

Proof. Let $\ell \in \mathbb{Z} \setminus \{0\}$ and $n \geq n_\lambda$. As usual let \square_0 denote the unique square of the skew diagram $\lambda^{(n+1)}/\lambda^{(n)}$. Directly from Proposition 3.10 we see

$$\begin{aligned} & \Phi_\lambda^{(n)} \left(\sum_{j=1}^{n+1} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)} \right) \\ &= \Phi_\lambda^{(n)} \left(\sum_{j=1}^n (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) + \Phi_\lambda^{(n)} \left((\theta_{n+1}^{(n+1)})^\ell - t^{\ell c(\square_0)} \right) \\ &= \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \Phi_\lambda^{(n)} + \Phi_\lambda^{(n)} \left((\theta_{n+1}^{(n+1)})^\ell - t^{\ell(n-|\lambda|)} \right). \end{aligned}$$

It follows from the relation $\Phi_\lambda^{(n)} \left(\theta_{n+1}^{(n+1)} - t^{n-|\lambda|} \right) = 0$ and the fact that $\theta_{n+1}^{(n+1)}$ is invertible on $V_{\lambda^{(n+1)}}$ that

$$\Phi_\lambda^{(n)} \left((\theta_{n+1}^{(n+1)})^\ell - t^{\ell(n-|\lambda|)} \right) = 0.$$

Therefore,

$$\Phi_\lambda^{(n)} \left(\sum_{j=1}^{n+1} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)} \right) = \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \Phi_\lambda^{(n)}.$$

□

4 Positive EHA Representations from Young Diagrams

4.1 The $\mathcal{D}_n^{\text{sph}}$ -modules $W_{\lambda^{(n)}}$

We now turn to the corresponding spherical DAHA modules and symmetric v.v. polynomials to the positive DAHA modules V_λ and the non-symmetric v.v. polynomials F_τ considered in the prior sections.

Definition 4.1. For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ define the $\mathcal{D}_n^{\text{sph}}$ -module $W_\lambda := \epsilon^{(n)}(V_\lambda)$.

The F_τ expansions of any symmetrized element of any \mathcal{A}_n submodule U_T satisfy a simple set of recurrence relations.

Lemma 4.2. Let $T \in \text{RSSYT}_{\geq 0}(\lambda)$ and $v \in \epsilon^{(n)}(U_T)$. Suppose that v has the following expansion into the F_τ basis:

$$v = \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \kappa_\tau F_\tau.$$

Then for each $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $1 \leq i \leq n-1$ such that $s_i(\tau) > \tau$ we have the relation

$$\kappa_{s_i(\tau)} = \left(\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \right) \kappa_\tau.$$

As a consequence, if $\kappa_{\text{top}(T)} \neq 0$ then each coefficient κ_τ is also nonzero.

Proof. Let $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ and $1 \leq i \leq n-1$ with $s_i(\tau) > \tau$. Note that $\mathbb{Q}(q, t)\{F_\tau, F_{s_i(\tau)}\}$ is a 2-dimensional submodule for $\mathbb{Q}(q, t)[T_i]$. The T_i -invariant subspace of $\mathbb{Q}(q, t)\{F_\tau, F_{s_i(\tau)}\}$ is given by $\mathbb{Q}(q, t)(1 + tT_i^{-1})F_\tau$. From Proposition 3.2 we find

$$\begin{aligned} (1 + tT_i^{-1})F_\tau &= F_\tau + tT_i^{-1}F_\tau \\ &= F_\tau + F_{s_i(\tau)} + \frac{(1-t)q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} F_\tau \\ &= F_{s_i(\tau)} + \frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} F_\tau. \end{aligned}$$

Since $v = \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \kappa_{\tau} F_{\tau}$ is T_i -invariant then we know that in particular $\kappa_{\tau} F_{\tau} + \kappa_{s_i(\tau)} F_{s_i(\tau)}$ is also T_i -invariant and therefore must be a scalar multiple of $(1 + tT_i^{-1})F_{\tau}$. Therefore,

$$\kappa_{\tau} F_{\tau} + \kappa_{s_i(\tau)} F_{s_i(\tau)} = \kappa_{s_i(\tau)} F_{s_i(\tau)} + \frac{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}}{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}} \kappa_{s_i(\tau)} F_{\tau}$$

and so

$$\kappa_{s_i(\tau)} = \left(\frac{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}}{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}} \right) \kappa_{\tau}.$$

□

Using the recurrence relations in Lemma 4.2 and the irreducibility of each of the \mathcal{A}_n submodules of V_{λ} we may determine which $T \in \text{RYT}_{\geq 0}(\lambda)$ have a non-zero space of \mathcal{H}_n -invariants $\epsilon^{(n)}(U_T)$.

Proposition 4.3. For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \text{RYT}_{\geq 0}(\lambda)$,

$$\dim_{\mathbb{Q}(q,t)} \epsilon^{(n)}(U_T) = \begin{cases} 1 & T \in \text{RSSYT}_{\geq 0}(\lambda) \\ 0 & T \notin \text{RSSYT}_{\geq 0}(\lambda). \end{cases}$$

Proof. By Proposition 3.3 each \mathcal{A}_n -module U_T is irreducible with simple $\theta^{(n)}$ spectrum. This implies that $\dim_{\mathbb{Q}(q,t)} \epsilon^{(n)}(U_T) \leq 1$ for any $T \in \text{RYT}_{\geq 0}(\lambda)$. Further, we have that $\epsilon^{(n)}(U_T)$ is zero if for any $\theta^{(n)}$ -weight vector v in U_T , $\epsilon^{(n)}(v)$ is zero. If $T \in \text{RYT}_{\geq 0}(\lambda) \setminus \text{RSSYT}_{\geq 0}(\lambda)$ then there exists a pair of boxes $\square_1, \square_2 \in \lambda$ with \square_1 directly above \square_2 such that $T(\square_1) = T(\square_2) = a$. Hence, $\text{top}(T)(\square_1) = iq^a$ and $\text{top}(T)(\square_2) = (i+1)q^a$ for some $1 \leq i \leq n-1$. Then $T_i(F_{\text{top}(T)}) = -tF_{\text{top}(T)}$ which implies that $\epsilon^{(n)}(F_{\text{top}(T)}) = 0$. Thus $\epsilon^{(n)}(U_T) = 0$.

Alternatively, now suppose $T \in \text{RSSYT}_{\geq 0}(\lambda)$. Following Lemma 4.2 we construct a vector $v \in U_T$ of the form

$$v = \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \kappa_{\tau} F_{\tau}$$

where $\kappa_{\text{top}(T)} = 1$ and if $s_i(\tau) > \tau$ then

$$\kappa_{s_i(\tau)} = \left(\frac{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}}{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}} \right) \kappa_{\tau}.$$

These coefficients κ_{τ} have the property that if $s_i(\tau) > \tau$ then

$$T_i(\kappa_{\tau} F_{\tau} + \kappa_{s_i(\tau)} F_{s_i(\tau)}) = \kappa_{\tau} F_{\tau} + \kappa_{s_i(\tau)} F_{s_i(\tau)}.$$

By construction $v \neq 0$ since $\frac{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}}{q^{w_{\tau}(i)} t^{c_{\tau}(i)} - q^{w_{\tau}(i+1)} t^{c_{\tau}(i+1)+1}} \neq 0$ whenever $s_i(\tau) > \tau$. We will show that $T_i(v) = v$ for all $1 \leq i \leq n-1$ and thus $\epsilon^{(n)}(U_T) \neq 0$.

We find that

$$\begin{aligned} T_i(v) &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \kappa_{\tau} T_i(F_{\tau}) \\ &= \sum_{\substack{(\tau, s_i(\tau)) \in \text{PSYT}_{\geq 0}(\lambda)^2 \\ s_i(\tau) > \tau}} T_i(\kappa_{\tau} F_{\tau} + \kappa_{s_i(\tau)} F_{s_i(\tau)}) + \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda) \\ i, i+1 \text{ same row of } \tau}} \kappa_{\tau} T_i(F_{\tau}) + \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda) \\ i, i+1 \text{ same column of } \tau}} \kappa_{\tau} T_i(F_{\tau}) \\ &= \sum_{\substack{(\tau, s_i(\tau)) \in \text{PSYT}_{\geq 0}(\lambda)^2 \\ s_i(\tau) > \tau}} (\kappa_{\tau} F_{\tau} + \kappa_{s_i(\tau)} F_{s_i(\tau)}) + \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda) \\ i, i+1 \text{ same row of } \tau}} \kappa_{\tau} F_{\tau} + \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda) \\ i, i+1 \text{ same column of } \tau}} (-t) \kappa_{\tau} F_{\tau}. \end{aligned}$$

Thus

$$T_i(v) - v = \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda) \\ i, i+1 \text{ same column of } \tau}} (1+t) \kappa_{\tau} F_{\tau}.$$

Lastly, since $T \in \text{RSSYT}_{\geq 0}(\lambda)$ there cannot be any $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $i, i+1$ occurring in the same column as necessarily this would imply that T would have redundant values in those boxes contradicting the fact that T is reverse semi-standard. Hence, the above sum vanishes and we find $T_i(v) = v$. \square

Finally, we are able to define the symmetric v.v. Macdonald polynomials following the conventions of this paper.

Definition 4.4. Let $T \in \text{RSSYT}_{\geq 0}(\lambda)$. Define $P_T \in \epsilon^{(n)}(U_T)$ to be the unique element of the form

$$P_T = F_{\text{top}(T)} + \sum_y \kappa_y F_y$$

where the sum above ranges over $y \in \text{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(y) = T$ and $y < \text{top}(T)$.

Now we are able to explicitly compute the F_τ expansion of each P_T using the recurrence relations found in Lemma 4.2.

Corollary 4.5. For all $T \in \text{RSSYT}_{\geq 0}(\lambda)$,

$$P_T = \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) F_\tau.$$

Proof. For $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ let

$$\kappa_\tau = \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right).$$

From Lemma 4.2 it suffices to show that

- $\kappa_{\text{top}(T)} = 1$
- If $s_i(\tau) > \tau$ then $\kappa_{s_i(\tau)} = \left(\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \right) \kappa_\tau$.

It is easy to see that $\text{Inv}(\text{top}(T)) = \emptyset$ so $\kappa_{\text{top}(T)} = 1$. Now suppose $s_i(\tau) > \tau$. Let $\square^{(i)}, \square^{(i+1)} \in \lambda$ denote the boxes of λ with $\tau(\square^{(i)}) = iq^a$ and $\tau(\square^{(i+1)}) = (i+1)q^b$ for some $a, b \geq 0$. It is straightforward to check that

$$\text{Inv}(s_i(\tau)) = \{(\square^{(i)}, \square^{(i+1)})\} \sqcup \text{Inv}(\tau).$$

Therefore,

$$\begin{aligned} & \kappa_{s_i(\tau)} \\ &= \prod_{(\square_1, \square_2) \in \text{Inv}(s_i(\tau))} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \\ &= \left(\frac{q^{T(\square^{(i)})} t^{c(\square^{(i)})+1} - q^{T(\square^{(i+1)})} t^{c(\square^{(i+1)})}}{q^{T(\square^{(i)})} t^{c(\square^{(i)})} - q^{T(\square^{(i+1)})} t^{c(\square^{(i+1)})}} \right) \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \\ &= \left(\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \right) \kappa_\tau. \end{aligned}$$

\square

We look now at the action of the special spherical DAHA elements $P_{0,\ell}^{(n)}$.

