COVARIANT SCHRÖDINGER OPERATOR AND L^2-VANISHING PROPERTY ON RIEMANNIAN MANIFOLDS

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ABSTRACT. Let M be a complete Riemannian manifold satisfying a weighted Poincaré inequality, and let $\mathcal{E}$ be a Hermitian vector bundle over M equipped with a metric covariant derivative ∇. We consider the operator $H_{X,V} = \nabla^{\dagger}\nabla + \nabla_X + V$, where $\nabla^{\dagger}$ is the formal adjoint of ∇ with respect to the inner product in the space of square-integrable sections of $\mathcal{E}$, X is a smooth (real) vector field on M, and V is a fiberwise self-adjoint, smooth section of the endomorphism bundle $\operatorname{End}\mathcal{E}$. We give a sufficient condition for the triviality of the L^2-kernel of $H_{X,V}$. As a corollary, putting $X \equiv 0$ and working in the setting of a Clifford bundle equipped with a Clifford connection ∇, we obtain the triviality of the L^2-kernel of D^2, where D is the Dirac operator corresponding to ∇. In particular, when $\mathcal{E} = \Lambda^k T^*M$ and D^2 is the Hodge–deRham Laplacian on k-forms, we recover some recent vanishing results for L^2-harmonic k-forms.

1. INTRODUCTION

For many years mathematicians have studied the triviality property of the space $\mathscr{K}_{\Delta}$ of L^2-harmonic k-forms on complete Riemannian manifolds without boundary,

$$\mathscr{K}_{\Delta} := \{\omega \in L^2 \colon \Delta\omega = 0\},$$

where $\Delta := d\delta + \delta d$ is the Hodge–deRham Laplacian acting k-forms (here, d and δ are the standard differential and codifferential).

Topological significance of $\mathscr{K}_{\Delta} = \{0\}$ on a compact Riemannian manifold M is clear if we remember that the space $\mathscr{K}_{\Delta}$ is isomorphic to the k-th de Rham cohomology group of M. While this isomorphism is generally not present in the setting of a non-compact Riemannian manifold M, it turns out that the triviality of $\mathscr{K}_{\Delta}$ may still offer some topological insights: for example, the authors of [19] showed that if M has no parabolic ends and $\mathscr{K}_{\Delta} = \{0\}$, where Δ is Hodge–deRham Laplacian acting on 1-forms, then M is connected at infinity.

Some forty years ago, the author of [7] introduced an elegant method for tackling the problem of triviality of $\mathscr{K}_{\Delta}$ pertaining to k-forms on a complete Riemannian manifold M. Using Weitzenböck formula (see (3.7) below) and a suitable sequence of cut-off functions (whose existence is guaranteed by the completeness of M; see section 4.2 below for details), the author

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of [7] showed, among other things, that if the volume of M is infinite and the Weitzenböck curvature operator $\mathscr{R}^W$ is non-negative definite, then $\mathscr{K}_\Delta = \{0\}$, generalizing an earlier result of [26] pertaining to 1-forms.

In subsequent years, a number of authors have refined the integration by parts technique from [7], aiming to accommodate various assumptions on M and $\mathscr{R}^W$ (in the case of 1-forms, $\mathscr{R}^W$ reduces to Ricci tensor Ric_M). About twenty years ago, in the context of 1-forms, the authors of [18], showed that if M (with $\dim M = n$) satisfies $\lambda_1(M) > 0$ and $\mathrm{Ric}_M \geq -\frac{n\lambda_1(M)}{n-1} + \varepsilon$, for some $\varepsilon > 0$, then $\mathscr{K}_\Delta = \{0\}$ (see (3.9) below for the definition of the first eigenvalue $\lambda_1(M)$). Later, this result was generalized in [16] to manifolds satisfying Poincaré inequality with a (continuous) weight $\rho \geq 0$:

$$\int_M \rho(x)|f(x)|^2\, d\nu_g(x) \leq \int_M |df(x)|^2 d\nu_g(x), \tag{1.1}$$

for all $f \in C_c^\infty(M)$, where $C_c^\infty(M)$ denotes smooth compactly supported functions on M and $d\nu_g$ is the volume element on M induced by the metric g. The author of [16] imposed a certain condition on the growth of ρ and the following condition on the Ricci tensor: $\mathrm{Ric}_M \geq -\frac{n\rho}{n-1} + \varepsilon$, for some $\varepsilon > 0$.

Subsequent to [16], the author of [25] proved (see theorem 5 there) that $\mathscr{K}_\Delta = \{0\}$ for k-forms under the following assumptions: M satisfies (1.1) (without growth-rate or sign restrictions on ρ), M has infinite volume or ρ is not identically equal to 0, and Weitzenböck curvature operator satisfies $\mathscr{R}^W \geq -a\rho$, where $a \in [0, a_0)$ is a constant (here, the constant a_0 comes from the refined Kato inequality for k-forms). A related vanishing result for L^q-harmonic $(0,k)$-tensors with $q \geq 2$ (here, "harmonic" is meant with respect to the Lichnerowicz Laplacian) was established by the authors of [4] under the following assumptions: M satisfies (1.1), M is non-parabolic, $\liminf\limits_{x\to\infty} \rho(x) > 0$, and the curvature condition $\mathscr{C} \geq -a\rho$, where $a \in [0, a_0)$, with a_0 depending (among other things) on q. (Here, $\mathscr{C}$ is a suitable curvature operator; see section 2 in [4] for details.) The paper [4] (see also [5] for the Kähler manifold setting) gives a number of vanishing results (for $(0,k)$-tensors and k-forms) in which the requirement $\mathscr{C} \geq -a\rho$ is replaced by more explicit conditions involving eigenvalues of $\mathscr{C}$.

By performing a careful analysis of the Weitzenböck curvature operator, the author of [20] established two types of vanishing results for k-forms: (i) theorems based on integral-type assumptions on the Weyl curvature tensor W, traceless Ricci tensor E, and scalar curvature, and (ii) theorems based on the assumption (1.1) and pointwise assumptions on W and E.

Over the last fifteen years, some authors have studied vanishing property assuming weighted Poincaré inequality for k-forms (with continuous weight $\rho \geq 0$):

$$\int_M \rho(x)|\omega(x)|^2\, d\nu_g(x) \leq \int_M (|d\omega(x)|^2 + |\delta\omega(x)|^2)\, d\nu_g(x), \tag{1.2}$$

for all ω smooth compactly supported k-forms ω.

Assuming (1.2) with some growth restrictions on ρ, the vanishing property for harmonic k-forms was established in [8] and, subsequently, in [10]. Recently, the author of [27] proved (see theorem 1.4 there) that $\mathscr{K}_{\Delta}=\{0\}$ under the following assumptions: M satisfies (1.2), ρ is not identically equal to 0, and Weitzenböck curvature operator satisfies $\mathscr{R}^W \geq -a\rho$, where $a \geq 0$ is a constant. Recently, the authors of [22] established several vanishing theorems for k-forms assuming (1.2) together with pointwise conditions on Weyl conformal curvature tensor and traceless Ricci tensor.

As can be seen from the preceding paragraphs, in recent years there has been quite a bit of activity on the L^2-vanishing property for harmonic k-forms. For the corresponding studies in the setting of p-harmonic k-forms, we refer the reader to [2, 9, 11, 21] and references therein. (Here, a k-form ω is p-harmonic, $p>1$, if $d\omega=0$ and $\delta(|\omega|^{p-2}\omega)=0$.) For L^q_f-vanishing results (in some papers $q=2$) in the context of 1-forms on smooth metric-measure spaces (Riemannian manifolds (M,g) with metric g and measure $e^{-f}\,d\nu_g$, where f is a smooth function on M and $d\nu_g$ is the volume measure induced by the metric g), see [3, 13, 24, 28, 29] and references therein.

