## ON THE ZEROS OF THE MILLER BASIS OF CUSP FORMS

#### ROEI RAVEH

ABSTRACT. We study the zeros of cusp forms in the Miller basis whose vanishing order at infinity is a fixed number m. We show that for sufficiently large weights, the finite zeros in the fundamental domain of such forms, all lie on the circular part of the boundary of the fundamental domain. We further show and quantify an effective bound for the weight, which is linear in terms of m.

#### 1. Introduction

Let  $k \geq 0$  be an even integer, and let  $M_k$  denote the linear space of modular forms of weight k for the modular group  $\Gamma = \operatorname{PSL}_2(\mathbb{Z})$ . Each modular form f has a g-expansion (i.e. Fourier series),

$$f(\tau) = \sum_{n=n_0}^{\infty} a_f(n) q^n, \quad q = e^{2\pi i \tau}$$

where  $\tau \in \mathbb{H}$ , and  $n_0 = \operatorname{ord}_{\infty}(f)$ .

For any nonzero  $f \in M_k$ , we have the valence formula (see [15]):

$$(1.1) \qquad \operatorname{ord}_{\infty}(f) + \frac{1}{2}\operatorname{ord}_{i}(f) + \frac{1}{3}\operatorname{ord}_{\rho}(f) + \sum_{z \in \mathcal{F} \setminus \{i, \rho\}} \operatorname{ord}_{z}(f) = \frac{k}{12},$$

where  $\mathcal{F}$  is the usual fundamental domain and  $\rho = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$ . Writing  $k = 12\ell + k'$  where  $k' \in \{0, 4, 6, 8, 10, 14\}$ , we obtain

$$(1.2) \ell > \operatorname{ord}_{\infty}(f).$$

Those formulas provide a powerful tool for studying modular forms via their zeros. (1.1) suggests that for a nonzero modular form of weight k, there are about  $\frac{k}{12} - \operatorname{ord}_{\infty}(f)$  zeros in the fundamental domain  $\mathcal{F}$ . The space  $M_k$  is finite-dimensional and is spanned by the Eisenstein series (2.4) and the space of cusp forms  $S_k$ , with  $\dim S_k = \ell$ . The zeros of the Eisenstein series were studied in 1970 by F. Rankin and F. Swinnerton-Dyer [9]. In their paper, they proved that all the zeros of the Eisenstein series in the fundamental domain lie on the arc  $\mathcal{A} = \left\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\right\}$  and become uniformly distributed in  $\mathcal{A}$  as  $k \to \infty$ . This argument of Rankin and Swinnerton-Dyer was used to prove similar results, for instance, [3], [10]. For different types of results about zeros of various modular forms, see [4], [7], [11], [12], [13], [14], [16].

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This paper will discuss the zeros of the Miller basis of modular forms. The elements of the Miller basis  $\{g_{k,m}\}_{m=0}^{\ell}$  are uniquely defined for every  $0 \leq m \leq \ell$  by requiring

$$g_{k,m}(\tau) = q^m + O\left(q^{\ell+1}\right).$$

The Miller basis forms a canonical basis of  $M_k$  in the sense that it is a basis of reduced row echelon form. In particular, we are interested in the cusp forms of the Miller basis  $g_{k,1}, \ldots, g_{k,\ell}$ .

W. Duke and P. Jenkins [1] showed that the "gap forms"  $g_{k,0} = 1 + O\left(q^{\ell+1}\right)$  has all its zeros on the arc  $\mathcal{A}$  in the fundamental domain. This is not generally true for  $g_{k,m}$  with  $m \geq 1$ , for example, as pointed out in [1],  $g_{132,9}$  doesn't have all its zeros on the arc  $\mathcal{A}$ . Nevertheless, an asymptotic result can be achieved:

**Theorem 1.1.** Fix  $m \geq 1$ . There exists  $\alpha, \beta > 0$  so that if  $\ell > \alpha m + \beta$ , then all the zeros of  $g_{k,m}$  in the fundamental domain lie on the arc  $\left\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\right\}$ , and become uniformly distributed on the arc.

In §2 we discuss some needed background on modular forms and the Miller basis. In §3 we will prove Theorem 1.1. In §4 we quantify those bounds and show we can choose  $\alpha = 4.5$  and  $\beta = 9.5$ . Finally, in §5 we investigate the behavior of  $g_{k,1}$ , and prove the following:

**Theorem 1.2.** For every  $\ell \geq 1$ , all the zeros of  $g_{k,1}(\tau) = q + O\left(q^{\ell+1}\right)$  in the fundamental domain lie on the arc.

# 2. Background and preliminaries on modular forms

2.1. **Definitions.** Let  $k \geq 0$  be an even integer, and let  $\mathbb{H} = \{\tau : \operatorname{Im}(\tau) > 0\}$  denote the upper half plane. Let  $f : \mathbb{H} \to \mathbb{C}$  be a holomorphic function; we say that f is a modular form of weight k if

(2.1) 
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2\left(\mathbb{Z}\right).$$

and f is bounded as  $\text{Im}(\tau) \to \infty$ . If f vanishes as  $\text{Im}(\tau) \to \infty$ , we say that f is a cusp form.

Remark. We can replace (2.1) with the following conditions:

$$(2.2) f(\tau) = f(\tau+1),$$

(2.3) 
$$f(\tau) = \tau^{-k} f(-1/\tau).$$

When  $k \geq 4$ , there exists a nonzero modular form in  $M_k$  known as the (normalized) Eisenstein series

(2.4) 
$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m\tau+n)^k} = 1 - \gamma_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where  $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ ,  $\gamma_k = \frac{2k}{B_k}$ , and  $B_k$  is the k-th Bernoulli number. One can also define the Eisenstein series of weight 2

$$(2.5) \quad E_2(\tau) = \frac{1}{2\zeta\left(2\right)} \sum_{n \neq 0} \frac{1}{n^2} + \frac{1}{2\zeta\left(2\right)} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{\left(m\tau + n\right)^2} = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

While  $E_2$  is not a modular form, it has some modular properties and transforms as

(2.6) 
$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{i\pi}.$$

The first cusp form we encounter is the Modular Discriminant,

(2.7) 
$$\Delta(\tau) = \frac{1}{1728} \left( E_4^3(\tau) - E_6^2(\tau) \right) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

The coefficients  $\tau(n)$  are known as Ramanujan's tau function and are all integers. Lastly, there is a meromorphic modular form of weight 0, Klein's absolute invariant, or the j-function:

(2.8) 
$$j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)} = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n,$$

where the coefficients c(n) are all integers.

2.2. The Miller Basis for Modular Forms. Let  $m \in \{1, ..., \ell\}$ , and denote:

$$e_{k,m} = \Delta^{\ell} E_{k'} j^{\ell-m}.$$

Notice that  $e_{k,m}$  has integer coefficients and that

$$e_{k,m} = (q^{\ell} + O(q^{\ell+1})) (q^{-\ell+m} + O(q^{-\ell+m+1})) = q^m + O(q^{m+1}).$$

Using Gaussian elimination we obtain a basis of reduced row echelon form,  $\{g_{k,m}\}_{m=1}^{\ell}$ . Moreover, for any  $m \in \{1, \dots, \ell\}$  there exists a unique polynomial  $F_{k,m} \in \mathbb{Z}[x]$  of degree  $\ell - m$ , so that

(2.9) 
$$g_{k,m} = \Delta^{\ell} E_{k'} F_{k,m}(j) = q^m + O\left(q^{\ell+1}\right).$$

Remark. Following each step in the Gaussian elimination process, we can see that  $F_{k,m}$  has integer coefficients. The polynomial  $F_{k,m}$  is the associated Faber polynomial of  $g_{k,m}$ ; Faber polynomials play a major role in the study of zeros of modular forms (see [14]), and we will discuss those in detail in §5.1.

2.3. Modular Forms on the Arc and Under Conjugation. Let  $f \in M_k$ , and consider its q-expansion  $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ . Suppose  $a_n$  are all real, then:

(2.10) 
$$\overline{f(\tau)} = \sum_{n=0}^{\infty} a_n \overline{e^{2\pi i n \tau}} = \sum_{n=0}^{\infty} a_n e^{2\pi i n (-\overline{\tau})} = f(-\overline{\tau}).$$

Suppose  $\tau = e^{i\theta}$ , with  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . We have  $\overline{\tau} = 1/e^{i\theta}$  and from (2.3) we get

$$\overline{f\left(e^{i\theta}\right)} = f\left(-1/e^{i\theta}\right) = e^{ik\theta}f\left(e^{i\theta}\right),\,$$

which yields,

(2.11) 
$$\overline{e^{ik\theta/2}f(e^{i\theta})} = e^{-ik\theta/2}\overline{f(e^{i\theta})} = e^{ik\theta/2}f(e^{i\theta}).$$

So,  $g(\theta) = e^{ik\theta/2} f(e^{i\theta})$  is real valued.

# 3. Proof of theorem 1.1

We begin with proving some bounds on  $\Delta$ .

## 3.1. Bounds and properties of $\Delta$ .

**Proposition 3.1.** For all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ ,

$$\left|\Delta\left(e^{i\theta}\right)\right| \ge \left|\Delta(i)\right| = \left(\frac{\varpi}{\sqrt{2}\pi}\right)^{12} = 0.00178537\dots$$

with  $\varpi = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.622057...$ , and

$$\left|\Delta\left(e^{i\theta}\right)\right| \le \left|\Delta(\rho)\right| = \frac{27}{256} \left(\frac{\varpi'}{\pi}\right)^{12} = 0.00480514\dots$$

with  $\varpi' = 2 \int_0^1 \frac{dx}{\sqrt{1-x^6}} = 2.42865...$ 

- (i) The function  $e_4\left(\theta\right)=e^{2i\theta}E_4\left(e^{i\theta}\right)$  is negative for all  $\theta\in$ 

  - $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$  and vanishes at  $\frac{2\pi}{3}$ .

    (ii) The function  $\delta(\theta) = e^{6i\theta}\Delta\left(e^{i\theta}\right)$  is negative for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ .

    (iii) The function  $e_2(\theta) = e^{i\theta}E_2\left(e^{i\theta}\right) + \frac{3}{i\pi}$  is real valued and is negative for all
- (i)  $E_4$  has a unique zero in the fundamental domain, at  $\rho = e^{2\pi i/3}$ . Proof. Therefore,  $e_4$  is continuous and nonzero for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$ . Hence, it is enough to show that  $e_4\left(\frac{\pi}{2}\right) < 0$ , and indeed

$$-e_4\left(\frac{\pi}{2}\right) = E_4(i) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) e^{-2\pi n} \ge 1 > 0.$$

(ii) We know that  $\Delta$  never vanishes, thus  $\delta$  never vanishes. Hence, it is enough to show that  $\delta\left(\frac{\pi}{2}\right) < 0$ . Indeed,

$$\delta\left(\frac{\pi}{2}\right) = -\Delta(i) = \frac{E_4(i)^3}{1728} > 0,$$

since  $E_4(i) > 0$ .

