# RIGIDITY MATROIDS AND LINEAR ALGEBRAIC MATROIDS WITH APPLICATIONS TO MATRIX COMPLETION AND TENSOR CODES 

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#### Abstract

We establish a connection between problems studied in rigidity theory and matroids arising from linear algebraic constructions like tensor products and symmetric products. A special case of this correspondence identifies the problem of giving a description of the correctable erasure patterns in a maximally recoverable tensor code with the problems of describing bipartite rigid graphs or low-rank completable matrix patterns. Additionally, we relate dependencies among symmetric products of generic vectors to graph rigidity and symmetric matrix completion. With an eye toward applications to computer science, we study the dependency of these matroids on the characteristic by giving new combinatorial descriptions in several cases, including the first description of the correctable patterns in an ( $m, n, a=2, b=2$ ) maximally recoverable tensor code.


## 1. Introduction

Given a graph $G$, the graph rigidity problem in $\mathbb{R}^{d}$ asks whether a generic embedding of the vertices of $G$ into $\mathbb{R}^{d}$ is rigid, i.e., whether every motion of $G$ which preserves the lengths of edges comes from a rigid motion of $\mathbb{R}^{d}$. This problem has been studied since the time of Maxwell Max64]. When $d=2$, the Pollaczek-Geiringer-Laman theorem PG27,Lam70 gives a simple characterization of rigid graphs. There is no known generalization to embeddings of graphs into $\mathbb{R}^{d}$ for any $d \geq 3$.

The rigid graphs on $n$ vertices form the spanning sets of a matroid on the ground set $\binom{[n]}{2}$, the set of edges of the complete graph with vertex set $[n]=\{1, \ldots, n\}$. The language of matroids gives a convenient framework and powerful tools for analyzing rigidity CJT22a, CJT22b, Gra91, Whi96. We refer to Oxl11. for undefined matroid terminology.

Several other matroids related to rigidity have been introduced, such as Kalai's hyperconnectivity matroid Kal85] $\mathcal{H}_{n}(d)$, for $0 \leq d \leq n$. This is a matroid whose vertex set is $\binom{[n]}{2}$ which has similar formal properties to the usual graph rigidity matroid. Kalai used the hyperconnectivity matroid to show the existence of highly connected subgraphs in graphs with a large number of edges. Hyperconnectivity was used to study polytopal realizations of certain simplicial spheres called higher associahedra CRS22a, CRS22b, CR23.

More recently introduced is the bipartite rigidity matroid $\mathcal{B}_{m, n}(a, b)$ of Kalai, Nevo, and Novak KNN16. This is a matroid on $[m] \times[n]$ which gives a version of rigidity for an embedding of a bipartite graph, where the parts have size $m$ and $n$, into $\mathbb{R}^{a} \oplus \mathbb{R}^{b}$ in such a way that it respects the direct sum structure. When $a=b$, the restriction of the hyperconnectivity matroid to the set of bipartite graphs coincides with the bipartite rigidity matroid. See Section 2.1 for precise definitions of these matroids.

The rigidity matroids mentioned above are closely related to low-rank rank matrix completion matroids. Given $d$ and $n$, the symmetric matrix completion matroid $\mathcal{S}_{n}(d)$ is a matroid on ground set $\binom{[n]}{2} \sqcup[n]$. A subset $S$ is independent if a matrix where the entries corresponding to $S$ have been filled in with generic complex numbers can be completed to a symmetric matrix of rank at most $d$. This is an algebraic matroid realized by the variety of $n \times n$ symmetric matrices of rank at most $d$. By [GS18, Theorem 2.4], the matroid describing graph rigidity in $\mathbb{R}^{d-1}$ is obtained by contracting the elements corresponding to the diagonal in $\mathcal{S}_{n}(d)$. In particular, a description of $\mathcal{S}_{n}(d)$ gives a description of graph rigidity in $\mathbb{R}^{d-1}$. The symmetric matrix completion matroid has been studied in connection with maximum likelihood problems in algebraic statistics BS19, BBL21. When $d=2$, it was studied from the perspective of tropical geometry in CLY24.

Similarly, there is a matrix completion matroid which describes when an $m \times n$ matrix which has been partially filled in with generic complex numbers can be completed to a matrix of rank at most $d$. The matrix completion matroid was studied in Ber17, Tsa24. This matroid is equal to the bipartite rigidity matroid $\mathcal{B}_{m, n}(d, d)$ SC09, Section 4]. As skew-symmetric matrices have even rank, one can only consider rank $d$ skew-symmetric matrix completion when $d$ is even. In this case, the analogously defined skew-symmetric matrix completion matroid coincides with $\mathcal{H}_{n}(d)$ CRS23, Proposition 3.1].

We will establish a connection between the above rigidity matroids and matroids arising from natural linear algebraic constructions. Suppose that we have $n$ generic vectors $v_{1}, \ldots, v_{n}$ in an $r$-dimensional vector space $V$ over an infinite field of characteristic $p \geq 0$. We assume $n \geq r$, so the vectors span $V$. Then we obtain $\binom{n+1}{2}$ vectors $v_{1}^{2}, v_{1} v_{2}, \ldots, v_{n-1} v_{n}, v_{n}^{2}$ in $\operatorname{Sym}^{2} V$. These vectors represent a matroid $\mathrm{S}_{n}(r, p)$ on the ground set $\binom{[n]}{2} \sqcup[n]$. This matroid depends only on the characteristic of the field, and in particular does not depend on the choice of vectors (provided they are sufficiently generic). We call this the symmetric power matroid.

Similarly, we obtain $\binom{n}{2}$ vectors $v_{1} \wedge v_{2}, \ldots, v_{n-1} \wedge v_{n}$ in $\wedge^{2} V$. These vectors represent a matroid $\mathrm{W}_{n}(r, p)$, which we call the wedge power matroid. If we also have $m$ generic vectors $u_{1}, \ldots, u_{m}$ in an $s$-dimensional vector space $U$ over the same field, for some $m \geq s$, then we obtain $m n$ vectors $u_{1} \otimes v_{1}, \ldots, u_{m} \otimes v_{n}$ in $U \otimes V$. These vectors represent a matroid $\mathrm{T}_{m, n}(s, r, p)$, which we call the tensor matroid. As with the symmetric power matroids, these matroids depend only on the characteristic of the field.

Similar constructions for arbitrary matroids have been been studied classically in the matroid literature Lov77.Mas81.LV81. The case of symmetric powers has attracted particular attention recently in connection with applications to tropical geometry [DR21,And23.

|  | Bipartite | Symmetric | Skew-symmetric |
| :---: | :---: | :---: | :---: |
| Rigidity matroids | Bipartite rigidity matroid | Graph rigidity matroid | Hyperconnectivity matroid |
|  | $\mathcal{B}_{m, n}(a, b)$ | $\mathcal{S}_{n}(d+1) /\{$ diags $\}$ | $\mathcal{H}_{n}(d)$ |
| Rank completion | Matrix completion matroid | Sym. matrix completion | Skew-sym. matrix completion |
| matroids | $\mathcal{B}_{m, n}(d, d)$ | $\mathcal{S}_{n}(d)$ | $\mathcal{H}_{n}(d), d$ is even |
| Linear algebraic | Tensor matroid | Sym. power matroid | Wedge power matroid |
| matroids | $\mathrm{T}_{m, n}(s, r, p)$ | $\mathrm{S}_{n}(r, p)$ | $\mathrm{W}_{n}(r, p)$ |

Table 1. A table of various matroids that have been mentioned. The matroids in the same column are related by Theorem 1.1.

Connections to Information Theory. The problem of understanding $\mathrm{T}_{m, n}(s, r, p)$ has been studied extensively in information theory in connection with tensor codes (c.f., e.g., MLR ${ }^{+} 14$ for a practical implementation). Consider a collection of servers arranged in an $m \times n$ grid. We view the data stored on each server as an element of some (large) finite field $\mathbf{k}$ of characteristic $p$. To ensure redundancy in this cluster, we constrain that each column lies in some fixed subspace of $\mathbf{k}^{m}$ of dimension $s$ and each row lies in some fixed subspace of $\mathbf{k}^{n}$ of dimension $r$ The goal of this redundancy is that if a (small) subset of the servers fail (also called erasures), we can completely recover the data using these various subspace constraints. The matroid of failures which which do not lead to data loss is sensitive to the choice of subspaces, but if everything is chosen generically, the "recoverability matroid" is the matroid dual of $\mathrm{T}_{m, n}(s, r, p)$.

[^0]Gopalan et al. $\mathrm{GHK}^{+} 17$ coined the term $(m, n, a, b)$-maximally recoverable tensor codes for realizations of $\mathrm{T}_{m, n}(m-a, n-b, p)$ as tensor products of vectors spaces over finite fields. Gopalan et al. gave an exponentialsized description of the spanning sets of $\mathrm{T}_{m, n}(s, r, p)$ when $m-s=1$ and conjectured a description in general. This conjecture was partially confirmed SRLS18 but was later refuted in general HPYWZ21. Overall, the focus of the information theory community has been on constructing maximally recoverable tensor codes over small fields (e.g., KMG21, HPYWZ21, Rot22, SL22, BDG23b, ACKL23), although exponential-sized lower bounds are now known BDG23b AGL24. For applications, it is important to understand the matroid realized by the tensor products of vectors which are not completely generic. For example, BDG23a, Section 1.3] asks if the tensors products of generic vectors on the moment curve give a maximally recoverable tensor code.

Within information theory, the study of higher order maximum-distance separable (MDS) codes BGM22, Rot22, BGM23 has shown that, in the $m-s=1$ case, essentially the same matroid arises in many other problems. Such scenarios include designing optimally list-decodable codes [ST23] and codes realizing particular zero patterns DSY14,YH19, Lov21,YH19, LWWZS23,BDG23a. Further, these equivalences imply that a construction of any one of these types of codes can be converted into the other types [BGM23], and they have led to many new constructions and analyses of near-optimal codes GZ23, AGL23, AGL24, BDGZ23, RZVW24. Very recently, these connections also led to a novel proof of the Pollaczek-Geiringer-Laman theorem BELL23].

Our first main result is a correspondence between the above linear algebraic matroids and rigidity matroids. Recall that the dual of a matroid M is the matroid whose bases are the complements of the bases of M .

Theorem 1.1. We have the following matroid dualities:
(1) The symmetric power matroid $\mathrm{S}_{n}(n-d, 0)$ is dual to the symmetric matrix completion matroid $\mathcal{S}_{n}(d)$.
(2) The wedge power matroid $\mathrm{W}_{n}(n-d, 0)$ is dual to the hyperconnectivity matroid $\mathcal{H}_{n}(d)$.
(3) The tensor matroid $\mathrm{T}_{m, n}(m-a, n-b, 0)$ is dual to the bipartite rigidity matroid $\mathcal{B}_{m, n}(a, b)$.

As a corollary, the correctable erasure patterns in a $m \times n$ maximally recoverable tensor code with $a$ column and $b$ row parity checks over a field of sufficiently large characteristic are precisely the independent sets in the bipartite rigidity matroid $\mathcal{B}_{m, n}(a, b)$.

Motivated by applications to information theory, we use Theorem[1.1(3) to study bipartite rigidity. This strategy is well-suited to understanding the case when $m-a$ is small. We give exact characterizations of the independent sets in $\mathrm{T}_{m, n}(s, r, p)$ for all $p$ when $s \leq 3$, see Section 3. When $m-a \leq 3$, this gives a characterization of the spanning sets in $\mathcal{B}_{m, n}(a, b)$. In particular, we are able to prove the following theorem.

Theorem 1.2. If $m-a \leq 3$, then there is an algorithm to compute the rank function of $\mathcal{B}_{m, n}(a, b)$ which runs in time polynomial in $m+n$.

Note that there are a few polynomial time algorithms to compute the rank function of $\mathcal{B}_{m, n}(a, b)$ when $a=1$, including ones based on the maximum flow problem [BGM22, total dual integral programs BGM23, and invariant theory BGM23. See also Whi89].

The rank of $\mathcal{B}_{m, n}(a, b)$ is $a n+b m-a b$. In particular, any graph with more than $a n+b m-a b$ edges must be dependent in $\mathcal{B}_{m, n}(a, b)$. Furthermore, for each subset $S \subseteq[m]$ and $T \subseteq[n]$, if we restrict $\mathcal{B}_{m, n}(a, b)$ to the edges $S \times T$, then we obtain $\mathcal{B}_{|S|,|T|}(a, b)$. See [KNN16, Lemma 3.7]. In particular, if $G$ is independent in $\mathcal{B}_{m, n}(a, b)$ and $|S| \geq a,|T| \geq b$, then $G$ must have at most $|T| a+|S| b-a b$ edges in $S \times T$. This observation gives rise to the following family of circuits, i.e., minimal dependent sets.

