HOMOTOPY RIGIDITY FOR QUASITORIC MANIFOLDS OVER A PRODUCT OF *d*-SIMPLICES

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ABSTRACT. For a fixed integer $d \ge 1$, we show that two quasitoric manifolds over a product of d-simplices are homotopy equivalent after appropriate localization, provided that their integral cohomology rings are isomorphic.

1. INTRODUCTION

A quasitoric manifold M is a smooth, compact 2n-dimensional manifold endowed with a locally standard T^n -action, such that the orbit space M/T^n is an n-dimensional simple polytope P. Here $T^n = (S^1)^n$ is the compact torus of rank n. The Cohomological Rigidity Problem in toric topology [MS] poses the question of whether two quasitoric manifolds are homeomorphic or diffeomorphic if their integral cohomology rings are isomorphic. Special cases have provided positive evidence, such as four-dimensional quasitoric manifolds [OR, Fre], Bott manifolds [CHJ], certain generalized Bott manifolds [CMS], quasitoric manifolds with second Betti number equal to 2 [CPS], and 6-dimensional quasitoric manifolds associated with Pogorelov polytopes [BEM⁺] or a 3-cube [Has].

As cohomology rings are homotopy invariant, it is more natural to ask whether two quasitoric manifolds are homotopy equivalent if their cohomology rings are isomorphic. In theory, this should be a more accessible problem, while providing a test as to whether two quasitoric manifolds with isomorphic cohomology rings are homeomorphic or diffeomorphic.

Question 1.1. Let M and N be two quasitoric manifolds. If their integral cohomology rings are isomorphic, are they homotopy equivalent?

In fact, one can ask the question above for a broader class of spaces with torus actions, such as toric orbifolds [DJ, Section 7]. No counterexamples are known while several affirmative results have been established. For instance, one can deduce a positive answer for weighted projective spaces from the work in [BFNR], and the first three authors proved an affirmative answer for four dimensional toric orbifolds whose homology groups have no 2-torsion [FSS].

In this paper we focus on 2*n*-dimensional quasitoric manifolds M with orbit space $P = \prod_{i=1}^{\ell} \Delta^d$, a product of *d*-dimensional simplices Δ^d , where $n = \ell d$. Notably, the class of quasitoric manifolds

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in this context includes certain generalised Bott manifolds B_{ℓ} for $d \geq 1$ that arise from a sequence

$$B_{\ell} \xrightarrow{\pi_{\ell}} B_{\ell-1} \to \dots \to B_1 \xrightarrow{\pi_1} B_0 = \text{ point},$$

where each $\pi_i \colon B_i \to B_{i-1}$ for $i = 1, \ldots, \ell$ is a $\mathbb{C}P^d$ -fibration. However, not every quasitoric manifold over $\prod_{i=1}^{\ell} \Delta^d$ is a generalized Bott manifold; see [Has, Section 5].

A simple polytope P can be associated with a moment-angle manifold Z_P . This manifold comes with a T^m -action, where m is the number of facets (that is codimension-one faces) of the polytope P. A quasitoric manifold M over P can be regarded as a quotient

$$M = \mathcal{Z}_P / T^{m-n}$$

by a freely acting subtorus T^{m-n} of T^m on \mathcal{Z}_P , where *n* is the dimension of *P*. This process results in a principal T^{m-n} -fibration

$$T^{m-n} \longrightarrow \mathcal{Z}_P \longrightarrow M.$$

Throughout the paper, $H^*(M)$ denotes the integral cohomology ring of M unless specified otherwise. Now, we introduce our main result as follows.

Theorem 1.2. Let M and N be 2n-dimensional quasi-toric manifolds with orbit space $\prod_{i=1}^{\ell} \Delta^d$ for some $d \ge 1$ and let \mathcal{P} be the set of primes $p \le n - d + 1$. If there is a ring isomorphism $H^*(M) \cong H^*(N)$ then, after localizing away from \mathcal{P} , there is a homotopy equivalence $M \simeq N$.

Two remarks should be made. First, if $\Phi: M \to N$ is a homotopy equivalence obtained from Theorem 1.2, then it induces an isomorphism $\Phi^*: H^*(N; \mathbb{Q}) \to H^*(M; \mathbb{Q})$ and hence is a rational homotopy equivalence. For specific values of d and ℓ , Theorem 1.2 shows that this rational equivalence occurs after localizing away from a small number of primes.

Second, Theorem 1.2 should be compared to the original cohomological rigidity results for Bott manifolds [CHJ], 2-stage generalized Bott manifolds [CMS], and quasitoric manifolds over a cube [Has]. While we impose stronger conditions by localizing away from certain primes and consider only homotopy equivalences, the benefit of Theorem 1.2 is that it works for a larger class of quasitoric manifolds. For instance, if d = 1 then $P = I^{\ell}$ is an ℓ -dimensional hypercube. Two quasitoric manifolds M and N over I^{ℓ} are homotopy equivalent if there is a ring isomorphism $H^*(M) \cong H^*(N)$ and localization occurs away from primes $p \leq \ell$. In particular, this works if M and N are Bott manifolds. Moreover, setting d = 1 and $\ell = 3$, Theorem 1.2 gives a homotopy version of Hasui's result [Has] after localizing away from 2 and 3.