(iii) First, we will show that  $e_2$  is real valued. Since  $E_2$  has real Fourier coefficients, we have  $\overline{E_2(\tau)} = E_2(-\overline{\tau})$ . Using (2.6) we obtain

$$\overline{e_{2}(\theta)} = e^{-i\theta} \overline{E_{2}(e^{i\theta})} - \frac{3}{i\pi} = e^{-i\theta} E_{2}(-e^{-i\theta}) - \frac{3}{i\pi}$$

$$= e^{-i\theta} \left( e^{2i\theta} E_{2}(e^{i\theta}) + \frac{6e^{i\theta}}{i\pi} \right) - \frac{3}{i\pi} = e^{i\theta} E_{2}(e^{i\theta}) + \frac{3}{i\pi} = e_{2}(\theta).$$

Using (2.6) again, we get  $E_2(i) = \frac{3}{\pi}$ , and thus

$$e_2\left(\frac{\pi}{2}\right) = e^{i\pi/2}E_2(i) + \frac{3}{i\pi} = i\frac{3}{\pi} - \frac{3}{\pi}i = 0.$$

We claim that  $e_2$  is decreasing on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ , which yields  $e_2(\theta) < 0$  for all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . We will show that  $\frac{de_2}{d\theta} < 0$ : Using an identity of Ramanujan [8], we know that

$$\frac{1}{2\pi i} \frac{dE_2}{d\tau} = \frac{E_2^2 - E_4}{12}.$$

Hence,

$$\begin{split} \frac{de_2}{d\theta}(\theta) &= \frac{d}{d\theta} \left( e^{i\theta} E_2(\theta) + \frac{3}{i\pi} \right) \\ &= i e^{i\theta} E_2\left( e^{i\theta} \right) + e^{i\theta} \frac{d}{d\theta} \left( E_2\left( e^{i\theta} \right) \right) \\ &= i \left( e_2(\theta) - \frac{3}{i\pi} \right) + e^{i\theta} \cdot i e^{i\theta} \frac{dE_2}{d\tau} \left( e^{i\theta} \right) \\ &= i e_2(\theta) - \frac{3}{\pi} - e^{2i\theta} \frac{\pi}{6} \left( E_2\left( e^{i\theta} \right)^2 - E_4\left( e^{i\theta} \right) \right) \\ &= i e_2(\theta) - \frac{3}{\pi} - \frac{\pi}{6} \left( \left( e^{i\theta} E_2\left( e^{i\theta} \right) \right)^2 - e_4(\theta) \right) \\ &= i e_2(\theta) - \frac{3}{\pi} + \frac{\pi}{6} e_4(\theta) + \frac{\pi}{6} \left( e_2(\theta) - \frac{3}{i\pi} \right)^2 \\ &= i e_2(\theta) - \frac{3}{\pi} + \frac{\pi}{6} e_4(\theta) - \frac{\pi}{6} e_2(\theta)^2 + \frac{e_2(\theta)}{i} + \frac{3}{2\pi} \\ &= -\frac{3}{2\pi} - \frac{\pi}{6} e_2(\theta)^2 + \frac{\pi}{6} e_4(\theta) \,. \end{split}$$

Since  $e_4(\theta) < 0$  for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$  we obtain  $\frac{de_2}{d\theta}(\theta) < 0$  for all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$ .

We now have the tools to prove the following lemma, which implies Proposition 3.1,

**Lemma 3.3.** The function  $\delta(\theta)$  is decreasing on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ .

*Proof.* Taking the logarithmic derivative of  $\Delta$ , we get

$$q\frac{\frac{d\Delta}{dq}}{\Delta} = 1 - 24\sum_{n=1}^{\infty} n\frac{q^{n}}{1 - q^{n}} = E_{2}\left(\tau\right),$$

therefore

$$\frac{d\Delta}{d\tau} = 2\pi i E_2 \Delta.$$

Hence.

$$\begin{split} \frac{d\delta}{d\theta}\left(\theta\right) &= 6ie^{6i\theta}\Delta\left(e^{i\theta}\right) + e^{6i\theta}\cdot ie^{i\theta}\frac{d\Delta}{d\tau}\left(e^{i\theta}\right) = 6i\delta\left(\theta\right) - 2\pi e^{6i\theta}\cdot e^{i\theta}E_{2}\left(e^{i\theta}\right)\Delta\left(e^{i\theta}\right) \\ &= 6i\delta\left(\theta\right) - 2\pi\left(e_{2}\left(\theta\right) - \frac{3}{i\pi}\right)\delta\left(i\theta\right) = -2\pi e_{2}\left(\theta\right)\delta\left(\theta\right). \end{split}$$

Since  $e_2 < 0$  for all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$  and  $\delta\left(\theta\right) < 0$  for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$  we get  $\frac{d\delta}{d\theta}\left(\theta\right) < 0$  for all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . Hence,  $\delta$  is decreasing on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ .

Proof of Proposition 3.1. Since  $|\Delta(e^{i\theta})|$  is increasing on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ , for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$  we have

$$\Delta\left(i\right) \leq \left|\Delta\left(e^{i\theta}\right)\right| \leq \left|\Delta\left(\rho\right)\right|.$$

Since  $\Delta = \frac{E_4^3 - E_6^2}{1728}$  and since  $E_4(\rho) = 0$ ,  $E_6(i) = 0$  we get

(3.1) 
$$\Delta(i) = \frac{E_4(i)^3}{1728}$$

$$\Delta\left(\rho\right) = -\frac{E_6\left(\rho\right)^2}{1728}$$

Hurwitz [5] showed that

$$\sum_{(m,n)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}}\frac{1}{(mi+n)}=\frac{\varpi^4}{15},$$

with  $\varpi = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.622057...$ , hence

$$E_4(i) = \frac{1}{2\zeta(4)} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mi+n)} = \frac{3\varpi^4}{\pi^4}.$$

Therefore,

$$\Delta(i) = \frac{E_4(i)^3}{1728} = \left(\frac{E_4(i)}{12}\right)^3 = \left(\frac{\varpi}{\sqrt{2}\pi}\right)^{12} = 0.00178537....$$

Katayama [6] gives a generalization of Hurwitz's formula

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\rho+n)^6} = \frac{\varpi'^6}{35},$$

with  $\varpi' = 2 \int_0^1 \frac{dx}{\sqrt{1-x^6}} = 2.42865...$ , hence

$$E_{6}\left(\rho\right) = \frac{1}{2\zeta\left(6\right)} \sum_{(m,n) \in \mathbb{Z}^{2} \setminus \{(0,0)\}} \frac{1}{\left(m\rho + n\right)^{6}} = \frac{945}{2\pi^{6}} \frac{\varpi'^{6}}{35} = \frac{27\varpi'^{6}}{2\pi^{6}}.$$

Therefore,

$$\Delta(\rho) = -\frac{E_6(\rho)^2}{1728} = -\frac{27}{256} \left(\frac{\varpi'}{\pi}\right)^{12} = -0.00480514\dots$$

**Proposition 3.4.** (i) For all  $\tau \in \mathbb{H}$  with  $e^{-2\pi \operatorname{Im}(\tau)} \leq \frac{1}{3}$ ,

$$|\Delta\left(\tau\right)| \ge e^{-2\pi \operatorname{Im}(\tau)} \left(1 - e^{-2\pi \operatorname{Im}(\tau)} - e^{-4\pi \operatorname{Im}(\tau)} - \frac{2e^{-4\pi \operatorname{Im}(\tau)}}{1 - e^{-2\pi \operatorname{Im}(\tau)}}\right)^{24}.$$

(ii) For all  $\tau \in \mathbb{H}$ 

$$|\Delta(\tau)| \le e^{-2\pi \operatorname{Im}(\tau)} \left( 1 + 2e^{-2\pi \operatorname{Im}(\tau)} + \frac{2e^{-8\pi \operatorname{Im}(\tau)}}{1 - e^{-4\pi \operatorname{Im}(\tau)}} \right)^{24}.$$

*Proof.* (i) Firstly, recall that by the pentagonal number theorem (Euler) for any |z| < 1 we have:

$$\prod_{n=1}^{\infty} (1 - z^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left( z^{\frac{k(3k+1)}{2}} + z^{\frac{k(3k-1)}{2}} \right)$$

and the series on the RHS converges absolutely.

Due to the absolute convergence, we can change the order of summation,

$$\prod_{n=1}^{\infty} (1 - z^n) = 1 + \sum_{k=1}^{\infty} \left( z^{k(6k+1)} + z^{k(6k-1)} - z^{(2k-1)(3k-1)} - z^{(2k-1)(3k-2)} \right) 
= 1 + z^7 + z^5 - z - z^2 + \sum_{k=2}^{\infty} z^{6k^2} \left( z^k + z^{-k} - z^{-5k+1} - z^{-7k+2} \right)$$
(\*)

Let  $z \in (0,1)$ . For all  $k \ge 1$ ,  $z^{-5k+1} < z^{-7k+2}$  and since  $0 \le 5k^2 - 7k + 2$  we also have  $z^{6k^2 - 7k + 2} \le z^{k^2}$ .

$$z^{6k^2} \left( z^k + z^{-k} - z^{-5k+1} - z^{-7k+2} \right) \ge -z^{6k^2} \left( z^{-5k+1} + z^{-7k+2} \right) > -2z^{6k^2} z^{-7k+2} > -2z^{k^2} \ge -2z^{k^2} z^{-7k+2} > -$$

$$\prod_{n=1}^{\infty} (1-z^n) \ge 1-z-z^2-2\sum_{k=1}^{\infty} z^k = 1-z-z^2-\frac{2z^2}{1-z}.$$

Thus, for any  $\tau \in \mathbb{H}$  with  $e^{-2\pi \operatorname{Im}(\tau)} \leq \frac{1}{3}$  we have:

$$\left| q \prod_{n=1}^{\infty} (1 - q^n)^{24} \right| \ge |q| \left( \prod_{n=1}^{\infty} (1 - |q|^n) \right)^{24} = e^{-2\pi \operatorname{Im}(\tau)} \left( \prod_{n=1}^{\infty} \left( 1 - e^{-2\pi n \operatorname{Im}(\tau)} \right) \right)^{24}$$

$$\ge e^{-2\pi \operatorname{Im}(\tau)} \left( 1 - e^{-2\pi \operatorname{Im}(\tau)} - e^{-4\pi \operatorname{Im}(\tau)} - \frac{2e^{-4\pi \operatorname{Im}(\tau)}}{1 - e^{-2\pi \operatorname{Im}(\tau)}} \right)^{24}.$$

(ii) By the pentagonal number theorem and the triangle inequality:

$$\left| \prod_{n=1}^{\infty} (1-z^n) \right| \le 1 + \sum_{k=1}^{\infty} \left| (-1)^k \left( z^{\frac{k(3k+1)}{2}} + z^{\frac{k(3k-1)}{2}} \right) \right| \le 1 + \sum_{k=1}^{\infty} |z|^{\frac{k(3k+1)}{2}} + |z|^{\frac{k(3k-1)}{2}}.$$

For all |z| < 1, we have  $|z|^{\frac{k(3k+1)}{2}} < |z|^{\frac{k(3k-1)}{2}}$ . In addition, since  $\frac{k(3k-1)}{2} \ge 2k$  for all  $k \ge 2$ , we obtain:

$$|z|^{\frac{k(3k+1)}{2}} + |z|^{\frac{k(3k-1)}{2}} \le 2|z|^{\frac{k(3k-1)}{2}} \le 2|z|^{2k}$$

Thus,

$$\left| \prod_{n=1}^{\infty} (1 - z^n) \right| \le 1 + 2|z| + \sum_{k=2}^{\infty} 2|z|^{2k} = 1 + |z| + \frac{2|z|^4}{1 - |z|^2},$$

which shows

$$|\Delta(\tau)| \le e^{-2\pi \operatorname{Im}(\tau)} \left( 1 + 2e^{-2\pi \operatorname{Im}(\tau)} + \frac{2e^{-8\pi \operatorname{Im}(\tau)}}{1 - e^{-4\pi \operatorname{Im}(\tau)}} \right)^{24},$$

for all  $\tau \in \mathbb{H}$ , and concludes our proof.