Definition 1.3. A circuit $C$ of $\mathcal{B}_{m, n}(a, b)$ is a Laman circuit if the edges of $C$ are contained in $S \times T \subseteq$ $[m] \times[n]$, and $C$ has more than $|T| a+|S| b-a b$ edges.

The name "Laman circuits" comes from the Pollaczek-Geiringer-Laman theorem PG27, Lam70, which shows that all circuits of the graph rigidity matroid in $\mathbb{R}^{2}$ are Laman circuits. Laman circuits are called "regularity" conditions in the information theory literature $\mathrm{GHK}^{+} 17$. These authors showed that the Laman circuits completely describe $\mathrm{T}_{m, n}(m-a, n-b, p)$ when $a=1$. For all three rigidity problems mentioned above, there are typically circuits which are not Laman circuits. Theorem 1.2 allows us to characterize exactly when all circuits of $\mathcal{B}_{m, n}(a, b)$ are Laman circuits.

Example 1.4. [KNN16, Example 5.5][HPYWZ21, Lemma 4] The subset of [5] $\times[5]$ depicted as blue $\star$ in Figure 1 is a circuit of $\mathcal{B}_{5,5}(2,2)$ which is not a Laman circuit. This is most easily seen from the perspective of low-rank matrix completion. If we fill in the entries corresponding to the elements of the circuit with generic complex numbers, then there will be two conflicting conditions on the $(3,3)$ entry. The complement of this circuit is dependent in $\mathrm{T}_{5,5}(3,3, p)$ for any $p$, as two $2 \times 2$-dimensional tensors in a $3 \times 3$-dimensional space necessarily intersect.


Figure 1. A circuit of $\mathcal{B}_{5,5}(2,2)$ which is not a Laman circuit. The squares of the $5 \times 5$ grid represent the ground set of the matroid, with the blue $\star$ squares representing the circuit elements. The red $\diamond$ squares form the corresponding circuit in $\mathrm{T}_{5,5}(3,3, p)$ for any $p$.

Corollary 1.5. All circuits in $\mathcal{B}_{m, n}(a, b)$ are Laman circuits if and only if at least one of the following holds: (1) $a \leq 1$, (2) $b \leq 1$, (3) $m-a \leq 2$, or (4) $n-b \leq 2$.

The case when $a \in\{0, m\}$ or $b \in\{0, n\}$ is trivial, and the case when $a=1$ or $b=1$ was proven in Whi89. and independently in GHK ${ }^{+17}$. Note that we are able to describe $\mathcal{B}_{m, n}(a, b)$ when $m-a=3$ even though the Laman condition usually fails. To do this, we find an additional family of combinatorial inequalities which rule out circuits such as the one in Example 1.4.

Applications to information theory make use of $\mathrm{T}_{m, n}(s, r, p)$ when $p>0$. While the bases of $\mathrm{T}_{m, n}(s, r, p)$ are bases of $\mathrm{T}_{m, n}(s, r, 0)$ (Proposition 4.1), it is not obvious that these matroids are equal. Indeed, And23, Example 2.18] shows that $\mathrm{S}_{4}(2,2) \neq \mathrm{S}_{4}(2,0)$, so the symmetric power matroid depends on the characteristic. Some of the proofs of the Pollaczek-Geiringer-Laman theorem, such as the ones in LY82, BELL23, show that the rank of the matrices considered in 2-dimensional graph rigidity do not depend on the characteristic. We show that the tensor matroid does not depend on the characteristic in several case.
Theorem 1.6. If $s \leq 3, m-s \leq 1$, or $m-s=n-r=2$, then $\mathrm{T}_{m, n}(s, r, p)=\mathrm{T}_{m, n}(s, r, 0)$.

One of the deepest results on bipartite rigidity is the following theorem of Bernstein. The proof crucially uses the interpretation of bipartite rigidity in terms of low-rank matrix completion to reformulate the problem in terms of tropical geometry. Bernstein then uses several ingenious ideas to obtain the following combinatorial characterization of the independent sets of the $(2,2)$ bipartite rigidity matroid.
Theorem 1.7. Ber17] A bipartite graph is independent in $\mathcal{B}_{m, n}(2,2)$ if and only if it has an edge orientation with no directed cycles or alternating cycles.

That is, $G$ is independent in $\mathcal{B}_{m, n}(2,2)$ if and only if $G$ has an acyclic orientation such that no cycle of $G$ is oriented so that the edges alternate in orientation. As part of our proof of Theorem 1.6, we give an elementary proof (Proposition 4.2) of the sufficiency part of Theorem 1.7, i.e., if a bipartite graph $G$ has an edge orientation with no directed cycles or alternating cycles, then it is independent in $\mathcal{B}_{m, n}(2,2)$. Unlike Bernstein's original proof, our argument establishes the stronger statement that $G$ is independent in the dual of $\mathrm{T}_{m, n}(m-2, n-2, p)$ for any $p$. Together with Proposition4.1, this proves the independence of the characteristic in this case.

As a consequence, we have a combinatorial description of correctable patterns in an ( $m, n, a=2, b=2$ ) maximally recoverable tensor code.

Proposition 1.8. Let $C \subseteq \mathbf{k}^{m \times n}$ be an $(m, n, a=2, b=2)$ maximally recoverable tensor code. An erasure pattern $E \subseteq C$ is correctable if and only if $E$, when viewed as a bipartite graph, has an edge orientation with no directed cycles or alternating cycles.

Despite the combinatorial nature of the description in Theorem 1.7 we do not know a polynomial time algorithm to check independence in $\mathcal{B}_{m, n}(2,2)$. We do not even know a coNP certificate, i.e., a certificate that a graph is not independent in $\mathcal{B}_{m, n}(2,2)$, that can be checked in polynomial time. A candidate coNP certificate for independence in $\mathcal{B}_{m, n}(a, b)$ is given in JT24, Conjecture 6.4].

Finally, we give a conjectural description of the bases of $\mathcal{B}_{m, n}(d, d)$ for all $d$, generalizing Theorem 1.7 (Conjecture 5.4). Using a "coning" operation (Proposition 3.10), this gives a description of $\mathcal{B}_{m, n}(a, b)$ for all $a, b$. We show that our conjecture implies that $\mathrm{T}_{m, n}(s, r, p)$ is independent of $p$.

| Cases | Description of $\mathrm{T}_{m, n}(s, r, p)$ |
| :---: | :---: |
| $s=1$ or $r=1$ | Proposition [3.2 |
| $s=2$ or $r=2$ | Proposition 3.3 |
| $s=3$ or $r=3$ | Proposition 3.4] |
| $m-s=1$ or $n-r=1$ | Whi89, Theorem 4.2] |
| GHK ${ }^{+}$17, Theorem 3.2] |  |
| $m-s=n-r=2$ | Ber17, Theorem 4.4] for $p=0$ |
| Proposition 4.2 for $p>0$ |  |

TABLE 2. Currently known cases of the structure of the matroid $\mathrm{T}_{m, n}(s, r, p)$.

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## 2. Rigidity matroids and their duals

In this section, we recall the definitions of the rigidity matroids. We then prove Theorem 1.1

### 2.1. Rigidity matroids.

2.1.1. Symmetric matrix completion. For $0 \leq d \leq n$, the symmetric matrix completion matroid $\mathcal{S}_{n}(d)$ is the algebraic matroid realized by the $\binom{n}{2}+n$ coordinate functions on the variety of $n \times n$ symmetric matrices of rank at most $d$. More precisely, let $K$ be the field of rational functions on the variety of $n \times n$ symmetric matrices of rank at most $d$ over $\mathbb{C}$. For each $i \leq j$, we have a coordinate function $x_{i j} \in K$. A subset $S \subseteq\binom{[n]}{2} \sqcup[n]$ is independent in $\mathcal{S}_{n}(d)$ if and only if the corresponding set of coordinate functions is algebraically independent. The dimension of the variety of $n \times n$ symmetric matrices of rank at most $d$ is $n d-\binom{d}{2}$, so the rank of $\mathcal{S}_{n}(d)$ is $n d-\binom{d}{2}$.

The restriction of $\mathcal{S}_{n}(d)$ to $\binom{[n]}{2}$ is the matroid denoted $\mathcal{I}_{n}^{d}$ in [JT24, Section 6.3], which was introduced in [Kal85, Section 8]. If $n=m+p$, then the restriction of $\mathcal{S}_{n}(d)$ to $[m] \times[p] \subseteq\binom{[n]}{2}$ is $\mathcal{B}_{m, p}(d, d)$. This is most easily seen using the description of $\mathcal{B}_{m, p}(d, d)$ as a matrix completion matroid: the projection of the variety of $n \times n$ symmetric matrices of rank at most $d$ onto the northeast $m \times p$ corner is the variety of $m \times p$ matrices of rank at most $d$.

We use a description of $\mathcal{S}_{n}(d)$ which was derived in KRT13, Section 3.2]. See Section 2.3 for a discussion of an analogous calculation.

Proposition 2.1. Consider the $n d \times\left(\binom{n}{2}+n\right)$ matrix $J_{\text {Sym }}$ over the field $\mathbb{C}\left(x_{i j}\right)_{1 \leq i \leq n, 1 \leq j \leq d}$ whose rows are labeled by pairs $(i, j)$ with $1 \leq i \leq n, 1 \leq j \leq d$, and whose columns are labeled by either $\{k, \ell\} \in\binom{[n]}{2}$ or $k \in n$. In each row labeled by $(i, j)$, we have an entry of $2 x_{i j}$ in the column labeled by $i \in[n]$, and we have an entry of $x_{k j}$ in the column labeled by $\{i, k\} \in\binom{[n]}{2}$. The other entries are 0 . A subset $S \subseteq\binom{[n]}{2} \sqcup[n]$ is independent in $\mathcal{S}_{n}(d)$ if and only the columns labeled by $S$ in $J_{\text {Sym }}$ are linearly independent.

In other words, $\mathcal{S}_{n}(d)$ is the column matroid of $J_{\text {Sym }}$.
2.1.2. Hyperconnectivity. For $0 \leq d \leq n$, the hyperconnectivity matroid $\mathcal{H}_{n}(d)$ is a matroid on $\binom{[n]}{2}$ of rank $d n-\binom{d+1}{2}$. It was defined in Kal85 in terms of algebraic shifting. It can equivalently be defined as a column matroid of an explicit matrix, see Definition 2.2 below. If $n=m+p$, then the restriction of $\mathcal{H}_{n}(d)$ to $[m] \times[p] \subseteq\binom{[n]}{2}$ is $\mathcal{B}_{m, p}(d, d)$.

Definition 2.2. Consider the $n d \times\binom{ n}{2}$ matrix over the field $\mathbb{C}\left(x_{i j}\right)_{1 \leq i \leq n, 1 \leq j \leq d}$ whose rows are labeled by pairs $(i, j)$ with $1 \leq i \leq n, 1 \leq j \leq d$, and whose columns are labeled by $\{k, \ell\} \in\binom{[n]}{2}$. The row corresponding corresponding to $(i, j)$ has $x_{k i}$ in the column labeled by $\{k, j\}$ if $k<j,-x_{k i}$ in the column labeled by $\{k, j\}$ if $k>j$, and is 0 otherwise. The hyperconnectivity matroid $\mathcal{H}_{n}(d)$ is the column matroid of this matrix.
2.1.3. Bipartite rigidity. For $0 \leq a \leq m$ and $0 \leq b \leq n$, the bipartite rigidity matroid $\mathcal{B}_{m, n}(a, b)$ is a matroid on $[m] \times[n]$ of rank $a n+b m-a b$. It was introduced in KNN16. See KNN16, Section 1.2] for a physical interpretation of bipartite rigidity in terms of embeddings of bipartite graphs on $[m] \sqcup[n]$ into $\mathbb{R}^{a} \oplus \mathbb{R}^{b}$, where $[m]$ is embedded into $\mathbb{R}^{a} \oplus 0$ and $[n]$ is embedded into $0 \oplus \mathbb{R}^{b}$. It can equivalently be defined as the column matroid of an explicit matrix KNN16, Proposition 3.3].