Proposition 4.6. Let $|\lambda| = n$. The set $\{P_T : T \in \text{RSSYT}_{\geq 0}(\lambda)\}$ is a $\mathbb{Q}(q, t)[\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}]^{\otimes n}$ -weight basis for W_λ . Further, for $\ell \in \mathbb{Z} \setminus \{0\}$

$$P_{0,\ell}^{(n)}(P_T) = \left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) P_T.$$

Consequently, $P_{0,1}^{(n)}$ acts on W_λ with simple spectrum

$$\left\{ \sum_{\square \in \lambda} q^{T(\square)} t^{c(\square)} : T \in \text{RSSYT}_{\geq 0}(\lambda) \right\}.$$

Proof. It follows directly from Proposition 3.3 and Proposition 4.3 that the set $\{P_T : T \in \text{RSSYT}_{\geq 0}(\lambda)\}$ is a linear basis for W_λ . We need to show that the P_T are $\mathbb{Q}(q, t)[\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}]^{\otimes n}$ -weight vectors. Let $T \in \text{RSSYT}_{\geq 0}(\lambda)$. Then from Definition 4.4 we know that

$$P_T = \beta \epsilon^{(n)}(F_{\text{top}(T)})$$

for some nonzero scalar β depending on T . Then for any $\ell \in \mathbb{Z} \setminus \{0\}$ we have that

$$\begin{aligned} & \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell \right) (P_T) \\ &= \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell \right) (\beta \epsilon^{(n)}(F_{\text{top}(T)})) \\ &= \beta \epsilon^{(n)} \left(\left(\sum_{j=1}^n (\theta_j^{(n)})^\ell \right) (F_{\text{top}(T)}) \right) \\ &= \beta \epsilon^{(n)} \left(\left(\sum_{j=1}^n q^{\ell w_{\text{top}(T)}(j)} t^{\ell c_{\text{top}(T)}(j)} \right) F_{\text{top}(T)} \right) \\ &= \beta \epsilon^{(n)} \left(\left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) F_{\text{top}(T)} \right) \\ &= \left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) \beta \epsilon^{(n)}(F_{\text{top}(T)}) \\ &= \left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) P_T. \end{aligned}$$

Hence, P_T is a $\mathbb{Q}(q, t)[\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}]^{\otimes n}$ -weight vector.

Now let $S \in \text{RSSYT}_{\geq 0}(\lambda)$ and suppose that

$$\sum_{\square \in \lambda} q^{T(\square)} t^{c(\square)} = \sum_{\square \in \lambda} q^{S(\square)} t^{c(\square)}.$$

Fix any $d \in \mathbb{Z}$. Since q and t are algebraically independent over \mathbb{Q} ,

$$\sum_{\substack{\square \in \lambda \\ c(\square)=d}} q^{T(\square)} = \sum_{\substack{\square \in \lambda \\ c(\square)=d}} q^{S(\square)}.$$

Since the labelling T is reverse semi-standard, the values of $T(\square)$ for $\square \in \lambda$ with $c(\square) = d$ are all distinct and strictly decreasing down the d -diagonal. Of course, the same is true for S . Therefore, the values of T and S agree along the d -diagonal of λ . As $d \in \mathbb{Z}$ was general it follows that $T = S$. Thus the spectrum of the operator $P_{0,1}^{(n)}$ on W_λ is simple. \square

As mentioned previously, the non-symmetric v.v. Macdonald polynomials do not align with those of Dunkl-Luque. However, we are able to show that, once symmetrized, the symmetrized v.v. polynomials agree.

Corollary 4.7. The symmetric vector valued Macdonald polynomials of Dunkl-Luque [DL11] agree with the P_T of this paper up to nonzero scalars.

Proof. The \mathcal{D}_n -modules V_λ in this paper are isomorphic (after aligning conventions) to the \mathcal{D}_n -modules \mathcal{M}_λ in Dunkl-Luque's paper. Dunkl and Luque characterize the symmetric vector valued Macdonald polynomials as eigenvectors with distinct eigenvalues for the operator $\xi_1^{(n)} + \dots + \xi_n^{(n)}$ acting on $\epsilon^{(n)}(\mathcal{M}_\lambda)$. Here $\xi_i^{(n)}$ are the standard Cherednik elements given in our conventions by $\xi_i^{(n)} = t^{n-i+1}T_{i-1} \dots T_1 \pi_n T_{n-1}^{-1} \dots T_i^{-1}$. A simple calculation shows that the spherical DAHA elements $\epsilon^{(n)}(\xi_1^{(n)} + \dots + \xi_n^{(n)})\epsilon^{(n)}$ and $\epsilon^{(n)}(\theta_1^{(n)} + \dots + \theta_n^{(n)})\epsilon^{(n)}$ are both nonzero scalar multiples of $\epsilon^{(n)}\pi_n\epsilon^{(n)}$. Since the spectrum of $\epsilon^{(n)}\pi_n\epsilon^{(n)}$ acting on W_λ is simple, it follows that the P_T are eigenvectors for $\epsilon^{(n)}(\xi_1^{(n)} + \dots + \xi_n^{(n)})\epsilon^{(n)}$ and hence agree with the symmetric vector valued Macdonald polynomials of Dunkl-Luque up to re-normalization. \square

4.2 Stable Limit of the $W_{\lambda^{(n)}}$

Finally, we identify a special stability property for the P_T elements.

Corollary 4.8. For $T \in \text{RSSYT}_{\geq 0}(\lambda^{(n)})$ let $T' \in \text{RSSYT}_{\geq 0}(\lambda^{(n+1)})$ be such that $T(\square) = T'(\square)$ for $\square \in \lambda^{(n)}$ and $T'(\square_0) = 0$ for $\square_0 \in \lambda^{(n+1)}/\lambda^{(n)}$. Then

$$\Phi_\lambda^{(n)}(P_{T'}) = P_T.$$

Proof. Note that restriction from $\lambda^{(n+1)}$ to $\lambda^{(n)}$ identifies $\text{PSYT}_{\geq 0}(\lambda^{(n)}; T)$ as the subset of $\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)}; T')$ with $\tau(\square_0) = n+1$. Thus by using Corollary 3.11 in conjunction with Corollary 4.5 we find that

$$\begin{aligned} & \Phi_\lambda^{(n)}(P_{T'}) \\ &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)}; T')} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) \Phi_\lambda^{(n)}(F_\tau) \\ &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)}; T') \\ \tau(\square_0) = n+1}} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) F_{\tau|_{\lambda^{(n)}}} \\ &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)}; T') \\ \tau(\square_0) = n+1}} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau|_{\lambda^{(n)}})} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) F_{\tau|_{\lambda^{(n)}}} \\ &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n)}; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) F_\tau \\ &= P_T. \end{aligned}$$

\square

This stability allows for the following definition.

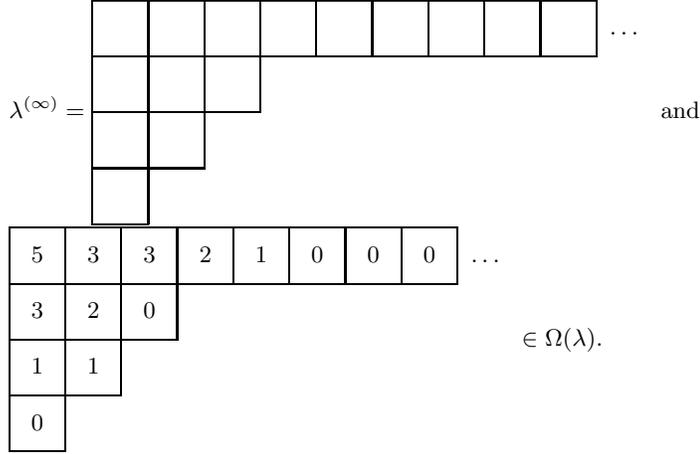
Definition 4.9. Let $\lambda \in \mathbb{Y}$. Define the infinite diagram $\lambda^{(\infty)} := \bigcup_{n \geq n_\lambda} \lambda^{(n)}$. Define $\Omega(\lambda)$ to be the set of all labellings $T : \lambda^{(\infty)} \rightarrow \mathbb{Z}_{\geq 0}$ such that

- $|\{\square \in \lambda^{(\infty)} : T(\square) \neq 0\}| < \infty$
- T decreases weakly across rows
- T decreases strictly down columns.

For $T \in \Omega(\lambda)$ we define the degree of T as $\deg(T) := \sum_{\square \in \lambda^{(\infty)}} T(\square)$. Define the rank of T , $\text{rk}(T)$, to be the minimal $n \geq n_\lambda$ such that $T|_{\lambda^{(\infty)} \setminus \lambda^{(n)}} = 0$.

Define the space $W_\lambda^{(\infty)}$ to be the inverse limit $\varprojlim W_{\lambda^{(n)}}$ with respect to the maps $\Phi_\lambda^{(n)}$. Let \widetilde{W}_λ be the subspace of all bounded X -degree elements of $W_\lambda^{(\infty)}$. For $T \in \Omega_\lambda$ define the *generalized Macdonald function* $\mathfrak{P}_T := \lim_n P_{T|_{\lambda^{(n)}}} \in \widetilde{W}_\lambda$.

Example 7. For $\lambda = (3, 2, 1)$



Remark 10. The degree of each \mathfrak{P}_T is given simply as

$$\deg(\mathfrak{P}_T) = \deg(T) = \sum_{\square \in \lambda^{(\infty)}} T(\square).$$

It is clear from definition that the set of all \mathfrak{P}_T for $T \in \Omega(\lambda)$ gives a $\mathbb{Q}(q, t)$ -basis of \widetilde{W}_λ .

Using the stability of the action of the $P_{0,\ell}^{(n)}$ operators we may define the following operators.

Definition 4.10. For $\ell \in \mathbb{Z} \setminus \{0\}$ define the operator $\widetilde{\Delta}_\ell^{(\infty)} : W_\lambda^{(\infty)} \rightarrow W_\lambda^{(\infty)}$ to be the stable-limit

$$\widetilde{\Delta}_\ell^{(\infty)} := \lim_n \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right).$$

A simple calculation shows the following:

Lemma 4.11. For all $\ell \in \mathbb{Z} \setminus \{0\}$ and $T \in \Omega(\lambda)$,

$$\widetilde{\Delta}_\ell^{(\infty)}(\mathfrak{P}_T) = \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) \mathfrak{P}_T.$$

Proof. Let $\ell \in \mathbb{Z} \setminus \{0\}$ and $T \in \Omega(\lambda)$. Then

$$\begin{aligned} & \widetilde{\Delta}_\ell^{(\infty)}(\mathfrak{P}_T) \\ &= \lim_n \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \left(\lim_n P_{T|_{\lambda^{(n)}}} \right) \\ &= \lim_n \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) (P_{T|_{\lambda^{(n)}}}) \\ &= \lim_n \left(\sum_{\square \in \lambda^{(n)}} q^{\ell T(\square)} t^{\ell c(\square)} - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} \\ &= \lim_n \left(\sum_{\square \in \lambda^{(n)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}}. \end{aligned}$$

Importantly, for $n \geq \text{rk}(T)$

$$\sum_{\square \in \lambda^{(n)}} (q^{\ell T(\square)} - 1)t^{\ell c(\square)} = \sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1)t^{\ell c(\square)}.$$

Therefore,

$$\begin{aligned} & \lim_n \left(\sum_{\square \in \lambda^{(n)}} (q^{\ell T(\square)} - 1)t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} \\ &= \lim_n \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1)t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} \\ &= \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1)t^{\ell c(\square)} \right) \lim_n P_{T|_{\lambda^{(n)}}} \\ &= \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1)t^{\ell c(\square)} \right) \mathfrak{P}_T. \end{aligned}$$

□

Corollary 4.12. For $\ell \in \mathbb{Z} \setminus \{0\}$ the operator $\tilde{\Delta}_\ell^{(\infty)}$ restricts to an operator on \widetilde{W}_λ .