Before describing the results of our article, we note the paper [1], which is situated in the setting of a Hermitian vector bundle $\mathcal{E}$ (over a complete Riemannian manifold M), equipped with a metric covariant derivative ∇; see section 2.3 below for details. Denoting by $\Gamma_{L^2}(\mathcal{E})$ the square integrable sections and by $\nabla^{\dagger}$ the formal adjoint of ∇ (with respect to the inner product in $\Gamma_{L^2}(\mathcal{E})$), the author of [1] considered the *covariant Schrödinger operator*

$$H_V=\nabla^{\dagger}\nabla+V,$$

where V is a fiberwise self-adjoint, smooth section of the endomorphism bundle $\operatorname{End}\mathcal{E}$. Let us the denote L^2-kernel of H_V by

$$\mathscr{K}_{H_V}:=\{u\in\Gamma_{L^2}(\mathcal{E})\colon H_Vu=0\}. \tag{1.3}$$

In the paper [1] the author observed that in the case $\mathcal{E}=\Lambda^kT^*M$, the Weitzenböck formula (see (3.7) below) leads to the following equality: $\mathscr{K}_{\Delta}=\mathscr{K}_{H_V}$, where $V=\mathscr{R}^W$ and Δ is the Hodge–deRham Laplacian on k-forms (here, $\mathscr{R}^W$ is the Weitzenböck curvature operator). Thus, establishing the L^2-vanishing property for k-forms amounts to proving that $\mathscr{K}_{H_V}=\{0\}$. In particular, the author of [1] showed that $\mathscr{K}_{H_V}=\{0\}$ provided that M satisfies a Sobolev p-type inequality with $p>2$, and that $|V_-|$ satisfies a certain integral-type condition (here, V_- is the negative part of V). In the recent years, the operator H_V has been studied extensively; see the book [6].

In our article we consider the operator

$$H_{X,V}=\nabla^{\dagger}\nabla+\nabla_X+V,$$

where ∇, $\nabla^{\dagger}$ are as in the preceding paragraph and X is a real, smooth (generally unbounded) vector field on M and V is a fiberwise self-adjoint, smooth section of the endomorphism bundle $\operatorname{End}\mathcal{E}$. We define $\mathscr{K}_{H_{X,V}}$ as in (1.3) with $H_{X,V}$ in place of H_V.

In theorem 3.1 we prove that $\mathscr{K}_{H_{X,V}} = \{0\}$ under the following assumptions: M satisfies (1.1), M has infinite volume or ρ is not identically equal to zero, $|X| \leq \hat{a}\sqrt{\rho}$ and $V - \operatorname{div} X \geq -a\rho$, with constants $0 \leq \hat{a} < 1$ and $0 \leq a < 1 - \hat{a}$ (here, $\operatorname{div} X$ is the divergence of X). As a corollary, putting $X \equiv 0$ and working in the setting of a Clifford bundle $\mathcal{E}$ equipped with a Clifford connection ∇ (see section 3.2 below for details) we get $\mathscr{K}_{D^2} = \{0\}$, where D is the Dirac operator corresponding to ∇ and $\mathscr{K}_{D^2}$ is the L^2-kernel of D^2. In particular, when $\mathcal{E} = \Lambda^k T^*M$ and D^2 is the Hodge–deRham Laplacian on k-forms, we recover theorem 5 of [25] with $\rho \geq 0$.

In theorem 3.6 we accomplish the same goal as in theorem 3.1, with the following hypotheses on X and V: $|X| \leq \hat{a}\sqrt{\rho}$ with $0 \leq \hat{a} < 1$, $V - \operatorname{div} X \geq -a\rho - b$ with $0 \leq a < 1 - \hat{a}$ and $b \geq 0$, and the condition $\lambda_1(M) > b/(1 - a - \hat{a})$ (see (3.9) for the definition of the first eigenvalue $\lambda_1(M)$). As a corollary, putting $X \equiv 0$ and specializing to k-forms, we recover theorem 6 of [25] with $\rho \geq 0$.

In theorems 3.9 and 3.11 we work in the setting of a Clifford bundle $\mathcal{E}$ (over a complete Riemannian manifold M) equipped with a Clifford connection ∇. We extend some vanishing results of [27] to D^2, the square of the Dirac operator D corresponding to ∇, assuming weighted Poincaré inequality for D (see (3.13) for precise formulation) and curvature conditions analogous to those in [27].

The paper is organized into five sections. After describing the notations, operators, and function spaces in section 2, we state the main results in section 3. The proofs of the main results are carried out in sections 4 and 5.

2. Description of Notations, Function Spaces, and Operators

2.1. **Basic Notations.** In this paper we work in the setting of a connected Riemannian n-manifold (M, g) without boundary. The symbol $d\nu_g$ denotes the volume measure on M: in local coordinates $x^1, x^2, \dots, x^n$, we have $d\nu_g = \sqrt{\det(g^{ij})}\, dx$, where (g^{ij}) is the inverse of the matrix $g = (g_{ij})$ and $dx = dx^1\, dx^2 \dots dx^n$ is the Lebesgue measure.

We use the notations TM, T^*M for tangent and cotangent bundles of M respectively. Additionally, $\Lambda^k T^*M$ stands for the k-th exterior power of the cotangent bundle T^*M. The metric $g = (g_{ij})$ on TM gives rise (in the usual way) to the Euclidean structure on T^*M, and this, in turn, leads to the Euclidean structure on the bundle $\Lambda^k T^*M$.

Throughout the paper, $\mathcal{E} \to M$ is a smooth Hermitian vector bundle over M equipped with a Hermitian structure $\langle \cdot, \cdot \rangle$, linear in the first and antilinear in the second variable. We use $|\cdot|_x$ to indicate the fiberwise norms on $\mathcal{E}_x$, usually writing just $|\cdot|$ to simplify the notations. The symbols $\Gamma_{C^\infty}(\mathcal{E})$ and $\Gamma_{C_c^\infty}(\mathcal{E})$ denote smooth sections of $\mathcal{E}$ and smooth compactly supported sections of $\mathcal{E}$, respectively. In particular, for smooth k-forms on M we use the symbol $\Gamma_{C^\infty}(\Lambda^k T^*M)$, and for their compactly supported analogues, the symbol $\Gamma_{C_c^\infty}(\Lambda^k T^*M)$. When talking about complex-valued functions on M, the corresponding spaces will be indicated by $C^\infty(M)$ and $C_c^\infty(M)$. Additionally, $C(M)$ denotes continuous (complex-valued) functions on M.

We will also use basic "musical" isomorphisms coming from g: for a vector field Y on M, the symbol $Y^\flat$ indicates the one-form associated to Y, while $\omega^\sharp$ refers to the vector field associated to the one-form ω.

Lastly, we recall that the Levi–Civita connection ∇^{LC} on M induces the Euclidean covariant derivative on $\Lambda^k T^*M$, which we also denote by ∇^{LC}.

2.2. **Description of L^p-spaces.** For $1 \leq p < \infty$, the notation $\Gamma_{L^p}(\mathcal{E})$ refers to the space of p-integrable sections of $\mathcal{E}$ with the norm

$$\|u\|_p^p := \int_M |u(x)|_x^p \, d\nu_g(x), \tag{2.1}$$

where $|\cdot|_x$ is the fiberwise norm in $\mathcal{E}_x$.