Corollary 3.5. For all  $\theta \in [\pi/2, 2\pi/3], x \in [-1/2, 1/2]$ :

(3.3) 
$$\left| \frac{\Delta(e^{i\theta})}{\Delta(x + 0.65i)} \right| < \frac{1}{2},$$

and

(3.4) 
$$\left| \frac{\Delta(e^{i\theta})}{\Delta(x+0.75i)} \right| \le \frac{7}{10}.$$

*Proof.* Substitute  $\tau = x + 0.65i$ ,

$$|\Delta(x+0.65i)| \ge e^{-\frac{13\pi}{10}} \left( 1 - e^{-\frac{13\pi}{10}} - e^{-\frac{13\pi}{5}} - \frac{2e^{-\frac{26\pi}{5}}}{1 - e^{-\frac{13\pi}{10}}} \right)^{24} > 0.01$$

Now, using Proposition 3.1, we know  $\left|\Delta(e^{i\theta})\right| < 0.005$  for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . For all  $\theta \in [\pi/2, 2\pi/3], \ x \in [-1/2, 1/2]$ :

$$\left| \frac{\Delta(e^{i\theta})}{\Delta(x+0.65i)} \right| \le \frac{0.005}{0.01} = \frac{1}{2}.$$

Similarly, we can get:

$$\left|\frac{\Delta(e^{i\theta})}{\Delta(x+0.75i)}\right| \le \frac{7}{10}.$$

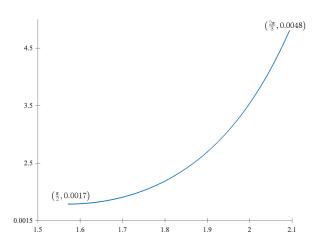


FIGURE 1.  $\left|\Delta(e^{i\theta})\right|$  on the interval  $\left[\frac{\pi}{2},\frac{2\pi}{3}\right]$ 

For the proof of Theorem 1.1 we will also need the following proposition:

**Proposition 3.6.** Fix  $m \ge 1$ . There exists  $c_1, c_2 > 0$  so that if  $\ell > c_1 m + c_2$ , then we have

$$\left| e^{ik\theta/2} e^{2\pi m \sin \theta} g_{k,m}(e^{i\theta}) - 2\cos\left(k\theta/2 + 2\pi m \cos \theta\right) \right| < 2$$

for all  $\theta \in (\frac{\pi}{2}, \frac{2\pi}{3})$ . Furthermore,  $c_1 = \frac{\pi}{2\log(10/7)}$ .

3.2. **Proof of Proposition 3.6.** We use the method introduced in the work of W. Duke and P. Jenkins in [1]. We begin with the statement and proof of the following version of Lemma 2 in [1]:

**Lemma 3.7.** Let R > 0. There exists A > 1 so that for all  $z \in \mathcal{F}$  with |j(z)| < R we have

$$g_{k,m}(z) = \int_{-1/2 + iA}^{1/2 + iA} \frac{\Delta^{\ell}(z) E_{k'}(z) E_{14 - k'}(\tau)}{\Delta^{\ell+1}(\tau) \left(j(\tau) - j(z)\right)} e^{2\pi i m \tau} d\tau.$$

Proof. We have  $j(\tau) = q^{-1} + 744 + P(q)$  where P(q) = O(|q|) as  $\text{Im}(\tau) \to \infty$ . Thus, there exists N > 0 so that for any  $\tau \in \mathbb{H}$  with  $\text{Im}(\tau) > N$ , we have |P(q)| < 744. Choose  $A > \max(N, 1)$  such that  $e^{2\pi A} > R + 1488$ . For any  $\text{Im}(\tau) \ge A$ ,

$$(3.5) |j(\tau)| \ge \frac{1}{|q|} - 744 - |P(q)| > e^{2\pi \operatorname{Im}(\tau)} - 1488 \ge e^{2\pi A} - 1488 > R.$$

Thus, there exists A>1 so that  $j\left(\left[-\frac{1}{2}+iA,\frac{1}{2}+iA\right]\right)\subset\mathbb{C}\setminus\{|z|< R\}$ . Denote  $\mathcal{C}=j\left(\left[-\frac{1}{2}+iA,\frac{1}{2}+iA\right]\right)$ , and  $\gamma:\left[-\frac{1}{2}+iA,\frac{1}{2}+iA\right]\to\mathbb{C}$  as the curve  $\gamma(\tau)=j(\tau)$ . Then  $\gamma$  is a closed, simple, and smooth curve onto  $\mathcal{C}$ , oriented clockwise. Let  $U\subset\mathbb{C}$  be the bounded connected component of  $\mathbb{C}\setminus\mathcal{C}$ , and let  $|\zeta|< R$ . So  $\zeta\in U$ , and the Faber polynomial  $F_{k,m}$  is holomorphic on  $\overline{U}$ . Thus, using Cauchy's formula,

$$F_{k,m}(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F_{k,m}(\xi)}{\xi - \zeta} d\xi = \frac{-1}{2\pi i} \int_{-1/2 + iA}^{1/2 + iA} \frac{F_{k,m}(j(\tau))}{j(\tau) - \zeta} \frac{dj}{d\tau}(\tau) d\tau.$$

To calculate  $\frac{dj}{dx}$ , we use the following identity, which is due to Ramanujan [8]:

$$q\frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3}.$$

Hence,

$$q\frac{dj}{dq} = \frac{3q\frac{dE_4}{dq}E_4^2\Delta - q\frac{d\Delta}{dq}E_4^3}{\Delta^2} = \frac{(E_2E_4 - E_6)E_4^2\Delta - E_2\Delta E_4^3}{\Delta^2} = -\frac{E_6E_4^2}{\Delta},$$

which, using the chain rule, yields:

(3.6) 
$$\frac{dj}{d\tau} = \frac{dj}{dq}\frac{dq}{d\tau} = 2\pi i q \frac{dj}{dq} = -2\pi i \frac{E_{14}}{\Delta}.$$

Therefore,

(3.7) 
$$F_{k,m}(\zeta) = \int_{-1/2+iA}^{1/2+iA} \frac{F_{k,m}(j(\tau))}{j(\tau) - \zeta} \frac{E_{14}(\tau)}{\Delta(\tau)} d\tau.$$

We have  $g_{k,m}(\tau) = q^m + q^{\ell+1}G(\tau)$ , where G is a holomorphic function and  $\operatorname{ord}_{\infty}G = 0$ . Therefore, we have:

$$(3.8) \ F_{k,m}(j(\tau)) = \frac{q^m}{\Delta^{\ell}(\tau)E_{k'}(\tau)} + \frac{q^{\ell+1}G(\tau)}{\Delta^{\ell}(\tau)E_{k'}(\tau)} = \frac{q^m}{\Delta^{\ell}(\tau)E_{k'}(\tau)} + q\frac{G(\tau)}{P^{\ell}(\tau)E_{k'}(\tau)},$$

where  $P(\tau) = \prod_{n>1} (1-q^n)^{24}$ . Plugging (3.8) in (3.7), we get:

$$F_{k,m}\left(\zeta\right) = \int_{-1/2+iA}^{1/2+iA} \frac{e^{2\pi im\tau}}{\Delta^{\ell}(\tau)E_{k'}(\tau)} \frac{E_{14}(\tau)}{\Delta(\tau)\left(j(\tau) - \zeta\right)} d\tau + \int_{-1/2+iA}^{1/2+iA} q \frac{G(\tau)}{P^{\ell}(\tau)E_{k'}(\tau)} \frac{E_{14}(\tau)}{\Delta(\tau)\left(j(\tau) - \zeta\right)} d\tau.$$

Under the change of variables  $\tau \to q$ , the contour of integration deforms into a circle of radius  $e^{-2\pi A}$  around the origin (oriented counter-clockwise). Doing so with the second integral gives us

$$F_{k,m}(\zeta) = \int_{-1/2+iA}^{1/2+iA} \frac{e^{2\pi i m \tau}}{\Delta^{\ell}(\tau) E_{k'}(\tau)} \frac{E_{14}(\tau)}{\Delta(\tau) (j(\tau) - \zeta)} d\tau + \frac{1}{2\pi i} \int_{\{|q| = e^{-2\pi A}\}} \frac{G(\tau)}{P^{\ell}(\tau) E_{k'}(\tau)} \frac{E_{14}(\tau)}{\Delta(\tau) (j(\tau) - \zeta)} dq.$$

The function  $q\mapsto \frac{G(\tau)}{P^\ell(\tau)E_{k'}(\tau)}\frac{F_{k,m}(j(\tau))}{j(\tau)-\zeta}\frac{E_{14}(\tau)}{\Delta(\tau)}$  is holomorphic on the disk  $\{|q|< e^{-2\pi A}\}$ , so by Cauchy's Theorem:

(3.9) 
$$F_{k,m}(\zeta) = \int_{-1/2+iA}^{1/2+iA} \frac{e^{2\pi i m \tau}}{\Delta^{\ell+1}(\tau) E_{k'}(\tau)} \frac{E_{14}(\tau)}{j(\tau) - \zeta} d\tau.$$

Now, let  $z \in \mathcal{F}$  and suppose that |j(z)| < R. Substituting  $\zeta = j(z)$  into (3.9) and multiplying by  $\Delta^{\ell}(z)E_{k'}(z)$ , we get: (3.10)

$$g_{k,m}(z) = \Delta^{\ell}(z)E_{k'}(z)F_{k,m}(j(z)) = \int_{-1/2+iA}^{1/2+iA} \frac{\Delta^{\ell}(z)E_{k'}(z)E_{14-k'}(\tau)}{\Delta^{\ell+1}(\tau)(j(\tau)-j(z))} e^{2\pi i m \tau} d\tau.$$

Equipped with Lemma 3.7, we will lower the contour of integration, collecting poles as we decrease it from its initial height.

Fix  $m \geq 1$ . For briefness, we denote

(3.11) 
$$G(\tau, z) = \frac{\Delta^{\ell}(z) E_{k'}(z) E_{14-k'}(\tau)}{\Delta^{\ell+1}(\tau) (j(\tau) - j(z))} e^{2\pi i m \tau}.$$

Using (3.6), and the identity  $E_{14-k'}E_{k'}=E_{14}$  we can write

(3.12) 
$$G(\tau, z) = \frac{e^{2\pi i m \tau}}{-2\pi i} \frac{\Delta^{\ell}(z) E_{k'}(z)}{\Delta^{\ell}(\tau) E_{k'}(\tau)} \frac{\frac{d}{d\tau} (j(\tau) - j(z))}{j(\tau) - j(z)},$$

which will be useful for calculating residues.

Assume  $z=e^{i\theta}$ . Since  $j(e^{i\theta})\in[0,1728]$ , we can choose R>1728 and get A>1 so that:

$$g_{k,m}(z) = \int_{-1/2+iA}^{1/2+iA} G(\tau, z) d\tau.$$

We begin to lower the contour of integration from its initial height A to some A' (which we will determine later).

When  $\frac{\sqrt{3}}{2} < A' < \sin \theta$ , we have two poles inside the region of integration at  $\tau = e^{i\theta}$ 

and  $\tau = e^{i(\pi - \theta)}$ , as demonstrated in Figure 2.

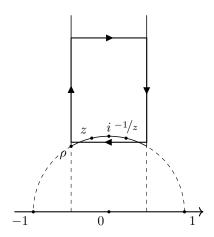


FIGURE 2. The contour of integration when  $\frac{\sqrt{3}}{2} < A' < \sin \theta$ .

Calculating the residues using (3.12), we obtain:

$$\mathop{\rm Res}_{\tau=z} G(\tau,z) = \frac{e^{2\pi i m z}}{-2\pi i}, \quad \mathop{\rm Res}_{\tau=-1/z} G(\tau,z) = \frac{z^{-k} e^{2\pi i m (-1/z)}}{-2\pi i}.$$

Therefore, we get:

$$\int_{-1/2+iA'}^{1/2+iA'} G(\tau,e^{i\theta}) d\tau = g_{k,m}(e^{i\theta}) - e^{2\pi i m e^{i\theta}} - e^{-ik\theta} e^{-2\pi i m e^{-i\theta}}.$$

Multiplying both sides by  $e^{ik\theta/2}e^{2\pi m\sin\theta}$  gives us:

(3.13) 
$$e^{ik\theta/2}e^{2\pi m\sin\theta} \int_{-1/2+iA'}^{1/2+iA'} G(\tau, e^{i\theta})d\tau$$
  
=  $e^{ik\theta/2}e^{2\pi m\sin\theta}g_m(e^{i\theta}) - 2\cos(k\theta/2 + 2\pi m\cos\theta)$ .