Proof. Let $\ell \in \mathbb{Z} \setminus \{0\}$. We know that the set $\{\mathfrak{P}_T | T \in \Omega(\lambda)\}$ is a basis for \widetilde{W}_λ . From Lemma 4.11 we further know that $\tilde{\Delta}_\ell^{(\infty)}$ acts diagonally on this basis. Therefore, $\tilde{\Delta}_\ell^{(\infty)}$ restricts to an operator on \widetilde{W}_λ . □

Example 8. For $T \in \Omega(3, 2, 1)$ as in Example 7,

$$\tilde{\Delta}_1^{(\infty)}(\mathfrak{P}_T) = ((q^5 - 1) + (q^3 - 1)(t^{-1} + t^1 + t^2) + (q^2 - 1)(t^0 + t^3) + (q - 1)(t^{-2} + t^{-1} + t^4)) \mathfrak{P}_T.$$

4.3 Positive Elliptic Hall Algebra Action on \widetilde{W}_λ

Combining every result of this paper thus far we are able to define a novel family of positive EHA representations.

Theorem 4.13. For $\lambda \in \mathbb{Y}$, \widetilde{W}_λ is a graded \mathcal{E}^+ -module with action determined for $\ell \in \mathbb{Z} \setminus \{0\}$ and $r > 0$ by

- $P_{r,0} \rightarrow q^r p_r^\bullet$
- $P_{0,\ell} \rightarrow \tilde{\Delta}_\ell^{(\infty)}$.

Further, \widetilde{W}_λ is spanned by a basis of eigenvectors $\{\mathfrak{P}_T\}_{T \in \Omega(\lambda)}$ with distinct eigenvalues for the Macdonald operator $\Delta = \tilde{\Delta}_1^{(\infty)}$.

Proof. It suffices to establish that the map $\mathcal{E}^+ \rightarrow \text{End}_{\mathbb{Q}(q,t)}(\widetilde{W}_\lambda)$ satisfies the generating relations of \mathcal{E}^+ . Any such relation is a non-commutative polynomial expression in \mathcal{E}^+ of the form

$$F(P_{0,-r}, \dots, P_{0,-1}P_{0,1}, \dots, P_{0,r}, P_{1,0}, \dots, P_{s,0}) = 0$$

for some $r > 0$ and $s > 0$. By an argument of Schiffmann-Vasserot (Lemma 1.3 in [SV13]), there are automorphisms $\Gamma^{(n)}$ of $\mathcal{D}_n^{\text{sph}}$ such that for all $\ell \in \mathbb{Z} \setminus \{0\}$ and $s > 0$, $\Gamma^{(n)}(P_{0,\ell}^{(n)}) = P_{0,\ell}^{(n)} - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)}$ and $\Gamma^{(n)}(P_{s,0}^{(n)}) = P_{s,0}^{(n)}$. By applying the canonical quotient maps $\Pi_n : \widetilde{W}_\lambda \rightarrow W_{\lambda^{(n)}}$ we see using Cor. 3.12 that as maps

$$\begin{aligned}
& \Pi_n F(P_{0,-r}, \dots, P_{0,-1}, P_{0,1}, \dots, P_{0,r}, P_{1,0}, \dots, P_{s,0}) \\
&= F(\Gamma^{(n)}(P_{0,-r}^{(n)}), \dots, \Gamma^{(n)}(P_{0,-1}^{(n)}), \Gamma^{(n)}(P_{0,1}^{(n)}), \dots, \Gamma^{(n)}(P_{0,r}^{(n)}), \Gamma^{(n)}(P_{1,0}^{(n)}), \dots, \Gamma^{(n)}(P_{s,0}^{(n)})) \Pi_n \\
&= \Gamma^{(n)}(F(P_{0,-r}^{(n)}, \dots, P_{0,-1}^{(n)}, P_{0,1}^{(n)}, \dots, P_{0,r}^{(n)}, P_{1,0}^{(n)}, \dots, P_{s,0}^{(n)})) \Pi_n = 0.
\end{aligned}$$

As this holds for all $n \geq n_\lambda$, it follows that $F(P_{0,-r}, \dots, P_{0,-1}, P_{0,1}, \dots, P_{0,r}, P_{1,0}, \dots, P_{s,0}) = 0$ in $\text{End}_{\mathbb{Q}(q,t)}(\widetilde{W}_\lambda)$ as desired. The last statement regarding the spectrum of Δ follows directly from Prop. 4.6 and Cor. 4.8. \square

Remark 11. For $T \in \Omega(\lambda)$ and $\ell \in \mathbb{Z} \setminus \{0\}$,

$$P_{0,\ell}(\mathfrak{P}_T) = \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) \mathfrak{P}_T.$$

Remark 12. For $\lambda = \emptyset$, $\widetilde{W}_\emptyset = \Lambda_{q,t}$ recovers the standard representation of \mathcal{E}^+ . In this case, $\Omega(\emptyset) = \mathbb{Y}$ and $\mathfrak{P}_\mu = P_\mu[X; q^{-1}, t]$ (up to nonzero scalar).

We identify a special element of each \widetilde{W}_λ .

Definition 4.14. For any $\lambda \in \mathbb{Y}$ define the labelling T_λ^{\min} of $\lambda^{(\infty)}$ by

$$T_\lambda^{\min}(\square) = \#\{\square' \in \lambda^{(\infty)} \mid \square' \text{ strictly below } \square\}.$$

Lemma 4.15. The labelling T_λ^{\min} is the unique element of $\Omega(\lambda)$ of lowest degree $d_\lambda := \sum_{i \geq 1} i \lambda_i$.

Proof. It is immediate that since λ is a partition $T_\lambda^{\min} \in \Omega(\lambda)$. Further, by construction each entry of T_λ^{\min} is chosen minimally in that for any $T \in \Omega(\lambda)$ and $\square \in \lambda^{(\infty)}$, $T_\lambda^{\min}(\square) \leq T(\square)$. To see this simply note that if $T \in \Omega(\lambda)$ and $\square \in \lambda^{(\infty)}$ then if \square' is the box directly below \square then $T(\square) > T(\square')$. Hence, $T(\square)$ must be at least as large as the number of boxes strictly below \square . Therefore, T_λ^{\min} has the minimal degree among all elements of $\Omega(\lambda)$. Lastly, the number of boxes $\square \in \lambda^{(\infty)}$ with $T_\lambda^{\min}(\square) = i$ is λ_i so $\deg(T_\lambda^{\min}) = d_\lambda$ as defined above. \square

Proposition 4.16. For $\lambda, \mu \in \mathbb{Y}$ distinct, $\widetilde{W}_\lambda \not\cong \widetilde{W}_\mu$ as graded \mathcal{E}^+ modules.

Proof. Let $\lambda, \mu \in \mathbb{Y}$ and suppose that $f : \widetilde{W}_\lambda \rightarrow \widetilde{W}_\mu$ is a graded \mathcal{E}^+ module isomorphism. Then by Lemma 4.15 we know that

$$f(\mathfrak{P}_{T_\lambda^{\min}}) = \alpha \mathfrak{P}_{T_\mu^{\min}}$$

for some nonzero scalar $\alpha \in \mathbb{Q}(q, t)$. Further,

$$\begin{aligned}
P_{0,1}(f(\mathfrak{P}_{T_\lambda^{\min}})) &= f(P_{0,1}(\mathfrak{P}_{T_\lambda^{\min}})) \\
&= f\left(\sum_{\square \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\square)} - 1) t^{c(\square)} \mathfrak{P}_{T_\lambda^{\min}}\right) \\
&= \left(\sum_{\square \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\square)} - 1) t^{c(\square)}\right) f(\mathfrak{P}_{T_\lambda^{\min}}) \\
&= \left(\sum_{\square \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\square)} - 1) t^{c(\square)}\right) \alpha \mathfrak{P}_{T_\mu^{\min}}.
\end{aligned}$$

On the other hand,

$$P_{0,1}(\alpha \mathfrak{P}_{T_\mu^{\min}}) = \left(\sum_{\square \in \mu^{(\infty)}} (q^{T_\mu^{\min}(\square)} - 1) t^{c(\square)}\right) \alpha \mathfrak{P}_{T_\mu^{\min}}.$$

By assumption $\alpha \neq 0$ so

$$\sum_{\square \in \lambda^{(\infty)}} (q^{T_{\lambda}^{\min(\square)}} - 1)t^{c(\square)} = \sum_{\square \in \mu^{(\infty)}} (q^{T_{\mu}^{\min(\square)}} - 1)t^{c(\square)}.$$

This gives

$$\sum_{\square \in \lambda^{(n,\lambda)}} (q^{T_{\lambda}^{\min(\square)}} - 1)t^{c(\square)} = \sum_{\square \in \mu^{(n,\mu)}} (q^{T_{\mu}^{\min(\square)}} - 1)t^{c(\square)}$$

which after limiting $q \rightarrow 0$ gives

$$\sum_{\square \in \lambda} t^{c(\square)} = \sum_{\square \in \mu} t^{c(\square)}.$$

By comparing the coefficient of t^d for all $d \in \mathbb{Z}$ on both sides of the above equality we see that λ and μ have the same number of boxes on each diagonal and are therefore equal. \square

5 Pieri Rule for Generalized Macdonald Functions

The goal of this section is to derive and utilize an explicit combinatorial formula for the action of the multiplication operators $e_r[X]^\bullet$ on \widehat{W}_λ . We will investigate the e_1 Pieri coefficients in more depth and show that they satisfy a simple non-vanishing condition. We will use this non-vanishing to prove that the \widehat{W}_λ modules are cyclic.

5.1 Pieri Rule Preliminaries

We begin first by establishing some useful lemmas.

Lemma 5.1. For $T \in \text{RYT}_{\geq 0}(\lambda)$

$$\epsilon^{(n)}(F_{\min(T)}) = \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}).$$

Proof. The result follows from the following simple calculation:

$$\begin{aligned} & \epsilon^{(n)}(F_{\min(T)}) \\ &= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} t^{\binom{n}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}) \\ &= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} \sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \ell(\sigma\gamma)} T_{\sigma\gamma}(F_{\min(T)}) \\ &= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} \sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \ell(\sigma) - \ell(\gamma)} T_\sigma T_\gamma(F_{\min(T)}) \\ &= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} \sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma)} t^{\binom{\mu(T)}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}) \\ &= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}) \left(\sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{\mu(T)}{2} - \ell(\sigma)} \right) \\ &= \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}). \end{aligned}$$

\square

Lemma 5.2. For $\text{RSSYT}_{\geq 0}(\lambda)$ and $\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$

$$T_\sigma(F_{\min(T)}) = F_{\sigma(\min(T))} + \sum_{\tau < \sigma(\min(T))} \kappa_\tau F_\tau$$

for some scalars κ_τ .

Proof. We will proceed by induction using the fact that $\text{PSYT}_{\geq 0}(\lambda; T)$ is isomorphic to $\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$ as posets which we saw in Remark 9. Certainly, the statement holds trivially for $\tau = \min(T)$. Take some $\sigma(\min(T)) = \tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $s_i(\tau) > \tau$ and suppose that

$$T_\sigma(F_{\min(T)}) = F_{\sigma(\min(T))} + \sum_{\tau' < \sigma(\min(T))} \kappa_{\tau'} F_{\tau'}$$

for some scalars $\kappa_{\tau'}$. Then using Proposition 3.2

$$\begin{aligned} & T_{s_i\sigma}(F_{\min(T)}) \\ &= T_i T_\sigma(F_{\min(T)}) \\ &= T_i F_\tau + \sum_{\tau' < \sigma(\min(T))} \kappa_{\tau'} T_i F_{\tau'} \\ &= F_{s_i(\tau)} + \frac{(1-t)q^{w_\tau(i)}t^{c_\tau(i)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} F_\tau + \sum_{\tau' < \sigma(\min(T))} \kappa_{\tau'} \left(F_{s_i(\tau')} + \frac{(1-t)q^{w_{\tau'}(i)}t^{c_{\tau'}(i)}}{q^{w_{\tau'}(i)}t^{c_{\tau'}(i)} - q^{w_{\tau'}(i+1)}t^{c_{\tau'}(i+1)}} F_{\tau'} \right) \\ &= F_{(s_i\sigma)(\min(T))} + \sum_{\tau' < (s_i\sigma)(\min(T))} \kappa'_{\tau'} F_{\tau'}. \end{aligned}$$

□

The above lemmas may now be used to compute the symmetrization of each F_τ in terms of the P_T basis.

