FRINGE TREES OF PATRICIA TRIES AND COMPRESSED BINARY SEARCH TREES

SVANTE JANSON

Abstract. We study the distribution of fringe trees in Patricia tries and compressed binary search trees; both cases are random binary trees that have been compressed by deleting vertices of outdegree 1 so that they are random full binary trees. The main results are central limit theorems for the number of fringe trees of a given type, which imply quenched and annealed limit results for the fringe tree distribution; for Patricia tries, this is complicated by periodic oscillations in the usual manner. We also consider extended fringe trees. The results are derived from earlier results for uncompressed tries and binary search trees. In the case of compressed binary search trees, it seems difficult to give a closed formula for the asymptotic fringe tree distribution, but we provide a recursion and give examples.

1. Introduction

We consider in this paper fringe tree distributions for some different types of random binary trees. It is well known that there are two types of tress that are called binary trees. We use the following (common, but not universal) notation. A *binary tree* is a rooted tree such that each child of a node is labelled either *left* or *right*, and each node has at most one left and at most one right child. A *full binary tree* is a binary tree where each node has outdegree 0 or 2. (In the latter case, it thus has one left and one right node.) A *leaf* is a node with no children (i.e., outdegree 0). The *size* $|T|$ of a tree T is its number of nodes. All trees in the paper are non-empty, finite, and binary (and thus rooted), except when we explicitly say otherwise.

Is is well known that a full binary tree has odd size, and that there is a bijection between binary trees of size $n \geqslant 1$ and full binary trees of size $2n+1$ defined as follows: given a binary tree, add new leaves at all possible places, i.e., add one new leaf to each node of outdegree 1, and two new leaves to each node of outdegree 0; conversely, given a full binary tree, delete all its leaves. For many purposes, the two types of binary trees are thus equivalent, but both types are important, and there are several reasons for studying both types of binary trees.

In the present paper we will mainly study another relation between binary trees and full binary trees. Given a binary tree T, its *compression* $\widehat{T}$ is the full binary tree obtained by deleting all nodes of outdegree 1 (and connecting the remaining nodes in the obvious way). This is obviously not a bijection; there is no way to reconstruct the binary tree without further information. Note also that the size of the compressed tree typically is smaller; we have $1 \leqslant |\widehat{T}| \leqslant |T|$, and every size in this range is possible.

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One well known example of compression is for tries: a *trie* is a binary tree constructed from a sequence of distinct infinite strings of 0 and 1 (see Section 2.2 for details), and its compression is known as a *Patricia trie*; see [29, Section 6.3] for the computer science background. We study in this paper tries and Patricia tries defined by independent random strings where all bits are independent. We denote the random trie defined by n random strings by Υ_n, and the corresponding Patricia trie by $\widehat{\Upsilon}_n$.

The *Binary Search Tree (BST)* is another commonly studied random binary tree. (See Section 2.3 for definition.) Just as tries and Patricia tries, it appears naturally in computer science in connection with sorting and searching, see e.g. [29, Section 6.2.2]. Unlike tries, the compressed version is perhaps not so interesting in that context, but we study here the different random binary trees from a purely mathematical perspective, and we find it natural to study also the *compressed Binary Search Tree*, and compare it to other random full binary trees. We denote the random BST with n nodes by $\mathcal{B}_n$, and its compression by $\widehat{\mathcal{B}}_n$.