Definition 2.3. Consider the $(a n+b m) \times m n$ matrix over the field $\mathbb{C}\left(x_{i j}, y_{k \ell}\right)$, where $(i, j) \in[m] \times[a]$ and $(k, \ell) \in[n] \times[b]$, whose rows are labeled by elements of $[a] \times[n]$ or $[b] \times[m]$, and whose columns are labeled by elements of $[m] \times[n]$. In the row labeled by $(j, k) \in[a] \times[n]$, we have $x_{p j}$ in the column labeled by $(p, k)$ for $k \in[m]$ and 0 in every other column. In the row labeled by $(\ell, i) \in[b] \times[m]$, we have $y_{p \ell}$ in the column labeled by $(p, i)$ for $i \in[n]$ and 0 in every other column. The bipartite rigidity matroid $\mathcal{B}_{m, n}(a, b)$ is the column matroid of this matrix.
2.2. Dualities. In this section, prove Theorem 1.1 For this, it is convenient to pass to the dual picture. Suppose we have a collection of $n$ vectors in an $d$-dimensional vector space $L$. If we choose a basis for $L$, then we obtain an $d \times n$ matrix $A$. A collection of vectors is independent in the matroid represented by the vector configuration if and only if the corresponding columns of $A$ are linearly independent. Whether a given set of columns is linearly independent depends only on the row span of $A$. In particular, we can replace $A$ by any matrix with the same row span (even if its rows are linearly dependent).

Using this formulation, we can describe $\mathrm{S}_{n}(r, p)$ as follows. Choose an infinite field $K$ of characteristic $p \geq 0$, and choose a generic linear subspace $L \subseteq K^{n}$ of dimension $r$. We then obtain a subspace $\operatorname{Sym}^{2} L \subseteq$ $\operatorname{Sym}^{2} K^{n}$. We have a canonical basis for $\operatorname{Sym}^{2} K^{n}$, given by the vectors $e_{1}^{2}, e_{1} e_{2}, \ldots, e_{n}^{2}$, where $e_{1}, \ldots, e_{n}$ is the standard basis of $K^{n}$. To compute the independent sets of $\mathrm{S}_{n}(r, p)$, we choose $m$ vectors in $\operatorname{Sym}^{2} K^{n}$ whose span is $\operatorname{Sym}^{2} L$, form the corresponding $m \times\binom{ n+1}{2}$ matrix, and check which columns are linearly independent. To compute the dual of $\mathrm{S}_{n}(r, p)$, we choose vectors which span $\left(\operatorname{Sym}^{2} L\right)^{\perp} \subseteq \operatorname{Sym}^{2} K^{n}$, where the orthogonal complement is taken with respect to the usual inner product on a vector space with a basis. We can calculate $\mathrm{W}_{n}(r, p)$ and $\mathrm{T}_{m, n}(s, r, p)$ in a similar way.

Proof of Theorem 1.1 (1). As $\mathrm{S}_{n}(n-d, 0)$ is independent of the choice of infinite field of characteristic 0 , we may work over $K=\mathbb{C}\left(x_{i j}\right)_{1 \leq i \leq n, 1 \leq j \leq d}$ and choose our generic linear subspace $L \subseteq K^{n}$ to be the orthogonal complement of the span of the vectors $\left(x_{11}, \ldots, x_{n 1}\right), \ldots,\left(x_{1 d}, \ldots, x_{n d}\right)$. In order to compute the dual of $\mathrm{S}_{n}(n-d, 0)$, we find vectors which span $\left(\mathrm{Sym}^{2} L\right)^{\perp}$. There is a surjective map $L^{\perp} \otimes K^{n} \rightarrow\left(\operatorname{Sym}^{2} L\right)^{\perp} \subseteq$ $\operatorname{Sym}^{2} K^{n}$ which sends $v \otimes w$ to $v w$. There is a basis for $L^{\perp} \otimes K^{n}$ given by vectors of the form $\left(x_{1 j}, \ldots, x_{n j}\right) \otimes e_{i}$ for $1 \leq i \leq n$ and $1 \leq j \leq d$. We form the matrix $A$ whose rows are given by the images of these vectors in $\operatorname{Sym}^{2} K^{n}$, written in the basis $e_{1}^{2}, e_{1} e_{2}, \ldots, e_{n}^{2}$. The dual of $\mathrm{S}_{n}(n-d, 0)$ records which columns of $A$ are linearly independent. We note that we obtain the matrix $J_{\text {Sym }}$ of Proposition 2.1 from $A$ after multiplying the columns labeled by $i \in[n]$ by 2 , proving the equivalence.

Proof of Theorem 1.1 (2). We work over $K=\mathbb{C}\left(x_{i j}\right)_{1 \leq i \leq n, 1 \leq j \leq d}$, and choose our generic linear subspace $L \subseteq K^{n}$ to be the orthogonal complement of the span of the vectors $\left(x_{11}, \ldots, x_{n 1}\right), \ldots,\left(x_{1 d}, \ldots, x_{n d}\right)$. There is a surjective map $L^{\perp} \otimes K^{n} \rightarrow\left(\wedge^{2} L\right)^{\perp} \subseteq \wedge^{2} K^{n}$. Using the basis $\left\{\left(x_{1 j}, \ldots, x_{n j}\right) \otimes e_{i}\right\}_{1 \leq i \leq n, 1 \leq j \leq d}$ for $L^{\perp} \otimes K^{n}$ and the basis $\left\{e_{i} \wedge e_{j}: i<j\right\}$ for $\wedge^{2} K^{n}$, we see that $\left(\wedge^{2} L\right)^{\perp}$ is the row span of the matrix appearing in Definition 2.2.

Proof of Theorem 1.1(3). We work over $K=\mathbb{C}\left(x_{i j}, y_{k \ell}\right)$, where $(i, j) \in[m] \times[a]$ and $(k, \ell) \in[n] \times[b]$. Set $L_{1} \subseteq K^{m}$ to be the orthogonal complement to the span of the vectors $\left(x_{11}, \ldots, x_{m 1}\right), \ldots,\left(x_{1 a}, \ldots, x_{m a}\right)$, and set $L_{2} \subseteq K^{n}$ to be the orthogonal complement to the span of the vectors $\left(y_{11}, \ldots, y_{n 1}\right), \ldots,\left(y_{1 b}, \ldots, y_{n b}\right)$. There is a surjective map $\left(L_{1}^{\perp} \otimes K^{n}\right) \oplus\left(K^{m} \otimes L_{2}^{\perp}\right) \rightarrow\left(L_{1} \otimes L_{2}\right)^{\perp} \subseteq K^{m} \otimes K^{n}$. This implies that $\left(L_{1} \otimes L_{2}\right)^{\perp}$ is the row span of the matrix appearing in Definition 2.3.

Example 2.4. Using Theorem 1.1 a special case of Kal02, Problem 3] which is given as a conjecture in CRS23, Conjecture 4.3] becomes the following. Let $G$ be a graph on $n$ vertices, and suppose that $E(G)$ is independent in $\mathrm{W}_{n}(r, 0)$. Then $E(G)$ is independent in $\mathrm{S}_{n}(r-1,0)$. We checked this for $r \leq 6$.
2.3. Matrix completion in positive characteristic. As mentioned in the introduction, the algebraic matroid of the variety of $n \times n$ skew-symmetric matrices of rank at most $2 d$ over $\mathbb{C}$ is the hyperconnectivity matroid $\mathcal{H}_{n}(2 d)$ CRS23, Proposition 3.1], and the algebraic matroid of $m \times n$ matrices of rank at most $d$ over $\mathbb{C}$ is the bipartite rigidity matroid $\mathcal{B}_{m, n}(d, d)$ [SC09, Section 4]. We briefly comment on the relationship between the linear algebraic matroids in characteristic $p$ and the low-rank matrix completion matroids in characteristic $p$. This section is not used in the rest of the paper and can be skipped by the uninterested reader. For the field theory facts used in this section, see Mat86, Section 26], especially Theorem 26.6.

We first sketch how one computes the low-rank matrix completion matroid. Let $Y_{d}$ be the subvariety of $\mathbb{C}^{m n}$ given by $m \times n$ matrices of rank at most $d$, and let $K\left(Y_{d}\right)$ be the function field of $Y_{d}$. The matrix completion matroid encodes when the coordinate functions $z_{i j} \in K\left(Y_{d}\right)$ are algebraically independent. Because we are over a field of characteristic 0 , some functions $\left\{z_{i j}\right\}$ are algebraically independent if and only if their differentials $\mathrm{d} z_{i j} \in \Omega_{K\left(Y_{d}\right) / \mathbb{C}}$ are linearly independent in the module of differentials of $K\left(Y_{d}\right)$. In order to make the module of differentials $\Omega_{K\left(Y_{d}\right) / \mathbb{C}}$ explicit, we use that every matrix of rank at most $d$ can be written as $A B$, where $A$ is an $m \times d$ matrix and $B$ is an $d \times n$ matrix. This means that there is a surjective map $\mathbb{C}^{m d+d n} \rightarrow Y_{d}$, which sends a coordinate function $z_{i j}$ to $\sum_{\ell=1}^{d} x_{i \ell y_{\ell j}}$. Because we are over a field of characteristic 0 , the pullback map $\Omega_{K\left(Y_{d}\right) / \mathbb{C}} \otimes_{K\left(Y_{d}\right)} \mathbb{C}\left(x_{i j}, y_{k \ell}\right) \rightarrow \Omega_{\mathbb{C}\left(x_{i j}, y_{k \ell}\right) / \mathbb{C}}$ is injective. There is a basis for $\Omega_{\mathbb{C}\left(x_{i j}, y_{k \ell}\right) / \mathbb{C}}$ given by $\left\{\mathrm{d} x_{i j}, \mathrm{~d} y_{k \ell}\right\}$. Then $\mathrm{d} z_{i j}$ pulls back to $\mathrm{d}\left(\sum_{\ell=1}^{r} x_{i \ell} y_{\ell j}\right)=\sum_{\ell=1}^{d}\left(x_{i \ell} \mathrm{~d} y_{\ell j}+y_{\ell j} \mathrm{~d} x_{i \ell}\right)$. The matrix whose columns are given by the pullbacks of the $\mathrm{d} z_{i j}$ is exactly the matrix defining $\mathcal{B}_{m, n}(d, d)$. A similar argument can be used to compute the symmetric matrix completion matroid, using that an $n \times n$ symmetric matrix of rank at most $d$ can be written as $A A^{t}$, where $A$ is an $n \times d$ matrix, or the skew-symmetric matrix completion matroid, using that an $n \times n$ skew symmetric matrix of rank at most $2 d$ can be written as $A B^{t}-B A^{t}$ for $A, B n \times d$ matrices.

This argument breaks down in positive characteristic due to the presence of inseparable extensions. Given a finitely generated extension of fields $K / L$, we say that $a_{1}, \ldots, a_{\ell}$ are separably algebraically independent if there are $b_{1}, \ldots, b_{p}$ such that $a_{1}, \ldots, a_{\ell}, b_{1}, \ldots, b_{p}$ is a separating transcendence basis for $K / L$, i.e., they are algebraically independent and $K / L\left(a_{1}, \ldots, b_{p}\right)$ is a finite separable extension. We say that $K / L$ is separable if it has a separating transcendence basis.

Over a field $\mathbf{k}$ of positive characteristic, it is no longer true that we can test algebraic independence of a collection $a_{1}, \ldots, a_{\ell}$ by checking the linear independence of $\mathrm{d} a_{1}, \ldots, \mathrm{~d} a_{\ell}$. Rather, $\mathrm{d} a_{1}, \ldots, \mathrm{~d} a_{\ell}$ are linearly independent if and only $a_{1}, \ldots, a_{\ell}$ are separably algebraically independent. Furthermore, pullback maps on differentials are no longer automatically injective; they are injective if and only if the corresponding field extension is separable. This fails for the variety of symmetric matrices of rank at most $d$ in characteristic 2 : the map

$$
\mathbf{k}^{n d} \rightarrow\{n \times n \text { symmetric matrices of rank } \leq d\}, \quad A \mapsto A A^{t}
$$

does not induce a separable extension of function fields. In all other case, the analogous map does induce a separable extension of function fields. This can be proved by verifying that the matrix defining $\mathcal{S}_{n}(d)$ has rank $n d-\binom{d}{2}$ in characteristic $p \neq 2$, the matrix defining $\mathcal{H}_{n}(2 d)$ has rank $2 d n-\binom{2 d+1}{2}$ in any characteristic, and the matrix defining $\mathcal{B}_{m, n}(d, d)$ has rank $d(m+n)-d^{2}$ in any characteristic. Then the proof of Theorem 1.1 then gives the following result.
Theorem 2.5. Let $\mathbf{k}$ be a field of characteristic $p$.
(1) If $p \neq 2$, then a collection of elements is independent in the dual of the symmetric power matroid $\mathrm{S}_{n}(n-d, p)$ if and only if the corresponding rational functions on the variety of $n \times n$ symmetric matrices of rank at most $d$ over $\mathbf{k}$ are separably algebraically independent.
(2) A collection of elements is independent in the dual of wedge power matroid $\mathrm{W}_{n}(n-2 d, p)$ if and only if the corresponding rational function on the variety of $n \times n$ skew-symmetric matrice ${ }^{2}$ of rank at most $2 d$ over $\mathbf{k}$ are separably algebraically independent.
(3) A collection of elements is independent in the dual of tensor matroid $\mathrm{T}_{m, n}(m-d, n-d, p)$ if and only if the corresponding rational function on the variety of $m \times n$ matrices of rank at most $d$ over k are separably algebraically independent.