In the case $p = 2$ we get a Hilbert space $\Gamma_{L^2}(\mathcal{E})$ with the inner product

$$(u, v) = \int_M \langle u(x), v(x)\rangle_x \, d\nu_g(x). \tag{2.2}$$

For simplicity, we often drop the subscript x from $\langle\cdot,\cdot\rangle_x$ and $|\cdot|_x$ and simply write $\langle\cdot,\cdot\rangle$ and $|\cdot|$.

For the L^p-space of (complex-valued) functions on M we use the symbol $L^p(M)$, and in the formulas (2.1) and (2.2) we replace the fiberwise norm by the absolute value, and $\langle u(x), v(x)\rangle_x$ by $u(x)\overline{v(x)}$, where $\overline{z}$ indicates the conjugate of a complex number z.

2.3. **Covariant Schrödinger Operator.** With the basic function spaces in place, we turn to differential operators. The first operator is $\nabla\colon \Gamma_{C^\infty}(\mathcal{E}) \to \Gamma_{C^\infty}(T^*M \otimes \mathcal{E})$, a (smooth) metric covariant derivative on $\mathcal{E}$. Next, we have $\nabla^\dagger\colon \Gamma_{C^\infty}(T^*M \otimes \mathcal{E}) \to \Gamma_{C^\infty}(\mathcal{E})$, the formal adjoint of ∇ with respect to $(\cdot,\cdot)$, the inner product (2.2). Composing the latter two operators produces the so-called *Bochner Laplacian* $\nabla^\dagger\nabla$. In the case of functions, we have the usual differential $d\colon C^\infty(M) \to \Gamma_{C^\infty}(T^*M)$ and its formal adjoint $d^\dagger\colon \Gamma_{C^\infty}(T^*M) \to C^\infty(M)$, understood with respect to the inner product $(\cdot,\cdot)$ in $L^2(M)$. The composition $d^\dagger d$, denoted by Δ_M, is known as *the scalar Laplacian* on M. We note that in our article $\nabla^\dagger\nabla$ and Δ_M are non-negative operators.

For a smooth vector field Y, we define the divergence of Y as

$$\operatorname{div} Y := -d^\dagger(Y^\flat). \tag{2.3}$$

Let $V \in \Gamma_{C^\infty}(\operatorname{End}\mathcal{E})$ such that $V(x)\colon \mathcal{E}_x \to \mathcal{E}_x$ is a self-adjoint operator for all $x \in M$, and let X be a smooth, real vector field on M. We consider the expression

$$H_{X,V}u := \nabla^\dagger\nabla u + \nabla_X u + Vu. \tag{2.4}$$

We call the operator H_V *covariant Schrödinger operator with potential V and drift X.*

2.4. **The Space $\mathscr{K}_{H_{X,V}}$.** We define

$$\mathscr{K}_{H_{X,V}} := \{u \in \Gamma_{L^2}(\mathcal{E})\colon H_{X,V}u = 0\}. \tag{2.5}$$

Since V and X are smooth and since $H_{X,V}$ is an elliptic operator, it follows (by local elliptic regularity) that $\mathscr{K}_{H_{X,V}} \subseteq \Gamma_{L^2}(\mathcal{E}) \cap \Gamma_{C^\infty}(\mathcal{E})$.

3. Statements of Results

In the first two theorems we assume that M satisfies a weighted Poincaré inequality, which we describe as follows:

3.1. **Hypothesis (P1).** Let $\rho\colon M \to \mathbb{R}$. Assume that

(P1a) ρ is a continuous function such that $\rho(x) \geq 0$ for all $x \in M$;
(P1b) for all $f \in C_c^\infty(M)$ we have

$$\int_M \rho(x)|f(x)|^2\, d\nu_g(x) \leq \int_M |df(x)|^2 d\nu_g(x), \tag{3.1}$$

where $|\cdot|$ on the right hand side is the fiberwise norm in T_x^*M.

We are ready to state the first result.

Theorem 3.1. *Assume that M is a geodesically complete Riemannian manifold without boundary. Assume that M satisfies the hypothesis (P1). Additionally, assume that one of the following two conditions is satisfied:*

(m1) $\rho(x)$ is not identically equal to 0;
(m2) the volume vol(M) is infinite.

Let $\mathcal{E}$ be a Hermitian vector bundle over M equipped with a metric covariant derivative ∇. Let X be a smooth, real vector field on M such that

$$|X(x)| \leq \hat{a}\sqrt{\rho(x)}, \tag{3.2}$$

for all $x \in M$, where $0 \leq \hat{a} < 1$ is a constant, and $|\cdot|$ is the norm in T_xM.

Let $V \in \Gamma_{C^\infty}(\operatorname{End}\mathcal{E})$ be a fiberwise self-adjoint endomorphism such that

$$V(x) - \operatorname{div} X \geq -a\rho(x)I_x, \tag{3.3}$$

for all $x \in M$, where $0 \leq a < 1 - \hat{a}$ is a constant, with $\hat{a}$ as in (3.2). (Here, $\operatorname{div} X$ is as in (2.3), and $I_x\colon \mathcal{E}_x \to \mathcal{E}_x$ is the identity endomorphism. The inequality (3.3) is understood in quadratic-form sense in $\mathcal{E}_x$.)

Then, the set $\mathscr{K}_{H_{X,V}}$ from (2.5) has the following property: $\mathscr{K}_{H_{X,V}} = \{0\}$.

As we will describe below, theorem 3.1, in conjunction with Weitzenböck formula, leads to a vanishing result for the kernel of the square of the Dirac operator.

3.2. **Clifford Bundle.** By a *Clifford bundle* we mean a Hermitian vector bundle $\mathcal{E}$ over M satisfying the following two properties:

(i) each fiber $\mathcal{E}_x$ is a module over the Clifford algebra $\mathcal{C}(T_xM)$ and

$$\langle \xi \bullet u, v\rangle \;=\; \langle u, \xi \bullet v\rangle, \quad \text{for all } \xi \in T_xM \text{ and all } u, v \in \mathcal{E}_x,$$

where $\langle \cdot, \cdot \rangle$ is the fiberwise inner product in $\mathcal{E}_x$ and "$\bullet$" is the Clifford action.

(ii) $\mathcal{E}$ is endowed with a metric connection ∇ satisfying the property

$$\nabla_X(Y \bullet s) \;=\; (\nabla^{LC}_X Y) \bullet s + Y \bullet (\nabla_X s),$$

for all $s \in \Gamma_{C^\infty}(\mathcal{E})$ and vector fields Y, $X \in \Gamma_{C^\infty}(TM)$, where "$\bullet$" is the Clifford action and ∇^{LC} is the Levi–Civita connection on M.

The composition

$$\Gamma_{C^\infty}(\mathcal{E}) \xrightarrow{\nabla} \Gamma_{C^\infty}(T^*M \otimes \mathcal{E}) \xrightarrow{g} \Gamma_{C^\infty}(TM \otimes \mathcal{E}) \xrightarrow{\bullet} \Gamma_{C^\infty}(\mathcal{E})$$

defines a first-order differential operator

$$D\colon \Gamma_{C^\infty}(\mathcal{E}) \to \Gamma_{C^\infty}(\mathcal{E}), \tag{3.4}$$

called the *Dirac operator* corresponding to the Clifford bundle $(\mathcal{E}, \nabla)$.