In this paper we focus on fringe trees of the various random trees. Recall that for a rooted tree T and a node $v \in T$, the *fringe tree* T^v is the subtree of T consisting of v and all its descendants; the fringe tree T^v is a rooted tree with root v, and if T is a binary tree or a full binary tree, then so is each of its fringe trees. If T and t are two binary trees, let $N_t(T)$ be the number of fringe trees of T that are equal to t (in the sense of isomorphic as binary trees), i.e.,

$$N_t(T) := \bigl|\{v \in T : T^v = t\}\bigr|. \tag{1.1}$$

Here t will always be a fixed tree (think of it as small), while T usually will be a (big) random tree; then $N_t(T)$ is a random variable. We consider also the random fringe tree T^* defined as T^v for a node $v \in T$ chosen uniformly at random. (If T is random, we first condition on T and then choose a node v in it.) Thus, for every fixed binary tree t and a non-random binary tree T,

$$\mathbb{P}(T^* = t) = \frac{N_t(T)}{|T|}, \tag{1.2}$$

for a random binary tree T, (1.2) holds conditionally on T, and thus

$$\mathbb{P}(T^* = t) = \mathbb{E}\,\frac{N_t(T)}{|T|}. \tag{1.3}$$

See [1] for a general study of fringe tree distributions, including explicit results for several classes of random trees.

We consider asymptotics as the size of the random tree tends to infinity. When T is a random trie Υ_n or a binary search tree $\mathcal{B}_n$, the asymptotic distribution of the random fringe tree T^* is given already by [1]. More precise result including asymptotic normality of the fringe tree counts $N_t(T)$ are proved in [28] (tries) and [9], [10] (BST), see also e.g. [13], [23], and [20]. Our main results below use these results to show corresponding results for Patricia tries and compressed binary search trees. In particular, in both cases the counts $N_t(T)$ are asymptotically normal.

For Patricia tries $\widehat{\Upsilon}_n$, we find explicitly the asymptotics of the mean and variance of $N_t(\widehat{\Upsilon}_n)$, and thus the asymptotic distribution of the random fringe tree $\widehat{\Upsilon}^*_n$. For the compressed BST $\widehat{\mathcal{B}}_n$, this seems more difficult. We show how to find explicitly the asymptotics of the mean $\mathbb{E}\,N_t(\widehat{\mathcal{B}}_n)$ and, equivalently, the asymptotic distribution of the random fringe tree $\widehat{\mathcal{B}}^*_n$, but this is done by a recursion and we cannot give a general formula. We leave as an open problem to find a formula for the asymptotic variance of $N_t(\widehat{\mathcal{B}}_n)$ for the compressed BST.

Remark 1.1. A *cladogram* is a full binary tree where we do not care about the orientations, i.e., we do not distinguish between left and right. (Formally, we may see a cladogram as an equivalence classe of full binary trees.) In particular, the Patricia tries and compressed binary search trees studied here may be regarded as random cladograms by forgetting all orientations. The results below yield as corollaries corresponding results for fringe trees (which now are random cladograms) of these; we omit the details. See also Remark A.1.

A very different model of random cladograms is studied by Aldous [3, 4]; the results obtained here may be compared to results for fringe trees in that model [in preparation]. $\triangle$

Remark 1.2. Related results of fringe trees in other classes of random trees have been given by many authors, see for example [1], [5], [12], [15], [7], [8], [16], [19], [20], [27], [21], [22], [28], [6], and the further references therein. $\triangle$

2. Preliminaries

2.1. **Notation.** For a rooted tree T, the set of leaves of T is denoted $\mathcal{L}(T)$. The leaves are also called *external nodes* and the other nodes, i.e., those with outdegree > 0, are called *internal nodes.* The number of nodes of T is denoted $|T|$, and the number of leaves (external nodes) by $|T|_{\mathsf{e}} := |\mathcal{L}(T)|$. Recall that in a binary tree T, the number of nodes of outdegree 2 is $|T|_{\mathsf{e}} - 1$, and thus the number of nodes of outdegree 1 is $|T| - 2|T|_{\mathsf{e}} + 1$. Hence, if T is a full binary tree, then

$$|T| = 2|T|_{\mathsf{e}} - 1. \tag{2.1}$$

The *root degree* $\rho(T)$ is the (out)degree of the root $o \in T$.

We let $\bullet$ denote the tree consisting of a root only, so $|\bullet| = |\bullet|_{\mathsf{e}} = 1$.