Proposition 5.3. For $T \in \text{RSSYT}_{\geq 0}(\lambda)$

$$\epsilon^{(n)}(F_{\min(T)}) = \frac{[\mu(T)]_t!}{[n]_t!} P_T.$$

Proof. Recall from Definition 4.4 that the coefficient of $F_{\text{top}(T)}$ in P_T is 1. We know that from the proof of Proposition 4.3 that since $T \in \text{RSSYT}_{\geq 0}(\lambda)$,

$$\epsilon^{(n)}(F_{\min(T)}) = \alpha P_T$$

for some nonzero scalar α . Let σ_0 denote the longest element of $\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$. Note that $\sigma_0(\min(T)) = \text{top}(T)$. We now use Lemmas 5.1 and 5.2 to compute the coefficient of $F_{\text{top}(T)}$ in $\epsilon^{(n)}(F_{\min(T)})$ determining α :

$$\begin{aligned} & \epsilon^{(n)}(F_{\min(T)}) \\ &= \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}) \\ &= \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma)} \left(F_{\sigma(\min(T))} + \sum_{\tau < \sigma(\min(T))} \kappa_\tau F_\tau \right) \\ &= \frac{[\mu(T)]_t!}{[n]_t!} F_{\sigma_0(\min(T))} t^{\binom{n}{2} - \binom{\mu(T)}{2} - \ell(\sigma_0)} + \sum_{\tau < \sigma_0(\min(T))} \kappa'_\tau F_\tau \\ &= \frac{[\mu(T)]_t!}{[n]_t!} F_{\text{top}(T)} + \sum_{\tau < \text{top}(T)} \kappa'_\tau F_\tau. \end{aligned}$$

Therefore, $\alpha = \frac{[\mu(T)]_t!}{[n]_t!}$.

□

Lemma 5.4. For $\tau \in \text{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T \in \text{RSSYT}_{\geq 0}(\lambda)$

$$\epsilon^{(n)}(F_\tau) = \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\text{top}(T)}).$$

Proof. Let $T \in \text{RSSYT}_{\geq 0}(\lambda)$ and $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $s_i(\tau) > \tau$. Then using Proposition 3.2 we see

$$\begin{aligned} & \epsilon^{(n)}(F_{s_i(\tau)}) \\ &= \epsilon^{(n)} \left(\left(T_i + \frac{(t-1)q^{w_\tau(i)} t^{c_\tau(i)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} \right) F_\tau \right) \\ &= \left(1 + \frac{(t-1)q^{w_\tau(i)} t^{c_\tau(i)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} \right) \epsilon^{(n)}(F_\tau) \\ &= \left(\frac{q^{w_\tau(i+1)} t^{c_\tau(i+1)} - q^{w_\tau(i)} t^{c_\tau(i)}}{q^{w_\tau(i+1)} t^{c_\tau(i+1)} - q^{w_\tau(i)} t^{c_\tau(i)}} \right) \epsilon^{(n)}(F_\tau). \end{aligned}$$

Now using an induction argument nearly identical to the proof of Corollary 4.5 we see that for any $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$

$$\epsilon^{(n)}(F_\tau) = \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\text{top}(T)}).$$

□

Corollary 5.5. For $\mathfrak{p}_\lambda(\tau) = T \in \text{RSSYT}_{\geq 0}(\lambda)$

$$\epsilon^{(n)}(F_\tau) = K_T(q, t) \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) P_T$$

where

$$K_T(q, t) := \frac{[\mu(T)]_t!}{[n]_t!} \prod_{(\square_1, \square_2) \in \text{Inv}(\min(T))} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right).$$

Proof. We begin by noting that from Lemma 5.4 applied to $\min(T)$:

$$\epsilon^{(n)}(F_{\min}) = \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\text{top}(T)}).$$

But from Proposition 5.3 we know that $\epsilon^{(n)}(F_{\min}) = \frac{[\mu(T)]_t!}{[n]_t!} P_T$ so

$$\prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\text{top}(T)}) = \frac{[\mu(T)]_t!}{[n]_t!} P_T.$$

Thus

$$\epsilon^{(n)}(F_{\text{top}(T)}) = K_T(q, t) P_T$$

as defined in the corollary statement above.

Lastly, we can now use Lemma 5.4 to finish the proof. □

The last lemma of this section relates the action of $e_r[X]^\bullet$ to the action of γ_n^r on symmetrized elements.

Lemma 5.6. For $1 \leq r \leq n$, $\epsilon^{(n)} e_r[X_1 + \dots + X_n] \epsilon^{(n)} = t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \epsilon^{(n)} \gamma_n^r \epsilon^{(n)}$.

Proof. First, we will show by induction that for $1 \leq r \leq n$

$$\gamma_n^r = t^{(n-1)+\dots+(n-r)}(T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) X_1 \cdots X_r.$$

For $r = 1$ we see that

$$\gamma_n = X_n T_{n-1} \cdots T_1 = t^{n-1} T_{n-1}^{-1} \cdots T_1^{-1} X_1.$$

Now suppose this equation holds for some $1 \leq r \leq n-1$. Then we have

$$\begin{aligned} \gamma_n^{r+1} &= \gamma_n^r \gamma_n \\ &= t^{(n-1)+\dots+(n-r)}(T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) X_1 \cdots X_r t^{n-1} T_{n-1}^{-1} \cdots T_1^{-1} X_1 \\ &= t^{(n-1)+\dots+(n-r)} t^{n-1} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) X_1 \cdots X_r T_{n-1}^{-1} \cdots T_1^{-1} X_1 \\ &= t^{(n-1)+\dots+(n-r)} t^{n-1} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) T_{n-1}^{-1} \cdots T_{r+1}^{-1} X_1 \cdots X_r T_r^{-1} \cdots T_1^{-1} X_1. \end{aligned}$$

A simple calculation verifies that

$$X_1 \cdots X_r T_r^{-1} \cdots T_1^{-1} = t^{-r} T_r \cdots T_1 X_2 \cdots X_{r+1}.$$

Therefore,

$$\begin{aligned} \gamma_n^{r+1} &= t^{(n-1)+\dots+(n-r)} t^{n-1} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) T_{n-1}^{-1} \cdots T_{r+1}^{-1} t^{-r} T_r \cdots T_1 X_1 X_2 \cdots X_{r+1} \\ &= t^{(n-1)+\dots+(n-r)+(n-(r+1))} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) (T_{n-1}^{-1} \cdots T_{r+1}^{-1} T_r \cdots T_1) X_1 \cdots X_{r+1} \end{aligned}$$

which is of the correct form.

Now we see that for any $1 \leq r \leq n$,

$$\begin{aligned} \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} &= \epsilon^{(n)} t^{(n-1)+\dots+(n-r)} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) X_1 \cdots X_r \epsilon^{(n)} \\ &= t^{(n-1)+\dots+(n-r)} \epsilon^{(n)} X_1 \cdots X_r \epsilon^{(n)}. \end{aligned}$$

Suppose that $1 = i_0 \leq i_1 < \dots < i_r \leq i_{r+1} = n$ with $i_j < i_{j+1} - 1$ for some $0 \leq j \leq r$. Then

$$\begin{aligned} &X_{i_1} \cdots X_{i_{j-1}} X_{i_j+1} X_{i_{j+1}} X_{i_{j+2}} \cdots X_{i_r} \\ &= X_{i_1} \cdots X_{i_{j-1}} (t^{-1} T_{i_j}^{-1} X_{i_j} T_{i_j}^{-1}) X_{i_{j+1}} X_{i_{j+2}} \cdots X_{i_r} \\ &= t T_{i_j}^{-1} X_{i_1} \cdots X_{i_{j-1}} X_{i_j} X_{i_{j+1}} X_{i_{j+2}} \cdots X_{i_r} T_{i_j}^{-1} \\ &= t T_{i_j}^{-1} X_{i_1} \cdots X_{i_r} T_{i_j}^{-1} \end{aligned}$$

which shows that

$$\epsilon^{(n)} X_{i_1} \cdots X_{i_{j-1}} X_{i_j+1} X_{i_{j+1}} X_{i_{j+2}} \cdots X_{i_r} \epsilon^{(n)} = t \epsilon^{(n)} X_{i_1} \cdots X_{i_r} \epsilon^{(n)}.$$

It follows that for any $1 \leq i_1 < \dots < i_r \leq n$

$$\epsilon^{(n)} X_{i_1} \cdots X_{i_r} \epsilon^{(n)} = t^{(i_r-r)+\dots+(i_1-1)} \epsilon^{(n)} X_1 \cdots X_r \epsilon^{(n)}.$$

Now we see

$$\begin{aligned}
& \epsilon^{(n)} e_r[X_1 + \dots + X_n] \epsilon^{(n)} \\
&= \epsilon^{(n)} \left(\sum_{1 \leq i_1 < \dots < i_r \leq n} X_{i_1} \dots X_{i_r} \right) \epsilon^{(n)} \\
&= \sum_{1 \leq i_1 < \dots < i_r \leq n} \epsilon^{(n)} X_{i_1} \dots X_{i_r} \epsilon^{(n)} \\
&= \sum_{1 \leq i_1 < \dots < i_r \leq n} t^{(i_r-r)+\dots+(i_1-1)} \epsilon^{(n)} X_1 \dots X_r \epsilon^{(n)} \\
&= \sum_{1 \leq i_1 < \dots < i_r \leq n} t^{(i_r-r)+\dots+(i_1-1)} t^{-((n-1)+\dots+(n-r))} \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} \\
&= t^{-((n-1)+\dots+(n-r))} \left(\sum_{1 \leq i_1 < \dots < i_r \leq n} t^{(i_1-1)+\dots+(i_r-r)} \right) \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} \\
&= t^{-((n-1)+\dots+(n-r))} e_r(1, \dots, t^{n-1}) \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} \\
&= t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \epsilon^{(n)} \gamma_n^r \epsilon^{(n)}.
\end{aligned}$$

□

5.2 Pieri Rule

Using the above lemmas, we may derive an explicit formula for the action of $e_r[X]^\bullet$ on the symmetric v.v. Macdonald polynomials in the finite variable situation. We will then use the stability of the P_T to derive a similar formula for the \mathfrak{P}_T .

Theorem 5.7. For $T \in \text{RSSYT}_{\geq 0}(\lambda)$ and $1 \leq r \leq n$ we have the expansion

$$e_r[X_1 + \dots + X_n] P_T = \sum_S d_{S,T}^{(r)} P_S$$

where

$$\begin{aligned}
\frac{d_{S,T}^{(r)}}{t^{\binom{r}{2}} e_r \left[\frac{1-t^n}{1-t} \right] K_S(q, t)} &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \\ \Psi^r(\tau) \in \text{PSYT}_{\geq 0}(\lambda; S)}} t^{c_\tau(1)+\dots+c_\tau(r)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \times \\
&\quad \prod_{(\square_1, \square_2) \in \text{Inv}(\Psi^r(\tau))} \left(\frac{q^{S(\square_1)} t^{c(\square_1)} - q^{S(\square_2)} t^{c(\square_2)}}{q^{S(\square_1)} t^{c(\square_1)} - q^{S(\square_2)} t^{c(\square_2)+1}} \right)
\end{aligned}$$

and T' ranges over all $T' \in \text{RSSYT}_{\geq 0}(\lambda)$ one can obtain from T by adding r to the boxes of T with at most one 1 being added to each box.