The main result also works for ℓ -stage generalized Bott manifolds B_{ℓ} that are constructed from iterated $\mathbb{C}P^d$ -fibrations $\pi_i \colon B_i \to B_{i-1}$ for $i = 1, \ldots, \ell$ starting from $B_0 = \{pt\}$. In this case, the associated simple polytopes are $P = \prod_{i=1}^{\ell} \Delta^d$. For instance, two 3-stage generalized Bott manifolds over $\prod_{i=1}^{3} \Delta^2$ having isomorphic cohomology rings are homotopy equivalent after localizing away from 2, 3 and 5. It is worth noting that the original cohomological rigidity problem in toric topology is true for 2-stage generalized Bott manifolds [CMS, Theorem 1.3], which are quasitoric manifolds over the product of two simplices. Theorem 1.2 provides positive evidence that this extends to higher-stage generalized Bott manifolds.

2. Preliminary information

2.1. A review of quasitoric manifolds. This section defines terms and makes some preliminary observations. We start with moment-angle manifolds from simple polytopes following Buchstaber and Panov [BP, Section 6.2]. See also [DJ, Section 4]. Let P be an n-dimensional simple polytope. In other words P is a convex polytope having exactly n facets intersecting at each vertex. Let

$$\mathcal{F}(P) = \{F_1, \dots, F_m\}$$

be the set of facets of P.

The moment-angle manifold \mathcal{Z}_P is the quotient space

$$\mathcal{Z}_P = P \times T^m /_{\sim}.$$

Here $(x,t) \sim (x',t')$ if and only if x = x' and $t^{-1}t' \in \prod_{i \in \mathcal{I}(x)} S_i^1$, where $\mathcal{I}(x) = \{i \mid x \in F_i\}$. There exists a T^m -action on \mathcal{Z}_P given by

(1)
$$T^m \times \mathcal{Z}_P \to \mathcal{Z}_P, \quad (g, [x, t]) \mapsto [x, gt]$$

for $g \in T^m$ and the equivalence class $[x, t] \in \mathcal{Z}_P$ of $(x, t) \in P \times T^m$.

Next, we define a quasitoric manifold following Davis and Januszkiewicz [DJ]. A 2n-dimensional manifold has a locally standard T^n -action if locally it is the standard action of T^n on \mathbb{C}^n . A quasitoric manifold over P is a closed, smooth 2n-dimensional manifold M that has a smooth locally standard T^n -action for which the orbit space M/T^n is homeomorphic to P as a manifold with corners.

A characteristic pair (P, λ) consists of an n-dimensional simple polytope P and a function

$$\lambda \colon \mathcal{F}(P) \to \mathbb{Z}^n$$

satisfying:

- $\lambda(F_i)$ is primitive for $i = 1, \ldots, m$;
- the set $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$ extends to a basis of \mathbb{Z}^n whenever $F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset$.

Such a function is called a *characteristic function*. For a face $F = F_{i_1} \cap \cdots \cap F_{i_k}$ of codimension k, let T_F denote the k-dimensional subtorus of T^n spanned by $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$. If F = P, then T_F is the trivial subgroup.

For a characteristic pair (P, λ) , define the quotient space

$$M(P,\lambda) = P \times T^n /_{\sim_\lambda}$$

by the equivalence relation: $(x,t) \sim_{\lambda} (x',t')$ if and only if x = x' and $t^{-1}t' \in T_F$, where F is the unique face such that x = x' lies in its relative interior. Notice that every quasitoric manifold M

over P can be constructed as such a quotient space. Here, T^n acts on the torus factor of $P \times T^n/_{\sim_{\lambda}}$, similarly to (1).

Moment-angle manifolds and quasitoric manifolds are linked. A characteristic function

$$\lambda: \mathcal{F}(P) \to \mathbb{Z}^n, \quad F_i \mapsto (\lambda_{1i}, \dots, \lambda_{ni}),$$

defines a linear map of lattices

$$\Lambda \colon \mathbb{Z}^m \to \mathbb{Z}^n, \quad e_i \mapsto (\lambda_{1i}, \dots, \lambda_{ni})$$

where $\{e_1, \ldots, e_m\}$ is the standard basis of \mathbb{Z}^m . Take the exponential of Λ to get a homomorphism exp $\Lambda: T^m \to T^n$ of tori sending (t_1, \ldots, t_m) to $(t_1^{\lambda_{11}} t_2^{\lambda_{12}} \cdots t_m^{\lambda_{1m}}, \ldots, t_1^{\lambda_{n1}} t_2^{\lambda_{n2}} \cdots t_m^{\lambda_{nm}})$. By [BP, Proposition 7.3.13], the kernel of exp Λ is isomorphic to T^{m-n} , which acts freely on \mathcal{Z}_P and a 2*n*dimensional quasitoric manifold M is T^n -equivariantly homeomorphic to the quotient $\mathcal{Z}_P/\ker \exp \Lambda$ equipped with the residual T^n -action. This implies that there is a principal T^{m-n} -fibration

$$T^{m-n} \longrightarrow \mathcal{Z}_P \longrightarrow M.$$

As this is principal, it is classified by a map $M \to BT^{m-n}$ and there is a homotopy fibration

(2)
$$\mathcal{Z}_P \longrightarrow M \longrightarrow BT^{m-n}.$$

The cohomology of M was calculated in [DJ]. They showed that there is a ring isomorphism

$$H^*(M) \cong \mathbb{Z}[x_1, \dots, x_m]/\mathcal{I} + \mathcal{J}$$

where each x_i has degree 2, the ideal \mathcal{I} is generated by monomials $x_{i_1} \cdots x_{i_k}$ for which the intersection of F_{i_1}, \ldots, F_{i_k} is empty, which is often called the *Stanley–Reisner ideal* of P, and \mathcal{J} is an ideal of linear relations $\lambda_{j_1}x_1 + \lambda_{j_2}x_2 + \cdots + \lambda_{j_m}x_m$ for $j = 1, \ldots n$. The cohomological properties of Mthat will be relevant to us are:

- $H^2(M)$ has rank m n;
- $H^*(M)$ is multiplicatively generated by degree-two elements.