Remark 3.2. Since M is geodesically complete, by theorem II.5.4 in [17], it follows that $D|_{\Gamma_{C_c^\infty}(\mathcal{E})}$ is an essentially self-adjoint operator in $\Gamma_{L^2}(\mathcal{E})$ whose self-adjoint closure in $\Gamma_{L^2}(\mathcal{E})$ we denote (again) by D.

Defining

$$\mathscr{K}_{D^2} := \{u \in \Gamma_{L^2}(\mathcal{E})\colon D^2u = 0\} \tag{3.5}$$

and using local elliptic regularity we see that $\mathscr{K}_{D^2} \subseteq \Gamma_{L^2}(\mathcal{E}) \cap \Gamma_{C^\infty}(\mathcal{E})$.

Referring again to theorem II.5.4 in [17], we have

$$\mathscr{K}_{D^2} = \mathscr{K}_D, \tag{3.6}$$

where

$$\mathscr{K}_D := \{u \in \Gamma_{L^2}(\mathcal{E})\colon Du = 0\}$$

Before stating a corollary of theorem 3.1, we recall a formula linking the Bochner Laplacian on a Clifford bundle $\mathcal{E}$ with the square D^2 of the corresponding Dirac operator D.

3.3. **Weitzenböck formula.** In the Clifford-bundle setting of section 3.2 we have (see proposition 10.4.1 in [23])

$$D^2u \;=\; \nabla^\dagger\nabla u + \mathscr{R}^W u, \tag{3.7}$$

and $\mathscr{R}^W \in \Gamma_{C^\infty}(\operatorname{End}\mathcal{E})$ is a fiberwise self-adjoint endomorphism.

More explicitly (see the formula (10.4.15) in [23]), if $\{e_j\}_{j=1}^n$ is a local orthonormal frame field and $\{v_j\}_{j=1}^n$ is the corresponding dual frame (here, $n = \dim M$), then

$$\mathscr{R}^W u = -\frac{1}{2}\sum_{j,k=1}^{n} v_j v_k R^{\nabla}(e_j, e_k)u,$$

for all $u \in \Gamma_{C^\infty}(\mathcal{E})$, where R^{∇} is the curvature tensor corresponding to the connection ∇.

For future reference, in this paper we call $\mathscr{R}^W$ *Weitzenböck curvature operator*.

The formula (3.7) and theorem 3.1 with $X \equiv 0$ and $V = \mathscr{R}^W$ lead to the following corollary:

Corollary 3.3. *Assume that M is a geodesically complete Riemannian manifold without boundary. Assume that M satisfies the hypothesis (P1). Additionally, assume that one of the following two conditions holds:*

(i) $\rho(x)$ is not identically equal to 0;
(ii) the volume vol(M) is infinite.

Let $\mathcal{E}$ be a Clifford bundle over M equipped with a Clifford connection ∇, and let D be the associated Dirac operator. Assume that the Weitzenböck curvature operator $\mathscr{R}^W$ satisfies the inequality

$$\mathscr{R}^W(x) \geq -a\rho(x)I_x, \tag{3.8}$$

for all $x \in M$, where $0 \leq a < 1$ is a constant. (Here, $I_x\colon \mathcal{E}_x \to \mathcal{E}_x$ is the identity endomorphism, and the inequality (3.8) is understood in quadratic-form sense in $\mathcal{E}_x$.)

Then, the set $\mathscr{K}_{D^2}$ from (3.5) has the following property: $\mathscr{K}_{D^2} = \{0\}$.

Remark 3.4. Corollary 3.3 can be applied in the setting of a spin manifold M and the associated spinor bundle $\mathcal{E}$, with D being the so-called *classical Dirac operator*. In this situation (see proposition 10.4.4 in [23]), $\mathscr{R}^W = \mathrm{scal}_M/4$ where scal_M is the scalar curvature of M (that is, the trace of the Ricci tensor). Corollary 3.3 can also be applied in the setting of the bundle $\mathcal{E} = \Lambda^k T^*M$ over an oriented Riemannian manifold M. As explained in section 10.1 of [23], the bundle $\mathcal{E} = \Lambda^k T^*M$ (with its natural metric and connection ∇^{LC}, as mentioned in section 2.1 above), has the structure of a Clifford bundle. In this situation, the associated Dirac operator D is the so-called *Gauss–Bonnet operator* operator $d + \delta$, where

$$d\colon \Gamma_{C^\infty}(\Lambda^k T^*M) \to \Gamma_{C^\infty}(\Lambda^{k+1} T^*M), \quad \delta\colon \Gamma_{C^\infty}(\Lambda^k T^*M) \to \Gamma_{C^\infty}(\Lambda^{k-1} T^*M).$$

are the standard differential and codifferential respectively.

In this setting, the operator D^2 becomes $D^2 = d\delta + \delta d$, the so-called Hodge–deRham Laplacian acting on k-forms, and the set $\mathscr{K}_{D^2}$ from (3.5) is known as the space of L^2*-harmonic k-forms*.

Furthermore, in the setting $\mathcal{E} = \Lambda^k T^*M$, the operator $\mathscr{R}^W$ depends on the Riemannian curvature tensor of M, and, by proposition 10.4.2 in [23], in the case $\mathcal{E} = \Lambda^1 T^*M$ we have $\mathscr{R}^W = \mathrm{Ric}_M$, where Ric_M is the Ricci tensor of M.

Thus, in the case $\mathcal{E} = \Lambda^k T^*M$ and the Gauss–Bonnet operator $D = d + \delta$, corollary 3.3 recovers theorem 5 from [25] with $\rho \geq 0$, a vanishing result concerning L^2-harmonic k-forms on M.

Before stating the second theorem, we recall the concept of the first eigenvalue of M:

3.4. **The First Eigenvalue of M.** The first eigenvalue of M, denoted as $\lambda_1(M)$, is defined as

$$\lambda_1(M) := \inf_{f \in C_c^\infty(M)} \frac{\int_M |df(x)|^2 \, d\nu_g(x)}{\int_M |f(x)|^2 d\nu_g(x)} \tag{3.9}$$

Remark 3.5. By (3.9) we have

$$\lambda_1(M) \int_M |f(x)|^2 d\nu_g(x) \leq \int_M |df(x)|^2 \, d\nu_g(x), \tag{3.10}$$

for all $f \in C_c^\infty(M)$, where $d\nu_g$ is the volume element on M corresponding to the metric g.

We now state the second theorem.

Theorem 3.6. *Assume that M is a geodesically complete Riemannian manifold without boundary. Assume that M satisfies the hypothesis (P1).*

Let $\mathcal{E}$ be a Hermitian vector bundle over M equipped with a metric covariant derivative ∇. Let X be a smooth, real vector field on M satisfying the condition (3.2).

Let $V \in \Gamma_{C^\infty}(\operatorname{End} \mathcal{E})$ be a fiberwise self-adjoint endomorphism such that

$$V(x) - \operatorname{div} X \geq -(a\rho(x) + b)I_x, \tag{3.11}$$

for all $x \in M$, where $0 \leq a < 1 - \hat{a}$ and $b \geq 0$ are constants, with $0 \leq \hat{a} < 1$ as in (3.2). (Here, $\operatorname{div} X$ is as in (2.3), and $I_x \colon \mathcal{E}_x \to \mathcal{E}_x$ is the identity endomorphism. The inequality (3.11) is understood in quadratic-form sense.)