For convenience, we denote:

(3.14) 
$$I(A') = e^{2\pi m \sin \theta} \int_{-1/2 + iA'}^{1/2 + iA'} |G(\tau, e^{i\theta})| d\tau.$$

Since  $G(\tau, z)$  has a simple pole whenever  $\tau \sim z$ , we need to check which poles are relevant as we continue to decrease A':

**Lemma 3.8.** The only possible poles in  $D = \left\{ \tau \in \mathbb{H} : |\text{Re}(\tau)| \leq \frac{1}{2}, \text{Im}(\tau) \geq \frac{2}{5} \right\}$  are  $\frac{-1}{z+1}, \frac{z}{z+1}$  and  $\frac{-1}{z-1}, \frac{z}{z-1}$ .

*Proof.* Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ , and suppose  $\gamma \neq \mathrm{id}$ . We divide into cases:

Case I. Suppose |cd| = 0:

If c=0, then  $\gamma=\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$ , and since  $\gamma\neq$  id we have  $|\mathrm{Re}(\gamma.z)|>\frac{1}{2}$ . If d=0, then -bc=1, and thus  $\gamma=\left(\begin{smallmatrix}*&-1\\1&0\end{smallmatrix}\right)$ , so again we have  $|\mathrm{Re}(\gamma.z)|>\frac{1}{2}$ . Hence,  $\gamma.z\notin D$ .

Case II. Suppose  $|cd| \ge 2$ : In this case,  $c^2 + d^2 \ge 5$ , and

$$\left|ce^{i\theta} + d\right|^2 = c^2 + d^2 + 2cd\cos\theta \ge c^2 + d^2 - |cb| \ge \frac{1}{2}\left(c^2 + d^2\right) \ge \frac{5}{2}$$

Hence.

$$\operatorname{Im}(\gamma.e^{i\theta}) = \frac{\operatorname{Im}(e^{i\theta})}{|ce^{i\theta} + d|^2} \le \frac{2}{5}.$$

Case III. Suppose |cd| = 1:

Since  $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ , we can assume WLOG that c = 1.

If d=1, then a-b=1, and thus  $\gamma.z=a+\frac{-1}{1+e^{i\theta}}$ . Therefore  $\operatorname{Re}(\gamma.z)=a-\frac{1}{2}$ . If d=-1, then a+b=1, and thus  $\gamma.z=a+\frac{-1}{1-e^{i\theta}}$ . Therefore  $\operatorname{Re}(\gamma.z)=$  $a + \frac{1}{2}$ .

In the third case, we can see that the only possible poles in D are  $\tau = \frac{-1}{z+1}$  and  $\tau = \frac{z}{z+1}$ , and later  $\tau = \frac{-1}{z-1}$  and  $\tau = \frac{z}{z-1}$ .

We see that

(3.15)

$$\operatorname{Im}\left(\frac{-1}{e^{i\theta}+1}\right) = \operatorname{Im}\left(\frac{e^{i\theta}}{e^{i\theta}+1}\right) = \frac{\sin\theta}{2+2\cos\theta} \ge \frac{\sin\theta}{2-2\cos\theta} = \operatorname{Im}\left(\frac{-1}{e^{i\theta}-1}\right)$$

and 
$$\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}+1}\right) = \operatorname{Re}\left(\frac{-1}{e^{i\theta}-1}\right) = \frac{1}{2}$$
.

This means that for values of  $\theta$  close to  $\pi/2$ , the two poles are too close to each other, so we divide them into cases:

3.2.1. Suppose  $\frac{\pi}{2} \le \theta < 1.9$ . We have  $\text{Im}\left(\frac{-1}{1+e^{i\theta}}\right) < 0.75$ , so we set A' = 0.75.

Thus, the quantity we need to bound is: (3.16)

$$I(0.75) \le \max_{|x| \le 1/2} e^{2\pi m(\sin \theta - 0.75)} \left| \frac{\Delta(e^{i\theta})}{\Delta(x + 0.75)} \right|^{\ell} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x + 0.75i)}{\Delta(x + 0.75i) \left(j(x + 0.75i) - j(e^{i\theta})\right)} \right|.$$

Lemma 3.8 and (3.15) yield that there exists a lower bound for the difference

$$|j(x+0.75i) - j(e^{i\theta})|.$$

We also have  $\Delta(z) \neq 0$  for all  $z \in \mathbb{H}$ , and since  $E_{k'}$ ,  $E_{14-k'}$ ,  $\Delta$ , and j are all holomorphic in  $\mathbb{H}$ , there exists  $B_1 \geq 0$  so that

(3.17) 
$$\max_{|x| \le \frac{1}{2}} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| < e^{B_1}.$$

Additionally, by (3.4), for all  $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$  and all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ ,

(3.18) 
$$\left| \frac{\Delta(e^{i\theta})}{\Delta(x + 0.75i)} \right| < e^{-\log(10/7)}.$$

Plugging (3.17) and (3.18) into (3.16), we obtain:

(3.19) 
$$I(0.75) \le e^{2\pi m(\sin \theta - 0.75) - \log(10/7)\ell + B_1}.$$

3.2.2. Suppose  $1.9 \le \theta \le \frac{2\pi}{3}$ . In this case, the pole at  $\tau = \frac{-1}{e^{i\theta}-1}$  is of height at most 0.4. Choosing A' = 0.65 and adding a little semicircle to our contours (see Figure 3) gives us:

$$(3.20) \quad e^{ik\theta/2} e^{2\pi m \sin \theta} \int_{-1/2}^{1/2} G(\tau + 0.65i, e^{i\theta}) d\tau$$

$$= e^{ik\theta/2} e^{2\pi m \sin \theta} \left( -2\pi i \operatorname{Res}_{\tau = \frac{-1}{1+z}} G(\tau, z) + \int_{-1/2}^{1/2} G(\tau + 0.65i, e^{i\theta}) d\tau \right).$$

Calculating the residue, we get:

$$\operatorname{Res}_{\tau = \frac{-1}{1+z}} G(\tau, z) = \frac{(z+1)^{-k} e^{2\pi i m \left(\frac{-1}{1+z}\right)}}{-2\pi i}.$$

Therefore,

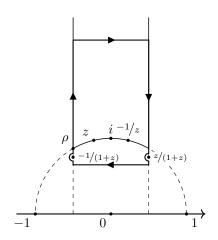


FIGURE 3. The contour of integration when  $\theta \in \left[1.9, \frac{2\pi}{3}\right]$ .

$$-2\pi i e^{ik\theta/2} e^{2\pi m \sin \theta} \operatorname{Res}_{\tau = \frac{-1}{1+z}} G(\tau, z) = \frac{e^{\pi m (2 \sin \theta - \tan(\theta/2))} e^{-2\pi i m \frac{\cos \theta}{2+2 \cos \theta}}}{(2 \cos(\theta/2))^k}.$$

Together with (3.20), we have:

$$(3.21) \quad \left| e^{ik\theta/2} e^{2\pi m \sin \theta} \int_{-1/2}^{1/2} G(\tau + 0.65i, e^{i\theta}) d\tau \right| \le \frac{e^{\pi m (2 \sin \theta - \tan(\theta/2))}}{(2 \cos(\theta/2))^k} + I(0.65).$$

Consider the derivative of the first term on the RHS:

$$\left(\frac{e^{\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k}\right)' = \frac{e^{\pi m(2\sin\theta - \tan(\theta/2))}\left((k\sin\theta)/2 + \pi m\left(4\cos^2(\theta/2)\cos\theta - 1\right)\right)}{2^{k+1}\cos^{k+2}(\theta/2)}.$$

Therefore, if  $k \geq \frac{8\pi}{\sqrt{3}}m$ , then for all  $\theta \in \left[\frac{\theta}{2}, \frac{2\pi}{3}\right]$ , we get:

$$(3.23) \left(\frac{e^{\pi m(2\sin\theta - \tan(\theta/2))}}{\left(2\cos(\theta/2)\right)^k}\right)' \ge \frac{e^{\pi m(2\sin\theta - \tan(\theta/2))}\left(\frac{\sqrt{3}k}{4}k + \pi m\left(-1 - 1\right)\right)}{2^{k+1}\cos^{k+2}(\theta/2)} \ge 0.$$

This shows that the first term on the RHS in (3.21) is increasing. So it is bounded by its value at  $\frac{2\pi}{3}$ , which means:

$$\frac{e^{\pi m(2\sin\theta - \tan(\theta/2))}}{\left(2\cos(\theta/2)\right)^k} \le 1$$

for all  $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . Thus, we are left to deal with (3.24)

$$I(0.65) \le \max_{|x| \le 1/2} e^{2\pi m(\sin \theta - 0.65)} \left| \frac{\Delta(e^{i\theta})}{\Delta(x + 0.65i)} \right|^{\ell} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x + 0.65i)}{\Delta(x + 0.65i) (j(x + 0.65i) - j(e^{i\theta}))} \right|.$$

As in the previous case, Lemma 3.8 and (3.15) yield that there exists a lower bound for the difference

$$|j(x+0.65i) - j(e^{i\theta})|.$$

This is sufficient to imply that there exists  $B_2 \geq 0$  so that

(3.25) 
$$\max_{|x| \le \frac{1}{2}} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| < e^{B_2}.$$

Additionally, by (3.3), for all  $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$  and all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ ,

(3.26) 
$$\left| \frac{\Delta(e^{i\theta})}{\Delta(x + 0.65i)} \right| < e^{-\log(2)}.$$

Plugging (3.25) and (3.26) into (3.24), we obtain:

(3.27) 
$$I(0.65) \le e^{2\pi m(\sin \theta - 0.65) - \log(2)\ell + B_2}.$$

Now, denote

(3.28) 
$$c_1 = \max\left(\frac{\pi}{2\log(10/7)}, \frac{7\pi}{10\log(2)}\right) = \frac{\pi}{2\log(10/7)},$$

and

(3.29) 
$$c_2 = \max\left(\frac{B_1 - \log(2)}{\log(10/7)}, \frac{B_2}{\log(2)}\right).$$

Suppose  $\ell > c_1 m + c_2$ , so by (3.19)

(3.30) 
$$I(0.75) \le e^{\frac{\pi m}{2} - \log(10/7)\ell + B_1} < e^{\log(2)} = 2.$$

By (3.27),

$$(3.31) 1 + I(0.65) \le 1 + e^{\frac{7\pi m}{10} - \log(2)\ell + B_2} < 1 + e^0 = 2.$$

Therefore,

$$(3.32) \max(I(0.65) + 1, I(0.75)) < 2.$$

Finally, using (3.13) and (3.21), we get:

(3.33)

$$\left| e^{ik\theta/2} e^{2\pi m \sin \theta} g_m(e^{i\theta}) - 2\cos(k\theta/2 + 2\pi m \cos \theta) \right| \le \max(I(0.65) + 1, I(0.75)),$$

which implies:

$$\left| e^{ik\theta/2} e^{2\pi m \sin \theta} g_{k,m}(e^{i\theta}) - 2\cos\left(k\theta/2 + 2\pi m \cos \theta\right) \right| < 2,$$

and completes our proof of Proposition 3.6.