Note that these two properties also imply that M is simply-connected. We record a simple property of M that follows immediately from the homotopy fibration (2) and the fact that T^{m-n} is an Eilenberg-MacLane space $K(\mathbb{Z}^{m-n}, 1)$.

Lemma 2.1. Let M be a quasitoric manifold associated to a characteristic pair (P, λ) . Then the map $\mathcal{Z}_P \to M$ in (2) induces an isomorphism $\pi_t(\mathcal{Z}_P) \cong \pi_t(M)$ for $t \ge 3$.

We will need an identification of Z_P in the special case when $P = P_1 \times P_2$ is a product of two simple polytopes P_1 and P_2 . The following statement can be found in [BP2, Proposition 6.4]; we give a brief proof. Lemma 2.2. There is a homeomorphism

$$\mathcal{Z}_{P_1 \times P_2} \cong \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$$

In particular, if $P = \prod_{i=1}^{\ell} \Delta^d$ is a product of d-simplices Δ^d , then $\mathcal{Z}_P \cong \prod_{i=1}^{\ell} S^{2d+1}$.

Proof. The first homeomorphism follows directly from the definition of a moment-angle manifold. In the simplex case we have $\mathcal{Z}_{\Delta^d} \cong S^{2d+1}$. Therefore $\mathcal{Z}_{\prod_{i=1}^{\ell} \Delta^d} \cong \prod_{i=1}^{\ell} S^{2d+1}$. \Box

2.2. Homotopy theory of quasitoric manifolds. Consider the quasitoric manifold M with orbit space $P = \prod_{i=1}^{\ell} \Delta^d$ as in Theorem 1.2. As $\mathcal{Z}_P \cong \prod_{i=1}^{\ell} S^{2d+1}$, by (2), there is a homotopy fibration

(3)
$$\prod_{i=1}^{\ell} S^{2d+1} \longrightarrow M \xrightarrow{\delta} BT^{\ell}.$$

Since M is simply-connected, it has a CW-structure in which each cell corresponds to a homology class. Fix such a CW-structure and for $1 \le t \le 2n$, let $\text{skel}_t(M)$ be the *t*-skeleton of M. Since the cohomology of M is concentrated in even degrees, there are homotopy equivalences

$$\operatorname{skel}_{2k+1}(M) \simeq \operatorname{skel}_{2k}(M)$$

for any k. Therefore, we may focus only on the skeletons indexed by even integers. Consider the homotopy cofibrations

(4)
$$\bigvee S^{2k-1} \xrightarrow{f_k} \operatorname{skel}_{2k-2}(M) \longrightarrow \operatorname{skel}_{2k}(M)$$

for $2 \le k \le n$. Note that $\operatorname{skel}_0(M) \simeq *$ and $\operatorname{skel}_{2n}(M) \simeq M$.

First we study the homotopy groups of $\operatorname{skel}_{2k}(M)$.

Lemma 2.3. Let M be a quasitoric manifold as in Theorem 1.2, and let \mathcal{P} be the set of primes $p \leq n - d + 1$. Fix an integer k such that $k \geq d + 1$. If $2d + 1 < i \leq 2k$ then the homotopy group $\pi_i(\text{skel}_{2k}(M))$ is a finite group of an order divisible only by primes in \mathcal{P} . Consequently, after localizing away from \mathcal{P} , there is an isomorphism

$$\pi_i(skel_{2k}(M)) \cong 0.$$

Proof. The fibration sequence (3) induces a long exact sequence of homotopy groups. Since $\pi_i(BT^\ell)$ is trivial for i > 2, we have

$$\pi_i(M) \cong \pi_i(\prod_{j=1}^{\ell} S^{2d+1}) \cong \bigoplus_{j=1}^{\ell} \pi_i(S^{2d+1}).$$

It is known that $\pi_i(S^{2d+1})$ is a finite group when $i \neq 2d + 1$, and the first nontrivial *p*-torsion element in $\pi_*(S^{2d+1})$ occurs in $\pi_{2(p+d-1)}(S^{2d+1})$ (see [To]). By definition, \mathcal{P} consists of all the primes that divide the order of $\pi_t(S^{2d+1})$ for any $2d + 1 < t \leq 2n$. Hence the lemma holds true for $\operatorname{skel}_{2n}(M) \simeq M$. Let k be such that $d + 1 \leq k < n$. Let F_k be the homotopy fibre of the skeletal inclusion skel_{2k}(M) \rightarrow M. Since the first cells in M that are not in skel_{2k}(M) occur in dimension 2k + 2, the space F_k is 2k-connected. Therefore, skel_{2k}(M) \rightarrow M induces an isomorphism $\pi_i(\text{skel}_{2k}(M)) \cong$ $\pi_i(M)$ for all $i \leq 2k$. In particular, for $2d + 1 < i \leq 2k$, as $\pi_i(M)$ is a finite group of an order divisible only by primes in \mathcal{P} , the same is true for $\pi_i(\text{skel}_{2k}(M))$.

Next we record two cohomological properties for the map $\delta \colon M \to BT^{\ell}$.

Lemma 2.4. The ring homomorphism $\delta^* \colon H^*(BT^{\ell}) \to H^*(M)$ is a surjection that induces an isomorphism in degree 2.