Let $\mathfrak{T}$ be the set of all binary trees, and $\widehat{\mathfrak{T}}$ the subset of all full binary trees, and $\mathfrak{T}'$ the set of all binary trees such that no leaf has a parent of outdegree 1; thus $\widehat{\mathfrak{T}} \subset \mathfrak{T}' \subset \mathfrak{T}$. Furthermore, for any set $S \subseteq \{0,1,2\}$, let $\mathfrak{T}^S := \{T \in \mathfrak{T} : \rho(T) \in S\}$ be the set of all binary trees with root degree in S. In particular, $\mathfrak{T}^{\{0\}} = \{\bullet\}$.

If t is a full binary tree, let $\check{\mathfrak{T}}_t$ be the set of all binary trees that can be obtained fron t by subdividing the edges, i.e., by replacing every edge by a path of $\ell \geqslant 1$ edges; each such path thus contains $\ell - 1$ new nodes of outdegree 1. Note that each new node has its (only) child as either a left or a right child. Let further $\check{\mathfrak{T}}_t^+$ be the set of all binary trees that compress to t; note that this is larger than $\check{\mathfrak{T}}_t$ since $\check{\mathfrak{T}}_t^+$ allows also adding a path from the root to a new root, but $\check{\mathfrak{T}}_t$ does not; in fact, $\check{\mathfrak{T}}_t = \check{\mathfrak{T}}_t^+ \cap \mathfrak{T}^{\{0,2\}}$. (If $t \neq \bullet$, then $\check{\mathfrak{T}}_t = \check{\mathfrak{T}}_t^+ \cap \mathfrak{T}^{\{2\}}$.)

If v is a node in a binary tree T, then the *left depth* $d_{\mathsf{L}}(v)$ is the number of edges that go from some node to its left child in the path from the root to v. The *right depth* $d_{\mathsf{R}}(v)$ is defined similarly. Note that $d_{\mathsf{L}}(v) + d_{\mathsf{R}}(v) = d(v)$, the depth of v. When necessary, we write $d_T(v)$ for the depth in T.

We define the *left external path length* $\mathsf{LPL}(T)$ and *right external path length* $\mathsf{RPL}(T)$ by

$$\mathsf{LPL}(T) = \sum_{v \in \mathcal{L}(T)} d_{\mathsf{L}}(v), \qquad \mathsf{RPL}(T) = \sum_{v \in \mathcal{L}(T)} d_{\mathsf{R}}(v). \tag{2.2}$$

Thus the sum $\mathsf{LPL}(T) + \mathsf{RPL}(T)$ equals the total external path length $\sum_{v \in \mathcal{L}(T)} d(v)$.

The fringe tree T^v and the fringe tree counts $N_t(T)$ are defined in the introduction.

We use $\xrightarrow{\mathrm{d}}$ and $\xrightarrow{\mathrm{p}}$ to denote convergence in distribution and probability, respectively, of random variables. We further say that $X_n \xrightarrow{\mathrm{d}} Y$ *with all moments* if $X_n \xrightarrow{\mathrm{d}} Y$ and also $\mathbb{E}\, X_n^r \to \mathbb{E}\, Y^r$ for every integer $r > 0$. We let $o_{\mathrm{p}}(1)$ denote any sequence of random variables X_n such that $X_n \xrightarrow{\mathrm{p}} 0$.

$N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance $\sigma^2 \geqslant 0$. $\mathrm{Po}(\lambda)$ denotes the Poisson distribution with parameter $\lambda \geqslant 0$. We thus have

$$\mathrm{Po}(\lambda; n) := \mathrm{Po}(\lambda)(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \qquad n = 0, 1, \dots \tag{2.3}$$

We may sometimes abbreviate "uniformly random" to "random". log denotes natural logarithms. Unspecified limits are as $n \to \infty$.

In this paper we discuss results for the random fringe tree T_n^* of a sequence of random trees T_n; we state results both conditioned on the tree T_n and unconditioned, and we use the standard terminology and call such results *quenched* and *annealed*, repectively.