Proof. Using Lemma 4.5 and Remark 8 we find

$$\begin{aligned}
& e_r[X_1 + \dots + X_n]P_T \\
&= \epsilon^{(n)} e_r[X_1 + \dots + X_n] \epsilon^{(n)}(P_T) \\
&= t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \epsilon^{(n)} \gamma_n^r \epsilon^{(n)}(P_T) \\
&= t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \epsilon^{(n)} \gamma_n^r(P_T) \\
&= t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \epsilon^{(n)} \gamma_n^r \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) F_\tau \\
&= t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \epsilon^{(n)} \gamma_n^r(F_\tau) \\
&= t^{-((n-1)+\dots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \epsilon^{(n)} (t^{r(n-1)} t^{c_\tau(1)+\dots+c_\tau(r)} F_{\Psi^r(\tau)}) \\
&= t^{\binom{r}{2}} e_r \left[\frac{1-t^n}{1-t} \right] \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} t^{c_\tau(1)+\dots+c_\tau(r)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \epsilon^{(n)}(F_{\Psi^r(\tau)}).
\end{aligned}$$

From Corollary 5.5,

$$\epsilon^{(n)}(F_{\Psi^r(\tau)}) = \mathbb{1}(\mathfrak{p}_\lambda(\Psi^r(\tau)) \in \text{RSSYT}_{\geq 0}(\lambda)) K_{\mathfrak{p}_\lambda(\Psi^r(\tau))}(q, t) \prod_{(\square_1, \square_2) \in \text{Inv}(\Psi^r(\tau))} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) P_{\mathfrak{p}_\lambda(\Psi^r(\tau))}.$$

Hence, by collecting coefficients around each P_S for $S \in \text{RSSYT}_{\geq 0}(\lambda)$ we see that

$$e_r[X_1 + \dots + X_n]P_T = \sum_S d_{S,T}^{(r)} P_S$$

where $d_{S,T}^{(r)}$ are as given in the theorem statement above.

Lastly, if $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ then the boxes of λ containing the labels $1, \dots, r$ (with some powers of q given by T) are exactly those boxes $\square \in \lambda$ with $\mathfrak{p}_\lambda(\Psi^r(\tau))(\square) = T(\square) + 1$. Thus if $S = \mathfrak{p}_\lambda(\Psi^r(\tau)) \in \text{RSSYT}_{\geq 0}(\lambda)$ then we may obtain S from T by adding the value 1 to r boxes of T with at most one 1 added to each box. □

Definition 5.8. For $S, T \in \Omega(\lambda)$ and $r \geq 1$ define $\mathfrak{d}_{S,T}^{(r)} \in \mathbb{Q}(q, t)$ by

$$e_r[X] \mathfrak{P}_T = \sum_{S \in \Omega(\lambda)} \mathfrak{d}_{S,T}^{(r)} \mathfrak{P}_S.$$

Remark 13. Note that from Theorem 5.7 it is clear for $T \in \Omega(\lambda)$ and $r \geq 1$ that each $S \in \Omega(\lambda)$ such that $\mathfrak{d}_{S,T}^{(r)} \neq 0$ will necessarily be obtained from T by adding r 1's to the boxes of T with at most one 1 being added to each box. As such the set of such S is finite. Further, any such S has $\text{rk}(S) \leq \text{rk}(T) + r$.

As an immediate consequence of Theorem 5.7 and the definition of \mathfrak{P}_T from Definition 4.9 we obtain the following result.

Corollary 5.9 (Pieri Rule). Let $S, T \in \Omega(\lambda)$ and $r \geq 1$. For all $n \geq \text{rk}(T) + r$

$$\mathfrak{d}_{S,T}^{(r)} = d_{S|_{\lambda^{(n)}}, T|_{\lambda^{(n)}}}^{(r)}.$$

5.3 Non-vanishing for e_1 Pieri Coefficients

In this section we will prove that if T, S satisfy a simple combinatorial relationship then $\mathfrak{d}_{T', T}^{(1)} \neq 0$. This will be instrumental in the proof that the modules \widetilde{W}_λ are cyclic.

Definition 5.10. Let $\lambda \in \mathbb{Y}$ and $T \in \text{RSSYT}_{\geq 0}(\lambda)$. A box \square_0 in λ is *T -raisable* if the labelling S defined by

$$S(\square) = \begin{cases} T(\square) & \square \neq \square_0 \\ T(\square) + 1 & \square = \square_0. \end{cases}$$

is also in $\text{RSSYT}_{\geq 0}(\lambda)$. We say that S is obtained by raising the box \square_0 of T . Further, we say that \square_0 is a *S -lowerable* box in λ .

We will write $T \uparrow S$ if S may be obtained by raising one box of T .

Remark 14. We may define a partial order \sqsubseteq on the set $\text{RSSYT}_{\geq 0}(\lambda)$ simply by

$$T \sqsubseteq S \leftrightarrow \forall \square \in \lambda^{(\infty)}, T(\square) \leq S(\square).$$

Then the relation $T \uparrow S$ defined in Definition 5.10 is simply the cover relation of \sqsubseteq . Lastly, we may extend the definitions of raisable/lowerable boxes and of the relation $T \uparrow S$ to $\Omega(\lambda)$ analogously in the obvious way.

We require the following lemmas.

Lemma 5.11. Let $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ for $T \in \text{RSSYT}_{\geq 0}(\lambda)$. If $(\square_1, \square_2) \in \text{Inv}(\tau)$ with $T(\square_1) = T(\square_2) = n$ then $c(\square_2) - c(\square_1) \geq 2$.

Proof. Since $T \in \text{RSSYT}_{\geq 0}(\lambda)$, for all $n \geq 0$ the set of boxes $\{\square \in \lambda \mid T(\square) = n\}$ is a skew-diagram consisting of a union of disjoint horizontal strips. Suppose $(\square_1, \square_2) \in \text{Inv}(\tau)$ with $T(\square_1) = T(\square_2) = n$. Then \square_1 and \square_2 cannot be in the same horizontal strip component of $\{\square \in \lambda \mid T(\square) = n\}$. Further, \square_1 must be to the left of \square_2 . Hence, $c(\square_2) - c(\square_1) \geq 2$. \square

Lemma 5.12. Let $T \in \text{RSSYT}_{\geq 0}(\lambda)$. Given a T -raisable box of λ , \square_0 , there exists a unique $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ such that

- $\tau(\square_0) = 1q^a$ for some $a \geq 0$
- $\text{inv}(\tau) = S(T)(\square_0) - 1$.

Proof. Since the count $\text{inv}(\tau) = S(T)(\square_0) - 1$ is tight there exists at most one such labelling. We may simply define $\tau \in \text{PSYT}_{\geq 0}$ by labelling the boxes $\square \in \lambda$ with $\square <_T \square_0$ with the labels $\{2, \dots, S(T)(\square_0) - 1\}$ following the box ordering $S(T)$. We then fill the boxes $\square >_T \square_0$ with the values $\{S(T)(\square_0), \dots, n\}$ following the box ordering $S(T)$. Thus τ has exactly $S(T)(\square_0) - 1$ inversion pairs. \square

Lemma 5.13. Let $T, T' \in \text{RSSYT}_{\geq 0}(\lambda)$ with $T \uparrow T'$. Let \square_0 be the box of λ on which T and T' differ. Let $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $\Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')$. Then we have the following:

- $\text{Inv}(\tau) = \{(\square_1, \square_2) \in \text{Inv}(\tau) \mid \square_i \neq \square_0\} \sqcup \{(\square, \square_0) \mid \square <_T \square_0\}$
- $\text{Inv}(\Psi(\tau)) = \{(\square_1, \square_2) \in \text{Inv}(\Psi(\tau)) \mid \square_i \neq \square_0\} \sqcup \{(\square_0, \square) \mid \square_0 <_{T'} \square\}$
- $\{(\square_1, \square_2) \in \text{Inv}(\tau) \mid \square_i \neq \square_0\} = \{(\square_1, \square_2) \in \text{Inv}(\Psi(\tau)) \mid \square_i \neq \square_0\}$.

Proof. This result follows by simple case work which we leave to the reader. \square

Putting these lemmas together we may show the following:

Theorem 5.14. Let $\lambda \in \mathbb{Y}$ and $T, T' \in \text{RSSYT}_{\geq 0}(\lambda)$ with $T \uparrow T'$. Then $d_{T', T}^{(1)} \neq 0$.

Proof. Let \square_0 be the T -raisable box on which T and T' differ. From 5.7 we see that

$$\begin{aligned} \frac{d_{T',T}^{(1)}}{\left(\frac{1-t^n}{1-t}\right)K_{T'}(q,t)} &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda;T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')}} t^{c_\tau(1)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \times \\ &\quad \prod_{(\square_1, \square_2) \in \text{Inv}(\Psi(\tau))} \left(\frac{q^{T'(\square_1)} t^{c(\square_1)} - q^{T'(\square_2)} t^{c(\square_2)}}{q^{T'(\square_1)} t^{c(\square_1)} - q^{T'(\square_2)} t^{c(\square_2)+1}} \right). \end{aligned}$$

Therefore, it suffices to show that the sum on the right hand side of the above equation is nonzero. If $\tau \in \text{PSYT}_{\geq 0}(\lambda;T)$ with $\Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')$ then $c_\tau(1) = c(\square_0)$. Hence, we may factor out the term $t^{c_\tau(1)} = t^{c(\square_0)}$ outside the sum. From Lemma 5.13 we have the following for any $\tau \in \text{PSYT}_{\geq 0}(\lambda;T)$ with $\Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')$:

$$\begin{aligned} &\prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \prod_{(\square_1, \square_2) \in \text{Inv}(\Psi(\tau))} \left(\frac{q^{T'(\square_1)} t^{c(\square_1)} - q^{T'(\square_2)} t^{c(\square_2)}}{q^{T'(\square_1)} t^{c(\square_1)} - q^{T'(\square_2)} t^{c(\square_2)+1}} \right) \\ &= \prod_{\square <_T \square_0} \left(\frac{q^{T(\square)} t^{c(\square)+1} - q^{T(\square_0)} t^{c(\square_0)}}{q^{T(\square)} t^{c(\square)} - q^{T(\square_0)} t^{c(\square_0)}} \right) \prod_{\square_0 <_{T'} \square} \left(\frac{q^{T(\square_0)+1} t^{c(\square_0)} - q^{T(\square)} t^{c(\square)}}{q^{T(\square_0)+1} t^{c(\square_0)} - q^{T(\square)} t^{c(\square)+1}} \right) \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \end{aligned}$$

The first two products above are nonzero and do not depend on τ and can therefore be brought outside the summation

$$\sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda;T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')}} .$$

Hence, it suffices to show that

$$\sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda;T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')}} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \neq 0.$$

Notice that we can rewrite the above product terms in the following way:

$$\begin{aligned} &\prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \\ &= t^{\text{inv}(\tau) - S(T)(\square_0)+1} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda;T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')}} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \\ &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda;T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda;T')}} t^{\text{inv}(\tau) - S(T)(\square_0)+1} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \end{aligned}$$