Proof. As $BT^{\ell} \simeq \prod_{i=1}^{\ell} \mathbb{C}P^{\infty}$, it follows that $H^*(BT^{\ell})$ is a polynomial algebra generated by degree 2 elements. By (3), the homotopy fibre of δ is at least 2-connected. Therefore δ induces an isomorphism on π_2 . Since M and BT^{ℓ} are both simply-connected, the Hurewicz Theorem implies that δ induces an isomorphism on H_2 . Simple-connectivity also implies, by the Universal Coefficient Theorem, that H^2 is the dual of H_2 . Therefore $\delta^* \colon H^2(BT^{\ell}) \to H^2(M)$ is an isomorphism. Since $H^*(M)$ is generated as an algebra by degree 2 elements, this implies that δ^* is surjective.

Lemma 2.5. Let δ_{2d} : $skel_{2d}(M) \to skel_{2d}(BT^{\ell})$ be the restriction of $M \xrightarrow{\delta} BT^{\ell}$ to 2d-skeletons. Then it is a homotopy equivalence and induces an isomorphism in cohomology.

Proof. In fibration sequence (3), since the base is 1-connected and the fibre is 2*d*-connected, the Serre exact sequence (see [A, Theorem 6.4.4], for example) implies that δ induces an isomorphism in homology in degrees $\leq 2d$. Thus δ_{2d} induces an isomorphism in homology in all degrees, implying that it is a homotopy equivalence by Whitehead's theorem.

3. The strategy for the proof of Theorem 1.2

For $d \ge 1$, let Δ^d be a *d*-simplex and let $P = \prod_{i=1}^{\ell} \Delta^d$ be a product of *d*-simplices. For two quasitoric manifolds M and N over P, assume that there is a ring isomorphism

(5)
$$\gamma \colon H^*(N;\mathbb{Z}) \to H^*(M;\mathbb{Z}).$$

The goal is to determine whether there is a homotopy equivalence $M \simeq N$. We will show that this is true after localizing away from \mathcal{P} . A similar local approach was taken in [Th] in the special case of Bott manifolds when d = 1; our methods are different and generalize the result to all quasitoric manifolds over products of d-simplices for $d \ge 1$.

The argument proceeds by an induction on skeletons. Consider the homotopy cofibrations

$$\bigvee S^{2k-1} \xrightarrow{f_k} \operatorname{skel}_{2k-2}(M) \longrightarrow \operatorname{skel}_{2k}(M)$$
$$\bigvee S^{2k-1} \xrightarrow{f'_k} \operatorname{skel}_{2k-2}(N) \longrightarrow \operatorname{skel}_{2k}(N)$$

for $2 \le k \le n$. We will construct a homotopy equivalence

$$\Phi_{2k}$$
: skel_{2k}(M) \longrightarrow skel_{2k}(N)

by different methods in two cases: (i) k = d and (ii) $d + 1 \le k \le n$.

Case 1: k = d. In Proposition 4.2, it will be shown that the ring isomorphism $\gamma \colon H^*(M) \to H^*(N)$ gives rise to a homotopy equivalence $\Phi_{2d} \colon \operatorname{skel}_{2d}(M) \longrightarrow \operatorname{skel}_{2d}(N)$. No localization is needed.

Case 2: $d+1 \leq k \leq n$. In Proposition 5.6 an induction on skeletons and novel methods relating the cohomology and homotopy groups of *CW*-complexes with only even dimensional cells will be used to show that, after localization away from \mathcal{P} , there is a homotopy equivalence Φ_{2k} : $\operatorname{skel}_{2k}(M) \to \operatorname{skel}_{2k}(N)$.

Granting Propositions 4.2 and 5.6, we can prove the main result in the paper.

Proof of Theorem 1.2. Since $\operatorname{skel}_{2n}(M) \simeq M$ and $\operatorname{skel}_{2n}(N) \simeq N$, the k = n instance of Case 2 implies that after localizing away from \mathcal{P} there is a homotopy equivalence $M \simeq N$.

4. Construction of the homotopy equivalence Φ_{2d}

Given a ring isomorphism $\gamma \colon H^*(N) \to H^*(M)$, its restriction to degree 2 is a group isomorphism

$$\gamma \colon H^2(N) \cong \mathbb{Z}^\ell \to H^2(M) \cong \mathbb{Z}^\ell.$$

Apply the classifying space functor B(-) twice to obtain a homotopy equivalence

(6)
$$\Gamma \colon BT^{\ell} \to BT^{\ell}$$

Recall from (3) that there are homotopy fibrations

$$\prod_{i=1}^{\ell} S^{2d+1} \longrightarrow M \xrightarrow{\delta} BT^{\ell} \quad \text{and} \quad \prod_{i=1}^{\ell} S^{2d+1} \longrightarrow N \xrightarrow{\delta'} BT^{\ell}.$$

Lemma 4.1. There is a commutative diagram

Proof. Observe that Diagram (7) commutes in degree 2 by definition of Γ . The fact that both $H^*(BT^{\ell})$ and $H^*(M)$ are generated as algebras by degree 2 elements then implies the diagram commutes in all degrees since all maps are algebra maps.

Restricting $BT^{\ell} \xrightarrow{\Gamma} BT^{\ell}$ to 2*d*-skeletons gives a map

$$\Gamma_{2d}$$
: skel $_{2d}(BT^{\ell}) \to$ skel $_{2d}(BT^{\ell})$.