2.2. **Tries.** A trie is a rooted tree constructed from a set of $n \geqslant 1$ distinct strings $\Xi^{(1)}, \Xi^{(2)}, \dots, \Xi^{(n)}$ in some alphabet $\mathcal{A}$; we consider here only the case $\mathcal{A} = \{0, 1\}$, and then the trie will be a binary tree. (See e.g. [29, Section 6.3] and [11, Section 1.4.4].) We assume for convenience that the strings are infinite; thus $\Xi^{(i)} = \xi_1^{(i)} \xi_2^{(i)} \cdots \in \{0,1\}^\infty$ for every i. The trie is constructed recursively. If $n = 0$, then the trie is the empty tree $\varnothing$. Otherwise, we begin with a root, and put every string in the root. If $n = 1$, then we stop there, so the trie equals $\bullet$. Otherwise, i.e., if $n \geqslant 2$, we pass all strings to new nodes; all strings beginning with 0 (if any) are passed to a left child of the root and all strings beginning with 1 (if any) are passed to a right child of the root. We continue recursively, the next time partitioning the strings according to the second letter, and so on, always looking at the first letter not yet inspected. At the end there is a tree with n leaves, each containing one string. Equivalently, each string $\Xi^{(i)}$ defines an infinite path Γ_i from the root in the infinite binary tree, by processing the letters in the string in order and going to a left child for every 0 and a right child for every 1. We then stop each path at first node that it does not share with any other of the paths Γ_j; these nodes are the leaves of the trie, and the trie is the union of the n stopped paths. Note that in a trie, a parent of a leaf must have outdegree 2, i.e., a trie belongs to the set $\mathfrak{T}'$.

We will consider the random trie Υ_n defined by n random strings $\Xi^{(1)}, \dots, \Xi^{(n)}$ where we assume that the strings are independent, and furthermore in each string the letters are independent and identically distributed with distribution $\mathrm{Be}(p)$ for some $p \in (0,1)$, i.e., each letter $\xi_k^{(i)}$ has $\mathbb{P}(\xi_k^{(i)} = 1) = p$ and $\mathbb{P}(\xi_k^{(i)} = 0) = q := 1 - p$. We omit the parameter p from the notation, but it is implicit when we discuss tries; we use always the notations p and $q = 1 - p$ in the sense above.

It is well known that many results for tries show (typically small) periodic oscillations instead of limits as $n \to \infty$, see e.g. [29; 31; 14; 17; 25; 18; 24; 28]. More precisely, if $\log p / \log q$ is irrational, then such oscillations do not occur, but if $\log p / \log q$ is rational they typically do. We call the case when $\log p / \log q \in \mathbb{Q}$ *periodic*; otherwise we have the *aperiodic* case. In the periodic case, if $\log p / \log q$ equals a/b in lowest terms (with $a, b \in \mathbb{N}$), then define

$$d = d_p := \frac{-\log p}{a} = \frac{-\log q}{b} > 0, \tag{2.4}$$

the greatest common divisor of $-\log p$ and $-\log q$.

The Patricia trie $\widehat{\Upsilon}_n$ is obtained by compressing Υ_n. Note that the trie Υ_n and the Patricia tree $\widehat{\Upsilon}_n$ both have exactly n leaves:

$$|\Upsilon_n|_{\mathsf{e}} = |\widehat{\Upsilon}_n|_{\mathsf{e}} = n. \tag{2.5}$$

The Patricia tree $\widehat{\Upsilon}_n$ thus has $n-1$ internal nodes, while the number of internal nodes in Υ_n is random.