[^1]Remark 2.6. A collection of coordinate functions $\left\{x_{i j}:(i, j) \in S \subseteq[m] \times[n]\right\} \subseteq K\left(Y_{d}\right)$ is algebraically independent if and only if the coordinate projections $Y_{d} \hookrightarrow \mathbf{k}^{m n} \rightarrow \mathbf{k}^{S}$ is dominant, i.e., its image contains a Zariski open set. The collection is separably algebraically independent if and only if the coordinate projection is generically smooth.

Remark 2.7. One can sometimes show that the rank completion matroid in characteristic $p$, i.e., the algebraic matroid of the rank at most $d$ locus, is independent of the characteristic by showing that the tropicalization of the rank at most $d$ locus does not depend on the characteristic. For example, this was shown for $d=2$ in DSS05, Section 6]. We note that results of this form do not imply that the tensor matroid is independent of the characteristic.

Remark 2.8. We are unaware of any case in which the symmetric matrix completion matroid, the skewsymmetric matrix completion matroid, or the matrix completion matroid depends on the characteristic. In particular, the $4 \times 4$ symmetric rank 2 matrix completion matroid is the same in all characteristics, even though $\mathrm{S}_{4}(2,0) \neq \mathrm{S}_{4}(2,2)$.

## 3. Bipartite rigidity when $m-a$ is small

In this section, we give a complete description of the matroid $\mathrm{T}_{m, n}(s, r, p)$ when $s \leq 3$. Our description is independent of the characteristic $p$. By Theorem 1.1(3), this gives a description of $\mathcal{B}_{m, n}(a, b)$ when $m-a \leq 3$. We use this description to prove Theorem 1.2 and Corollary 1.5 ,
3.1. The disjoint case and $s=1$. Given an infinite field $\mathbf{k}$ of characteristic $p$, consider $m$ generic vectors $u_{1}, \ldots, u_{m}$ in $\mathbf{k}^{s}$ and $n$ generic vectors in $v_{1}, \ldots, v_{n}$ in $\mathbf{k}^{r}$. By definition, the independent sets of the matroid $\mathrm{T}_{m, n}(s, r, p)$ are given by sets $E \subseteq[m] \times[n]$ such that $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$ is linearly independent. We give a characteristic-independent description of the independent sets when $s \leq 3$.

We write $E=\bigcup_{i=1}^{m}\{i\} \times A_{i}, A_{i} \subseteq[n]$. We first show that when the $A_{i}$ are disjoint a simple dimension/size criteria is equivalent to independence.

Lemma 3.1 (The disjoint case). Let $A_{1}, \ldots A_{m} \subseteq[n]$ be disjoint sets. Let $E=\bigcup_{i=1}^{m}\{i\} \times A_{i} \subseteq[m] \otimes[n]$. Then $E$ is independent in $\mathrm{T}_{m, n}(s, r, p)$ if and only if $\left|A_{i}\right| \leq r$ for all $i$ and $\sum_{i=1}^{m}\left|A_{i}\right| \leq s r$.
Proof. The "only if" conditions follow from comparing dimensions. For the "if" direction, it suffices to give a choice of the $u_{i}$ and $v_{j}$ such that $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$ is a set of linearly independent vectors. Let $e_{1}, \ldots, e_{r}$ be a basis of $\mathbf{k}^{r}$. Set each $v_{j} \in A_{1}$ to be one of the basis vectors. Keep going, and set each $v_{j} \in A_{2}$ to be the "next" basis vectors and so forth. Each $v_{j}$ will be set to some basis vector, no two vectors in some $A_{i}$ are set to the same basis vector, and each basis vector is used at most $s$ times. For each $j \in[r]$, let $C_{j}$ be the set of $i$ for which $e_{j}$ is used for some vector in $A_{i}$.

Pick the $u_{i}$ to be generic vectors, so every $s$ of them are linearly independent. Consider a linear dependence among $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$. That is, some $\lambda_{i, j} \in \mathbf{k}$ for which

$$
\sum_{(i, j) \in E} \lambda_{i, j} u_{i} \otimes v_{j}=0
$$

In particular, we have that

$$
\sum_{j \in[r]} \sum_{i \in C_{j}} \lambda_{i, j^{\prime}} u_{i} \otimes e_{j}=0
$$

where $j^{\prime}$ is shorthand for the $j^{\prime} \in A_{i}$ for which $v_{j^{\prime}}=e_{j}$. Since the $e_{j}$ form a basis, we have that, for all $j \in[r]$,

$$
\sum_{i \in C_{j}} \lambda_{i, j^{\prime}} u_{i}=0
$$

Since $\left|C_{j}\right| \leq s$ for each $j$ and the $u_{i}$ are generic, we have that $\lambda_{i, j^{\prime}}=0$ for all $\left(i, j^{\prime}\right) \in E$. Thus, the vectors $u_{i} \otimes v_{j}$ are linearly independent, as desired.

We can use the above result to address the case $s=1$.
Corollary 3.2. Let $A_{1}, \ldots A_{m} \subseteq[n]$, and set $E=\bigcup_{i=1}^{m}\{i\} \times A_{i} \subseteq[m] \otimes[n]$. Then $E$ is independent in $\mathrm{T}_{m, n}(1, r, p)$ if and only if

$$
\begin{align*}
\left|A_{i} \cap A_{j}\right| & =0  \tag{1}\\
\sum_{i=1}^{m}\left|A_{i}\right| & \leq r .
\end{align*} \quad \text { for all distinct } i, j \in[m]
$$

Proof. First, we prove that (11) and (22) are necessary. If there exists some $k \in A_{i} \cap A_{j}$, then $u_{i} \otimes v_{k}$ and $u_{j} \otimes v_{k}$ are linearly dependent, so (11) is necessary. The necessity of (2) is obvious, as $\operatorname{dim}\left(\mathbf{k} \otimes \mathbf{k}^{r}\right)=r$.

To prove sufficiency, note that (11) implies we are in the disjoint case. Thus, we can apply Lemma 3.1 with $s=1$.

### 3.2. The case $s=2$.

Proposition 3.3 (Case $s=2$ ). Let $A_{1}, \ldots A_{m} \subseteq[n]$ be sets of size at most $r$, and set $E=\bigcup_{i=1}^{m}\{i\} \times A_{i} \subseteq$ $[m] \otimes[n]$. Then $E$ is independent in $\mathrm{T}_{m, n}(2, r, p)$ if and only if

$$
\begin{array}{rlr}
\left|A_{i} \cap A_{j} \cap A_{k}\right| & =0 & \text { for all distinct } i, j, k \in[m] \\
\sum_{1 \leq i<j \leq m}\left|A_{i} \cap A_{j}\right|+\left|A_{k} \backslash \bigcup_{i \in[m] \backslash\{k\}} A_{i}\right| \leq r & \text { for all } k \in[m] \\
\sum_{i=1}^{m}\left|A_{i}\right| \leq 2 r . & \tag{5}
\end{array}
$$

Proof. For any set $A \subseteq[n]$, let $V_{A}$ be the subspace of $\mathbf{k}^{r}$ spanned by the generic vectors $v_{i}$ for $i \in A$.
The fact that these inequalities are necessary is not hard to see. The last inequality is equivalent to requiring $|E| \leq 2 r=\operatorname{dim}\left(\mathbf{k}^{2} \otimes \mathbf{k}^{r}\right)$. For (4), let $W=\sum_{(i, j) \in E} u_{i} \otimes v_{j}$. Observe that, for all $i<j \in[m]$, $\mathbf{k}^{2} \otimes V_{A_{i} \cap A_{j}} \subseteq W$. In particular, $u_{k} \otimes V_{A_{i} \cap A_{j}} \subseteq W$. Also note that $u_{k} \otimes V_{A_{k} \backslash \cup_{i \neq k} A_{i}} \subseteq W$. For these to be linearly independent we must have (4).

If $\ell \in A_{i} \cap A_{j} \cap A_{k}$ for some distinct $i, j, k \in[m]$, then we have linearly dependent vectors $u_{i} \otimes v_{\ell}, u_{j} \otimes$ $v_{\ell}, u_{k} \otimes v_{\ell} \in \mathbf{k}^{2} \otimes \mathbf{k}^{r}$. This shows that (3) is necessary.

If the above inequalities are satisfied, then we show that $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$ are independent for a semi-explicit choice of the $u_{i} \in \mathbf{k}^{2}$ and $v_{j} \in \mathbf{k}^{r}$. Let $e_{1}, \ldots, e_{r}$ be a basis of $\mathbf{k}^{r}$. Let $S_{1}$ be the elements of $\bigcup_{i=1}^{m} A_{i}$ that appear in exactly one $A_{i}$. Let $S_{2}$ be the elements which appear in exactly two $A_{i}$. By (3), we have that $S_{1} \cup S_{2}=\bigcup_{i=1}^{m} A_{i}$. Observe then that (44) and (5) can thus be re-written as

$$
\begin{array}{rlr}
\left|A_{i} \cap S_{1}\right|+\left|S_{2}\right| & \leq r & \text { for all } i \\
\left|S_{1}\right|+2\left|S_{2}\right| & \leq 2 r & \tag{7}
\end{array}
$$

Pick $u_{1}, \ldots, u_{m}$ to be generic vectors in $\mathbf{k}^{2}$, so any pair of the vectors are linearly independent. For each $j \in S_{2}$, set $v_{j}$ to be a separate $e_{i}$ for $i \in\left\{1,2, \ldots,\left|S_{2}\right|\right\}$. This is possible by (44) as $\left|S_{2}\right| \leq r$. Finally, set the $v_{i}$ for $i \in S_{1}$ to be generic vectors in the space spanned by $\left\{e_{i}: i \in\left\{\left|S_{2}\right|+1, \ldots, r\right\}\right\}$.

It is not hard to see from these choices that $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$ are independent if and only if $\left\{u_{i} \otimes\right.$ $\left.v_{j}:(i, j) \in E, j \in S_{1}\right\}$ are independent, as the vectors in $S_{2}$ and $S_{1}$ are in disjoint subspaces. Thus a non-zero linear combination of $\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \in S_{2}\right\}$ cannot intersect with the subspace spanned
by $\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \in S_{1}\right\}$. As $i \in S_{2}$ only appears in two of the $A_{j}$, we also see that the set $\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \in S_{2}\right\}$ is linearly independent and indeed spans $\mathbf{k}^{2} \otimes \mathbf{k}^{\left|S_{2}\right|}$.

Then $\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \in S_{1}\right\}$ are vectors in the subspace $\mathbf{k}^{2} \otimes \mathbf{k}^{r-\left|S_{2}\right|}$. The result follows from applying Lemma 3.1 with the parameters $(m, n, s, r)=\left(m, n-\left|S_{2}\right|, 2, r\right)$ and the sets $A_{i} \cap S_{1}$ for $i \in[m]$. The inequalities needed to invoke Lemma 3.1 are exactly (6) and (7).

### 3.3. The case $s=3$.

Proposition 3.4 (Case $s=3$ ). Let $A_{1}, \ldots A_{m} \subseteq[n]$ be sets of size at most $r$, and set $E=\bigcup_{i=1}^{m}\{i\} \times A_{i} \subseteq$ $[m] \otimes[n]$. For $k \in\{1,2,3\}$, let $S_{k}$ be the set of $j \in[n]$ which appear in exactly $k$ of the $A_{i}$. Then $E$ is independent in $\mathrm{T}_{m, n}(3, r, p)$ if and only if

$$
\begin{array}{rlrl}
\left|A_{i} \cap A_{j} \cap A_{k} \cap A_{\ell}\right| & =0 & & \text { for all distinct } i, j, k, \ell \in[m] \\
\left|A_{i} \backslash S_{3}\right|+\left|S_{3}\right| \leq r & & \text { for all } i \in[m] \\
\left|\left(A_{i} \cap A_{j}\right) \backslash S_{3}\right|+\left|\left(A_{k} \cap A_{\ell}\right) \backslash S_{3}\right|+\left|S_{3}\right| \leq r & & \text { for all distinct } i, j, k, \ell \in[m] \\
\left|A_{i} \backslash S_{3}\right|+\left|A_{j} \backslash S_{3}\right|+\left|S_{2} \backslash\left(S_{3} \cup A_{i} \cup A_{j}\right)\right|+2\left|S_{3}\right| \leq 2 r & & \text { for all distinct } i, j \in[m] \\
\left|S_{1}\right|+2\left|S_{2}\right|+3\left|S_{3}\right| \leq 3 r . & & \tag{12}
\end{array}
$$

Proof. Let $W:=\sum_{(i, j) \in E} u_{i} \otimes v_{j}$. For any set $A \subseteq[n]$, let $V_{A}$ be the subspace of $\mathbf{k}^{r}$ spanned by the generic vectors $v_{i}$ for $i \in A$.