Furthermore, assume that

$$\lambda_1(M) > \frac{b}{1 - a - \hat{a}}, \tag{3.12}$$

where $\lambda_1(M)$ is as in (3.9) and $0 \leq \hat{a} < 1$ is as in (3.2).

Then, the set $\mathscr{K}_{H_{X,V}}$ from (2.5) has the following property: $\mathscr{K}_{H_{X,V}} = \{0\}$.

The formula (3.7) and Theorem 3.6 with $X \equiv 0$ and $V = \mathscr{R}^W$ lead to the following corollary:

Corollary 3.7. *Assume that M is a geodesically complete Riemannian manifold without boundary. Assume that M satisfies the hypothesis (P1).*

Let $\mathcal{E}$ be a Clifford bundle over M equipped with a Clifford connection ∇, and let D be the associated Dirac operator. Assume that the inequality (3.11) is satisfied with $X \equiv 0$ and the Weitzenböck curvature operator $\mathscr{R}^W$ in place of V. Furthermore, assume that

$$\lambda_1(M) > \frac{b}{1-a}.$$

Then, the set $\mathscr{K}_{D^2}$ from (3.5) has the following property: $\mathscr{K}_{D^2} = \{0\}$.

Remark 3.8. In the case $\mathcal{E} = \Lambda^k T^*M$ and the Gauss–Bonnet operator $D = d + \delta$ (see remark 3.4 for the notations), corollary 3.7 recovers theorem 6 from [25] with $\rho \geq 0$, a vanishing result concerning L^2-harmonic k-forms on M.

For the remainder of this section, $\mathcal{E}$ is a Clifford vector bundle over M equipped with a Clifford connection ∇, and D is the associated Dirac operator.

In the next two theorems we make the following assumption on D:

3.5. **Hypothesis (P2).** Let $\rho\colon M \to \mathbb{R}$ be a continuous function. Assume that

(P2a) $\rho(x) \geq 0$ and $\rho(x)$ is not identically equal to 0;
(P2b) for all $u \in \Gamma_{C_c^\infty}(\mathcal{E})$ we have

$$\int_M \rho(x)|u(x)|^2 \, d\nu_g(x) \leq \int_M |Du(x)|^2 d\nu_g(x), \tag{3.13}$$

where $|\cdot|$ is the fiberwise norm in $\mathcal{E}_x$.

Theorem 3.9. *Assume that M is a geodesically complete Riemannian manifold without boundary. Let $\mathcal{E}$ be a Clifford bundle over M equipped with a Clifford connection ∇, and let D be the associated Dirac operator. Assume that the hypothesis (P2) is satisfied.*

Furthermore, assume that the Weitzenböck curvature operator $\mathscr{R}^{\mathrm{W}}$ satisfies the inequality

$$\mathscr{R}^{\mathrm{W}}(x) \geq -a\rho(x)I_x, \tag{3.14}$$

for all $x \in M$, where $a \geq 0$ is a constant. (Here, $I_x\colon \mathcal{E}_x \to \mathcal{E}_x$ is the identity endomorphism, and the inequality (3.14) is understood in quadratic-form sense in $\mathcal{E}_x$.)

Then, the set $\mathscr{K}_{D^2}$ from (3.5) has the following property: $\mathscr{K}_{D^2} = \{0\}$.

Remark 3.10. In the case $\mathcal{E} = \Lambda^k T^*M$ and the Gauss–Bonnet operator $D = d + \delta$ (see remark 3.4 for the notations), theorem 3.9 recovers theorem 1.4 from [27], a vanishing result concerning L^2-harmonic k-forms on M.

Theorem 3.11. *Assume that M is a geodesically complete Riemannian manifold without boundary. Let $\mathcal{E}$ be a Clifford bundle over M equipped with a Clifford connection ∇, and let D be the associated Dirac operator. Assume that the hypothesis (P2) is satisfied. Furthermore, assume that the Weitzenböck curvature operator $\mathscr{R}^{\mathrm{W}}$ satisfies the inequality*

$$\mathscr{R}^{\mathrm{W}}(x) \geq -(a\rho(x) + b)I_x, \tag{3.15}$$

for all $x \in M$, where $a \geq 0$ and $b \geq 0$ are constants. (Here, $I_x\colon \mathcal{E}_x \to \mathcal{E}_x$ is the identity endomorphism, and the inequality (3.15) is understood in quadratic-form sense in $\mathcal{E}_x$.)

Furthermore, assume that

$$\lambda_1(M) > b, \tag{3.16}$$

where $\lambda_1(M)$ is as in (3.9).

Then, the set $\mathscr{K}_{D^2}$ from (3.5) has the following property: $\mathscr{K}_{D^2} = \{0\}$.

Remark 3.12. In the case $\mathcal{E} = \Lambda^k T^*M$ and the Gauss–Bonnet operator $D = d+\delta$ (see remark 3.4 for the notations), theorem 3.11 recovers theorem 4.1 from [27], a vanishing result concerning L^2-harmonic k-forms on M.

4. Proofs of Theorems 3.1 and 3.6

We begin with a description of Sobolev spaces on M.

4.1. **Sobolev Space Notations.** We define

$$W^{1,2}(M) := \{v \in L^2(M) \colon dv \in \Gamma_{L^2}(\Lambda^1 T^*M)\}.$$

A local Sobolev space $W^{1,2}_{\mathrm{loc}}(M)$ consists of distributions v on M such that $\psi v \in W^{1,2}(M)$, for all $\psi \in C_c^\infty(M)$. The space of compactly supported elements of $W^{1,2}_{\mathrm{loc}}(M)$ will be indicated by $W^{1,2}_{\mathrm{comp}}(M)$.

Remark 4.1. Let $\mathcal{E}$ be a Hermitian vector bundle over M. The following observation will be used in the sequel: if $u \in \Gamma_{C^\infty}(\mathcal{E})$ then $|u| \in C(M) \cap W^{1,2}_{\mathrm{loc}}(M)$, where $C(M)$ stands for continuous functions on M and $|u(x)|$ is the fiberwise norm in $\mathcal{E}_x$.

We also need a sequence of cut-off functions:

4.2. **Cut-Off Functions.** On a geodesically complete Riemannian manifold M without boundary, there exists (see theorem III.3(a) in [6]) a sequence of functions $\chi_k \in C_c^\infty(M)$ with the following properties:

- (c1) for all $x \in M$, we have $0 \leq \chi_k(x) \leq 1$;
- (c2) for all compact sets $G \subset M$, there exists $n_0(G) \in \mathbb{N}$ such that for all $k > n_0$, we have $\chi_k|_G \equiv 1$;
- (c3) $\sup\limits_{x\in M} |d\chi_k(x)| \leq \dfrac{C}{k}$, where $C > 0$ is a constant independent of k, and $|\cdot|$ is the fiberwise norm in $T^*_x M$.

Remark 4.2. From the property (c2) it follows that $\lim\limits_{k\to\infty} \chi_k(x) = 1$, for all $x \in M$.

We now state a key lemma whose parts (ii) and (iii), in the presence of a vector field X, extend lemmas 1 and 2 from [25].

Lemma 4.3. *Assume that M is a geodesically complete Riemannian manifold without boundary. Let $\rho\colon M \to \mathbb{R}$ be a continuous function such that $\rho(x) \geq 0$ for all $x \in M$. Furthermore, let X be a smooth, real vector filed on M such that*

$$|X(x)| \leq \hat{a}\sqrt{\rho(x)}, \tag{4.1}$$

for all $x \in M$, where $0 \le \hat{a} < 1$ is a constant.