2.3. **Binary search trees.** Binary search trees may be constructed in different (but equivalent) ways; we will use the following, closely connected to the sorting algorithm *Quicksort* (see e.g. [29, Section 6.2.2] and [11, Section 1.4.1]): Consider a set of n distinct items, which we may assume are real numbers $x_1, \dots, x_n$. If $n = 0$, the BST $\mathcal{B}_0$ is the empty tree $\varnothing$, and if $n = 1$, $\mathcal{B}_1 := \bullet$. If $n \geqslant 2$, pick one of the n items at random, and call it the *pivot*. Compare all other elements to the pivot, and let L be the set of all x_i that are smaller than the pivot, and let R be the set of all items x_i that are greater than the pivot. (Thus, $|L| + |R| = n - 1$.) The BST $\mathcal{B}_n$ is defined as the binary tree with the root having left and right subtrees that are constructed recursively from the sets L and R, respectively. It is easily seen, by induction, that $|\mathcal{B}_n| = n$; in fact, it is natural to label the root by the pivot, and then during the recursion each node becomes labelled by exactly one of the n numbers x_i. Note that both $|\mathcal{B}_n|_{\mathsf{e}} = |\widehat{\mathcal{B}}_n|_{\mathsf{e}}$ and $|\widehat{\mathcal{B}}_n|$ are random. It is shown by Aldous [1, Example 3.3] and Devroye [9, Theorem 2], that, as $n \to \infty$,

$$|\mathcal{B}_n|_{\mathsf{e}}/n = |\widehat{\mathcal{B}}_n|_{\mathsf{e}}/n \xrightarrow{\mathrm{p}} 1/3 \tag{2.6}$$

and thus, see (2.1),

$$|\widehat{\mathcal{B}}_n|/n \xrightarrow{\mathrm{p}} 2/3. \tag{2.7}$$

More precisely [9],

$$\frac{|\mathcal{B}_n|_{\mathsf{e}} - n/3}{\sqrt{n}} = \frac{|\widehat{\mathcal{B}}_n|_{\mathsf{e}} - n/3}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0, 2/45) \tag{2.8}$$

and thus $\frac{|\widehat{\mathcal{B}}_n| - 2n/3}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0, 8/45)$ by (2.1); see also [30] for expectations and e.g. [11, Theorem 6.9] for a more general result.

2.4. **Fringe trees of compressed trees.** Let T be a binary tree and $\widehat{T}$ its compression. Note first that the leaves of $\widehat{T}$ are precisely the leaves of T, while the roots may differ. (In general, there may in T be a path of nodes of outdegree 1 from the root of T to the root of $\widehat{T}$.) Thus, $|\widehat{T}|_{\mathsf{e}} = |T|_{\mathsf{e}}$.

Let t be a full binary tree and consider the nodes $v \in \widehat{T}$ such that $\widehat{T}^v = t$. We consider two cases separately:

(i) If $t = \bullet$, then: $v \in \widehat{T}$ and $\widehat{T}^v = t \iff v$ is a leaf in $\widehat{T} \iff v$ is a leaf in T $\iff v \in T$ and $T^v = t$.

(ii) If $|t| > 1$, then the root of t has outdegree 2, and:
$v \in \widehat{T}$ and $\widehat{T}^v = t \iff v$ has outdegree 2 in T and t is the contraction of T^v.

Consequently, in both cases,

$$v \in \widehat{T} \text{ and } \widehat{T}^v = t \iff v \in T \text{ and } T^v \in \breve{\mathfrak{T}}_t, \tag{2.9}$$

where we recall that $\breve{\mathfrak{T}}_t$ is the set of all binary trees that can be obtained fron t by subdividing the edges.

Define the functional φ on binary trees by

$$\varphi(T) := \mathbf{1}\{T \in \breve{\mathfrak{T}}_t\}, \tag{2.10}$$

and define the corresponding additive functional

$$\Phi(T) := \sum_{v \in T} \varphi(T^v). \tag{2.11}$$

Then, by (2.9),

$$N_t(\widehat{T}) = \sum_{v \in \widehat{T}} \mathbf{1}\{\widehat{T}^v = t\} = \sum_{v \in T} \mathbf{1}\{T^v \in \breve{\mathfrak{T}}_t\} = \sum_{v \in T} \varphi(T^v) = \Phi(T). \tag{2.12}$$