Using the homotopy equivalences $\operatorname{skel}_{2d}(M) \xrightarrow{\delta_{2d}} \operatorname{skel}_{2d}(BT^{\ell})$ and $\operatorname{skel}_{2d}(N) \xrightarrow{\delta'_{2d}} \operatorname{skel}_{2d}(BT^{\ell})$ from Lemma 2.5, define the map Φ_{2d} by the composite

(8)
$$\Phi_{2d} \colon \operatorname{skel}_{2d}(M) \xrightarrow{\delta_{2d}} \operatorname{skel}_{2d}(BT^{\ell}) \xrightarrow{\Gamma_{2d}} \operatorname{skel}_{2d}(BT^{\ell}) \xrightarrow{(\delta'_{2d})^{-1}} \operatorname{skel}_{2d}(N).$$

By definition of Φ_{2d} , there is a commutative diagram

(9)

$$\begin{array}{ccc}
H^*(\operatorname{skel}_{2d}(BT^{\ell})) & \xrightarrow{(\delta'_{2d})^*} & H^*(\operatorname{skel}_{2d}(N)) \\
& & & \downarrow^{\Gamma^*_{2d}} & & \downarrow^{\Phi^*_{2d}} \\
& & & H^*(\operatorname{skel}_{2d}(BT^{\ell})) & \xrightarrow{\delta^*_{2d}} & H^*(\operatorname{skel}_{2d}(M)).
\end{array}$$

Proposition 4.2. The map $skel_{2d}(M) \xrightarrow{\Phi_{2d}} skel_{2d}(N)$ is a homotopy equivalence.

Proof. Recall from (6) that Γ is a homotopy equivalence, so Γ^* is an isomorphism and so is Γ_{2d}^* . As $H^*(BT^{\ell})$ is concentrated in even degrees, the Universal Coefficient Theorem implies that $H_*(BT^{\ell})$ is dual to $H^*(BT^{\ell})$, and therefore $(\Gamma_{2d})_*$ is also an isomorphism. Hence Γ_{2d} is a homotopy equivalence by Whitehead's Theorem. By Lemma 2.5 both δ_{2d} and δ'_{2d} are homotopy equivalences. Therefore the composite Φ_{2d} is a homotopy equivalence.

5. Construction of the homotopy equivalence Φ_{2k} for $k \ge d+1$

We begin with a general lemma that is of interest in its own right.

Lemma 5.1. Let X be a connected CW complex having only even dimensional cells. For each $k \ge 1$ there is a group homomorphism

$$g_X \colon H_{2k+2}(X) \to \pi_{2k+1}(skel_{2k}(X))$$

sending the homology class of a (2k+2)-cell $[e_{\alpha}]$ to the homotopy class of its attaching map $f_{\alpha}: \partial e_{\alpha} \rightarrow$ skel_{2k}(X). This satisfies the following properties:

- (a) if the standard maps of pairs $H_{2k+2}(X) \to H_{2k+2}(X, skel_{2k}(X))$ and $\pi_{2k+2}(X, skel_{2k}(X)) \to \pi_{2k+1}(skel_{2k}(X))$ are both isomorphisms then so is g_X ;
- (b) if Y is another CW-complex having only even dimensional cells and f: X → Y is a map then there is a commutative diagram

(10)
$$H_{2k+2}(X) \xrightarrow{g_X} \pi_{2k+1}(skel_{2k}(X))$$
$$\downarrow^{f_*} \qquad \qquad \downarrow^{(f_{2k})_*} H_{2k+2}(Y) \xrightarrow{g_Y} \pi_{2k+1}(skel_{2k}(Y)).$$

Proof. First we define g_X . The pair $(X, \text{skel}_{2k}(X))$ induces a long exact sequence of relative homotopy groups

$$\cdots \longrightarrow \pi_j(\operatorname{skel}_{2k}(X)) \longrightarrow \pi_j(X) \xrightarrow{j_\pi} \pi_j(X, \operatorname{skel}_{2k}(X)) \xrightarrow{\partial_\pi} \pi_{j-1}(\operatorname{skel}_{2k}(X)) \longrightarrow \cdots$$

and a long exact sequence of relative homology groups

$$\cdots \longrightarrow H_j(\operatorname{skel}_{2k}(X)) \longrightarrow H_j(X) \xrightarrow{j_H} H_j(X, \operatorname{skel}_{2k}(X)) \xrightarrow{\partial_H} H_{j-1}(\operatorname{skel}_{2k}(X)) \longrightarrow \cdots$$

Since X has cells only in even dimensions, the pair $(X, \text{skel}_{2k}(X))$ is (2k+1)-connected and $\text{skel}_{2k}(X)$ is simply-connected. Therefore, by [Hat, Proposition 4.28] for example, there is an isomorphism

$$\pi_{2k+2}(X, \operatorname{skel}_{2k}(X)) \cong \pi_{2k+2}(X/\operatorname{skel}_{2k}(X))$$

On the other hand, the usual isomorphism $H_m(X, A) \cong \widetilde{H}_m(X/A)$ for pairs of spaces (X, A) and any $m \ge 0$ implies in our case that there is an isomorphism

$$H_{2k+2}(X, \operatorname{skel}_{2k}(X)) \cong H_{2k+2}(X/\operatorname{skel}_{2k}(X)).$$

Observe that the inclusion $\bigvee S^{2k+2} \hookrightarrow X/\operatorname{skel}_{2k}(X)$ of the bottom cells is a (2k+3)-equivalence, implying that there are isomorphisms

$$\pi_{2k+2}(X/\operatorname{skel}_{2k}(X)) \cong \pi_{2k+2}(\bigvee S^{2k+2}) \text{ and } H_{2k+2}(X/\operatorname{skel}_{2k}(X)) \cong H_{2k+2}(\bigvee S^{2k+2}).$$