Assume that $h\colon M \to \mathbb{R}$ is a function belonging to $C(M) \cap W^{1,2}_{\mathrm{loc}}(M) \cap L^2(M)$ and satisfying the distributional inequality

$$h\Delta_M h \le -(Xh)h - (\operatorname{div} X)h^2 + a\rho h^2 + bh^2, \tag{4.2}$$

where $a \ge 0$ and $b \ge 0$ are constants. (Here, Δ_M is the non-negative Laplacian acting on functions, and the notation Xh means $dh(X)$, the action of dh on X.)

Then, the following hold:

(i) If h satisfies (4.2) with $X \equiv 0$, then

$$\int_M |dh(x)|^2 \, d\nu_g(x) \le a \int_M \rho(x) h^2(x) \, d\nu_g(x) + b \int_M h^2(x) \, d\nu_g(x). \tag{4.3}$$

(ii) Assume, in addition, that the hypothesis (P1) is satisfied. Furthermore, assume that $0 \le a < 1 - \hat{a}$, with $\hat{a}$ as in (4.1). Then,

$$\int_M |dh(x)|^2 \, d\nu_g(x) \le \frac{b}{1 - a - \hat{a}} \int_M h^2(x) \, d\nu_g(x). \tag{4.4}$$

(iii) Assume, in addition, that the hypothesis (P1) is satisfied. Furthermore, assume that $0 \le a < 1 - \hat{a}$, with $\hat{a}$ as in (4.1). Assume also that h is not identically equal to 0 and that h satisfies (4.2) with $b = 0$. Then, M has finite volume and ρ is identically equal to 0.

Proof. The assertion (i) was proved in lemma 2.4 in [27]. We remark that in lemma 2.4 of [27] the author assumes $h \in C^\infty(M)$ and

$$\int_{B(x_0,R)} h^2 d\nu_g = o(R^2),$$

as $R \to \infty$, where $B(x_0, R)$ is the geodesic open ball centered at $x_0 \in M$ with radius R.

An inspection of the arguments used in the quoted lemma reveals that they work without changes under the hypothesis $h \in C(M) \cap L^2(M) \cap W^{1,2}_{\mathrm{loc}}(M)$.

We now prove the assertion (ii). As in lemmas 1 and 2 of [25], we use the integration-by-parts method, modified to account for the presence of the vector field X. Using the cut-off functions $\{\chi_k\}$ from section 4.2, we multiply both sides of (4.2) by χ_k^2 and integrate each term over M.

In particular, remembering that the scalar Laplacian $\Delta_M w = d^\dagger dw$ is a non-negative operator and performing integration by parts on the left hand side of (4.2) we have, after using the product rule on $d(\chi_k^2 h)$,

$$\int_M |dh|^2 \chi_k^2 \, d\nu_g + 2 \int_M \langle h d\chi_k, \chi_k dh \rangle \, d\nu_g \tag{4.5}$$

where $\langle \cdot, \cdot \rangle$ is the fiberwise inner product in $T_x^* M$.

Furthermore, performing integration by parts in the term with integrand $-(Xh)h\chi_k^2$ on the right hand side of (4.2) and using the formula (see proposition 1.4 in appendix C of [23])

$$X^\dagger w = -Xw - (\operatorname{div} X)w,$$

where $X^\dagger$ is the formal adjoint of the action of X on a function w, the right hand side of (4.2) becomes

$$\begin{aligned}
&\int_M [X(\chi_k^2 h)]h\, d\nu_g + \int_M (\operatorname{div} X)\chi_k^2 h^2\, d\nu_g - \int_M (\operatorname{div} X)\chi_k^2 h^2\, d\nu_g \\
&+ a\int_M \rho(\chi_k h)^2\, d\nu_g + b\int_M (\chi_k h)^2\, d\nu_g \\
&= 2\int_M (X\chi_k)\chi_k h^2\, d\nu_g + \int_M (Xh)\chi_k^2 h\, d\nu_g \\
&+ a\int_M \rho(\chi_k h)^2\, d\nu_g + b\int_M (\chi_k h)^2\, d\nu_g,
\end{aligned} \tag{4.6}$$

where we used the product rule

$$X(\chi_k^2 h) = 2\chi_k(X\chi_k)h + \chi_k^2 Xh.$$

Remembering that (4.5) is less than or equal to (4.6), we obtain after some rearranging

$$\begin{aligned}
&\int_M |dh|^2\chi_k^2\, d\nu_g \le -2\int_M \langle hd\chi_k, \chi_k dh\rangle\, d\nu_g \\
&+ 2\int_M (X\chi_k)\chi_k h^2\, d\nu_g + \int_M (Xh)\chi_k^2 h\, d\nu_g + a\int_M \rho(\chi_k h)^2\, d\nu_g \\
&+ b\int_M (\chi_k h)^2\, d\nu_g.
\end{aligned} \tag{4.7}$$

Our next goal is to use the hypotheses of part (ii) of the lemma to estimate (from above) the terms on the right hand side of (4.7), and if, as a result of those estimates, we get terms with integrand $|dh|^2\chi_k^2$, make sure that those terms have as small coefficients as possible (with a total sum less than 1), so that after transferring those terms to the left hand side we a get a positive coefficient in front of the integral of $|dh|^2\chi_k^2$.

Before doing this we record a useful inequality for (real) numbers α, β, and $\varepsilon > 0$:

$$\alpha\beta \le \frac{\varepsilon\alpha^2}{2} + \frac{\beta^2}{2\varepsilon} \tag{4.8}$$

Using (4.8) and the (fiberwise) inequality (for one-forms ω and η)

$$|\langle \omega, \eta\rangle| \le |\omega||\eta|,$$

we estimate the first term on the right hand side of (4.7) as

$$
\begin{aligned}
&-2\int_M \langle hd\chi_k, \chi_k dh\rangle \, d\nu_g \\
&\leq \varepsilon \int_M |dh|^2\chi_k^2 \, d\nu_g + \varepsilon^{-1}\int_M |d\chi_k|^2 h^2 \, d\nu_g.
\end{aligned} \tag{4.9}
$$

Using the hypothesis (4.1), the inequality (here, f is a function)

$$
|Xf| \leq |X||df|,
$$

and (4.8), we estimate the second term on the right hand side of (4.7) as

$$
\begin{aligned}
&2\int_M [X(\chi_k)]\chi_k h^2 \, d\nu_g \leq 2\hat{a}\int_M \sqrt{\rho}|d\chi_k|\chi_k h^2 \\
&\leq \hat{a}\varepsilon \int_M \rho(\chi_k h)^2 \, d\nu_g + \hat{a}\varepsilon^{-1}\int_M |d\chi_k|^2 h^2 \, d\nu_g
\end{aligned} \tag{4.10}
$$

Using the hypothesis (4.1) and the inequality (4.8) with $\varepsilon = 1$, we estimate the third term on the right hand side of (4.7) as

$$
\begin{aligned}
&\int_M (Xh)\chi_k^2 h \, d\nu_g \leq \hat{a}\int_M \sqrt{\rho}|dh|\chi_k^2 h d\nu_g \\
&\leq \frac{\hat{a}}{2}\int_M \rho(\chi_k h)^2 \, d\nu_g + \frac{\hat{a}}{2}\int_M \chi_k^2|dh|^2 \, d\nu_g
\end{aligned} \tag{4.11}
$$

We keep the fourth and the fifth term on the right hand side of (4.7) in their present form.