2.5. **Extended fringe trees.** Let T be a rooted tree. The fringe tree T^v defined in the introduction consists of a node v and all its descendants. Aldous [1] introduces also the *extended fringe tree* by also going up from the chosen node v to ancestors and then taking descendants, thus including siblings, cousins, and so on; we may formally define the extended fringe tree as the nested sequence of fringe trees $(T^{v_i})_{i=0}^{d(v)}$ where v_i is the ith ancestor of v. (See [1, Section 4] for details.) It is shown in [1, Proposition 11] that if the random fringe trees of some sequence of random trees T_n converge in distribution, then so do the random extended fringe trees. The limit then is an infinite nested sequence of random trees $(T_\infty^{(i)})_{i \geqslant 0}$.

Remark 2.1. Assume also the technical condition (satisfied for the random trees studied here) that the depth of a random node in T_n tends to infinity in probability. Then the random limit sequence $(T_\infty^{(i)})_{i \geqslant 0}$ can be combined to a random infinite tree, called *sin-tree*, with a single infinite path from o consisting of the roots of the trees $T_\infty^{(i)}$, see [1]. The results below can be translated to this, more intuitive, description of the limit of the extended fringe tree, but we leave that to the reader. △

We consider in the present paper the extended fringe tree $T^{**} = (T^{**(i)})_i$ of a uniformly random *leaf* in the tree T (instead of a random node); this also converges in distribution when the random fringe tree does, since it can be regarded as the random extended fringe tree conditioned on the fringe tree being $\bullet$.

The distribution of this random extended fringe tree $T^{**} = (T^{**(i)})_i$ can be described by the probabilities, for pairs (t, ℓ) of a tree t and a marked leaf $\ell \in t$, and any (deterministic or random) tree T, letting L be a uniformly random leaf in T,

$$q(T; t, \ell) := \mathbb{P}\bigl(\exists w \in T : L \in T^w \text{ and } (T^w, L) \text{ is isomorphic to } (t, \ell)\bigr). \tag{2.13}$$

In fact, it is easy to see that for any such pair (t, ℓ) and $i \geqslant 0$, if o is the root of T^{**} (defined as the root of $T^{**(0)}$), then

$$\mathbb{P}\bigl((T^{**(i)}, o) = (t, \ell)\bigr) = q(T; t, \ell)\mathbf{1}\{i = d_t(\ell)\}. \tag{2.14}$$

We define also the simpler

$$q(T; t) := \mathbb{P}\bigl(\text{a random leaf } v \text{ lies in some fringe tree } T^w \text{ isomorphic to } t\bigr). \tag{2.15}$$

Note that, if $|T| > 1$, any leaf lies in several fringe trees of different sizes; hence $\sum_t q(T;t) > 1$ so $q(T;\cdot)$ is not a probability distribution on trees. In fact, trivially $q(T;\bullet) = 1$ for every tree T.

For a random tree T we define also $q(T;t,\ell \mid T)$ and $q(T;t \mid T)$ by (2.13) and (2.15) conditioned on T. Thus $q(T;t \mid T)$ is a functional of T, and thus a random variable, while $q(T;t)$ is a number depending on the distribution of T only, and similarly for $q(T;t,\ell)$. We have, as always for conditional expectations,

$$q(T;t,\ell) = \mathbb{E}\, q(T;t,\ell \mid T), \qquad q(T;t) = \mathbb{E}\, q(T;t \mid T). \tag{2.16}$$