3.3. **Proof of Theorem 1.1.** Choose  $\alpha = 4.5 \ge c_1$  and  $\beta = c_2$ . Let  $k \ge 0$  be an even integer with  $\ell > \alpha m + \beta$ . Denote  $D = \ell - m$  and  $h(\theta) = \frac{k\theta}{2} + 2\pi m \cos \theta$ , consider the derivative of h:

$$h'(\theta) = k/2 - 2\pi m \sin \theta = (k - 4\pi m \sin \theta)/2 \ge (k - 4\pi m)/2.$$

Since  $k \geq 12\ell > 12\alpha m \geq 4\pi m$ , we have  $h'(\theta) \geq 0$ . Therefore, h increases from  $\frac{\pi k}{4} = 3\pi \ell + \frac{\pi k'}{4}$  to  $\frac{\pi k}{3} - \pi m = 3\pi \ell + \frac{\pi k'}{3} + \pi D$ , passing through D+1 consecutive integer multiples of  $\pi$ . Let  $n_0$  be the least integer such that  $\pi n_0 \in \left[\frac{\pi k}{4}, \frac{\pi k}{3} - \pi m\right]$ . Thus, there exists  $\frac{\pi}{2} \leq \theta_0 < \theta_1 < \ldots < \theta_D \leq \frac{2\pi}{3}$  such that  $h\left(\theta_j\right) = \pi n_0 + \pi j$  for all  $0 \leq j \leq D$ . We have  $\cos\left(h\left(\theta_j\right)\right) = (-1)^{j+n_0}$ . So, by Proposition 3.6 we get:

$$2+2\cdot(-1)^{j+n_0} > e^{ik\theta_j/2}e^{2\pi m\sin\theta_j}g_{k,m}(e^{i\theta_j}) > -2+2\cdot(-1)^{j+n_0}$$

Hence, the function  $\theta \mapsto e^{ik\theta/2}e^{2\pi m\sin\theta}g_{k,m}(e^{i\theta})$  is continuous and changes its sign in the interval  $(\theta_{j-1},\theta_j)$ , showing that it attains the value zero at least once in each of those intervals. Since there are D intervals, we deduce that  $g_{k,m}(e^{i\theta})$  has D zeros. Hence, the form  $g_{k,m}$  has  $D = \ell - m$  zeros on the arc  $\mathcal{A}$ , in addition to the m zeros of  $g_{k,m}$  at infinity. In other words, all of the finite zeros of  $g_{k,m}$  in the fundamental domain are on the arc  $\mathcal{A}$ . We are left to show that the zeros of  $g_{k,m}$  are uniformly distributed on the arc:

Let  $z_1, \ldots, z_D \in [\pi/2, 2\pi/3]$  be the D zeros of the form  $g_{k,m}$  under the parametrization  $\theta \mapsto e^{i\theta}$  and let  $[a,b] \subset [\pi/2, 2\pi/3]$ . Since h is increasing, we have  $z_j \in [a,b]$  if and only if  $h(z_j) \in [h(a), h(b)]$ . By definition  $h(z_j)$  is between two consecutive integer multiples of  $\pi$ , therefore the number of zeros in the interval [a,b] is roughly the number of integer multiples in the interval [h(a), h(b)], i.e.

# 
$$\{1 \le j \le D : z_j \in [a, b]\}$$
  
= #  $\{n \in \mathbb{Z} : \pi n \in [h(a), h(b)]\} + O(1) = \left| \frac{h(b) - h(a)}{\pi} \right| + O(1).$ 

Hence,

$$\begin{split} \frac{\#\left\{1 \leq j \leq D : z_{j} \in [a,b]\right\}}{D} &= \frac{h\left(b\right) - h\left(a\right)}{\pi D} + O\left(\frac{1}{D}\right) \\ &= \frac{k\left(b - a\right)}{2\pi D} + 2\pi m \frac{\cos\left(b\right) - \cos\left(a\right)}{D} + O\left(\frac{1}{D}\right) \\ &= \frac{\left(12\ell + k'\right)\left(b - a\right)}{2\pi\left(\ell - m\right)} + O\left(\frac{1}{D}\right) \xrightarrow{\ell \to \infty} \frac{6\left(b - a\right)}{\pi} = \frac{b - a}{\frac{2\pi}{3} - \frac{\pi}{2}}. \end{split}$$

Thus, the zeros become uniformly distributed which implies Theorem 1.1.

## 4. Quantifying the bounds

In this section, we will quantify the constants  $c_1$  and  $c_2$  in Proposition 3.6, which will determine  $\alpha$  and  $\beta$  in Theorem 1.1. Specifically, we will prove:

**Theorem 4.1.** Fix  $m \ge 1$ . For all  $k = 12\ell + k'$ , if  $\ell > 4.5m + 9.5$  then all the zeros of  $g_{k,m}$  in the fundamental domain lie on the arc  $\left\{e^{i\theta}: \frac{\pi}{2} \le \theta \le \frac{2\pi}{3}\right\}$ .

4.1. Outline of the Proof of Theorem 4.1. As shown in Proposition 3.6, we have  $c_1 = \frac{\pi}{2\log(10/7)} \le 4.5$ . Thus, we are left to quantify  $c_2$ . To determine  $c_2$ , we need to quantify the bounds  $B_1$  in (3.17)

(3.17) 
$$\max_{|x| \le \frac{1}{2}} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| < e^{B_1},$$

and  $B_2$  in (3.25)

$$\max_{|x| \le \frac{1}{2}} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x+0.65i)}{\Delta(x+0.65i) \left( j(x+0.65i) - j(e^{i\theta}) \right)} \right| < e^{B_2}.$$

We will do so in the following manner:

(1) In §4.2, we will find a lower bound to the differences

$$\max_{|x| \le \frac{1}{2}} \left| j\left(x + i0.75\right) - j\left(e^{i\theta}\right) \right|$$

where  $\theta \in [\pi/2, 1.9]$ , and

$$\max_{|x| \leq \frac{1}{2}} \left| j\left(x + i0.65\right) - j\left(e^{i\theta}\right) \right|$$

where  $\theta \in [1.9, 2\pi/3]$ .

(2) In §4.3, §4.4 and §4.5, we will find an upper bounds for  $E_4$  and  $E_6$  on the arc  $\mathcal{A}$ , and on the horizontal lines of height 0.65 and 0.75. Those bounds, together with the following identities,

$$E_8 = E_4^2,$$
  
 $E_{10} = E^4 E_6,$   
 $E_{14} = E_4^2 E_6,$ 

would yield upper bounds for  $E_{k'}$  for all  $k' \in \{0, 4, 6, 8, 10, 14\}$ .

(3) We will use Proposition 3.4 to obtain a lower bound on  $\Delta(x+i0.65)$  and  $\Delta(x+i0.75)$ .

Lastly, we will substitute the bounds above in (3.17) and (3.25) and get  $B_1 = 4.04$  and  $B_2 = 5.2$ .

## 4.2. Bounds for the *j*-function.

**Lemma 4.2.** (i) For all  $\theta \in \left[\frac{\pi}{2}, 1.9\right]$  we have

(4.1) 
$$\max_{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} \left| j(x + 0.75i) - j(e^{i\theta}) \right| \ge 158.$$

(ii) For all  $\theta \in \left[1.9, \frac{2\pi}{3}\right]$  we have

(4.2) 
$$\max_{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} \left| j(x + 0.65i) - j(e^{i\theta}) \right| \ge 299.$$

*Proof.* We begin with a few basic facts on the *j*-function:

- (1) The q-expansion of j has integer coefficients, thus j(x+iy) = j(-x+iy).
- (2) The j-function is injective on the fundamental domain and onto  $\mathbb{C}$ .
- (3)  $j(\tau)$  is real if and only if  $\tau$  lies on the boundary of the fundamental domain or the imaginary line. Specifically, j maps the line  $\left\{-\frac{1}{2}+it:t>\frac{\sqrt{3}}{2}\right\}$  onto  $(-\infty,0)$ , the arc  $\mathcal{A}=\left\{e^{i\theta}:\frac{\pi}{2}\leq\theta\leq\frac{2\pi}{3}\right\}$  onto the interval [0,1728], and the line  $\{it:t>1\}$  onto  $(1728,\infty)$ .

(4) For all  $n \ge 1$  we have  $c(n) \le e^{4\pi\sqrt{n}}$ , where c(n) are the coefficients of the j-function (for the proof of this fact see [2] or A.3).

Since the j-function is real on the arc, the function  $x \mapsto |j(x+iy)-j(e^{i\theta})|$  is an even function. Thus,

$$\max_{|x| \leq \frac{1}{2}} \left| j(x+iy) - j(e^{i\theta}) \right| = \max_{x \in [0, \frac{1}{2}]} \left| j(x+iy) - j(e^{i\theta}) \right|.$$

Furthermore, since j on the fundamental domain is injective and real on its boundary, the function  $\theta \mapsto j(e^{i\theta})$  is decreasing on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$  (since j(i) = 1728 and  $j(\rho) = 0$ ).

(i) The case  $y = \frac{3}{4}$  and  $\theta \in \left[\frac{\pi}{2}, 1.9\right]$ : In this case  $j(e^{i\theta}) \in \left[j(e^{1.9i}), 1728\right]$  and a simple MATLAB calculation yields  $j(e^{1.9i}) \geq 271$ . Examining the plot of the real and imaginary parts of j(x+0.75i) on  $\left[0,\frac{1}{2}\right]$  (see figure 4), we can see that a bound to (4.3) is achievable by subdividing the interval  $\left[0,\frac{1}{2}\right]$ .

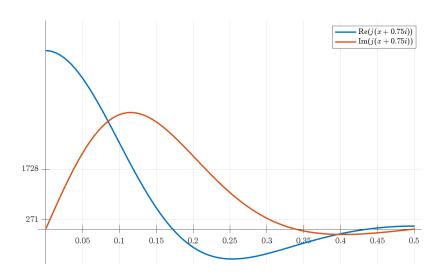


FIGURE 4. The real and imaginary parts of j(x+0.75i) on  $\left[0,\frac{1}{2}\right]$ 

For any 
$$n \geq 8$$
 we have  $4\pi\sqrt{n} - \frac{2\pi}{3}n \leq -\frac{\pi}{100}n$ , thus

$$c(n)e^{-\frac{2\pi}{3}n} \le e^{4\pi\sqrt{n} - \frac{2\pi}{3}n} \le e^{-\frac{\pi}{100}n}.$$

Denote  $f(x) = \sum_{n=-1}^{7} c(n)e^{-\frac{2\pi}{3}n}e^{2\pi i n x}$ , then:

$$|j(x+0.75i) - f(x)| \le \sum_{n=8}^{\infty} c(n)e^{-\frac{2\pi}{3}n} \le \sum_{n=8}^{\infty} e^{-\frac{\pi}{100}n} = \frac{e^{-\frac{8\pi}{100}}}{1 - e^{-\frac{\pi}{100}}} < 26.$$

Let us subdivide the interval into three parts:

•  $x \in [0, 0.1]$ : After plotting Re(f(x)) in MATLAB (see figure 5), we can conclude that Re(f(x)) > 2000, which yields

$$Re(i(x+0.75i)) > Re(f(x)) - 26 > 2000 - 26 = 1974.$$

Hence, for all  $x \in [0, 0.1]$ :

$$\begin{aligned} \left| j(x+0.75i) - j(e^{i\theta}) \right| &\geq \left| \text{Re}(j(x+0.75i) - j(e^{i\theta})) \right| \\ &\geq \text{Re}(j(x+0.75i)) - j(e^{i\theta}) \geq 1974 - 1728 = 246. \end{aligned}$$

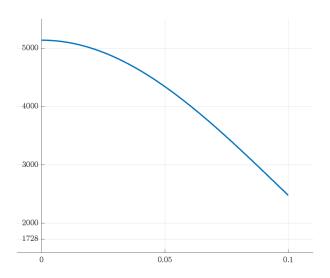


FIGURE 5. The real part of f(x) on [0,0.1]

•  $x \in [0.1, 0.2]$ : Here, we will plot Im(f(x)) in MATLAB (see figure 6), and conclude that Im(f(x)) > 2000, which yields

$$\operatorname{Im}(j(x+0.75i)) \ge \operatorname{Im}(f(x)) - 26 > 2000 - 26 = 1974.$$

Hence, for all  $x \in [0.1, 0.2]$ :

$$|j(x+0.75i) - j(e^{i\theta})| \ge \operatorname{Im}(j(x+0.75i)) \ge 1974.$$

•  $x \in [0.2, 0.5]$ : We will plot Re(f(x)) in MATLAB (see Figure 7) again, we can conclude that Re(f(x)) < 87, which yields

$$Re(j(x+0.75i)) \le Re(f(x)) + 26 < 87 + 26 = 113.$$

Hence, for all  $x \in [0.2, 0.5]$ :

$$|j(x+0.75i) - j(e^{i\theta})| \ge |\operatorname{Re}(j(x+0.75i) - j(e^{i\theta}))|$$
$$\ge j(e^{i\theta}) - \operatorname{Re}(j(x+0.75i)) \ge 271 - 113 = 158.$$

Finally, we can conclude that for all  $\theta \in \left[\frac{\pi}{2}, 1.9\right]$ :

(4.1) 
$$\max_{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} \left| j(x + 0.75i) - j(e^{i\theta}) \right| \ge 158.$$