The necessity of (8) and (12) are obvious. To show the necessity of (9), observe that $\mathbf{k}^{3} \otimes V_{S_{3}} \subseteq W$. Thus, $u_{i} \otimes V_{S_{3}} \subseteq W$. Also note that $u_{i} \otimes V_{A_{i} \backslash S_{3}} \subseteq W$. For these two subspaces to be linearly independent we must have $\left|A_{i} \backslash S_{3}\right|+\left|S_{3}\right| \leq r$.

Next, we now show (10) is necessary. Consider distinct $i, j, k, \ell \in[m]$ and let $u$ be a nontrivial vector in $\operatorname{span}\left\{u_{i}, u_{j}\right\} \cap \operatorname{span}\left\{u_{k}, u_{\ell}\right\}$. As before, $u \otimes V_{S_{3}} \subseteq W$. Further, $u \otimes V_{\left(A_{i} \cap A_{j}\right) \backslash S_{3}} \subseteq W$ and $u \otimes V_{\left(A_{k} \cap A_{\ell}\right) \backslash S_{3}} \subseteq W$. If these subspaces are linearly independent, then (10) must hold.

To finish the proof of the necessity of the inequalities, we show (11) is necessary. Consider distinct $i, j \in[m]$. For each $k \in S_{2} \backslash\left(S_{3} \cup A_{i} \cup A_{j}\right)$, let $u_{k}^{\prime}$ be a nontrivial vector in $\operatorname{span}\left\{u_{i}, u_{j}\right\} \cap \operatorname{span}\left\{u_{a}, u_{b}\right\}$, where $a$ and $b$ are the two sets such that $k \in A_{a} \cap A_{b}$. Observe that $u_{i} \otimes V_{A_{i} \backslash S_{3}}, u_{j} \otimes V_{A_{j} \backslash S_{3}}, \sum_{k \in S_{2} \backslash\left(S_{3} \cup A_{i} \cup A_{j}\right)} u_{k}^{\prime} \otimes v_{k}$, and $\operatorname{span}\left\{u_{i}, u_{j}\right\} \otimes V_{S_{3}}$ are linearly independent subspaces of $W \cap\left(\operatorname{span}\left\{u_{i}, u_{j}\right\} \otimes \mathbf{k}^{s}\right)$. This proves the necessity of (11).

Next, we prove the sufficiency of these equations by specializing some of the vectors. The strategy is to induct by finding a nice subspace to quotient by. We first use this strategy to reduce to the case $S_{3}=\emptyset$.

Claim 3.5. $E$ is independent in $\mathrm{T}_{m, n}(3, r, p)$ if and only if $E \backslash\left\{(i, j): j \in S_{3}\right\}$ is independent in $\mathrm{T}_{m, n}(3, r-$ $\left.\left|S_{3}\right|, p\right)$.

Proof. Let $e_{1}, \ldots, e_{r}$ be a basis of $\mathbf{k}^{r}$. For each $i \in S_{3}$, we set $v_{i}$ to a different basis element (which is possible as $\left|S_{3}\right| \leq r$ by (12)). For $j \in[n] \backslash S_{3}$ we choose the $v_{j}$ to be generic vectors in the subspace spanned $e_{\left|S_{3}\right|+1}, \ldots, e_{r}$. We have that $E \backslash\left\{(i, j): j \in S_{3}\right\}$ is independent in $\mathrm{T}_{m, n}\left(3, r-\left|S_{3}\right|, p\right)$ if and only if $\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \notin S_{3}\right\}$ is independent in $\mathrm{T}_{m, n}(3, r, p)$. We see that

$$
\operatorname{span}\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \in S_{3}\right\} \cap \operatorname{span}\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \notin S_{3}\right\}=0
$$

As the $u_{i}$ are generic, $\left\{u_{i} \otimes v_{j}:(i, j) \in E, j \in S_{3}\right\}$ is independent in $\mathrm{T}_{m, n}(3, r, p)$ if and only if $E \backslash\{(i, j)$ : $\left.j \in S_{3}\right\}$ is independent in $\mathrm{T}_{m, n}\left(3, r-\left|S_{3}\right|, p\right)$, so this implies the result.

With the assumption $S_{3}=\emptyset$, we have a much simpler set of inequalities:

$$
\begin{align*}
\left|A_{i}\right| \leq r & \text { for all } i \in[n]  \tag{13}\\
\left|A_{i} \cap A_{j}\right|+\left|A_{k} \cap A_{\ell}\right| \leq r & \text { for all distinct } i, j, k, \ell \in[n]  \tag{14}\\
\left|A_{i}\right|+\left|A_{j}\right|+\left|S_{2} \backslash\left(A_{i} \cup A_{j}\right)\right| \leq 2 r & \text { for all distinct } i, j \in[n]  \tag{15}\\
\left|S_{1}\right|+2\left|S_{2}\right| \leq 3 r . & \tag{16}
\end{align*}
$$

We will proceed via induction on $m$ and considering several cases.
Case 1, (13) is tight: Without loss of generality, assume that $i=1$ in (13). Since $\left|A_{1}\right|=r$, we have that $V_{A_{1}}=\mathbf{k}^{r}$. For $i \geq 2$, let $u_{i}^{\prime}$ be the image of $u_{i}$ in $\mathbf{k}^{3} /\left\langle u_{1}\right\rangle$. Then the $u_{i}^{\prime}$ are generic vectors in $\mathbf{k}^{3} /\left\langle u_{1}\right\rangle$.

We claim that the $(i, j) \in E$ with $i \geq 2$ give an independent set in $\mathrm{T}_{m-1, n}(2, r, p)$. This follows by checking the inequalities of Lemma (3.3 (3) follows because $\left|S_{3}\right|=0$, (4) follows from (15) for $A_{1}$ and $A_{i}$, and (5) follows from (16). This means $\left\{u_{i}^{\prime} \otimes v_{j}:(i, j) \in \bigcup_{k=2}^{m}\{k\} \times A_{k}\right\}$ is linearly independent, which implies $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$ is linearly independent.
Case 2, (14) is tight: Without loss of generality, assume that $(i, j, k, \ell)=(1,2,3,4)$. As the $u_{i}$ are generic vectors, any three of them are linearly independent and that there is no common non-zero vector in the span of any three disjoint pairs $\left(u_{i_{1}}, u_{i_{2}}\right),\left(u_{i_{3}}, u_{i_{4}}\right),\left(u_{i_{5}}, u_{i_{6}}\right)$.

Suppose $\left|A_{1} \cap A_{2}\right|+\left|A_{3} \cap A_{4}\right|=r$, so $\mathbf{k}^{r}=V_{A_{1} \cap A_{2}}+V_{A_{3} \cap A_{4}}$. Let $u$ be a non-zero vector in $\operatorname{span}\left\{u_{1}, u_{2}\right\} \cap$ $\operatorname{span}\left\{u_{3}, u_{4}\right\}$. We quotient $\mathbf{k}^{3} \otimes \mathbf{k}^{r}$ by $u \otimes V_{A_{1} \cap A_{2}}+u \otimes V_{A_{3} \cap A_{4}}=u \otimes \mathbf{k}^{r}$. Let $u_{j}^{\prime}$ be the image of $u_{j}$ in $\mathbf{k}^{3} /\langle u\rangle$.

We set $A_{1}^{\prime}=A_{1} \cup A_{2}$ and $A_{2}^{\prime}=A_{3} \cup A_{4}$. We note $u_{1}^{\prime}, u_{2}^{\prime}$ are scalar multiples of each other and so are $u_{3}^{\prime}, u_{4}^{\prime}$. Observe that $\left\{u_{1}^{\prime}, u_{3}^{\prime}, u_{5}^{\prime}, \ldots, u_{m}^{\prime}\right\}$ is a set of generic vectors, i.e., any pair is linearly independent.

Let $E^{\prime}=\left(\{1\} \times A_{1}^{\prime}\right) \cup\left(\{3\} \times A_{2}^{\prime}\right) \cup \bigcup_{i=5}^{m}\left(\{i\} \times A_{i}\right)$. We claim that $\left\{u_{i}^{\prime} \times v_{j}:(i, j) \in E^{\prime}\right\}$ is linearly independent in $\left(\mathbf{k}^{3} /\langle u\rangle\right) \otimes \mathbf{k}^{r}$. This follows by checking the conditions in Lemma 3.3 for $A_{1}^{\prime}, A_{2}^{\prime}, A_{5}, \ldots, A_{m}$. Indeed, (3) easily follows. Let $B=\left(A_{1} \cap A_{2}\right) \cup\left(A_{3} \cap A_{4}\right)$, which has size exactly $r$ because $A_{1} \cap A_{2} \cap A_{3} \cap A_{4}=\emptyset$. We see that $\left|A_{1}^{\prime}\right|+\left|A_{2}^{\prime}\right|+\sum_{i=5}^{m}\left|A_{i}\right|=\sum_{i=1}^{m}\left|A_{i}\right|-|B| \leq 2 r$, so (5) holds. To check (41), we use (15) for $A_{1}, \ldots, A_{m}$ and the fact that the indices which appear in two sets for $A_{1}^{\prime}, A_{2}^{\prime}, A_{5}, \ldots, A_{m}$ are precisely $S_{2} \backslash B$, and $\left|S_{2} \backslash B\right|=\left|S_{2}\right|-|B|=\left|S_{2}\right|-r$.

If we have a non-zero linear combination of $\left\{u_{i} \times v_{j}:(i, j) \in E\right\}$ which is zero, then quotienting by $u \otimes \mathbf{k}^{r}$ and using the linear independence of $\left\{u_{i}^{\prime} \times v_{j}:(i, j) \in E^{\prime}\right\}$ gives a contradiction.
Case 3, (15) is tight: Without loss of generality, assume that $\left|A_{1}\right|+\left|A_{2}\right|+\left|S_{2} \backslash\left(A_{1} \cup A_{2}\right)\right|=2 r$. For each $i \in S_{2} \backslash\left(A_{1} \cup A_{2}\right)$, let $u_{i}^{\prime}=\operatorname{span}\left\{u_{1}, u_{2}\right\} \cap \operatorname{span}\left\{u_{j}, u_{k}\right\}$, where $j, k$ are the two sets such that $i \in A_{j} \cap A_{k}$.

Claim 3.6. The three subspaces $\left\langle u_{1}\right\rangle \otimes V_{A_{1}},\left\langle u_{2}\right\rangle \otimes V_{A_{2}}$, and $\sum_{i \in S_{2} \backslash\left(A_{1} \cup A_{2}\right)}\left\langle u_{i}^{\prime}\right\rangle \otimes\left\langle v_{i}\right\rangle$ spans $\operatorname{span}\left\{u_{1}, u_{2}\right\} \otimes \mathbf{k}^{r}$.
Proof. We see that $\left\langle u_{1}\right\rangle \otimes V_{A_{1}},\left\langle u_{2}\right\rangle \otimes V_{A_{2}}$, and $\sum_{i \in S_{2} \backslash\left(A_{1} \cup A_{2}\right)}\left\langle u_{i}^{\prime}\right\rangle \otimes\left\langle v_{i}\right\rangle$ spans a subspace of span $\left\{u_{1}, u_{2}\right\} \otimes \mathbf{k}^{r}$. We will show it spans the whole space by proving that $\left\{u_{1} \otimes v_{i}: i \in A_{1}\right\},\left\{u_{2} \otimes v_{j}: j \in A_{2}\right\}$, and $\left\{u_{k}^{\prime} \otimes v_{l}: k \in S_{2} \backslash\left(A_{1} \cup A_{2}\right), l \in A_{k}\right\}$ are linearly independent.

We will use Lemma 3.3 for this. First note that, because the $u_{i}$ are generic, the vectors $\left\{u_{1}, u_{2}\right\} \cup\left\{u_{k}^{\prime}: k \in\right.$ $\left.S_{2} \backslash\left(A_{1} \cup A_{2}\right)\right\}$ are generic vectors in span $\left\{u_{1}, u_{2}\right\}$. In the collection of sets $\left\{A_{1}, A_{2}\right\} \cup\left\{\{i\}: i \in S_{2} \backslash\left(A_{1} \cup A_{2}\right)\right\}$, the only non-empty pairwise intersection is between $A_{1}$ and $A_{2}$. As $\left|A_{1} \cap A_{2}\right|<r$, we can use this to check (3), (4), and (5).