The Hurewicz homomorphism $\pi_{2k+2}(X/\operatorname{skel}_{2k}(X)) \to H_{2k+2}(X/\operatorname{skel}_{2k}(X))$ is therefore an isomorphism. Combining these isomorphisms then gives an isomorphism

$$hur: \pi_{2k+2}(X, \operatorname{skel}_{2k}(X)) \to H_{2k+2}(X, \operatorname{skel}_{2k}(X)).$$

Now define $g_X \colon H_{2k+2}(X) \to \pi_{2k+1}(\operatorname{skel}_{2k}(X))$ by the composite

$$g_X \colon H_{2k+2}(X) \xrightarrow{\mathcal{I}H} H_{2k+2}(X, \operatorname{skel}_{2k}(X)) \xrightarrow{hur^{-1}} \pi_{2k+2}(X, \operatorname{skel}_{2k}(X)) \xrightarrow{\partial_{\pi}} \pi_{2k+1}(\operatorname{skel}_{2k}(X)).$$

Since g_X is the composite of three group homomorphisms it too is a homomorphism, and by construction g_X sends the homology class of a (2k+2)-cell e_α to its attaching map $f_\alpha \in \pi_{2k+1}(\operatorname{skel}_{2k}(X))$.

For part (a), the composite defining g_X implies that if both j_H and ∂_{π} are isomorphisms then, as *hur* is also an isomorphism, so is g_X .

For part (b), consider the diagram

$$\begin{array}{cccc} H_{2k+2}(X) & \xrightarrow{\mathcal{I}_{H}} & H_{2k+2}(X, \operatorname{skel}_{2k}(X)) & \xrightarrow{hur^{-1}} & \pi_{2k+2}(X, \operatorname{skel}_{2k}(X)) & \xrightarrow{\partial_{\pi}} & \pi_{2k+1}(\operatorname{skel}_{2k+2}(X)) \\ & & \downarrow f_{*} & & \downarrow f_{*} & & \downarrow (f_{2k+2})_{*} \\ & H_{2k+2}(Y) & \xrightarrow{\mathcal{I}_{H}} & H_{2k+2}(Y, \operatorname{skel}_{2k}(Y)) & \xrightarrow{hur^{-1}} & \pi_{2k+2}(Y, \operatorname{skel}_{2k}(Y)) & \xrightarrow{\partial_{\pi}} & \pi_{2k+1}(\operatorname{skel}_{2k+2}(Y)). \end{array}$$

The left, middle and right squares commute by the naturality of ∂_{π} , j_H and *hur* respectively. The composites along the top and bottom rows are the definitions of g_X and g_Y respectively. Thus we obtain the commutativity of (10).

We apply Lemma 5.1 to the case $M \xrightarrow{\delta} BT^{\ell}$ in (3). Recall that \mathcal{P} is the set of primes $p \leq n-d+1$.

Corollary 5.2. For $k \geq 1$, there is a commutative diagram

(11)
$$H_{2k+2}(M) \xrightarrow{g_M} \pi_{2k+1}(skel_{2k}(M))$$
$$\downarrow^{\delta_*} \qquad \qquad \downarrow^{(\delta_{2k})_*} \\H_{2k+2}(BT^{\ell}) \xrightarrow{g_{BT^{\ell}}} \pi_{2k+1}(skel_{2k}(BT^{\ell})),$$

where g_M is an injection and $g_{BT^{\ell}}$ is an isomorphism. Furthermore, if k > d then

$$g_M \colon H_{2k+2}(M) \to \pi_{2k+1}(skel_{2k}(M))$$

is an isomorphism after localizing away from \mathcal{P} .

Proof. The commutativity of (11) is immediate from Lemma 5.1 by taking f to be $\delta: M \to BT^{\ell}$.

To show that $g_{BT^{\ell}}$ is an isomorphism, we consider the pair $(BT^{\ell}, \text{skel}_{2k}(BT^{\ell}))$, which induces an isomorphism $H_{2k+2}(BT^{\ell}) \to H_{2k+2}(BT^{\ell}, \text{skel}_{2k}(BT^{\ell}))$ for skeletal reasons. It also induces a long exact sequence of relative homotopy groups

$$\cdots \to \pi_j(BT^{\ell}) \to \pi_j(BT^{\ell}, \operatorname{skel}_{2k}(BT^{\ell})) \to \pi_{j-1}(\operatorname{skel}_{2k}(BT^{\ell})) \to \pi_{j-1}(BT^{\ell}) \to \cdots.$$

For $k \ge 1$, both $\pi_{2k+2}(BT^{\ell})$ and $\pi_{2k+1}(BT^{\ell})$ are trivial, which implies that $\pi_{2k+2}(BT^{\ell}, \text{skel}_{2k}(BT^{\ell}))$ and $\pi_{2k+1}(\text{skel}_{2k}(BT^{\ell}))$ are isomorphic. Thus, the map $g_{BT^{\ell}}$ is an isomorphism by Lemma 5.1(a).

Next, the map $\delta_* \colon H_{2k+2}(M) \to H_{2k+2}(BT^{\ell})$ in (11) is an injection by Lemma 2.4. Therefore, the commutativity of (11) implies that g_M is an injection.