(ii) The case  $y=\frac{13}{20}$  and  $\theta\in\left[1.9,\frac{2\pi}{3}\right]$ : In this case  $j(e^{i\theta})\in\left[0,j(e^{1.9i})\right]$  and a simple calculation shows  $j(e^{1.9i})\leq 272$ . For any  $n\geq 10$  we have  $4\pi\sqrt{n}-\frac{13\pi}{10}n\leq -\frac{\pi}{100}n$ , thus

$$c(n)e^{-\frac{13\pi}{10}n} \le e^{4\pi\sqrt{n} - \frac{13\pi}{10}n} \le e^{-\frac{\pi}{100}n}.$$

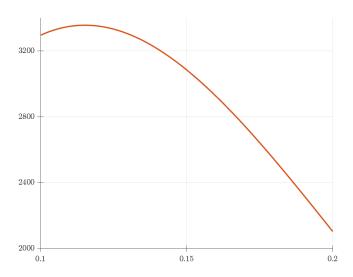


Figure 6. The imaginary part of f(x) on [0.1, 0.2]

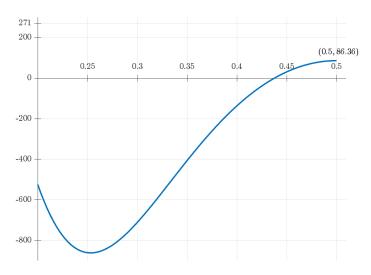


FIGURE 7. The real part of f(x) on [0.2, 0.5]

Denote  $g(x) = \sum_{n=-1}^{9} c(n) e^{-\frac{13\pi}{10}n} e^{2\pi i n x}$ , then:

$$||j(x+0.65i)| - |g(x)|| \le |j(x+0.65i) - g(x)| \le \sum_{n=8}^{\infty} e^{-\frac{\pi}{100}n} = \frac{e^{-\frac{\pi}{10}}}{1 - e^{-\frac{\pi}{100}}} < 24.$$

Which shows

$$|j(x+0.65i)| > |g(x)| - 24.$$

Plotting |g(x)| in MATLAB (see Figure 8) we can deduce  $|g(x)| \geq 595$ . Hence

$$(4.2) \quad \left|j(x+0.65i)-j(e^{i\theta})\right| \geq \left|j(x+0.65i)\right| - j(e^{i\theta}) > 595 - 24 - 272 = 299.$$

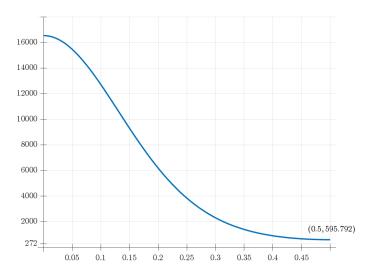


FIGURE 8. |g(x)| on [0, 0.5]

4.3. Monotonicity of  $|E_4(e^{i\theta})|$  and  $|E_6(e^{i\theta})|$ . For k=4,6, we define

$$e_k(\theta) = e^{\frac{ik\theta}{2}} E_k(e^{i\theta}),$$

and for k = 2, we define

$$e_2(\theta) = e^{i\theta} E_2(e^{i\theta}) + \frac{3}{i\pi}.$$

**Lemma 4.3.** (i) The function  $e_6(\theta) = e^{3i\theta} E_6(\theta)$  is positive for all  $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$  and vanishes at  $\frac{\pi}{2}$ .

- (ii) The function  $|E_4(e^{i\theta})|$  is a decreasing function on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . In particular, for all  $\theta \in \left[\frac{\pi}{2}, 1.9\right]$  we have
- (4.4)  $\left| E_4 \left( e^{i\theta} \right) \right| \le E_4 \left( i \right) = \frac{3\varpi^4}{\pi^4} = 1.455761 \dots,$

with  $\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.622057\ldots$ , and for all  $\theta \in \left[1.9, \frac{2\pi}{3}\right]$ 

(4.5) 
$$|E_4(e^{i\theta})| \le |E_4(e^{1.9i})| = 0.900253....$$

- (iii) The function  $|E_6(e^{i\theta})|$  is an increasing function on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ . In particular, for all  $\theta \in \left[\frac{\pi}{2}, 1.9\right]$  we have
- (4.6)  $\left| E_6 \left( e^{i\theta} \right) \right| \le \left| E_6 \left( e^{1.9i} \right) \right| = 1.980151...,$  and for all  $\theta \in \left[ 1.9, \frac{2\pi}{3} \right]$

(4.7) 
$$|E_6(e^{i\theta})| \le |E_6(\rho)| = \frac{27\varpi'^6}{2\pi^6} = 2.881536\dots$$

$$with \ \varpi' = 2\int_0^1 \frac{dx}{\sqrt{1-x^6}} = 2.622057\dots$$

Proof. (i)  $E_6$  has a unique zero in the fundamental domain, at  $i = e^{\pi i/2}$ . Therefore,  $e_4$  is continuous and nonzero for all  $\theta \in (\frac{\pi}{2}, \frac{2\pi}{3}]$ . Hence, it is enough to show that  $e_6(\frac{2\pi}{3}) > 0$ . As we previously saw, by [6]

$$e_{6}\left(\frac{2\pi}{3}\right) = e^{2\pi i} E_{6}\left(\rho\right) = \frac{27\varpi'^{6}}{2\pi^{6}} > 0.$$

(ii) Since  $e_4$  is negative, we have  $\left|E_4\left(e^{i\theta}\right)\right|=-e_4\left(\theta\right)$ . Hence, it is enough to show  $\frac{de_4}{d\theta}>0$ . Indeed, recall the following Ramanujan's identity,

$$\frac{1}{2\pi i} \frac{dE_4}{d\tau} = \frac{E_2 E_4 - E_6}{3}.$$

Then,

$$\begin{split} \frac{de_4}{d\theta} \left( \theta \right) &= \frac{d}{d\theta} \left( e^{2i\theta} E_4 \left( e^{i\theta} \right) \right) \\ &= 2i e^{2i\theta} E_4 \left( e^{i\theta} \right) + e^{2i\theta} \frac{d}{d\theta} \left( E_4 \left( e^{i\theta} \right) \right) \\ &= 2i e_4 \left( \theta \right) - 2\pi e^{2i\theta} e^{i\theta} \frac{dE_4}{d\tau} \left( e^{i\theta} \right) \\ &= 2i e_4 \left( \theta \right) - \frac{2\pi}{3} e^{2i\theta} e^{i\theta} \left( E_2 \left( e^{i\theta} \right) E_4 \left( e^{i\theta} \right) - E_6 \left( e^{i\theta} \right) \right) \\ &= 2i e_4 \left( \theta \right) - \frac{2\pi}{3} e_4 \left( \theta \right) \left( e_2 \left( \theta \right) - \frac{3}{i\pi} \right) + \frac{2\pi}{3} e_6 \left( e^{i\theta} \right) \\ &= \frac{2\pi}{3} \left( e_6 \left( \theta \right) - e_4 \left( \theta \right) e_2 \left( \theta \right) \right) \end{split}$$

By (i),  $e_6 > 0$  and as we saw in Lemma 3.2,  $e_2 < 0$  and  $e_4 < 0$  which shows  $\frac{de_4}{d\theta} > 0$ , and thus  $e_4$  is an increasing function. The bound (4.4) follows from a result which is due to Hurwitz [5],

$$E_4\left(i\right) = \frac{3\varpi^4}{\pi^4}.$$

The bound (4.5) follows from a simple calculation:

$$e_4(1.9) = -0.900253...$$

(iii) Since  $e_6$  is positive, we have  $\left|E_6\left(e^{i\theta}\right)\right|=e_6\left(\theta\right)$ . Hence, it is enough to show  $\frac{de_6}{d\theta}>0$ . Indeed, recall the following Ramanujan's identity,

$$\frac{1}{2\pi i} \frac{dE_6}{d\tau} = \frac{E_2 E_6 - E_4^2}{2}.$$

Then,

$$\frac{de_6}{d\theta}(\theta) = \frac{d}{d\theta} \left( e^{3i\theta} E_6 \left( e^{i\theta} \right) \right) 
= 3ie^{3i\theta} E_4 \left( e^{i\theta} \right) + e^{3i\theta} \frac{d}{d\theta} \left( E_6 \left( e^{i\theta} \right) \right) 
= 3ie_6(\theta) - 2\pi e^{3i\theta} e^{i\theta} \frac{dE_6}{d\tau} \left( e^{i\theta} \right) 
= 3ie_6(\theta) - \pi e^{3i\theta} e^{i\theta} \left( E_2 \left( e^{i\theta} \right) E_6 \left( e^{i\theta} \right) - E_4 \left( e^{i\theta} \right)^2 \right) 
= 3ie_6(\theta) - \pi e_6(\theta) \left( e_2(\theta) - \frac{3}{i\pi} \right) + \pi e_4 \left( e^{i\theta} \right)^2 
= \pi \left( e_4(\theta)^2 - e_6(\theta) e_2(\theta) \right)$$

By (i),  $e_6 > 0$  and as we saw in Lemma 3.2,  $e_2 < 0$ . Hence,  $\frac{de_6}{d\theta} > 0$ , which shows that  $e_6$  is an increasing function. The bound (4.6) follows from a simple calculation:

$$e_6(1.9) = 1.980151...$$

The bound (4.5) follows from a generalization of Hurwitz result, which is due to Katayama [6],

$$E_6\left(\rho\right) = \frac{27\varpi^6}{2\pi^6}.$$

### 4.4. Bounds for $E_4$ .

**Lemma 4.4.** (i) For all  $x \in [-\frac{1}{2}, \frac{1}{2}]$ ,

$$(4.8) |E_4(x+0.65i)| < 5.9.$$

(ii) For all  $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ ,

$$(4.9) |E_4(x+0.75i)| < 3.45.$$

Proof. Recall,

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n.$$

(i) An upper bound for  $|E_4(x+0.65i)|$  For all  $n \geq 3$  we have  $n^4 e^{-\frac{13\pi}{20}n} \leq \frac{3}{10}$ , and thus

$$\left| |E_4 (x + 0.65i)| - \left| 1 + 240e^{-\frac{13\pi}{10}} e^{2\pi i x} + 2160e^{-\frac{13\pi}{5}} e^{4\pi i x} \right| \right| \le 240 \sum_{n=3}^{\infty} \sigma_3(n) e^{-\frac{13\pi}{10}n}$$

$$\le 240 \sum_{n=3}^{\infty} n^4 e^{-\frac{13\pi}{10}n} \le 240 \cdot \frac{3}{10} \sum_{n=3}^{\infty} e^{-\frac{13\pi}{20}n} = 72 \frac{e^{-\frac{39\pi}{20}}}{1 - e^{-\frac{13\pi}{20}}} < \frac{1}{5} = 0.2.$$

Now, we have

$$\left|1 + 240e^{-\frac{13\pi}{10}}e^{2\pi ix} + 2160e^{-\frac{13\pi}{5}}e^{4\pi ix}\right| \le 1 + 240e^{-\frac{13\pi}{10}} + 2160e^{-\frac{13\pi}{5}} < 5.7$$
 Hence,

$$(4.8) |E_4(x+0.65i)| \le 5.9.$$

(ii) An upper bound for  $|E_4(x+0.75i)|$ : For all  $n \geq 3$  we have  $n^4 e^{-\frac{3\pi}{4}} \leq \frac{1}{5}$ , and thus

$$\left| |E_4 (x + 0.75i)| - \left| 1 + 240e^{-\frac{2\pi}{3}}e^{2\pi ix} + 2160e^{-3\pi ix} \right| \right| \le 240 \sum_{n=3}^{\infty} \sigma_3(n)e^{-\frac{2\pi}{3}n}$$

$$\le 240 \sum_{n=7}^{\infty} n^4 e^{-\frac{2\pi}{3}n} \le 240 \cdot \frac{1}{5} \sum_{n=3}^{\infty} e^{-\frac{3\pi}{4}n} = 48 \frac{e^{-\frac{9\pi}{4}}}{1 - e^{-\frac{3\pi}{4}}} < \frac{1}{20} = 0.05.$$