Suppose there is a dependence relation in $\left\{u_{i} \otimes v_{j}:(i, j) \in E\right\}$ :

$$
\begin{equation*}
\sum_{(i, j) \in E} \lambda_{i, j} u_{i} \otimes v_{j}=0 . \tag{17}
\end{equation*}
$$

Consider the image of this relation in $\mathbf{k}^{3} / \operatorname{span}\left\{u_{1}, u_{2}\right\} \otimes \mathbf{k}^{r}$. All $u_{i}, i \geq 3$ are projected to non-zero scalar multiples, say $\alpha_{i}$, of the same vector $u$. Then (17) becomes,

$$
\begin{equation*}
\sum_{i=3}^{m} \sum_{j \in A_{i} \backslash S_{2}} \alpha_{i} \lambda_{i, j} u \otimes v_{j}+\sum_{i \neq j \in[m] \backslash\{1,2\}} \sum_{k \in A_{i} \cap A_{j}}\left(\alpha_{i} \lambda_{i, k}+\alpha_{j} \lambda_{j, k}\right) u \otimes v_{k}=0 . \tag{18}
\end{equation*}
$$

Note $\left\{v_{j}: j \in A_{i} \backslash S_{2}, i \geq 3\right\} \cup\left\{v_{k}: k \in A_{i} \cap A_{j}, i \neq j \in[m] \backslash\{1,2\}\right\}$ is a set of at most $r$ distinct vectors (we use $S_{1}+2 S_{2} \leq 3 r$ and $\left|A_{1}\right|+\left|A_{2}\right|+\left|S_{2} \backslash\left(A_{1} \cup A_{2}\right)\right|=2 r$ ). Thus $\lambda_{i, j}=0$ for $i \geq 3, j \in A_{i} \backslash S_{2}$ and $\alpha_{i} \lambda_{i, k}+\alpha_{j} \lambda_{j, k}=0$ for $i \neq j \in[m] \backslash\{1,2\}, k \in A_{i} \cap A_{j}$.

Using this in (17) gives us a linear dependence which contradicts Claim 3.6,
Case 4, $\left|S_{2}\right|=0$ : This is just the disjoint case (Lemma 3.1).
Case 5, $\left|S_{1}\right| \neq 0,\left|S_{2}\right| \neq 0$ and (13),(14),(15) are not tight: As $m \geq 3$, without loss of generality we can assume that $A_{1} \cap S_{1} \neq \emptyset$ and $A_{2} \cap A_{3} \neq \emptyset$. After relabeling, let $i \in A_{1}$ be in $S_{1}$ and $j \in A_{2} \cap A_{3}$. We now set $v_{i}=v_{j}=e_{1}$ and quotient by $\mathbf{k}^{3} \otimes e_{1} \cong \operatorname{span}\left\{u_{1}, u_{2}, u_{3}\right\} \otimes e_{1}$. Set $A_{1}^{\prime}=A_{1} \backslash\{i\}, A_{2}^{\prime}=A_{2} \backslash\{j\}$, $A_{3}^{\prime}=A_{3} \backslash\{j\}$, and $A_{j}^{\prime}=A_{j}$ for $j \geq 4$. Then $A_{1}^{\prime}, \ldots, A_{m}^{\prime}$ satisfy the inequalities (13), (14), (16) for $\mathbf{k}^{3} \otimes \mathbf{k}^{r-1}$. The only non-trivial check is for (15), but if we re-write (15) as $\left|S_{2} \cup A_{1} \cup A_{2}\right|+\left|A_{1} \cap A_{2}\right| \leq 2 r-1$ (as (15)) is not tight) then we see $A_{i}^{\prime}$ satisfy the needed inequality.
Case 6, $\left|S_{1}\right|=0,\left|S_{2}\right| \neq 0$ and (13),(14),(15) are not tight: If (16) is not tight then $2\left|S_{2}\right| \leq 3 r-1$. Choose $i \in S_{2}$ and quotient by $\mathbf{k}^{3} \otimes\left\langle v_{i}\right\rangle$. The new sets will satisfy the inequalities for $\mathbf{k}^{3} \otimes \mathbf{k}^{r-1}$.

From now we assume (16) is tight, which means $\left|S_{2}\right|=3 r / 2$ (which also implies that $r$ is even). In that case (15) ) simplifies even further to $\left|A_{i} \cap A_{j}\right| \leq r / 2-1$ (as (15) is not tight), which also makes (14) redundant.

We claim that if $r \geq 1$, we can always pick $i_{1}, i_{2}, i_{3} \in S_{2}$ with the following properties. Let $j_{1}, j_{1}^{\prime}, j_{2}, j_{2}^{\prime}, j_{3}, j_{3}^{\prime} \in$ [ $m$ ] be such that $i_{1} \in A_{j_{1}} \cap A_{j_{1}^{\prime}}$, and so forth. Further, set $A_{i}^{\prime}=A_{i} \backslash\left\{i_{1}, i_{2}, i_{3}\right\}$ for all $i \in[m]$. It is not hard to see that $\left|A_{1}^{\prime}\right|+\cdots+\left|A_{m}^{\prime}\right|=3(r-2)$. We seek claim that the following holds:

- If $e_{1}, e_{2} \in \mathbf{k}^{2}$ are standard basis vectors, then $\operatorname{span}\left\{u_{j_{1}}, u_{j_{1}^{\prime}}\right\} \otimes e_{1}+\operatorname{span}\left\{u_{j_{2}}, u_{j_{2}^{\prime}}\right\} \otimes e_{2}+\operatorname{span}\left\{u_{j_{3}}, u_{j_{3}^{\prime}}\right\} \otimes$ $\left(e_{1}+e_{2}\right)=\mathbf{k}^{3} \otimes \mathbf{k}^{2}$.
- $\left|A_{i}^{\prime}\right| \leq r-2$ for all $i \in[m]$.
- $\left|A_{i}^{\prime} \cap A_{j}^{\prime}\right| \leq(r-2) / 2$ for $i \neq j \in[m]$.

If all these properties are satisfied, then we can use induction to finish this case. By setting $v_{i_{1}}=e_{1}, v_{i_{2}}=$ $e_{2}$, and $v_{i_{3}}=e_{1}+e_{2}$ and quotienting out $\mathbf{k}^{3} \otimes \operatorname{span}\left\{e_{1}, e_{2}\right\}$, we reduce to a smaller case on $\mathbf{k}^{3} \otimes \mathbf{k}^{r-2}$.

The third condition follows directly as $\left|A_{i} \cap A_{j}\right| \leq r / 2-1$. One can check that the first condition is satisfied when the pairs $\left\{j_{1}, j_{1}^{\prime}\right\},\left\{j_{2}, j_{2}^{\prime}\right\},\left\{j_{3}, j_{3}^{\prime}\right\}$ are distinct and no index appears in the multiset $\left\{j_{1}, j_{1}^{\prime}, j_{2}, j_{2}^{\prime}, j_{3}, j_{3}^{\prime}\right\}$ more than twice ${ }^{3}$ As $\left|A_{i} \cap A_{j}\right| \leq r / 2-1$ for every $i, j$ and $S_{2}=3 r / 2$, we have at least three pairs to choose from. We would be forced to pick the same index thrice if there are only 4 sets $A_{1}, A_{2}, A_{3}, A_{4}$ and one of them intersects with the rest but the rest do not intersect with each other, but that would contradict that $S_{2}=3 r / 2$.

For the second condition, if $\left|A_{i}\right|>r-2$, then $i$ must appear at least $\left|A_{i}\right|-(r-2)$ times in $\left\{j_{1}, j_{1}^{\prime}, j_{2}, j_{2}^{\prime}, j_{3}, j_{3}^{\prime}\right\}$. As $\left|A_{i}\right| \leq r-1$ we only have to ensure $\left|A_{i}\right|=r-1$ happens at most 6 times. If it happens 7 times, say for $A_{1}, \ldots, A_{7}$ then $\left|A_{i} \cap A_{j}\right| \geq r / 2-2$ for $i \neq j=1, \ldots, 7$. This gives us that $21 r / 2-42 \leq\left|S_{2}\right|=3 r / 2$ which implies $r \leq 42 / 9$. As $r$ is even we have $r \in\{2,4\}$. In either case we have $\left|S_{2}\right| \leq 6$ and $\left|\bar{A}_{i}\right|=r-1$ for $i=1, \ldots, 7$, so the fact that $\left|S_{3}\right|=\left|S_{1}\right|=0$ leads to a contradiction.

Proof of Theorem 1.2, The description of $\mathrm{T}_{m, n}(s, r, 0)$ when $s \leq 3$ above shows that we can check independence in $\mathrm{T}_{m, n}(s, r, 0)$ by checking polynomially many conditions (the case $s=0$ is immediate). By Theorem 1.1, $\mathrm{T}_{m, n}(s, r, 0)$ is dual to $\mathcal{B}_{m, n}(m-s, n-r)$, so a set is independent in $\mathrm{T}_{m, n}(s, r, 0)$ if and only

[^2]if its complement is spanning in $\mathcal{B}_{m, n}(m-s, n-r)$. By HK81, there is a polynomial time algorithm to compute the rank function of a matroid by checking if polynomially many sets are spanning.
3.4. Characterizing the Laman condition. In this section, we prove Corollary 1.5, One consequence of Corollary 1.5 is that, when $m-a \leq 2$, the Laman circuits are the circuits of a matroid; Example 1.4 shows that this is false in general. It will be necessary to prove this directly in order to prove Corollary 1.5

Any subset $U$ of $[m] \times[n]$ is contained in some minimal rectangle $S \times T$. We say that $U$ violates the Laman condition if there is some rectangle $S \times T$ with $|S| \geq a,|T| \geq b$ such that $|U \cap S \times T|>|S| b+|T| a-a b$. Clearly any subset which violates the Laman condition contains a minimal subset which violates the Laman condition. In general, a minimal subset which violates the Laman condition need not contain a Laman circuit (which are, by definition, minimal dependent sets).

Lemma 3.7. Let $C_{1}, C_{2}$ be distinct minimal sets which violate the Laman condition whose minimal rectangles are $S_{1} \times T_{1}$ and $S_{2} \times T_{2}$, respectively. Suppose that $\left|S_{1} \cap S_{2}\right| \geq a$. Then, for any $e \in C_{1} \cup C_{2}, C_{1} \cup C_{2} \backslash e$ violates the Laman condition.

Proof. We have $\left|C_{1}\right| \geq b\left|S_{1}\right|+a\left|T_{1}\right|-a b+1$ and $\left|C_{2}\right| \geq b\left|S_{2}\right|+a\left|T_{2}\right|-a b+1$. Therefore

$$
\left|C_{1} \cup C_{2} \backslash e\right| \geq b\left(\left|S_{1}\right|+\left|S_{2}\right|\right)+a\left(\left|T_{1}\right|+\left|T_{2}\right|\right)-2 a b+1-\left|C_{1} \cap C_{2}\right|
$$

We claim that we have the bound

$$
b\left|S_{1} \cap S_{2}\right|+a\left|T_{1} \cap T_{2}\right|-a b \geq\left|C_{1} \cap C_{2}\right|
$$

Given this inequality, we see that $\left|C_{1} \cup C_{2} \backslash e\right| \geq b\left|S_{1} \cup S_{2}\right|+a\left|T_{1} \cup T_{2}\right|-a b+1$. As $C_{1} \cup C_{2} \backslash e$ is contained in the rectangle $\left(S_{1} \cup S_{2}\right) \times\left(T_{1} \cup T_{2}\right)$, this implies the result.

First suppose that $\left|T_{1} \cap T_{2}\right| \geq b$. Note that, because $C_{1} \neq C_{2}, C_{1} \cap C_{2}$ does not violate the Laman condition and is contained in $\left(S_{1} \cap S_{2}\right) \times\left(T_{1} \cap T_{2}\right)$. The claimed inequality follows.

Now suppose that $\left|T_{1} \cap T_{2}\right| \leq b$. Write $\left|S_{1} \cap S_{2}\right|=a+\ell$ for some $\ell \geq 0$. Then we have that

$$
b\left|S_{1} \cap S_{2}\right|+a\left|T_{1} \cap T_{2}\right|-a b=b \ell+a\left|T_{1} \cap T_{2}\right|
$$

The bound $\left|C_{1} \cap C_{2}\right| \leq\left|S_{1} \cap S_{2}\right| \cdot\left|T_{1} \cap T_{2}\right|=(a+\ell)\left|T_{1} \cap T_{2}\right|$ implies the claim, as $b \ell \geq \ell\left|T_{1} \cap T_{2}\right|$.
Lemma 3.8. For $m, n, a, b$ with $m-a \leq 2$, the minimal sets which violate the Laman condition in $[m] \times[n]$ form the circuits of a matroid.