We now look at the right hand side of (4.7) and the estimates (4.10) and (4.11). Adding the coefficients of the terms with integrand $\rho(\chi_k h)^2$, we get

$$
\hat{a}\varepsilon + 2^{-1}\hat{a} + a. \tag{4.12}
$$

As $h \in W^{1,2}_{\mathrm{loc}}(M)$ and $\chi_k \in C_c^\infty(M)$, we have $(\chi_k h) \in W^{1,2}_{\mathrm{comp}}(M)$. Thus, using Friedrichs mollifiers (with the help of a finite partition of unity, we may assume that $\chi_k h$ is supported in a coordinate neighborhood) and the hypothesis (3.1), it follows that

$$
\int_M \rho|\chi_k h|^2 \, d\nu_g \leq \int_M \rho|d(\chi_k h)|^2 d\nu_g.
$$

The latter inequality, together with the estimate,

$$
\begin{aligned}
&|d(\chi_k h)|^2 = |\chi_k dh + hd\chi_k|^2 \leq |\chi_k dh|^2 + 2|hd\chi_k||\chi_k dh| + |hd\chi_k|^2 \\
&\leq (1+\varepsilon)\chi_k^2|dh|^2 + (1+\varepsilon^{-1})|d\chi_k|^2 h^2,
\end{aligned}
$$

where we used (4.8), yield

$$
\int_M \rho|\chi_k h|^2 \, d\nu_g \leq (1+\varepsilon)\int_M \chi_k^2|dh|^2\nu_g + (1+\varepsilon^{-1})\int_M |d\chi_k|^2 h^2 \, d\nu_g. \tag{4.13}
$$

We now go back to (4.7), refer to the estimates (4.9), (4.10), (4.11) and (4.13), remembering the coefficient (4.12) in front of the sum of the terms with integrand $\rho(\chi_k h)^2$. As a result, we obtain, after moving all terms with integrand $\chi_k^2|dh|^2$ to the left hand side,

$$\begin{aligned}&[1-\varepsilon-2^{-1}\hat{a}-(\hat{a}\varepsilon+2^{-1}\hat{a}+a)(1+\varepsilon)]\int_M |dh|^2\chi_k^2\,d\nu_g\\&\le[(\hat{a}\varepsilon+2^{-1}\hat{a}+a)(1+\varepsilon^{-1})+(\hat{a}+1)\varepsilon^{-1}]\int_M |d\chi_k|^2h^2\,d\nu_g+b\int_M(\chi_k h)^2\,d\nu_g.\end{aligned}\tag{4.14}$$

Since $0\le a<1-\hat{a}$, we can choose a small enough $\varepsilon>0$ so that

$$1-\varepsilon-2^{-1}\hat{a}-(\hat{a}\varepsilon+2^{-1}\hat{a}+a)(1+\varepsilon)>0.\tag{4.15}$$

Letting $k\to\infty$ in (4.14) and using the properties of χ_k from section 4.2, together with the assumption $h\in L^2(M)$, we get

$$\begin{aligned}&[1-\varepsilon-2^{-1}\hat{a}-(\hat{a}\varepsilon+2^{-1}\hat{a}+a)(1+\varepsilon)]\int_M |dh|^2\,d\nu_g\\&\le b\int_M h^2\,d\nu_g.\end{aligned}\tag{4.16}$$

Finally, letting $\varepsilon\to 0$, we obtain (4.4).

We now prove the assertion (iii). With (4.16) at our disposal, we can repeat the argument from the end of the proof of lemma 1 in [25]. Putting $b=0$ in (4.16) and keeping in mind (4.15), we get

$$\int_M |dh|^2\,d\nu_g\le 0.$$

This shows that exists $\tilde{c}\in\mathbb{R}$ such that $h(x)=\tilde{c}$ for all $x\in M$. By assumption in part (iii) of the lemma we have $\tilde{c}\neq 0$ and $h\in L^2(M)$. The only way the last sentence can be true is that $\mathrm{vol}(M)$ be finite. Furthermore, using (3.1) with $f=\chi_k$, we have

$$\int_M \rho\chi_k^2\,d\nu_g\le\int_M |d\chi_k|^2\,d\nu_g.$$

Letting $k\to\infty$ in the latter inequality and using the properties of χ_k from section 4.2, we obtain (remembering that $\mathrm{vol}(M)$ is finite)

$$\int_M \rho\,d\nu_g\le 0,$$

which, together with the hypothesis $\rho(x)\ge 0$, tells us that $\rho(x)=0$ for all $x\in M$. This concludes the proof of assertion (iii) of the lemma. □

4.3. **Bochner Formula.** Before moving forward, we record the following Bochner formula: For $u\in\Gamma_{C^\infty}(\mathcal{E})$ we have

$$\Delta_M\left(\frac{|u|^2}{2}\right)=\langle\nabla^\dagger\nabla u,u\rangle-|\nabla u|^2,\tag{4.17}$$

where Δ_M is the non-negative Laplacian (acting on functions), $\langle\cdot,\cdot\rangle$ is the fibrewise inner product in $\mathcal{E}_x$, $|\cdot|$ on the left hand side is the norm in $\mathcal{E}_x$, and $|\cdot|$ on the right hand side is the norm in $(T^*M\otimes\mathcal{E})_x$.

Using the formula

$$\Delta_M(f\circ w) = -f''(w)|dw|^2 + f'(w)\Delta_M w \tag{4.18}$$

with real-valued functions $w\in W^{1,2}_{\mathrm{loc}}(M)$ and $f\in C^\infty(\mathbb{R})$, we rewrite (4.17) as

$$|u|\Delta_M|u| - |d|u||^2 = \langle\nabla^\dagger\nabla u, u\rangle - |\nabla u|^2. \tag{4.19}$$

4.4. **Proof of Theorem 3.1.** Starting with $u\in\mathscr{K}_{H_{X,V}}$, that is,

$$\nabla^\dagger\nabla u = -\nabla_X u - Vu,$$

and using (4.19), we obtain

$$\begin{aligned} |u|\Delta_M|u| &= -\langle\nabla_X u, u\rangle - \langle Vu, u\rangle \\ &\quad + |d|u||^2 - |\nabla u|^2. \end{aligned} \tag{4.20}$$

Taking the real part on both sides of (4.20), keeping in mind that X is real, and using the property

$$X(|u|^2) = X\langle u,u\rangle = \langle\nabla_X u, u\rangle + \langle u, \nabla_X u\rangle = 2\,\mathrm{Re}\langle\nabla_X u, u\rangle, \tag{4.21}$$

together with the chain rule (here Xf means $df(X)$),

$$X(|u|^2) = 2|u|(X|u|), \tag{4.22}$$

we can rewrite (4.20) as

$$\begin{aligned} |u|\Delta_M|u| &= -(X|u|)|u| - \langle Vu, u\rangle \\ &\quad + |d|u||^2 - |\nabla u|^2. \end{aligned} \tag{4.23}$$

Using the hypothesis (3.3) and the so-called Kato's inequality (see formula (1.32) in [12] or formula (1) in [15])

$$|d|u(x)|| \le |(\nabla u)(x)|, \tag{4.24}$$

the formula (4.23) leads to

$$|u|\Delta_M|u| \le -(X|u|)|u| - (\operatorname{div} X)|u|^2 + a\rho|u|^2.$$

The last inequality and remark 4.1 tell us that the function $h(x) := |u(x)|$ satisfies the hypotheses of part (iii) of lemma 4.3. Hence, looking at the conditions (m1) and (m2) of theorem 3.1, we infer that $|u(x)| = 0$ for all $x\in M$. This shows that $u=0$, that is, $\mathscr{K}_{H_{X,V}} = \{0\}$. □