Finally, suppose that k > d. By Lemma 2.3, after localizing away from \mathcal{P} , homotopy groups $\pi_s(M)$ are trivial for $2d + 1 < s \leq 2n$. As d < k < n we have $2d + 1 < 2k + 1 \leq 2n$, so both $\pi_{2k+1}(M)$ and $\pi_{2k+2}(M)$ are trivial. Therefore the boundary map $\pi_{2k+2}(M, \text{skel}_{2k}(M)) \to \pi_{2k+1}(\text{skel}_{2k}(M))$ in the long exact sequence of relative homotopy groups is an isomorphism. Since $\text{skel}_{2k}(M)$ has trivial homology in degrees larger than 2k, the map $H_{2k+2}(M) \to H_{2k+2}(M, \text{skel}_{2k}(M))$ in the long exact sequence of relative homology groups is also an isomorphism. Therefore, by Lemma 5.1 (a), g_M is an isomorphism.

Lemma 5.3. If there is a ring isomorphism $\gamma \colon H^*(N) \to H^*(M)$ then there is a commutative diagram

where γ^{\vee} is the dual of γ .

Proof. As in (6), γ induces a map $\Gamma \colon BT^{\ell} \to BT^{\ell}$. Consider the diagram

$$(13) \qquad \begin{array}{c} H_{2d+2}(BT^{\ell}) & \xrightarrow{g_{BT^{\ell}}} & \pi_{2d+1}(\operatorname{skel}_{2d}(BT^{\ell})) \\ & & & & \\ &$$

Observe that Diagram (12) is located in the inner square. The outer square is obtained by applying Lemma 5.1 to the map $\Gamma: BT^{\ell} \to BT^{\ell}$, and hence it is commutative. Diagram (A) is the dual of (7), and hence it is also homotopy commutative. Diagrams (B) and (D) are commutative by Corollary 5.2. Diagram (C) commutes by the definition of Φ_{2d} , see (8). Therefore, from the commutativity of (13), we obtain

$$(\delta'_{2d})_* \circ g_N \circ \gamma^{\vee} = g_{BT^{\ell}} \circ \Gamma_* \circ \delta_* = (\Gamma_{2d})_* \circ g_{BT^{\ell}} \circ \delta_* = (\delta'_{2d})_* \circ (\Phi_{2d})_* \circ g_M.$$

Since δ'_{2d} is a homotopy equivalence by Lemma 2.5, we then obtain $g_N \circ \gamma^{\vee} = (\Phi_{2d})_* \circ g_M$, which is the equality asserted by the lemma.

To construct homotopy equivalences Φ_{2k} : skel $_{2k}(M) \to \text{skel }_{2k}(N)$ for $k \ge d+1$, we prepare more explicit notations for the map $g_M: H_{2k+2}(M) \to \pi_{2k+1}(\text{skel }_{2k}(M))$ of Lemma 5.1. For each $k \ge 1$, enumerate (2k + 2)-cells of M as e_1, \ldots, e_{s_k} , whose attaching maps are f_1, \ldots, f_{s_k} , respectively. To each e_i there is a homology class $[e_i] \in H_{2k+2}(M)$, giving a group isomorphism $H_{2k+2}(M) \cong \mathbb{Z}\langle [e_1], \ldots, [e_{s_k}] \rangle$. Define a linear map

(14)
$$\mathbb{Z}\langle f_1, \dots, f_{s_k} \rangle \longrightarrow \mathbb{Z}\langle [e_1], \dots, [e_{s_k}] \rangle \cong H_{2k+2}(M)$$

by sending f_i to $[e_i]$. Since g_M sends $[e_i]$ to the homotopy class of its attaching map, the composite

(15)
$$g: \mathbb{Z}\langle f_1, \dots, f_{s_k} \rangle \longrightarrow \mathbb{Z}\langle [e_1], \dots, [e_{s_k}] \rangle \cong H_{2k+2}(M) \xrightarrow{g_M} \pi_{2k+1}(\operatorname{skel}_{2k}(M))$$

sends f_i to its homotopy class.

The same logic applies to N, so that we have a group isomorphism $H_{2k+2}(N) \cong \mathbb{Z} \langle [e'_1], \ldots, [e'_{s_k}] \rangle$ where $[e'_1], \ldots, [e'_{s_k}]$ are homology classes of cycles representing (2k+2)-cells e'_1, \ldots, e'_{s_k} of N. The composite

(16)
$$g' \colon \mathbb{Z}\langle f'_1, \dots, f'_{s_k} \rangle \longrightarrow \mathbb{Z}\langle [e'_1], \dots, [e'_{s_k}] \rangle \cong H_{2k+2}(N) \xrightarrow{g_N} \pi_{2k+1}(\operatorname{skel}_{2k}(N))$$

sends f'_i to its homotopy class.

We are now ready to construct homotopy equivalences.

Lemma 5.4. Given $k \ge 1$, consider the maps g and g' defined above and suppose there is a homotopy equivalence Φ_{2k} : $skel_{2k}(M) \rightarrow skel_{2k}(N)$. If there is a linear isomorphism $A: \mathbb{Z}\langle f_1, \ldots, f_{s_k} \rangle \rightarrow \mathbb{Z}\langle f'_1, \ldots, f'_{s_k} \rangle$ making the diagram

(17)
$$\mathbb{Z}\langle f_1, \dots, f_{s_k} \rangle \xrightarrow{g} \pi_{2k+1}(skel_{2k}(M))$$
$$\downarrow_A \qquad \qquad \downarrow^{(\Phi_{2k})_*}$$
$$\mathbb{Z}\langle f'_1, \dots, f'_{s_k} \rangle \xrightarrow{g'} \pi_{2k+1}(skel_{2k}(N))$$

commute, then there is a homotopy equivalence Φ_{2k+2} : $skel_{2k+2}(M) \rightarrow skel_{2k+2}(N)$.

