CAPS AND WICKETS

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Abstract. Let $H_n^{(3)}$ be a 3-uniform linear hypergraph, i.e. any two edges have at most one vertex common. A special hypergraph, *wicket*, is formed by three rows and two columns of a 3×3 point matrix. In this note, we give a new lower bound on the Turán number of wickets using estimates on cap sets.

1. Introduction

Gyárfás and Sárközy asked for upper and lower bounds on the Turán number of a linear hypergraph called the wicket. Although it was not obvious in the original formulation, it seems this problem is at the crossing point of important questions in additive combinatorics and extremal hypergraph theory. To formulate the problem, let's begin with the basic definitions.

Definition. *A hypergraph is linear if two edges intersect in at most one vertex.*

We will work with 3-uniform linear hypergraphs, i.e., where every edge has three vertices.

Definition. *The Turán number of a linear 3-uniform hypergraph F, denoted by $ex_L(n, F)$, is the maximum number of edges of a 3-uniform linear hypergraph not containing a subgraph isomorphic to F.*

In [6] Gyárfás and Sárközy investigated $ex_L(n, F)$ for any F with at most five edges. They had good estimates except for one configuration, called *wicket*. The wicket, denoted by W, is formed by three rows and two columns of a 3×3 point matrix (fig. 1). Wickets are important structures in extremal hypergraph theory. Using a classical technique introduced by Ruzsa and Szemerédi in [9], we show the connection between its Turán number and solution sets of equations like in (1). There are also connections to a conjecture of Gowers and Long which we will describe later.

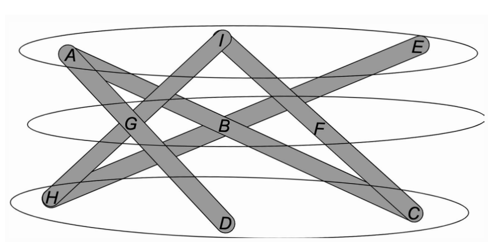

Figure 1. The wicket is drawn as a three-partite hypergraph. If we add the edge spanned by vertices D, E, F, it is isomorphic to a 3×3 grid.

In a recent paper [10], it was proved that $ex_L(n,W) = o(n^2)$. Only $ex_L(n,W) \geq cn^{3/2}$ type bounds were known for the lower bound [6]. Our goal is to improve this bound.

2. The Main Result

The wicket has cycles of length four, so hypergraphs avoiding quadrilaterals are wicket-free, giving the $ex_L(n,W) \geq cn^{3/2}$ bound [7]. In our improvement, the new construction follows the steps of the classical work of Ruzsa and Szemerédi [9]. In their construction, they defined a three-partite 3-uniform hypergraph. The three vertex classes, A, B, C, are three copies of $\mathbb{Z}/n\mathbb{Z}$. Let us suppose $S \subset \mathbb{Z}/n\mathbb{Z}$ is AP_3-free. In the hypergraph, three vertices $a \in A, b \in B$ and $c \in C$ are connected by an edge if there is an $s \in S$ such that $b = a + s$ and $c = a + 2s$. In this setting, no six vertices carry three edges since it would rise to a solution of $s + t = 2h$ (see in fig. 2).

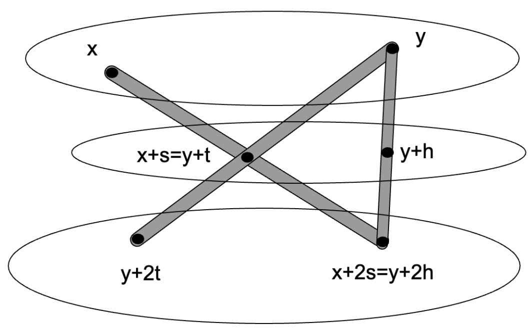

Figure 2. After eliminating x and y we get $s + t = 2h$

Let's follow the same method as above, but here the three vertex classes, A, B, C, are three copies of $\mathbb{F}_3^n$. Let $S \subset \mathbb{F}_3^n$ denote a maximal subset without 3-term arithmetic progressions (cap sets). It was proved recently in [2] that there are cap sets of size 2.2202^n, improving earlier bounds in [3, 11]. The 2.756^n upper bound on the size of the largest cap set was proved in [4], following bounds on AP_3-free sets in $\mathbb{F}_4^n$ in [1]. In the hypergraph, denoted by $\mathcal{H}^{(3)}(A, B, C)$, three vertices $a \in A, b \in B$ and $c \in C$ are connected by an edge if there is an $s \in S$ such that $b = a + s$ and $c = a + 2s$. A wicket would define four linear equations, as illustrated in fig. 3.

$$\begin{aligned} x + s &= y + t \\ x + 2s &= z + 2v \\ y + u &= z + v \\ x + 2w &= y + 2u \end{aligned}$$

After eliminating x, y, z and s in $\mathbb{F}_3^n$, we get the $w + v = 2t$ equation. This equation has no solution for distinct t, v, w values in S since it is a cap set in $\mathbb{F}_3^n$. The hypergraph $\mathcal{H}^{(3)}(A, B, C)$ has 3^{n+1} vertices and $(3 \cdot 2.2202)^n = 6.6606^n$ edges. The construction improves the bound to $ex_L(m, W) \geq m^{1.726}$. It also avoids the $(6, 3)$-configuration from fig. 2.

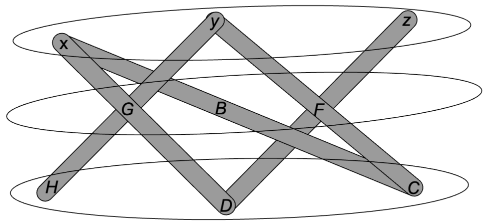

FIGURE 3. In the wicket, $G = x + s = y + t, D = x + 2s = z + 2v, F = y + u = z + v$ and $C = x + 2w = y + 2u$.

Gowers and Long conjectured (Conjecture 4.6 in [5]) that there is a $c > 0$ such that if a 3-uniform linear hypergraph on n vertices has no nine vertices spanning at least five edges, then its number of edges is $O(n^{2-c})$. By losing at most a constant fraction of the edges, every 3-uniform hypergraph can be arranged into a 3-partite hypergraph. In the previous construction, $\mathcal{H}^{(3)}(A, B, C)$ has at most four edges on any nine vertices. To see that, note that a 3-partite linear hypergraph with five edges on nine vertices contains a triangle (fig. 2) or a wicket. So, in the Gowers-Long conjecture, the constant can't be too large for the five-edge-nine-vertex case. It is $c < 0.274$ (improving the earlier known $c \leq 0.5$ bound).

3. Remarks

The hypergraph construction was based on the observation that the

$$3x + y = 2z + 2w \tag{1}$$

equation has no non-trivial solutions in cap sets in $\mathbb{F}_3^n$. This equation is a classic example from Ruzsa's work on solutions of linear equations in integers [8]. It is not known what is the size of the largest subset of the first n natural numbers without non-trivial solutions to (1). It is possible that there are sets $S \subset [n]$ without non-trivial solutions, with $|S| = n^{1-o(1)}$. It would give the $ex_L(m, W) = m^{2-o(1)}$ bound as conjectured in [10]. But for further improvements, it would be enough to find any abelian group with a large subset without a non-trivial solution to (1) or a similar linear equation.

4. Acknowledgements

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