Now we have by definition for any inversion pair (\square_1, \square_2) that $T(\square_2) - T(\square_1) \leq 0$. Therefore, by limiting $q \rightarrow \infty$ we see that

$$\begin{aligned} & \lim_{q \rightarrow \infty} \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')}} t^{\text{inv}(\tau) - S(T)(\square_0) + 1} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \\ &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')}} t^{\text{inv}(\tau) - S(T)(\square_0) + 1} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0 \\ T(\square_1) = T(\square_2)}} \left(\frac{1 - t^{c(\square_2) - c(\square_1) - 1}}{1 - t^{c(\square_2) - c(\square_1) + 1}} \right). \end{aligned}$$

By Lemma 5.11 we see that for each of the inversion pairs $(\square_1, \square_2) \in \text{Inv}(\tau)$ for $\tau \in \text{PSYT}_{\geq 0}(\lambda; T)$ with $\Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')$ and $T(\square_1) = T(\square_2)$ that $c(\square_2) - c(\square_1) - 1 \geq 1$. Therefore, if we limit $t \rightarrow 0$

$$\begin{aligned} & \lim_{t \rightarrow 0} \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')}} t^{\text{inv}(\tau) - S(T)(\square_0) + 1} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0 \\ T(\square_1) = T(\square_2)}} \left(\frac{1 - t^{c(\square_2) - c(\square_1) - 1}}{1 - t^{c(\square_2) - c(\square_1) + 1}} \right) \\ &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')}} \mathbb{1}(\text{inv}(\tau) = S(T)(\square_0) - 1) \lim_{t \rightarrow 0} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0 \\ T(\square_1) = T(\square_2)}} \left(\frac{1 - t^{c(\square_2) - c(\square_1) - 1}}{1 - t^{c(\square_2) - c(\square_1) + 1}} \right) \\ &= \sum_{\substack{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \\ \text{s.t.} \\ \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T')}} \mathbb{1}(\text{inv}(\tau) = S(T)(\square_0) - 1) \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ \square_i \neq \square_0 \\ T(\square_1) = T(\square_2)}} (1) \\ &= \#\{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \mid \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T'), \text{inv}(\tau) = S(T)(\square_0) - 1\}. \end{aligned}$$

By Lemma 5.12 $\#\{\tau \in \text{PSYT}_{\geq 0}(\lambda; T) \mid \Psi(\tau) \in \text{PSYT}_{\geq 0}(\lambda; T'), \text{inv}(\tau) = S(T)(\square_0) - 1\} = 1$ which in particular is not 0. Therefore, $d_{T', T}^{(1)} \neq 0$. \square

Using stability we find the following:

Corollary 5.15. Let $\lambda \in \mathbb{Y}$ and $T, T' \in \Omega(\lambda)$ with $T \uparrow T'$. Then $\mathfrak{d}_{T', T}^{(1)} \neq 0$.

Proof. From Corollary 5.9 we know that for all $n \geq \text{rk}(T) + 1$

$$\mathfrak{d}_{T', T}^{(1)} = d_{T'|\lambda^{(n)}, T|\lambda^{(n)}}^{(1)}.$$

Since T' is obtained from T by increasing the value of a single box of T by 1 we know that the same must be true for $T'|\lambda^{(n)}$ and $T|\lambda^{(n)}$ for all $n \geq \text{rk}(T) + 1$. Therefore, from Theorem 5.14 we conclude that $\mathfrak{d}_{T', T}^{(1)} = d_{T'|\lambda^{(n)}, T|\lambda^{(n)}}^{(1)} \neq 0$. \square

The non-vanishing of the e_1 Pieri coefficients is sufficient to prove that the \widetilde{W}_λ are cyclic \mathcal{E}^+ -modules.

Corollary 5.16. For $\lambda \in \mathbb{Y}$, \widetilde{W}_λ is a cyclic \mathcal{E}^+ -module.

Proof. We will show that $\mathfrak{P}_{T_\lambda^{\min}}$ is a cyclic vector for \widetilde{W}_λ i.e. $\mathcal{E}^+ \mathfrak{P}_{T_\lambda^{\min}} = \widetilde{W}_\lambda$. It suffices to show that for every $T \in \Omega(\lambda)$ there exists some $X \in \mathcal{E}^+$ with $X(\mathfrak{P}_{T_\lambda^{\min}}) = \mathfrak{P}_T$. Notice that given any $T \in \Omega(\lambda)$ we may choose any lowerable box \square_1 of T and obtain a labelling $T_1 \in \Omega(\lambda)$ by subtracting the value of 1 from \square_1 in the labelling T . Continuing in this process yields a sequence of labellings T_1, T_2, \dots with $T_{i+1} \uparrow T_i$ which must eventually terminate as $\text{deg}(T_i) = \text{deg}(T) - i$. It is easy to verify that the only element of $\Omega(\lambda)$ without any lowerable boxes is T_λ^{\min} so the sequence T_1, T_2, \dots must end at T_λ^{\min} . Reversing this process shows that any $T \in \Omega(\lambda)$ may be obtained from T_λ^{\min} by a sequence $T_\lambda^{\min} = T_1, \dots, T_n = T$ with

$T_i \uparrow T_{i+1}$. Hence, by induction it suffices to show that if $T \uparrow T'$ then there exists $X \in \mathcal{E}^+$ such that $X(\mathfrak{P}_T) = \mathfrak{P}_{T'}$.

Let $T, T' \in \Omega(\lambda)$ with $T \uparrow T'$. Consider the element $X \in \mathcal{E}^+$ defined by

$$X := \prod_{\substack{T \uparrow S \\ S \neq T'}} \left(\frac{P_{0,1} - \sum_{\square \in \lambda^{(\infty)}} (q^{S(\square)} - 1)t^{c(\square)}}{\sum_{\square \in \lambda^{(\infty)}} (q^{T'(\square)} - q^{S(\square)})t^{c(\square)}} \right).$$

The denominator of the above product is nonzero since $P_{0,1}$ acts with simple spectrum on \widetilde{W}_λ . Further, as mentioned before the set of $S \in \Omega(\lambda)$ with $T \uparrow S$ is finite so the above product is finite. We have that for $T \uparrow V$

$$\begin{aligned} X(\mathfrak{P}_V) &= \prod_{\substack{T \uparrow S \\ S \neq T'}} \left(\frac{P_{0,1} - \sum_{\square \in \lambda^{(\infty)}} (q^{S(\square)} - 1)t^{c(\square)}}{\sum_{\square \in \lambda^{(\infty)}} (q^{T'(\square)} - q^{S(\square)})t^{c(\square)}} \right) (\mathfrak{P}_V) \\ &= \prod_{\substack{T \uparrow S \\ S \neq T'}} \left(\frac{\sum_{\square \in \lambda^{(\infty)}} (q^{V(\square)} - q^{S(\square)})t^{c(\square)}}{\sum_{\square \in \lambda^{(\infty)}} (q^{T'(\square)} - q^{S(\square)})t^{c(\square)}} \right) \mathfrak{P}_V \\ &= \delta_{V,T'} \mathfrak{P}_V. \end{aligned}$$

From Corollary 5.15 we know that $\mathfrak{d}_{T',T}^{(1)} \neq 0$. Therefore, we may consider the element $X' \in \mathcal{E}^+$ defined by

$$X' := \frac{q^{-1}}{\mathfrak{d}_{T',T}^{(1)}} X P_{1,0}.$$

We find that

$$\begin{aligned} X'(\mathfrak{P}_T) &= \frac{q^{-1}}{\mathfrak{d}_{T',T}^{(1)}} X P_{1,0}(\mathfrak{P}_T) \\ &= \frac{q^{-1}}{\mathfrak{d}_{T',T}^{(1)}} X q e_1^\bullet(\mathfrak{P}_T) \\ &= \frac{1}{\mathfrak{d}_{T',T}^{(1)}} X \left(\sum_{T \uparrow S} \mathfrak{d}_{S,T}^{(1)} \mathfrak{P}_S \right) \\ &= \frac{1}{\mathfrak{d}_{T',T}^{(1)}} \sum_{T \uparrow S} \mathfrak{d}_{S,T}^{(1)} \delta_{S,T'} \mathfrak{P}_S \\ &= \mathfrak{P}_{T'}. \end{aligned}$$

□

6 Family of (q, t) Product-Sum Identities

In the final section of this paper we will investigate an interesting family of (q, t) product-sum identities which are derived using the combinatorics underpinning the structure of the generalized symmetric Macdonald functions \mathfrak{P}_T along with some elementary non-archimedean analysis.

Definition 6.1. A non-negative asymptotic periodic standard Young tableau with base shape $\lambda \in \mathbb{Y}$ is a labelling $\tau : \lambda^{(\infty)} \rightarrow \{iq^a : i \geq 1, a \geq 0\}$ such that

- τ is strictly increasing along rows and columns
- The set of boxes $\square \in \lambda^{(\infty)}$ such that $\tau(\square) = iq^a$ for some $i \geq 1$ and $a > 0$ is finite
- For all $i \geq 1$ there exists a unique $\square \in \lambda^{(\infty)}$ such that $\tau(\square) = iq^a$ for some $a \geq 0$.

We will write $\text{APSYT}_{\geq 0}(\lambda)$ for the set of all non-negative asymptotic periodic standard Young tableaux with base shape $\lambda \in \mathbb{Y}$. If $\tau \in \text{APSYT}_{\geq 0}(\lambda)$ has that for every $\square \in \lambda$, $\tau(\square) = iq^0$ for some $i \geq 1$ then we will call τ an asymptotic standard Young tableau with base shape λ . We will write $\text{ASYT}(\lambda)$ for the set of asymptotic standard Young tableau with base shape λ . As an abuse of notation will write $\mathfrak{p}_\lambda : \text{APSYT}_{\geq 0}(\lambda) \rightarrow \Omega(\lambda)$ for the map given on $\tau \in \text{APSYT}_{\geq 0}(\lambda)$ by $\mathfrak{p}_\lambda(\tau)(\square) = a$ whenever $\tau(\square) = iq^a$ for some $i \geq 1$. We will let $\text{APSYT}_{\geq 0}(\lambda; T)$ denote the set of all $\tau \in \text{APSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T$.

Definition 6.2. For $T \in \Omega(\lambda)$ define $S(T) \in \text{ASYT}(\lambda)$ by ordering the boxes of $\lambda^{(\infty)}$ according to $\square_1 \leq \square_2$ if and only if

- $T(\square_1) > T(\square_2)$ or
- $T(\square_1) = T(\square_2)$ and \square_1 comes before \square_2 in the column-standard labelling of $\lambda^{(\infty)}$.

Let $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$. An ordered pair of boxes $(\square_1, \square_2) \in \lambda^{(\infty)} \times \lambda^{(\infty)}$ is called an inversion pair of τ if $S(T)(\square_1) < S(T)(\square_2)$ and $\tau(\square_1) = iq^a$, $\tau(\square_2) = jq^b$ for some $i > j$ and $a, b \geq 0$. The set of all inversion pairs of τ will be denoted by $\text{Inv}(\tau)$. We will write $\text{inv}(\tau) = |\text{Inv}(\tau)|$. Define $\text{rk}(\tau)$ to be the minimal $n \geq n_\lambda$ such that $\tau|_{\lambda^{(\infty)}/\lambda^{(n)}}$ has consecutive labels.

Example 9. Consider $T \in \Omega(3, 2, 1)$ from Example 7. Then

$$\tau = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1q^3 & 2q^3 & 3q^3 & 5q^2 & 7q^2 & 12q^1 & 13q^0 & 14q^0 & 15q^0 & \dots \\ \hline 4q^2 & 6q^2 & 11q^1 & & & & & & & \\ \hline 8q^1 & 9q^1 & & & & & & & & \\ \hline 10q^0 & & & & & & & & & \\ \hline \end{array} \in \text{APSYT}_{\geq 0}(3, 2, 1; T),$$

$$S(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 7 & 11 & 13 & 14 & 15 & \dots \\ \hline 4 & 5 & 10 & & & & & & & \\ \hline 8 & 9 & & & & & & & & \\ \hline 12 & & & & & & & & & \\ \hline \end{array},$$

and $\text{rk}(T) = 12$.