Proof. We must check the circuit elimination axiom. Let $C_{1}, C_{2}$ be distinct minimal sets which violate the Laman condition, and let $S_{1} \times T_{1}, S_{2} \times T_{2}$ be the minimal rectangles containing them. We have $\left|S_{1}\right| \geq a+1$ and $\left|S_{2}\right| \geq a+1$ : if $\left|S_{i}\right|=a$, then no subset of $S_{i} \times T_{i}$ is large enough to violate the Laman condition. As $m-a \leq 2$, this implies that $\left|S_{1} \cap S_{2}\right| \geq a$. It then follows from Lemma 3.7 that, for any $e \in C_{1} \cup C_{2}$, $C_{1} \cup C_{2} \backslash e$ violates the Laman condition.

Proposition 3.9. If $m-a \leq 2$, then the circuits of $\mathcal{B}_{m, n}(a, b)$ are the minimal sets which violate the Laman condition.

Proof. We focus on the case $m-a=2$; the cases $m=a$ and $m=a+1$ are straightforward or can be deduced from the case $m-a=2$. Let $\mathrm{M}_{\text {Lam }}$ be the matroid on $[m] \times[n]$ whose circuits are the minimal sets which satisfy the Laman condition. Note that every basis of $\mathcal{B}_{m, n}(a, b)$ is independent in $\mathrm{M}_{\text {Lam }}$. The Laman condition implies that the rank of $\mathrm{M}_{\text {Lam }}$ is at most $n a+m b-a b=2 b+n a$, which is the rank of $\mathcal{B}_{m, n}(a, b)$, so the rank of $\mathrm{M}_{\text {Lam }}$ is the same as the rank of $\mathcal{B}_{m, n}(a, b)$. It therefore suffices to show that if $F \subseteq[m] \times[n]$ is independent in $\mathrm{M}_{\text {Lam }}$ with $|F|=2 b+n a$, then the complement $F^{c}$ is independent in $\mathrm{T}_{m, n}(2, n-b, 0)$. Set $F^{c}=\cup_{i}\{i\} \times A_{i}$.

We check that $F^{c}$ satisfies the inequalities in Proposition 3.3 (with $r=n-b$ ). That $\left|A_{i}\right| \leq n-b$ follows from the Laman condition applied to $([m] \backslash\{i\}) \times[n]$ and the fact that $|F|=2 b+n a$.

Condition (5) holds: it is equivalent to requiring that $|F| \geq 2 b+n a$.
Condition (3) holds: suppose $i \in A_{j} \cap A_{k} \cap A_{\ell}$, i.e., $|F \cap[m] \times\{i\}|<a$. The Laman condition implies that $|F \cap[m] \times([n] \backslash i)| \leq 2 b+a(n-1)$. But this implies that $|F|<2 b+a n$, a contradiction.

Condition (4) holds: let $S_{1}$ be the set of elements in [ $n$ ] which occur in exactly 1 of the $A_{i}$, let $S_{2}$ be the elements in $[n]$ which occur in exactly two of the $A_{i}$, and suppose $\left|A_{i} \cap S_{1}\right|+\left|S_{2}\right|>n-b$. Consider the complement of the first row. The Laman condition implies that there can at most $b$ columns $j$ where $F$ contains $([m] \backslash i) \times\{j\}$. The other $n-b$ columns have at least one entry missing from $([m] \backslash i) \times\{j\}$. We see that $\left|S_{1}\right|-\left|A_{i} \cap S_{1}\right|$ of these columns can arise from elements of $S_{1}$, and $\left|S_{2}\right|$ of these columns can arise from elements of $S_{2}$. Therefore

$$
n-b \leq\left(\left|S_{1}\right|-\left|A_{i} \cap S_{1}\right|\right)+\left|S_{2}\right| .
$$

Adding this to the equation $\left|A_{i} \cap S_{1}\right|+\left|S_{2}\right|>n-b$, we see that $2(n-b)<\left|S_{1}\right|+2\left|S_{2}\right|$. But $2(n-b)=\left|S_{1}\right|+2\left|S_{2}\right|$ because $|F|=2 b+n a$.

The circuits which are not Laman circuits that we will use to prove Corollary 1.5 will be built out Example 1.4 using the following result.

Proposition 3.10. For $i \in[m]$, let $S=\{(1,1), \ldots,(1, n)\}$. For any $p \geq 0$ and $0<s<m$, the contraction $\mathrm{T}_{m, n}(s, r, p) / S$ is $\mathrm{T}_{m-1, n}(s-1, r, p)$, and the deletion $\mathrm{T}_{m, n}(s, r, p) \backslash S$ is $\mathrm{T}_{m-1, n}(s, r, p)$.

Proof. Let $U, V$ be vector spaces over an infinite field $\mathbf{k}$ of characteristic $p$ of dimensions $s, r$ respectively. Choose generic vectors $u_{1}, \ldots, u_{m} \in U$ and $v_{1}, \ldots, v_{n} \in V$. The contraction $\mathrm{T}_{m, n}(s, r, p) / S$ is represented by the vector configuration $u_{2} \otimes v_{1}, u_{2} \otimes v_{2}, \ldots, u_{m} \otimes v_{n} \in U /\left(\mathbf{k} \cdot u_{1}\right) \otimes V$, and the deletion $\mathrm{T}_{m, n}(s, r, p) \backslash S$ is represented by $u_{2} \otimes v_{1}, u_{2} \otimes v_{2}, \ldots, u_{m} \otimes v_{n} \in U \otimes V$.

Using Proposition 3.10 and Theorem 1.1(3), we obtain a simple proof of the "cone lemma" KNN16, Lemma 3.12] for bipartite rigidity.

Corollary 3.11. Let $G$ be a bipartite graph on $[m] \times[n]$. Let $C_{L} G$ (respectively $C_{R} G$ ) be the bipartite graph on $[m+1] \times[n]$ (resp. $[m] \times[n+1]$ ) obtained by adding a vertex to $G$ on the left (resp. right) and then connecting it to everything in $[n]$ (resp. $[m]$ ). Then $G$ is independent in $\mathcal{B}_{m, n}(a, b)$ if and only if $C_{L} G$ is independent in $\mathcal{B}_{m+1, n}(a+1, b)$ (resp. $C_{R} G$ is independent in $\mathcal{B}_{m, n+1}(a, b+1)$ ).

Proof of Corollary 1.5. Suppose $2 \leq a<m-2$ and $2 \leq b<n-2$. Let $G$ be the graph described in Example 1.4, which is dependent in $\mathcal{B}_{5,5}(2,2)$ but does not contain a Laman circuit. If we perform $a-2$ left cones and $b-2$ right cones, then we obtain a graph $\mathcal{H}$ which is dependent in $\mathcal{B}_{a+3, b+3}(a, b)$ but does not contain a Laman circuit. The restriction of $\mathcal{B}_{m, n}(a, b)$ to $[a+3] \times[b+3]$ is $\mathcal{B}_{a+3, b+3}(a, b)$, e.g. by Proposition 3.10, so $\mathcal{H}$ also gives a dependent set which does not contain a Laman circuit for $\mathcal{B}_{m, n}(a, b)$.

If $a \leq 1$ or $b \leq 1$, then the result follows KNN16. Theorem 5.4], which is based on Whi89. If $m-a \leq 2$ or $b \leq n-2$, then the result follows from Proposition 3.9
3.5. Challenges for $s=4$. Consider $m=n=7$ and $s=r=4$. Consider the pattern

$$
E=\left(\{1,2,3\}^{2} \backslash\{(3,3)\}\right) \cup\{4,5\}^{2} \cup\{6,7\}^{2}
$$

This is illustrated in Figure2 Although $E$ is of basis size and its complement satisfies the Laman conditions, it is not a basis of the matroid $\mathrm{T}_{7,7}(4,4, p)$ for any $p$.

Proposition 3.12. For all $p, E$ is not a basis of $\mathrm{T}_{7,7}(4,4, p)$.


Figure 2. (red $\diamond)$ A circuit of $\mathrm{T}_{7,7}(4,4, p)$. (blue $\star$ ) The corresponding circuit in $\mathcal{B}_{7,7}(3,3)$. See Figure 1 for how to interpret.

Proof. Let $u_{1}, \ldots, u_{7}, v_{1}, \ldots, v_{7} \in \mathbf{k}^{4}$ be generic. Further define

$$
\begin{aligned}
& W_{0}=\operatorname{span}\left\{u_{i} \otimes v_{j}:(i, j) \in\{1,2,3\}^{2}\right\} \\
& W_{1}=\operatorname{span}\left\{u_{i} \otimes v_{j}:(i, j) \in\{1,2,3\}^{2} \backslash\{(3,3)\}\right\} \\
& W_{2}=\operatorname{span}\left\{u_{i} \otimes v_{j}:(i, j) \in\{4,5\}^{2}\right\} \\
& W_{3}=\operatorname{span}\left\{u_{i} \otimes v_{j}:(i, j) \in\{6,7\}^{2}\right\}
\end{aligned}
$$

Assume for sake of contradiction that $\operatorname{dim}\left(W_{1}+W_{2}+W_{3}\right)=16$. Note that there exists $w_{2} \in W_{0} \cap W_{2}$ and $w_{3} \in W_{0} \cap W_{3}$. Since $W_{1}, W_{2}$, and $W_{3}$ are linearly independent, then $W_{1}, w_{2}, w_{3}$ are linearly independent. This implies that $\operatorname{dim}\left(W_{1}+\operatorname{span}\left\{w_{2}\right\}+\operatorname{span}\left\{w_{3}\right\}\right)=10$, which contradicts the fact that $W_{1}+\operatorname{span}\left\{w_{2}\right\}+$ $\operatorname{span}\left\{w_{3}\right\} \subseteq W_{0}$.

Note that the example of Proposition 3.12 is an obstruction to extending the proof of Proposition 3.4 to $s=4$. In the proof of Proposition 3.4 once we reduce to the $S_{3}=\emptyset$ case, we show (through six cases) that a set $E \subseteq[m] \times[n]$ is dependent if and only if there is a tensor product $R=L \otimes \mathbf{k}^{r}$ for some $L \subseteq \mathbf{k}^{s}$ and a partition of $E_{1}, E_{2}, \ldots, E_{\ell}$ of $E$ such that $\sum_{i=1}^{\ell} \operatorname{dim}\left(\left(\operatorname{span} E_{i}\right) \cap R\right)>r \operatorname{dim} L$. This gives a family of inequalities that are satisfied by independent sets, and by this method we obtains all the inequalities in Proposition 3.4. To prove the sufficiency of these inequalities, we use an inductive argument: when one such inequality is tight then we can quotient by the tensor subspace $L \otimes \mathbf{k}^{r}$ and work over the smaller space $\left(\mathbf{k}^{s} / L\right) \otimes \mathbf{k}^{r}$. In the example of Proposition 3.12, the dependence is detected by looking at a subspace $L \otimes L$ in $\mathbf{k}^{4} \otimes \mathbf{k}^{4}$, where $\operatorname{dim} L=3$. As $\left(\mathbf{k}^{4} \otimes \mathbf{k}^{4}\right) /(L \otimes L)$ is not naturally the tensor product of two spaces, the simple inductive argument of Proposition 3.4 will not work.

## 4. Characteristic independence

In this section, we prove Theorem [1.6. The cases $s=0$ and $m=s$ are trivial. The descriptions of $\mathrm{T}_{m, n}(s, r, p)$ when $s \leq 3$ (Section 3) or when $\left.m-s=1\left(\mathrm{GHK}^{+} 17\right]\right)$ are independent of the characteristic, so it remains to do the case when $m-s=n-r=2$. For this we use Bernstein's description of the independent sets of $\mathcal{B}_{m, n}(2,2)$, Theorem 1.7, which gives a description of $\mathrm{T}_{m, n}(m-2, n-2,0)$. The following result shows that the dependent sets of $\mathrm{T}_{m, n}(m-2, n-2,0)$ are also dependent in $\mathrm{T}_{m, n}(m-2, n-2$, $p$ ), so we only need to show that the bases of $\mathrm{T}_{m, n}(m-2, n-2,0)$ are bases of $\mathrm{T}_{m, n}(m-2, n-2, p)$.

Proposition 4.1. Any independent set of $\mathrm{T}_{m, n}(s, r, p)$ is an independent set of $\mathrm{T}_{m, n}(s, r, 0)$.
Proof. The proof of Theorem 1.1 shows that one can check if a set is independent in $\mathrm{T}_{m, n}(s, r, 0)$ in terms of the rank of a matrix whose entries are polynomials with integer coefficients in $\left\{x_{i j}, y_{k \ell}\right\}$. We can check if a set is independent in $\mathrm{T}_{m, n}(s, r, p)$ by taking the same matrix and computing the rank over a field of characteristic $p$.

We now show that the independent sets of $\mathcal{B}_{m, n}(2,2)$, as described in Theorem 1.7, are still independent in the dual of $\mathrm{T}_{m, n}(m-2, n-2, p)$ for any $p$.