Now, we have

$$\left| 1 + 240e^{-\frac{2\pi}{3}}e^{2\pi ix} + 2160e^{-3\pi}e^{4\pi ix} \right| \le 1 + 240e^{-\frac{2\pi}{3}} + 2160e^{-3\pi} < 3.4$$

Hence,

$$(4.9) |E_4(x+0.75i)| < 3.45.$$

4.5. Bounds for  $E_6$ .

**Lemma 4.5.** (i) For all  $x \in [-\frac{1}{2}, \frac{1}{2}]$ ,

$$(4.10) |E_6(x+0.65i)| < 14.26.$$

(ii) For all 
$$x \in [-\frac{1}{2}, \frac{1}{2}]$$
,

$$(4.11) |E_6(x+0.75i)| < 5.25.$$

Proof. Recall

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

(i) An upper bound for  $|E_6(x+0.65i)|$ : For all  $n \geq 3$  we have  $n^6 e^{-\frac{13\pi}{20}n} \leq \frac{8}{5}$ , and thus

$$\left| |E_6(x+0.65i)| - \left| 1 - 504e^{-\frac{13\pi}{10}}e^{2\pi ix} - 16632e^{-\frac{13\pi}{5}}e^{4\pi ix} \right| \right| \le 504 \sum_{n=3}^{\infty} \sigma_6(n)e^{-\frac{13\pi}{10}n}$$

$$\le 504 \sum_{n=3}^{\infty} n^6 e^{-\frac{13\pi}{10}n} \le 504 \cdot \frac{8}{5} \sum_{n=3}^{\infty} e^{-\frac{13\pi}{20}n} = \frac{4032}{5} \frac{e^{-\frac{39\pi}{20}}}{1 - e^{-\frac{13\pi}{20}}} < 2.05.$$

Now, we have

$$\left| 1 - 504e^{-\frac{13\pi}{10}}e^{2\pi ix} - 16632e^{-\frac{13\pi}{5}}e^{4\pi ix} \right| = \left| \left( 1 - 252e^{-\frac{13\pi}{10}}e^{2\pi ix} \right)^2 - 80136e^{-\frac{13\pi}{5}}e^{4\pi ix} \right|$$

$$\leq \left| \left( 1 - 252e^{-\frac{13\pi}{10}} \right)^2 - 80136e^{-\frac{13\pi}{5}} \right| < 12.21.$$

Hence,

$$(4.10) |E_6(x+0.65i)| \le 14.26.$$

(ii) An upper bound for  $|E_6(x+0.75i)|$ : For all  $n \ge 1$  we have  $n^6 e^{-\frac{3\pi}{4}n} \le \frac{7}{10}$ , and thus

$$\left| |E_6(x+0.75i)| - \left| 1 - 504e^{-\frac{2\pi}{3}}e^{2\pi ix} - 16632e^{-3\pi}e^{4\pi ix} \right| \right| \le 504 \sum_{n=3}^{\infty} \sigma_6(n)e^{-\frac{2\pi}{3}n}$$

$$\le 504 \sum_{n=3}^{\infty} n^6 e^{-\frac{2\pi}{3}n} \le 504 \cdot \frac{7}{10} \sum_{n=3}^{\infty} e^{-\frac{3\pi}{4}n} = \frac{4032}{5} \frac{e^{-\frac{9\pi}{4}}}{1 - e^{-\frac{3\pi}{4}}} < 0.35.$$

Now, we have

$$\left| 1 - 504e^{-\frac{2\pi}{3}}e^{2\pi ix} - 16632e^{-3\pi}e^{4\pi ix} \right| = \left| \left( 1 - 252e^{-\frac{2\pi}{3}}e^{2\pi ix} \right)^2 - 80136e^{-3\pi}e^{4\pi ix} \right|$$

$$\leq \left| \left( 1 - 252e^{-\frac{2\pi}{3}} \right)^2 - 80136e^{-3\pi} \right| < 4.9$$

Hence,

$$(4.11) |E_6(x+0.75i)| < 5.25.$$

4.6. **Proof of Theorem 4.1.** We consider both cases for all six different choices of k'.

Case I.  $y = \frac{3}{4}$  and  $\theta \in \left[\frac{\pi}{2}, 1.9\right]$ : By Proposition 3.4 we get  $|\Delta(x+0.75i)| > 0.007$ . Now, For k' = 0:

$$\max_{|x| \le 1/2} \left| \frac{E_{14}(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| \\
= \max_{|x| \le 1/2} \left| \frac{E_{6}(x+0.75i) \cdot \left( E_{4}(x+0.75i) \right)^{2}}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| \\
\le \frac{5.25 \cdot (3.45)^{2}}{0.007 \cdot 158} < 56.5.$$

For k'=4:

$$\max_{|x| \le 1/2} \left| \frac{E_4(e^{i\theta}) E_{10}(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| \\
= \max_{|x| \le 1/2} \left| \frac{E_4(e^{i\theta}) E_6(x+0.75i) E_4(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| \\
\le \frac{1.46 \cdot 5.25 \cdot 3.45}{0.007 \cdot 158} < 24.$$

For k' = 6:

$$\max_{|x| \le 1/2} \left| \frac{E_6(e^{i\theta}) E_8(x + 0.75i)}{\Delta(x + 0.75i) \left( j(x + 0.75i) - j(e^{i\theta}) \right)} \right| 
= \max_{|x| \le 1/2} \left| \frac{E_6(e^{i\theta}) \left( E_4(x + 0.75i) \right)^2}{\Delta(x + 0.75i) \left( j(x + 0.75i) - j(e^{i\theta}) \right)} \right| 
\le \frac{1.99 \cdot (3.45)^2}{0.007 \cdot 158} < 21.42.$$

For k' = 8:

$$\max_{|x| \le 1/2} \left| \frac{E_8(e^{i\theta}) E_6(x + 0.75i)}{\Delta(x + 0.75i) (j(x + 0.75i) - j(e^{i\theta}))} \right| 
= \max_{|x| \le 1/2} \left| \frac{\left(E_4(e^{i\theta})\right)^2 E_6(x + 0.75i)}{\Delta(x + 0.75i) (j(x + 0.75i) - j(e^{i\theta}))} \right| 
\le \frac{(1.46)^2 \cdot 5.25}{0.007 \cdot 158} < 10.12.$$

For k' = 10:

$$\max_{|x| \le 1/2} \left| \frac{E_{10}(e^{i\theta}) E_4(x + 0.75i)}{\Delta(x + 0.75i) \left( j(x + 0.75i) - j(e^{i\theta}) \right)} \right| \\
= \max_{|x| \le 1/2} \left| \frac{E_6(e^{i\theta}) E_4(e^{i\theta}) E_4(x + 0.75i)}{\Delta(x + 0.75i) \left( j(x + 0.75i) - j(e^{i\theta}) \right)} \right| \\
\le \frac{1.99 \cdot 1.46 \cdot 3.45}{0.007 \cdot 158} < 9.07.$$

For k' = 14:

$$\max_{|x| \le 1/2} \left| \frac{E_{14}(e^{i\theta})}{\Delta(x + 0.75i) \left( j(x + 0.75i) - j(e^{i\theta}) \right)} \right| \\
= \max_{|x| \le 1/2} \left| \frac{E_{6}(e^{i\theta}) \cdot \left( E_{4}(e^{i\theta}) \right)^{2}}{\Delta(x + 0.75i) \left( j(x + 0.75i) - j(e^{i\theta}) \right)} \right| \\
\le \frac{1.99 \cdot (1.46)^{2}}{0.007 \cdot 158} < 3.84.$$

Therefore, for any  $k' \in \{0, 4, 6, 8, 10, 14\},\$ 

$$\max_{|x| \le \frac{1}{2}} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x+0.75i)}{\Delta(x+0.75i) \left( j(x+0.75i) - j(e^{i\theta}) \right)} \right| < e^{4.04}.$$

Hence, we got  $B_1 = 4.04$ .

Case II.  $y = \frac{13}{20}$  and  $\theta \in \left[1.9, \frac{2\pi}{3}\right]$ : By Proposition 3.4 we get  $|\Delta(x+0.65i)| > 0.01$ . Now, For k' = 0:

$$\begin{split} \max_{|x| \leq 1/2} \left| \frac{E_{14}(x + 0.65i)}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &= \max_{|x| \leq 1/2} \left| \frac{E_{6}(x + 0.65i) \cdot \left( E_{4}(x + 0.65i) \right)^{2}}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &\leq \frac{14.26 \cdot 36}{299 \cdot 0.01} < 171.7. \end{split}$$

For 
$$k'=4$$
:

$$\begin{aligned} \max_{|x| \le 1/2} \left| \frac{E_4(e^{i\theta}) E_{10}(x + 0.65i)}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &= \max_{|x| \le 1/2} \left| \frac{E_4(e^{i\theta}) E_6(x + 0.65i) E_4(x + 0.65i)}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &\le \frac{0.9022 \cdot 14.26 \cdot 6}{2.99} < 25.82. \end{aligned}$$

For k' = 6:

$$\max_{|x| \le 1/2} \left| \frac{E_6(e^{i\theta}) E_8(x + 0.65i)}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\
= \max_{|x| \le 1/2} \left| \frac{E_6(e^{i\theta}) \left( E_4(x + 0.65i) \right)^2}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\
\le \frac{2.89 \cdot 36}{2.99} < 34.8.$$

For k' = 8:

$$\max_{|x| \le 1/2} \left| \frac{E_8(e^{i\theta}) E_6(x + 0.65i)}{\Delta(x + 0.65i) (j(x + 0.65i) - j(e^{i\theta}))} \right| \\
= \max_{|x| \le 1/2} \left| \frac{\left(E_4(e^{i\theta})\right)^2 E_6(x + 0.65i)}{\Delta(x + 0.65i) (j(x + 0.65i) - j(e^{i\theta}))} \right| \\
\le \frac{0.9022^2 \cdot 14.26}{2.99} < 3.9.$$

For k' = 10:

$$\begin{split} \max_{|x| \le 1/2} \left| \frac{E_{10}(e^{i\theta}) E_4(x + 0.65i)}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &= \max_{|x| \le 1/2} \left| \frac{E_6(e^{i\theta}) E_4(e^{i\theta}) E_4(x + 0.65i)}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &\le \frac{2.89 \cdot 0.9022 \cdot 6}{2.99} < 5.24. \end{split}$$

For k' = 14:

$$\begin{split} \max_{|x| \le 1/2} \left| \frac{E_{14}(e^{i\theta})}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &= \max_{|x| \le 1/2} \left| \frac{E_{6}(e^{i\theta}) \cdot \left( E_{4}(e^{i\theta}) \right)^{2}}{\Delta(x + 0.65i) \left( j(x + 0.65i) - j(e^{i\theta}) \right)} \right| \\ &\le \frac{2.89 \cdot 0.9022^{2}}{2.99} < 1. \end{split}$$

Therefore, for any  $k' \in \{0, 4, 6, 8, 10, 14\}$ ,

(4.13) 
$$\max_{|x| \le \frac{1}{2}} \left| \frac{E_{k'}(e^{i\theta}) E_{14-k'}(x+yi)}{\Delta(x+yi) \left( j(x+yi) - j(e^{i\theta}) \right)} \right| < e^{5.2}.$$

Hence, we got  $B_2 = 5.2$ .

Finally, substituting  $B_1 = 4.04$  and  $B_2 = 5.2$  in (3.29) we obtain

$$c_2 = \max\left(\frac{4.04 - \log(2)}{\log(10/7)}, \frac{5.2}{\log(2)}\right) = \frac{4.04 - \log(2)}{\log(10/7)} = 9.384... \le 9.5.$$

Which proves Theorem 4.1.