Recall Corollary 5.5 for the definition of $K_T(q, t)$.

Proposition 6.3. For $T \in \text{RSSYT}_{\geq 0}(\lambda)$

$$\frac{1}{K_T(q, t)} = \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right).$$

Proof. Using Corollary 4.5 and Corollary 5.5 we find

$$\begin{aligned} P_T &= \epsilon^{(n)}(P_T) \\ &= \epsilon^{(n)} \left(\sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1) + 1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) F_\tau \right) \\ &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1) + 1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) \epsilon^{(n)}(F_\tau) \\ &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1) + 1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}} \right) K_T(q, t) \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2) + 1}} \right) P_T \\ &= K_T(q, t) \left(\sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1) + 1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2) + 1}} \right) \right) P_T. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{K_T(q, t)} &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)+1}} \right) \\ &= \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right). \end{aligned}$$

□

Our goal now is to compute the limit of both sides of the equation in Proposition 6.3 along sequences of the form $(\lambda^{(n)})_{n \geq n_\lambda}$. One side gives an infinite product and the other a power series which are dealt with separately. We require the following straightforward lemmas.

Lemma 6.4. For $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$, $\text{rk}(\tau) - \text{rk}(T) \leq \text{inv}(\tau) \leq \binom{\text{rk}(\tau)}{2}$.

Proof. Any inversion pair $(\square_1, \square_2) \in \text{Inv}(\tau)$ has $\square_1, \square_2 \in \lambda^{(\text{rk}(\tau))}$. Therefore, trivially $\text{inv}(\tau) \leq \binom{\text{rk}(\tau)}{2}$.

For the other side of the inequality, we only need to consider the case when $\text{rk}(\tau) > \text{rk}(T)$ since $\text{inv}(\tau) \geq 0$. Let \square_0 be the unique square of $\lambda^{(\text{rk}(\tau))} / \lambda^{(\text{rk}(T))}$. Then by of the definition of rank $\tau(\square_0) \neq \text{rk}(\tau)$. Further, for any $\square \in \lambda^{(\text{rk}(\tau))} / \lambda^{(\text{rk}(T))}$ we must have that $\tau(\square) \neq \text{rk}(\tau)$ as τ must be strictly increasing to the right along the horizontal strip $\lambda^{(\text{rk}(\tau))} / \lambda^{(\text{rk}(T))}$. Therefore, if \square_1 is the box of $\lambda^{(\text{rk}(T))}$ with $\tau(\square_1) = \text{rk}(\tau) q^a$ for some $a \geq 0$ then for all $\square \in \lambda^{(\text{rk}(\tau))} / \lambda^{(\text{rk}(T))}$ we find that $(\square_1, \square) \in \text{Inv}(\tau)$. Therefore, $\text{inv}(\tau) \geq \text{rk}(\tau) - \text{rk}(T)$. □

Lemma 6.5. For $k \geq 0$ there are only finitely many $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$ with $\text{rk}(\tau) \leq k$.

Proof. The map $\{\tau \in \text{APSYT}_{\geq 0}(\lambda; T) \mid \text{rk}(\tau) \leq k\} \rightarrow \text{PSYT}_{\geq 0}(\lambda^{(k)}; T)$ given by $\tau \rightarrow \tau|_{\lambda^{(k)}}$ is easily seen to be a bijection. Since $\text{PSYT}_{\geq 0}(\lambda^{(k)}; T)$ is a finite set we are done. □

Corollary 6.6. For $k \geq 0$ there are only finitely many $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$ with $\text{inv}(\tau) \leq k$.

Proof. If $\text{inv}(\tau) \leq k$ then by Lemma 6.4 we know that $\text{rk}(\tau) \leq k + \text{rk}(T)$. Thus by Lemma 6.5

$$\#\{\tau \mid \text{inv}(\tau) \leq k\} \leq \#\{\tau \mid \text{rk}(\tau) \leq k + \text{rk}(T)\} < \infty.$$

□

Lemma 6.7. For $T \in \text{RSSYT}_{\geq 0}(\lambda)$, the set $I(T) = \text{Inv}(\min(T))$ consists of all pairs of boxes $(\square_1, \square_2) \in \lambda \times \lambda$ with $\square_1 <_T \square_2$ except those pairs with $T(\square_1) = T(\square_2)$ and \square_1 before \square_2 in the same row.

Proof. This follows immediately from the definition of $\min(T)$. □

Now we deal with the limit of products.

Proposition 6.8. Let $T \in \Omega(\lambda)$. The sequence $(K_{T|_{\lambda^{(n)}}}(q, t))_{n \geq n_\lambda}$ converges with respect to the t -adic topology on $\mathbb{Q}(q)((t))$ to

$$\frac{(1-t)^{\text{rk}(T)} [\mu(T|_{\lambda^{(\text{rk}(T))}})] t!}{\prod_{\square \in \lambda^{(\text{rk}(T))}} (1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)})} \prod_{(\square_1, \square_2) \in I(T|_{\lambda^{(\text{rk}(T))}})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right).$$

Proof. Let $n \geq \text{rk}(T)$. From Lemma 6.7 we know

$$I(T|_{\lambda^{(n)}}) = I(T|_{\lambda^{(\text{rk}(T))}}) \sqcup \{(\square_1, \square_2) \mid \square_1 \in \lambda^{(\text{rk}(T))}, \square_2 \in \lambda^{(n)} / \lambda^{(\text{rk}(T))}\}.$$

Therefore,

$$\begin{aligned}
& \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(n)})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \\
&= \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(\text{rk}(T))})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \prod_{\square_1 \in \lambda(\text{rk}(T))} \prod_{\square_2 \in \lambda(n)/\lambda(\text{rk}(T))} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \\
&= \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(\text{rk}(T))})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \prod_{\square \in \lambda(\text{rk}(T))} \prod_{i = \text{rk}(T) - |\lambda|}^{n - |\lambda| - 1} \left(\frac{1 - q^{-T(\square)} t^{i - c(\square) + 1}}{1 - q^{-T(\square)} t^{i - c(\square)}} \right).
\end{aligned}$$

Note that the following product telescopes:

$$\begin{aligned}
& \prod_{i = \text{rk}(T) - |\lambda|}^{n - |\lambda| - 1} \left(\frac{1 - q^{-T(\square)} t^{i - c(\square) + 1}}{1 - q^{-T(\square)} t^{i - c(\square)}} \right) \\
&= \left(\frac{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square) + 1}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)}} \right) \left(\frac{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square) + 2}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square) + 1}} \right) \cdots \left(\frac{1 - q^{-T(\square)} t^{(n - |\lambda| - 1) - c(\square) + 1}}{1 - q^{-T(\square)} t^{(n - |\lambda| - 1) - c(\square)}} \right) \\
&= \left(\frac{1 - q^{-T(\square)} t^{n - |\lambda| - c(\square)}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)}} \right).
\end{aligned}$$

Thus

$$\begin{aligned}
& \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(n)})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \\
&= \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(\text{rk}(T))})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \prod_{\square \in \lambda(\text{rk}(T))} \left(\frac{1 - q^{-T(\square)} t^{n - |\lambda| - c(\square)}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)}} \right).
\end{aligned}$$

Now $\mu(T_{\lambda(n)}) = \mu(T_{\lambda(\text{rk}(T))}) * (n - \text{rk}(T))$ so

$$[\mu(T_{\lambda(n)})]_t! = [\mu(T_{\lambda(\text{rk}(T))})]_t! \cdot [n - \text{rk}(T)]_t!.$$

Putting this together gives

$$\begin{aligned}
& K_{T|_{\lambda(n)}}(q, t) \\
&= \frac{[\mu(T_{\lambda(n)})]_t!}{[n]_t!} \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(n)})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \\
&= [\mu(T_{\lambda(\text{rk}(T))})]_t! \frac{[n - \text{rk}(T)]_t!}{[n]_t!} \prod_{(\square_1, \square_2) \in \mathbf{I}(T|_{\lambda(\text{rk}(T))})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \prod_{\square \in \lambda(\text{rk}(T))} \left(\frac{1 - q^{-T(\square)} t^{n - |\lambda| - c(\square)}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)}} \right).
\end{aligned}$$

From here it is simple to see

$$\begin{aligned}
& \lim_{n \rightarrow \infty} K_{T|_{\lambda(n)}}(q, t) \\
&= \lim_{n \rightarrow \infty} [\mu(T_{\lambda(\text{rk}(T))})]_t! \frac{[n - \text{rk}(T)]_t!}{[n]_t!} \prod_{(\square_1, \square_2) \in \text{I}(T)|_{\lambda(\text{rk}(T))}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \prod_{\square \in \lambda(\text{rk}(T))} \left(\frac{1 - q^{-T(\square)} t^{n - |\lambda| - c(\square)}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)}} \right) \\
&= [\mu(T_{\lambda(\text{rk}(T))})]_t! \prod_{(\square_1, \square_2) \in \text{I}(T)|_{\lambda(\text{rk}(T))}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) \lim_{n \rightarrow \infty} \frac{[n - \text{rk}(T)]_t!}{[n]_t!} \prod_{\square \in \lambda(\text{rk}(T))} \left(\frac{1 - q^{-T(\square)} t^{n - |\lambda| - c(\square)}}{1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)}} \right) \\
&= [\mu(T_{\lambda(\text{rk}(T))})]_t! \prod_{(\square_1, \square_2) \in \text{I}(T)|_{\lambda(\text{rk}(T))}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right) (1 - t)^{\text{rk}(T)} \prod_{\square \in \lambda(\text{rk}(T))} \left(1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)} \right)^{-1} \\
&= \frac{(1 - t)^{\text{rk}(T)} [\mu(T_{\lambda(\text{rk}(T))})]_t!}{\prod_{\square \in \lambda(\text{rk}(T))} (1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)})} \prod_{(\square_1, \square_2) \in \text{I}(T)|_{\lambda(\text{rk}(T))}} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right).
\end{aligned}$$

□

We will now deal with the series side. For this we need the following lemmas. Here we write $|f(q, t)|$ for the t -adic norm of $f(q, t) \in \mathbb{Q}(q)((t))$ normalized so that $|t^n| = 2^{-n}$.

Lemma 6.9. For $a \neq 0$ and $b \in \mathbb{Z}$

$$\left| \frac{1 - q^a t^{b-1}}{1 - q^a t^{b+1}} \right| = \begin{cases} 1 & b \geq 1 \\ 2 & b = 0 \\ 4 & b \leq -1. \end{cases}$$

Proof. We proceed in cases. If $b \geq 1$ then

$$\left| \frac{1 - q^a t^{b-1}}{1 - q^a t^{b+1}} \right| = \frac{|1 - q^a t^{b-1}|}{|1 - q^a t^{b+1}|} = 1.$$

If $b = 0$,

$$\left| \frac{1 - q^a t^{-1}}{1 - q^a t} \right| = \left| t^{-1} \frac{-q^a + t}{1 - q^a t} \right| = |t^{-1}| \frac{|-q^a + t|}{|1 - q^a t|} = 2.$$

Lastly, if $b \leq -1$ then

$$\left| \frac{1 - q^a t^{b-1}}{1 - q^a t^{b+1}} \right| = \frac{|1 - q^a t^{b-1}|}{|1 - q^a t^{b+1}|} = \frac{2^{-b+1}}{2^{-b-1}} = 4.$$

□

Lemma 6.10. Let $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$. If $(\square_1, \square_2) \in \text{Inv}(\tau)$ with $c(\square_2) - c(\square_1) \leq 0$ then $\square_1, \square_2 \in \lambda^{(\text{rk}(T))}$.