Proposition 4.2. If a bipartite graph has an edge orientation with no directed cycles or alternating cycles, then the graph is independent in the dual of $\mathrm{T}_{m, n}(m-2, n-2, p)$ for any $p$.

Proof. Let $G$ be a bipartite graph on $[m] \sqcup[n]$ which has an edge orientation with no directed cycles or alternating cycles. Let $M_{G}$ be the $2|V(G)| \times|E(G)|$ matrix obtained by taking the columns of the matrix in Definition 2.3 (with $a=b=2$ ) indexed by edges of $G$. Instead of using the variables $\left\{x_{i j}, y_{k \ell}\right\}_{(i, j) \in[m] \times[2],(k, \ell) \in[n] \times[2]}$, it will be convenient to use the variables $\left\{x_{v c}\right\}_{v \in V(G), c \in\{1,2\}}$. Set $2 V(G)=\left\{x_{v c}: v \in V(G), c \in\{1,2\}\right\}$ to be the set of variables. We will show that there a maximal minor of $M_{G}$ for which some monomial occurs with coefficient $\pm 1$, so $M_{G}$ has the same rank in any characteristic. The proof of Theorem 1.1(3) then implies the result.

For each injective map $\sigma: E(G) \rightarrow 2 V(G)$, we set $M_{\sigma}:=\prod_{e \in E(G)} M_{\sigma(e), e}$. This is one term in the expansion of a maximal minor of $M_{G}$. Since the column of $M_{G}$ corresponding to an edge $e=(u, v)$ has only 4 non-zero entries $x_{u 1}, x_{u 2}, x_{v 1}$, and $x_{v 2}$, there are only a few $M_{\sigma}$ that are non-zero. A non-zero $M_{\sigma}$ can be represented by a 2 -colored directed version of $G$ (denoted $G_{\sigma}$ ): if $\sigma(e)=x_{u c}$, we color $(u, v)$ with color $c$ and direct $u \rightarrow v$. Set $\operatorname{indeg}_{c, G_{\sigma}}(v)$ to be the number of edges directed towards $v$ in $G_{\sigma}$ of color $c$, and similarly define outdeg ${ }_{c, G_{\sigma}}$. In order for the map $\sigma$ to be injective, the outdegree of each vertex in $G_{\sigma}$ must be at most 1 per color. In fact outdeg ${ }_{c, G_{\sigma}}(v)=1$ if $x_{v c} \in \sigma(E(G))$ and 0 otherwise. We have

$$
M_{\sigma}=\prod_{v \in V(G), c \in\{1,2\}} x_{v c}^{\operatorname{indeg}_{c, G_{\sigma}}(v)}
$$

Using the acyclic orientation of $G$ with no alternating cycles, we construct a $G_{\sigma}$ for which the corresponding monomial $M_{\sigma}$ does not occur for any other injective map $\sigma^{\prime}: E(G) \rightarrow 2 V(G)$. We color the edges of $G$ with color 1 if they are oriented from $[m]$ to $[n]$, and we use color 2 if they are oriented from $[n]$ to $[m]$. This coloring has no monochromatic cycles or cycles which alternate in color. Now we pick out a root for every monochromatic connected tree, and we direct the edges towards the root. In this way, we obtain a directed 2-coloring of $G$, say $G_{\sigma}$. As outdeg ${ }_{c, G_{\sigma}}(v) \leq 1$ for each $v$ and $c$, this $G_{\sigma}$ does indeed arise from an injective $\operatorname{map} \sigma: E(G) \rightarrow 2 V(G)$. Note that directions on the edges are not the same as the orientation we used to construct $G_{\sigma}$.

We claim that the monomial $M_{\sigma}$ only appears once in the determinant of the minor of $M_{G}$ with rows indexed by $\sigma(E(G))$, which is a maximal minor of $M_{G}$. Otherwise, there is another $G_{\sigma^{\prime}}$ with the same
monomial weight. Therefore

$$
\operatorname{indeg}_{c, G_{\sigma}}(v)=\operatorname{indeg}_{c, G_{\sigma^{\prime}}}(v), \text { for any } v \in V(G) \text { and each color } c \in\{1,2\}
$$

Note that outdeg ${ }_{c, G_{\sigma}}(v)=$ outdeg $_{c, G_{\sigma^{\prime}}}(v)$ for all $c$, $v$ since $\sigma(E(G))=\sigma^{\prime}(E(G))$ as they appear in the same maximal minor and the range of $\sigma$ determines which vertices have outdegree 1 for each color. Therefore

$$
\operatorname{deg}_{c, G_{\sigma}}(v)=\operatorname{deg}_{c, G_{\sigma^{\prime}}}(v), \text { for any } v \in V(G) \text { and color } c \in\{1,2\}
$$

Note that $G_{\sigma}$ and $G_{\sigma^{\prime}}$ have to give different colors to at least one edge. Indeed, suppose they give the same color to each edge. Since $G_{\sigma}$ has no monochromatic cycles, every monochromatic strongly connected component of $G_{\sigma}$ is a tree. Each tree has a unique vertex with outdeg ${ }_{c, G_{\sigma}}=0$. We make that vertex the root of the tree and direct all edges of the tree towards the root. This recovers the directions for the edges of $G_{\sigma}$, as this is how $G_{\sigma}$ was defined. But the same process can be followed for $G_{\sigma^{\prime}}$ to arrive at the same directions for the edges, i.e., once the outdegrees for each color are fixed, there is only one way to direct the graph. This implies that $\sigma=\sigma^{\prime}$, so $G_{\sigma}$ and $G_{\sigma^{\prime}}$ have to differ in at least one color if they are distinct.

However, consider the following process. We start with a vertex $v_{0}$ that is adjacent to an edge $e=\left(v_{0}, v_{1}\right)$ which is given a different color in $G_{\sigma}$ and $G_{\sigma^{\prime}}$. We can assume $e$ has color 1 in $G_{\sigma}$ and color 2 in $G_{\sigma^{\prime}}$. Since $\operatorname{deg}_{c, G_{\sigma}}\left(v_{1}\right)=\operatorname{deg}_{c, G_{\sigma^{\prime}}}\left(v_{1}\right)$, there must be another edge $e^{\prime}=\left(v_{1}, v_{2}\right)$ adjacent to $v_{1}$ that has color 2 in $G_{\sigma}$ and color 1 in $G_{\sigma^{\prime}}$. Now we can use the same argument repeatedly find a path $v_{0} \rightarrow v_{1} \rightarrow v_{2} \rightarrow \cdots$ which eventually self-intersects and forms a loop. However, this loop will be a cycle which alternates in color in $G_{\sigma}$, but there are no such cycles in $G_{\sigma}$.
Remark 4.3. The argument above can also be applied to rank 2 skew-symmetric matrix completion. A 2-coloring of the edges of a graph $G$ is unbalanced if it has no monochromatic cycles or trails which alternate in color. An acyclic orientation of the edges of $G$ is unbalanced if it has no alternating trail. When $G$ is bipartite, an unbalanced coloring is equivalent to an unbalanced acyclic orientation, so Theorem 1.7 states that $G$ is independent in $\mathcal{B}_{m, n}(2,2)$ if and only if it has an unbalanced coloring.

In Ber17, Bernstein proved that a graph $G$ has an unbalanced acyclic orientation if and only if it is independent in $\mathcal{H}_{n}(2)$. Extending the proof of Proposition4.2, one can show that a graph with an unbalanced coloring is independent in the dual of $\mathrm{W}_{n}(n-2, p)$ for any $p$. This implies that a graph with an unbalanced coloring has an unbalanced acyclic orientation, but we do not know a combinatorial proof of this fact. We do not know if the converse holds.
Proof of Theorem 1.6. The case when $s=0$ or $s=m$ is trivial. The case when $m-s=1$ is proven in GHK ${ }^{+17] \text {. The case when } 1 \leq s \leq 3 \text { is proven in Corollary 3.2, Proposition 3.3, and Proposition 3.4. The }}$ case when $m-s=n-r=2$ is proven in Proposition4.2.

## 5. Conjectural Description of the bipartite rigidity matroid

We now give a conjectural description of the independent sets of $\mathcal{B}_{m, n}(d, d)$ for all $d$. Using Proposition 3.10, this gives a description of the independent sets of $\mathcal{B}_{m, n}(a, b)$ for all $a$ and $b$. Our conjecture is inspired by Bernstein's proof of Theorem 1.7 using tropical geometry Ber17. We show that the sets we describe are in fact independent in $\mathcal{B}_{m, n}(d, d)$, and moreover are independent in the dual of $\mathrm{T}_{m, n}(m-d, n-d, p)$ for all $p$. In particular, our conjecture implies that tensor matroid is independent of the characteristic.
Definition 5.1. Given a bipartite graph $G$ and an integer $d \geq 0$, a $d$-coloring of the edges of $G$ is $d$ Bernstein if there are no monochromatic cycles and there exists a labeling $c: V(G) \rightarrow \mathbb{R}^{d}$ which sends $v \mapsto\left(c_{1}(v), c_{2}(v), \ldots, c_{d}(v)\right)$ that satisfies the following conditions:
(1) $c_{1}(v)+c_{2}(v)+\cdots+c_{d}(v)=0$ for any $v \in V(G)$;
(2) for every edge $(u, v) \in E(G)$ with color $i, c_{i}(u)+c_{i}(v)>c_{j}(u)+c_{j}(v)$ for any $j \in[d] \backslash\{i\}$.

Remark 5.2. When $d=2$, we recover Bernstein's condition in Theorem 1.7] if we orient the edges with color 1 from $[m]$ to $[n]$ and orient the edges with color 2 from $[n]$ to $[m]$.
Proposition 5.3. If a bipartite graph $G$ admits a d-coloring that is $d$-Bernstein, then $G$ is independent in the dual of $\mathrm{T}_{m, n}(m-d, n-d, p)$ for all $p$.
Proof. The proof is almost identical to the proof of Proposition 4.2 except in the last step. The statement reduces to proving that if there is a $d$-Bernstein coloring of $G$ say $G_{\sigma}$, then there does not exist a another $d$-coloring of $G$ say $G_{\sigma^{\prime}}$, where $G_{\sigma}$ and $G_{\sigma^{\prime}}$ differ in at least one edge color, and

$$
\operatorname{deg}_{c, G_{\sigma}}(v)=\operatorname{deg}_{c, G_{\sigma^{\prime}}}(v), \text { for any } v \in V(G) \text { and color } c \in[d]
$$

Consider the following equality:

$$
\sum_{v \in V(G), c \in[d]} \operatorname{deg}_{c, G_{\sigma}}(v) c_{i}(v)=\sum_{e=(u, v) \in E(G)}\left(c_{\sigma(e)}(u)+c_{\sigma(e)}(v)\right), \text { where } \sigma(e) \text { is the color of } e \text { in } G_{\sigma} .
$$

This equality also holds when we replace $\sigma$ with $\sigma^{\prime}$. However, as we change from $\sigma$ to $\sigma^{\prime}$, the left hand side does not change, but the right hand side strictly decreases because of condition (2) in Definition 5.1 and $\sigma, \sigma^{\prime}$ give different colors to at least one edge.

Conjecture 5.4. If $G$ is independent in $\mathcal{B}_{m, n}(d, d)$, then $G$ admits d-coloring that is $d$-Bernstein.
When $d=1$, this holds because $\mathcal{B}_{m, n}(1,1)$ is the graphical matroid of the complete bipartite graph $\mathrm{K}_{m, n}$. When $d=2$, this is proven in Ber17. We have checked this conjecture for $\mathcal{B}_{5,5}(3,3)$ and for $\mathcal{B}_{m, n}(d, d)$ when $d \geq m-1$. This conjecture and Proposition 3.10 imply that $\mathrm{T}_{m, n}(s, r, 0)=\mathrm{T}_{m, n}(s, r, p)$ for all $p$.
Remark 5.5. The $d$-Bernstein condition is closely related to the notion of Barvinok rank for tropical matrices DSS05. Conjecture 5.4 is equivalent to saying that the Barvinok rank $d$ cones, which are a subset of cones in the tropical determinantal variety, determine the matroid $\mathcal{B}_{m, n}(d, d)$.

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[^0]:    ${ }^{1}$ More commonly, information theorists care about the codimension of these spaces, often denoted by $a=m-s$ and $b=n-r$, respectively, as that is a measure of the redundancy of the encoding.

[^1]:    ${ }^{2}$ In characteristic 2, we say that a matrix is skew-symmetric if it is symmetric with zeroes on the diagonal. It remains true that, over an algebraically closed field, an $n \times n$ matrix can be written as $A B^{t}-B A^{t}$ with $A, B n \times d$ matrices if and only if it is skew-symmetric and has rank at most $2 d$.

[^2]:    ${ }^{3}$ And this is not a characteristic-dependent condition.