4.5. **Proof of Theorem 3.6.** Starting with $u \in \mathscr{K}_{H_{X,V}}$ and arguing as in the proof of theorem 3.1, we obtain (4.23). The latter formula, together with the inequality (4.24) and the hypothesis (3.11), lead to

$$|u|\Delta_M|u| \leq -(X|u|)|u| - (\operatorname{div} X)|u|^2 + a\rho|u|^2 + b|u|^2.$$

Referring to remark 4.1, the last inequality tells us that the function $h(x) := |u(x)|$ satisfies the hypotheses of part (ii) of lemma 4.3. Therefore, by (4.4) we have

$$\int_M |d|u||^2 \, d\nu_g \leq \frac{b}{1-a-\hat{a}} \int_M |u|^2 \, d\nu_g. \tag{4.25}$$

In particular, this estimate tells us $|u| \in W^{1,2}(M)$. As M is geodesically complete, the space $C_c^\infty(M)$ is dense in $W^{1,2}(M)$; see theorem 3.1 in [14]. Thus, the inequality (3.10) holds with $|u|$ in place of f. Consequently, combining (3.10) and (4.25) yields

$$\lambda_1(M) \int_M |u|^2 \, d\nu_g \leq \frac{b}{1-a-\hat{a}} \int_M |u|^2 \, d\nu_g,$$

which upon rearranging leads to

$$\left(\lambda_1(M) - \frac{b}{1-a-\hat{a}}\right) \int_M |u|^2 \, d\nu_g \leq 0.$$

The latter inequality and the hypothesis (3.12) lead to $|u(x)| = 0$ for all $x \in M$, that is, $\mathscr{K}_{H_{X,V}} = \{0\}$. □

5. Proofs of Theorems 3.9 and 3.11

In this section we work in the context of a Clifford bundle $\mathcal{E}$ over M, equipped with a Clifford connection ∇ and the associated Dirac operator D. We first recall the product rule for D.

5.1. **Product Rule.** By lemma II.5.5 in [17], for all $u \in \Gamma_{C^\infty}(\mathcal{E})$ and all $\psi \in C^\infty(M)$, we have

$$D(\psi u) = (d\psi)^\sharp \bullet u + \psi D u, \tag{5.1}$$

where "$\bullet$" is the Clifford multiplication and $(d\psi)^\sharp$ is the vector field corresponding to $d\psi$ via the metric g.

The following lemma is a Dirac-operator analogue of lemma 2.2 from [27].

Lemma 5.1. *Assume that M is a geodesically complete Riemannian manifold without boundary. Let $\mathcal{E}$ be a Clifford bundle over M equipped with a Clifford connection ∇, and let D be the associated Dirac operator. Assume that D satisfies the hypothesis (P2). Furthermore, assume that $u \in \Gamma_{C^\infty}(\mathcal{E}) \cap \Gamma_{L^2}(\mathcal{E})$ is a solution of the equation $Du = 0$. Then,*

$$\int_M \rho |u|^2 \, d\nu_g \leq 0. \tag{5.2}$$

Proof. Let $\{\chi_k\}$ be as in section 4.2 and let $u \in \Gamma_{C^\infty}(\mathcal{E})$. Then, $\chi_k u \in \Gamma_{C_c^\infty}(\mathcal{E})$, and we can use (3.13) to get

$$\int_M \rho\chi_k^2|u|^2\,d\nu_g \leq \|D(\chi_k u)\|_2^2$$
$$= \|(d\chi_k)^\sharp \bullet u + \chi_k Du\|_2^2 = \|(d\chi_k)^\sharp \bullet u\|_2^2 \leq \frac{C}{k^2}\|u\|_2^2, \tag{5.3}$$

where $C > 0$ is a constant and $\|\cdot\|_2$ is the norm in $\Gamma_{L^2}(\mathcal{E})$. Here, in the first equality we used (5.1), in the second equality we used the assumption $Du = 0$, and in the third inequality we used the property (c3) from section 4.2. Letting $k \to \infty$ in (5.3) we obtain (5.2). $\square$

Our last ingredient is a geometric formula.

5.2. **Bochner-Weitzenböck Formula.** Combining (4.19) and (3.7) we get

$$|u|\Delta_M|u| - |d|u||^2 = \langle D^2u, u\rangle - \langle \mathscr{R}^W u, u\rangle - |\nabla u|^2. \tag{5.4}$$

We are now ready to prove theorem 3.9.

5.3. **Proof of Theorem 3.9.** Starting with $u \in \mathscr{K}_{D^2}$ and using (5.4), we get

$$|u|\Delta_M|u| = -\langle \mathscr{R}^W u, u\rangle + |d|u||^2 - |\nabla u|^2$$
$$\leq a\rho|u|^2, \tag{5.5}$$

where the last estimate (with $a \geq 0$) follows from the hypothesis (3.14) and the inequality (4.24).

Looking at the inequality (5.5) and recalling remark 4.1 we can see that the function $h(x) := |u(x)|$ satisfies the hypotheses of part (i) of lemma 4.3 with $b = 0$. Therefore, appealing to (4.3) with $b = 0$, we get

$$\int_M |d|u||^2\,d\nu_g \leq a\int_M \rho|u|^2\,d\nu_g,$$

which in combination with lemma 5.1 (remember (3.6), that is, $Du = 0$) yields

$$\int_M |d|u||^2\,d\nu_g \leq a\int_M \rho|u|^2\,d\nu_g \leq 0.$$

Thus, there exists a number $\tilde{c} \geq 0$ such that $|u(x)| = \tilde{c}$ for all $x \in M$. Assume for a moment that $\tilde{c} > 0$.

Since (see hypothesis (P2a)) the function ρ is not identically equal to 0, we have

$$\int_M \rho\,d\nu_g > 0.$$

On the other hand (5.2) yields

$$0 \geq \int_M \rho|u|^2\,d\nu_g = \tilde{c}\int_M \rho\,d\nu_g,$$

that is (since we assumed $\tilde{c} > 0$),

$$\int_M \rho \, d\nu_g \leq 0.$$

The obtained contradiction says that $\tilde{c}$ must equal 0, that is, $\mathscr{K}_{D^2} = \{0\}$. $\square$

5.4. **Proof of Theorem 3.11.** Starting with $u \in \mathscr{K}_{D^2}$ and using (5.4), we get

$$\begin{aligned} |u|\Delta_M|u| &= -\langle \mathscr{R}^W u, u\rangle + |d|u||^2 - |\nabla u|^2 \\ &\leq a\rho|u|^2 + b|u|^2, \end{aligned} \tag{5.6}$$

where the last estimate (with $a \geq 0$) follows from the hypothesis (3.15) and the inequality (4.24).

The estimate (5.6) and remark 4.1 tell us that the function $h(x) := |u(x)|$ satisfies the hypotheses of part (i) of lemma 4.3. Therefore (4.3) gives

$$\int_M |d|u||^2 \, d\nu_g \leq a \int_M \rho|u|^2 \, d\nu_g + b \int_M |u|^2 \, d\nu_g.$$

Remembering (3.6), that is, $Du = 0$, and using lemma 5.1, the last estimate leads to

$$\int_M |d|u||^2 \, d\nu_g \leq b \int_M |u|^2 \, d\nu_g.$$

From hereon, we use (3.16) and argue in the same way as in the last stage of the proof of theorem 3.6 to infer that $\mathscr{K}_{D^2} = \{0\}$. $\square$

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