## 5. Proof of theorem 1.2

In this section, we investigate the zeros of the first element of the Miller basis, i.e.  $g_{k,1}$ . Using Theorem 4.1, we conclude that for any  $k = 12\ell + k'$  with  $\ell > 14$ , the form  $g_{k,1}$  has all of its zeros in the fundamental domain on the arc  $\mathcal{A}$ . Therefore, we need to show that the zeros of the forms  $g_{k,1}$  such that  $\ell \leq 14$  are all on the arc  $\mathcal{A}$ . The proof goes through Faber polynomials:

5.1. **Faber Polynomials.** For any nonzero modular form  $f \in M_k$ , we associate a polynomial  $F_f \in \mathbb{C}[x]$  of degree  $k - \operatorname{ord}_{\infty}(f)$  such that  $f = \Delta^{\ell} E_{k'} F_f(j)$ , where  $k = 12\ell + k'$  and  $k' \in \{0, 4, 6, 8, 10, 14\}$ , as before. The polynomial  $F_f$  is uniquely determined and is called the *Faber polynomial of f*. The valence formula (1.1) implies that f attains the zeros of  $E_{k'}$ ; we call these zeros trivial. The roots of  $F_f$  account for all the nontrivial zeros of f, i.e. for any  $\tau \not\sim i, \rho$ , we have  $f(\tau) = 0$  if and only if  $F_f(j(\tau)) = 0$ . Together with the fact that f is injective and maps the arc f and f are f and f and only if f and only if f has all its zeros in the fundamental domain on the arc f if and only if f has all of its roots in the interval f and f and only if f has all of its roots in the interval f and f and only if f has all of its roots in the interval f and f and only if f has all of its roots in the interval f and f and only if f has all of its roots in the interval f and f and f and only if f has all of its roots in the interval f and f are f and f and f and f are f and f and f are f and f and f and f are f and f and f are f and f are f and f are f and f and f and f are f are f and f are f are f and f are f are f and f are f and f are f and f are f and f are f are f and f are f are f and f are

*Remark.* Given some modular form f, it is not hard to compute its Faber polynomial (for more information see §3 in [14]).

Hence, it is sufficient to consider the Faber polynomials of the forms  $g_{k,1}$  and compute their roots.

# 5.2. Some Examples.

5.2.1. k = 48. The Faber polynomial of  $g_{48,1}$  is:

$$F_{48,1}(t) = t^3 - 2136t^2 + 931860t - 24903328,$$

and its roots are 28.5703, 565.1814,  $1542.2483 \in [0, 1728]$ . See Figure 9.

5.2.2. k = 124. The Faber polynomial of  $g_{124,1}$  is:

$$F_{124,1}(t) = t^9 - 6696t^8 + 18182340t^7 - 25703594848t^6$$

$$+ 20207360640402t^5 - 8750844530401680t^4$$

$$+ 1942806055074346280t^3 - 188671766710386398400t^2$$

$$+ 5718177043459037019855t^2 - 21437679033112542689512.$$

Its roots are 4.3445, 44.3322, 153.6441, 350.0448, 628.6821, 959.1844, 1289.5802, 1557.8272, 1708.3603  $\in$  [0, 1728]. See Figures 10,11.

Using this method, numerically computing the roots of the polynomial  $F_{k,1}$  with  $1 \leq \ell \leq 14$ , we can verify that for any even integer k > 14, with  $1 \leq \ell \leq 14$  the zeros in the fundamental domain of  $g_{k,1}$  are all on the arc  $\mathcal{A}$ , which concludes the proof of Theorem 1.2.

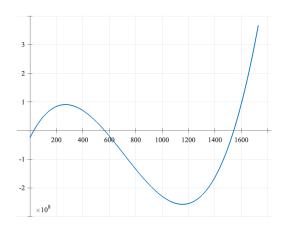


FIGURE 9.  $F_{48,1}$  on the interval [0, 1728].

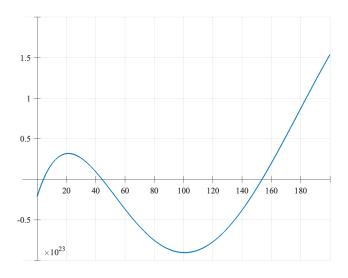


FIGURE 10.  $F_{124,1}$  on the interval [0, 200].

### Appendix A.

**Proposition A.1.** (1) Let  $\alpha > 0$  and let  $d \geq 1$  be an integer. The function  $f: x \mapsto x^d e^{-\alpha x}$ , is decreasing on  $\left[\frac{d}{\alpha}, \infty\right)$ . (2) For all  $n \geq 1$  and all  $x \in \left[0, \frac{1}{2}\right]$  we have  $n \sin(2\pi x) \geq |\sin(2\pi nx)|$ .

(1) Consider the derivative of f: Proof.

$$f'(x) = (d - \alpha x) x^{d-1} e^{-\alpha x}.$$

f' is non-positive on  $\left[\frac{d}{\alpha},\infty\right)$ , and therefore f is decreasing on  $\left[\frac{d}{\alpha},\infty\right)$ .

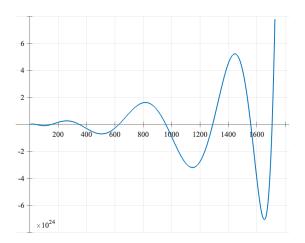


FIGURE 11.  $F_{124,1}$  on the interval [0, 1728].

(2) Let 
$$n \ge 1$$

$$\sum_{k=0}^{n-1} e^{4\pi i k x} = \frac{e^{4\pi i n x} - 1}{e^{4\pi i x} - 1} = \frac{e^{2\pi i n x}}{e^{2\pi i x}} \frac{e^{2\pi i n x} - e^{-2\pi i n x}}{e^{2\pi i x} - e^{-2\pi i x}} = e^{2\pi i (n-1) x} \frac{\sin(2\pi n x)}{\sin(2\pi x)}.$$
Thus,
$$n \ge \frac{|\sin(2\pi n x)|}{\sin(2\pi x)}.$$

**Proposition A.2.** Let  $\alpha \in \left(0, e^{\pi\sqrt{3}}\right)$  and denote  $S(\theta) = \left|1 + \alpha e^{2\pi i e^{\theta}}\right|^2$ , then S is decreasing on  $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ .

Proof.

$$S(\theta) = \left| 1 + \alpha e^{2\pi i e^{\theta}} \right|^2 = \left( 1 + \alpha e^{-2\pi \sin \theta} \cos(2\pi \cos \theta) \right)^2 + \left( \alpha e^{-2\pi \sin \theta} \sin(2\pi \cos \theta) \right)^2$$

$$= 1 + 2\alpha e^{-2\pi \sin \theta} \cos(2\pi \cos \theta)$$

$$+ \alpha^2 e^{-4\pi \sin \theta} \left( \cos^2(2\pi \cos \theta) + \sin^2(2\pi \cos \theta) \right)$$

$$= 1 + 2\alpha e^{-2\pi \sin \theta} \cos(2\pi \cos \theta) + \alpha^2 e^{-4\pi \sin \theta}$$

To show that S is non-increasing, consider its derivative, and see that it is non-positive.

$$S'(\theta) = -4\alpha\pi e^{-2\pi\sin\theta} \left(\cos\theta\cos(2\pi\cos\theta) - \sin\theta\sin(2\pi\cos\theta)\right) - 4\pi\alpha^2 e^{-4\pi\sin\theta}\cos\theta$$
$$= -4\pi\alpha e^{-2\pi\sin\theta}\cos(\theta + 2\pi\cos\theta) - 4\pi\alpha^2 e^{-4\pi\sin\theta}\cos\theta$$
$$= -4\pi\alpha e^{-2\pi\sin\theta} \left(\cos(\theta + 2\pi\cos\theta) + \alpha e^{-2\pi\sin\theta}\cos\theta\right)$$

On the interval  $[\pi/2, 2\pi/3]$ ,  $\theta \mapsto \cos \theta$  is decreasing from 0 to  $-\frac{1}{2}$ , in particular it is non-positive.

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Hence,

$$\alpha e^{-2\pi \sin \theta} \cos \theta < \alpha e^{-\pi \sqrt{3}} \cos \theta < \cos \theta$$

for all  $\theta \in [\pi/2, 2\pi/3]$ .

Therefore,

$$S'(\theta) = -4\pi\alpha e^{-2\pi\sin\theta} \left(\cos(\theta + 2\pi\cos\theta) + \alpha e^{-2\pi\sin\theta}\cos\theta\right)$$

$$\leq -4\pi\alpha e^{-2\pi\sin\theta} \left(\cos(\theta + 2\pi\cos\theta) + \cos\theta\right)$$

$$= -4\pi\alpha e^{-2\pi\sin\theta} \left(2\cos(\theta + \pi\cos\theta)\cos(\pi\cos\theta)\right)$$

$$-8\pi\alpha e^{-2\pi\sin\theta}\cos(\theta + \pi\cos\theta)\cos(\pi\cos\theta)$$

Now,  $\pi \cos \theta \in [-\pi/2, 0]$  and thus  $\cos(\pi \cos \theta)$  is non-negative. Furthermore,  $\theta + \pi \cos \theta \in [\pi/6, \pi/2]$  and thus  $\cos(\theta + \pi \cos \theta)$  is non-negative. Overall,  $S'(\theta)$  is non-positive, which proves our proposition.

The following proposition (and proof) were taken from [2].

**Proposition A.3.** Let c(n) be the coefficients in the q-expansion of the j-function, then  $c(n) < e^{4\pi\sqrt{n}}$  for all  $n \ge 1$ .

*Proof.* Let  $t \in (0,1)$ , by (2.3) we have  $j(it) = j(\frac{i}{t})$ . Expanding j to its q-series we get,

$$(\mathrm{A.1}) \ \ j \ (it) = e^{2\pi t} + 744 + \sum_{n=1}^{\infty} c(n) e^{-2\pi nt} = e^{\frac{2\pi}{t}} + 744 + \sum_{n=1}^{\infty} c(n) e^{-\frac{2\pi n}{t}} = j \left(\frac{i}{t}\right).$$

Hence,

(A.2) 
$$\sum_{n=1}^{\infty} c(n) \left( e^{-2\pi nt} - e^{-\frac{2\pi n}{t}} \right) = e^{\frac{2\pi}{t}} - e^{2\pi t}.$$

Since  $t \in (0,1)$  and c(n) > 0, the terms on the left-hand side are all positive, it follows for all  $n \ge 1$ :

(A.3) 
$$c(n)\left(e^{-2\pi nt} - e^{-\frac{2\pi n}{t}}\right) < e^{\frac{2\pi}{t}} - e^{2\pi t}.$$

Let  $n \geq 2$ , substitute  $t = \frac{1}{\sqrt{n}}$  and obtain:

(A.4) 
$$c(n) < \frac{e^{2\pi\sqrt{n}} - e^{\frac{2\pi}{\sqrt{n}}}}{e^{-2\pi\sqrt{n}} - e^{-2\pi n\sqrt{n}}}.$$

Now, we claim that

(A.5) 
$$\frac{e^{2\pi\sqrt{n}} - e^{\frac{2\pi}{\sqrt{n}}}}{e^{-2\pi\sqrt{n}} - e^{-2\pi n\sqrt{n}}} < e^{4\pi\sqrt{n}}.$$

Indeed, the inequality above is equivalent to

$$e^{2\pi\sqrt{n}} - e^{\frac{2\pi}{\sqrt{n}}} < e^{2\pi\sqrt{n}} - e^{4\pi\sqrt{n} - 2\pi n\sqrt{n}},$$

which is equivalent to

$$e^{\frac{2\pi}{\sqrt{n}}} > e^{4\pi\sqrt{n} - 2\pi n\sqrt{n}}.$$

Now, since  $n \ge 2$  we have  $\frac{2\pi}{\sqrt{n}} > 4\pi\sqrt{n} - 2\pi n\sqrt{n}$ . Thus, (A.5) holds for all  $n \ge 2$ . For n = 1, we have  $c(1) = 196884 < e^{4\pi}$ . Hence,  $c(n) < e^{4\pi\sqrt{n}}$  for all  $n \ge 1$ .

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School of Mathematical Sciences,, Tel Aviv University,, Tel Aviv 69978, Israel *Email address*: roeiraveh@mail.tau.ac.il