Proof. The linear isomorphism $A: \mathbb{Z}\langle f_1, \ldots, f_{s_k} \rangle \to \mathbb{Z}\langle f'_1, \ldots, f'_{s_k} \rangle$ can be geometrically realized by a homotopy equivalence $\varphi_A: \bigvee_{i=1}^{s_k} S^{2k+1} \to \bigvee_{i=1}^{s_k} S^{2k+1}$ as follows. If $A(f_i) = \sum_{j=1}^{s_k} a_{ij} f'_j$ for coefficients $a_{ij} \in \mathbb{Z}$, then define φ_A on the i^{th} -wedge summand to be the sum $\sum_{j=1}^{s_k} a_{ij} \iota_j$ where a_{ij} is a map of degree a_{ij} and $\iota_j: S^{2k+1} \to \bigvee_{l=1}^{s_k} S^{2k+1}$ is the inclusion of the j^{th} -wedge summand. We claim that (17) is then geometrically realized by a homotopy commutative diagram

To see this, restrict to the i^{th} sphere of $\bigvee_{i=1}^{s_k} S^{2k+1}$. By (17), we have $((\Phi_{2k})_* \circ g)(f_i) = (g' \circ A)(f_i)$. Since $A(f_i) = \sum_{j=1}^{s_k} a_{ij} f'_j$ and g' is linear, we obtain $((\Phi_{2k})_* \circ g)(f_i) = \left(\sum_{j=1}^{s_k} a_{ij} g'\right)(f_j)$. Since g sends f_i to its homotopy class and g' sends each f'_j to its homotopy class, we obtain $\Phi_{2k} \circ f_i \simeq \sum_{j=1}^{s_k} a_{ij} f'_j$. The right side may be rewritten as $\left(\bigvee_{j=1}^{s_k} f'_j\right) \circ \left(\sum_{j=1}^{s_k} a_{ij} \iota_j\right)$, which by definition of φ_A then equals $\left(\bigvee_{j=1}^{s_k} f'_j\right) \circ \varphi_A \circ \iota_i$. Thus (18) homotopy commutes when restricted to the i^{th} -wedge summand. As i was arbitrary, (18) homotopy commutes.

The homotopy commutativity of (18) implies that there is a homotopy cofibration diagram

where Φ_{2k+2} is an induced map of cofibres. Since φ_A and Φ_{2k} induce isomorphisms in homology, so does Φ_{2k+2} by the Five Lemma. Since all spaces are simply-connected, Φ_{2k+2} is therefore a homotopy equivalence by Whitehead's Theorem.

Remark 5.5. There is a local version of Lemma 5.4: in the statement and proof simply localize spaces away from \mathcal{P} and change all instances of \mathbb{Z} to the integers localized away from \mathcal{P} .

Proposition 5.6. Let M and N be 2n-dimensional quasitoric manifolds with orbit space $P = \prod_{i=1}^{\ell} \Delta^{d}$. If there is a ring isomorphism $H^{*}(M) \cong H^{*}(N)$, then after localizing away from \mathcal{P} , there is a homotopy equivalence Φ_{2k} : $skel_{2k}(M) \to skel_{2k}(N)$ for each $d + 1 \le k \le n$.

Proof. Suppose that k = d + 1. Define A by the composite

$$A \colon \mathbb{Z}\langle f_1, \dots, f_s \rangle \xrightarrow{\cong} H_{2d+2}(M) \xrightarrow{\gamma^{\vee}} H_{2d+2}(N) \xrightarrow{\cong} \mathbb{Z}\langle f'_1, \dots, f'_s \rangle$$

where the left map is from (14), γ^{\vee} is the dual of γ , and the right map is the inverse of (14) with respect to N. As A is a composite of linear isomorphisms, it too is a linear isomorphism. Now consider the diagram

The left square commutes by definition of A and the right square commutes by (12). The composites along the top and bottom rows are the definitions of g and g' respectively. By Proposition 4.2, Φ_{2d} is a homotopy equivalence. Thus the outer rectangle satisfies the hypotheses of Lemma 5.4, implying that there is a homotopy equivalence Φ_{2d+2} : skel_{2d+2}(M) \rightarrow skel_{2d+2}(N).

For k > d + 1, assume inductively that there is a homotopy equivalence Φ_{2k} : $\operatorname{skel}_{2k}(M) \to \operatorname{skel}_{2k}(N)$. Consider the diagram

where A will be defined momentarily. Localize away from \mathcal{P} . Then the maps g_M and g_N are isomorphisms by Corollary 5.2. The top and bottom rows are the definitions of g and g' as in (15) and (16) respectively, so they are both isomorphisms. Define A by the composite

$$A: \mathbb{Z}\langle f_1, \dots, f_{s_k} \rangle \xrightarrow{g} \pi_{2k+1}(\operatorname{skel}_{2k}(M)) \xrightarrow{(\Phi_{2k})_*} \pi_{2k+1}(\operatorname{skel}_{2k}(N)) \xrightarrow{(g')^{-1}} \mathbb{Z}\langle f'_1, \dots, f'_{s_k} \rangle.$$

Then A is a linear isomorphism and it makes (19) commute. Thus (19) satisfies the hypotheses of the local version of Lemma 5.4 discussed in Remark 5.5, implying that there is a homotopy equivalence Φ_{2k+2} : $\operatorname{skel}_{2k+2}(M) \to \operatorname{skel}_{2k+2}(N)$. This completes the induction. \Box

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