Proof. Suppose $(\square_1, \square_2) \in \text{Inv}(\tau)$ with either $\square_1 \in \lambda^{(\infty)}/\lambda^{(\text{rk}(T))}$ or $\square_2 \in \lambda^{(\infty)}/\lambda^{(\text{rk}(T))}$. Then since $\lambda^{(\infty)}/\lambda^{(\text{rk}(T))}$ is a horizontal strip necessarily $\square_2 \in \lambda^{(\infty)}/\lambda^{(\text{rk}(T))}$ and $\square_1 \in \lambda^{(\text{rk}(T))}$. Thus $c(\square_2) \geq c(\square_1) + 1$. □

Using these lemmas gives the following:

Proposition 6.11. Let $T \in \Omega(\lambda)$. The sequence of sums

$$\left(\sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda(n)})} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right)_{n \geq n_\lambda}$$

converges with respect to the t -adic topology on $\mathbb{Q}(q)((t))$ to the series

$$\sum_{\tau \in \text{APSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \in \mathbb{Q}(q)((t)).$$

Proof. Our method will be to first verify that the above infinite series over $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$ is convergent in $\mathbb{Q}(q)((t))$ and then argue that the above sums over $\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})$ converge to the same element of $\mathbb{Q}(q)((t))$.

We begin by noting that from Lemma 6.10 we have the (sufficient but egregiously unoptimal) upper bound

$$\#\{(\square_1, \square_2) \in \text{Inv}(\tau) | c(\square_1) - c(\square_2) \leq -1\} \leq \binom{\text{rk}(T)}{2}.$$

Recall that if $T(\square_1) = T(\square_2)$ then by Lemma 5.11 $c(\square_2) - c(\square_1) \geq 2$. Thus using Lemma 6.9 we find

$$\left| \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \leq 4^{\binom{\text{rk}(T)}{2}}$$

and hence

$$\left| t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \leq 2^{-\text{inv}(\tau)} 4^{\binom{\text{rk}(T)}{2}}.$$

Recall that (from the strong triangle inequality) if $(f_m(q, t))_{m \geq 1}$ is any sequence in $\mathbb{Q}(q)((t))$ then the series $\sum_{m=0}^{\infty} f_m(q, t)$ is convergent in $\mathbb{Q}(q)((t))$ if and only if $\lim_{m \rightarrow \infty} |f_m(q, t)| = 0$. In turn, this is equivalent to the property that for every $r \geq 0$ there are only finitely many $m \geq 1$ with $|f_m(q, t)| \geq 2^{-r}$. From Corollary 6.6 we find that for any $r \geq 0$ there are only finitely many $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$ with

$$\text{inv}(\tau) \leq 2 \binom{\text{rk}(T)}{2} + r \implies 2^{-\text{inv}(\tau)} 4^{\binom{\text{rk}(T)}{2}} \geq 2^{-r}.$$

Thus there are only finitely many $\tau \in \text{APSYT}_{\geq 0}(\lambda; T)$ with

$$\left| t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \geq 2^{-r}.$$

We conclude that the series

$$S := \sum_{\tau \in \text{APSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right)$$

is convergent in $\mathbb{Q}(q)((t))$.

Now let $n \geq \text{rk}(T)$:

$$\begin{aligned} & \left| S - \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\ &= \left| \sum_{\substack{\tau \in \text{APSYT}_{\geq 0}(\lambda; T) \\ \text{rk}(\tau) > n}} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\ &\leq \max_{\substack{\tau \in \text{APSYT}_{\geq 0}(\lambda; T) \\ \text{rk}(\tau) > n}} \left| t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\ &\leq 2^{-(n+1 - \text{rk}(T))} 4^{\binom{\text{rk}(\tau)}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| S - \sum_{\tau \in \text{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\ & \leq \lim_{n \rightarrow \infty} 2^{-(n+1 - \text{rk}(T))} 4^{\binom{\text{rk}(T)}{2}} \\ & = 0. \end{aligned}$$

□

We immediately arrive at the following product-series formula:

Theorem 6.12. For $T \in \Omega(\lambda)$ we have the following equality in $\mathbb{Q}(q)((t))$:

$$\begin{aligned} & \frac{\prod_{\square \in \lambda(\text{rk}(T))} \left(1 - q^{-T(\square)} t^{\text{rk}(T) - |\lambda| - c(\square)} \right)}{(1-t)^{\text{rk}(T)} [\mu(T|_{\lambda(\text{rk}(T))})]_t!} \prod_{(\square_1, \square_2) \in \text{I}(\text{rk}(T))} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \\ & = \sum_{\tau \in \text{APSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right). \end{aligned}$$

Remark 15. Note that the powers of q appearing in the Theorem 6.12 are all non-positive i.e. the sum and product are elements of $\mathbb{Q}[q^{-1}](t)$. In particular, we may limit $q \rightarrow \infty$ to obtain the prod-sum equality in $\mathbb{Q}(t)$:

$$\begin{aligned} & \frac{\prod_{\substack{\square \in \lambda(\text{rk}(T)) \\ T(\square)=0}} \left(1 - t^{\text{rk}(T) - |\lambda| - c(\square)} \right)}{(1-t)^{\text{rk}(T)} [\mu(T|_{\lambda(\text{rk}(T))})]_t!} \prod_{\substack{(\square_1, \square_2) \in \text{I}(\lambda(\text{rk}(T))) \\ T(\square_1)=T(\square_2)}} \left(\frac{1 - t^{c(\square_2) - c(\square_1)}}{1 - t^{c(\square_2) - c(\square_1) + 1}} \right) \\ & = \sum_{\tau \in \text{APSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{\substack{(\square_1, \square_2) \in \text{Inv}(\tau) \\ T(\square_1)=T(\square_2)}} \left(\frac{1 - t^{c(\square_2) - c(\square_1) - 1}}{1 - t^{c(\square_2) - c(\square_1) + 1}} \right). \end{aligned}$$

By noting that the product term in Theorem 6.12 is a *finite* product of rational terms we observe the following:

Corollary 6.13. For $T \in \Omega(\lambda)$,

$$\sum_{\tau \in \text{APSYT}_{\geq 0}(\lambda; T)} t^{\text{inv}(\tau)} \prod_{(\square_1, \square_2) \in \text{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \in \mathbb{Q}(q, t).$$

Example 10. Here we give a few simple examples of this (q, t) identity. Consider $\lambda = \emptyset$ and $T =$

1	0	0
---	---	---

$\dots \in \Omega(\emptyset)$. Then we get

$$\frac{1 - q^{-1}t}{1 - t} = \sum_{k=0}^{\infty} t^k \prod_{j=1}^k \left(\frac{1 - q^{-1}t^{j-1}}{1 - q^{-1}t^{j+1}} \right).$$

Now consider $\lambda = (1)$ and $T = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & & \\ \hline \end{array} \dots \in \Omega(1)$. In this case we get

$$\frac{(1 - q^{-1}t)(1 - t^2)(1 - q^{-1}t^{-1})}{(1 - t)^2(1 - q^{-1})} = \sum_{i,j=1}^{\infty} t^{i+j-3} \prod_{k=1}^{i-2} \left(\frac{1 - q^{-1}t^{k-1}}{1 - q^{-1}t^{k+1}} \right) \prod_{k=2}^{j-1} \left(\frac{1 - t^{k-1}}{1 - t^{k+1}} \right) \left(\mathbb{1}(j \leq i - 1)t \left(\frac{1 - q^{-1}t^{-2}}{1 - q^{-1}} \right) + \mathbb{1}(i + 1 \leq j) \left(\frac{1 - q^{-1}}{1 - q^{-1}t^2} \right) \right).$$

References

- [Ber+99] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler. “Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions”. In: vol. 6. 3. Dedicated to Richard A. Askey on the occasion of his 65th birthday, Part III. 1999, pp. 363–420. DOI: 10.4310/MAA.1999.v6.n3.a7. URL: <https://doi.org/10.4310/MAA.1999.v6.n3.a7>.
- [BG99] F. Bergeron and A. M. Garsia. “Science fiction and Macdonald’s polynomials”. In: *Algebraic methods and q-special functions (Montréal, QC, 1996)*. Vol. 22. CRM Proc. Lecture Notes. Amer. Math. Soc., Providence, RI, 1999, pp. 1–52. DOI: 10.1090/crpm/022/01. URL: <https://doi.org/10.1090/crpm/022/01>.
- [BS12] Igor Burban and Olivier Schiffmann. “On the Hall algebra of an elliptic curve, I”. In: *Duke Math. J.* 161.7 (2012), pp. 1171–1231. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-1593263. URL: <https://doi.org/10.1215/00127094-1593263>.
- [CDL22] Laura Colmenarejo, Charles F. Dunkl, and Jean-Gabriel Luque. “Connections between vector-valued and highest weight Jack and Macdonald polynomials”. In: *Ann. Inst. Henri Poincaré D* 9.2 (2022), pp. 297–348. ISSN: 2308-5827,2308-5835. DOI: 10.4171/aihpd/119. URL: <https://doi.org/10.4171/aihpd/119>.
- [Che03] Ivan Cherednik. “Double affine Hecke algebras and difference Fourier transforms”. In: *Invent. Math.* 152.2 (2003), pp. 213–303. ISSN: 0020-9910. DOI: 10.1007/s00222-002-0240-0. URL: <https://doi.org/10.1007/s00222-002-0240-0>.
- [DL11] C. F. Dunkl and J.-G. Luque. “Vector valued Macdonald polynomials”. In: *Sém. Lothar. Combin.* 66 (2011/12), Art. B66b, 68. ISSN: 1286-4889.
- [Dun19] Charles F. Dunkl. “A positive-definite inner product for vector-valued Macdonald polynomials”. In: *Sém. Lothar. Combin.* 80 ([2019–2021]), Art. B80a, 26. ISSN: 1286-4889.
- [Fei+11] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. “Quantum continuous gl_{∞} : semiinfinite construction of representations”. In: *Kyoto J. Math.* 51.2 (2011), pp. 337–364. DOI: 10.1215/21562261-1214375.
- [Fei+12] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. “Quantum toroidal gl_1 -algebra: plane partitions”. In: *Kyoto J. Math.* 52.3 (2012), pp. 621–659. DOI: 10.1215/21562261-1625217.
- [Hai01] Mark Haiman. “Hilbert schemes, polygraphs and the Macdonald positivity conjecture”. In: *J. Amer. Math. Soc.* 14.4 (2001), pp. 941–1006. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-01-00373-3. URL: <https://doi.org/10.1090/S0894-0347-01-00373-3>.
- [HHL08] J. Haglund, M. Haiman, and N. Loehr. “A combinatorial formula for nonsymmetric Macdonald polynomials”. In: *Amer. J. Math.* 130.2 (2008), pp. 359–383. ISSN: 0002-9327. DOI: 10.1353/ajm.2008.0015. URL: <https://doi.org/10.1353/ajm.2008.0015>.
- [Mac88] Ian Macdonald. “A New Class of Symmetric Functions”. In: *Sém. Lothar. Combin.* B20a (1988).

- [Mur38] F. D. Murnaghan. “The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group”. In: *Amer. J. Math.* 60.3 (1938), pp. 761–784. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2371610. URL: <https://doi.org/10.2307/2371610>.
- [SV13] Olivier Schiffmann and Eric Vasserot. “The elliptic Hall algebra and the K -theory of the Hilbert scheme of \mathbb{A}^2 ”. In: *Duke Math. J.* 162.2 (2013), pp. 279–366. ISSN: 0012-7094. DOI: 10.1215/00127094-1961849. URL: <https://doi.org/10.1215/00127094-1961849>.