Let t be a fixed tree with a marked leaf ℓ, and let T be a (deterministic or random) tree. For every fringe tree T^w that is isomorphic to t, there is exactly one leaf L in T such that $L \in T^w$ and $(T^w, L) = (t, \ell)$. Since all copies of t in T are disjoint, these leaves L are distinct for different fringe trees $T^w = t$, and thus (for a random tree T)

$$q(T;t,\ell \mid T) = \frac{N_t(T)}{|T|_{\mathsf{e}}} \tag{2.17}$$

and hence

$$q(T;t,\ell) = \mathbb{E}\, \frac{N_t(T)}{|T|_{\mathsf{e}}}. \tag{2.18}$$

Consequently, recalling (2.14), the distribution of the random extended fringe tree T^{**} is determined by the fringe subtree counts $N_t(T)$. Furthermore, again since the copies of t in T are disjoint, the number of leaves that lie in some copy of t equals $N_t(T)|t|_{\mathsf{e}}$, and hence, for any (random) tree T,

$$q(T;t \mid T) = \frac{N_t(T)|t|_{\mathsf{e}}}{|T|_{\mathsf{e}}} \tag{2.19}$$

and

$$q(T;t) = |t|_{\mathsf{e}}\, \mathbb{E}\, \frac{N_t(T)}{|T|_{\mathsf{e}}}. \tag{2.20}$$

Consequently, for every leaf $\ell \in t$,

$$q(T;t,\ell \mid T) = \frac{1}{|t|_{\mathsf{e}}} q(T;t \mid T), \qquad q(T;t,\ell) = \frac{1}{|t|_{\mathsf{e}}} q(T,t). \tag{2.21}$$

Hence, it suffices to study the simpler $q(T;t \mid T)$ and $q(T;t)$ without marked leaves.

For a sequence of (deterministic or random) trees T_n, it follows from (2.14) and (2.21) that the extended fringe trees $T_n^{**} = (T_n^{**(i)})_i$ converge in distribution, to some sequence of (random) trees $T_\infty^{**} = (T_\infty^{**(i)})_{i\geqslant 0}$, if and only if $q(T_n;t)$ converges for every fixed tree t, and in this case the limit distribution is determined by the limits

$$q(T_\infty^{**};t) := \lim_{n\to\infty} q(T_n;t). \tag{2.22}$$

More precisely, if these limits exist, then it follows from (2.14) and (2.21) that if o is the root of T_∞^{**} (i.e., the root of $T_\infty^{**(0)}$), then for every tree t and leaf $\ell \in t$ with $d_t(\ell) = i$,

$$\mathbb{P}\big((T_\infty^{**(i)}, o) = (t,\ell)\big) = \frac{1}{|t|_{\mathsf{e}}} \lim_{n\to\infty} q(T_n;t) = \frac{1}{|t|_{\mathsf{e}}} q(T_\infty^{**};t). \tag{2.23}$$

Note also that for any (deterministic) tree T, (2.19) and (1.2) yield

$$q(T;t)=\frac{|t|_{\mathsf{e}}|T|\,\mathbb{P}(T^*=t)}{|T|_{\mathsf{e}}}=|t|_{\mathsf{e}}\frac{|T|}{|T|_{\mathsf{e}}}\,\mathbb{P}(T^*=t)=|t|_{\mathsf{e}}\frac{\mathbb{P}(T^*=t)}{\mathbb{P}(T^*=\bullet)}. \tag{2.24}$$

This leads to the following version of the result by Aldous [1, Proposition 11] that, as said above, if a sequence of random fringe trees converge in distribution, then so do the extended fringe trees. We also obtain a formula for the limits (2.22).

Lemma 2.2. *Let (T_n) be a sequence of random trees such that $\mathbb{P}(T_n^*=t\mid T_n)\xrightarrow{\mathrm{p}}\mathbb{P}(T_\infty^*=t)$ for every fixed tree t and some random tree T_∞^* (a quenched limiting fringe tree). Then*

$$q(T_n;t\mid T_n)\xrightarrow{\mathrm{p}}q(T_\infty^{**};t)=\kappa|t|_{\mathsf{e}}\mathbb{P}(T_\infty^*=t), \tag{2.25}$$

where (with the limit in probability, in general)

$$\kappa:=\frac{1}{\mathbb{P}(T_\infty^*=\bullet)}=\lim_{n\to\infty}\frac{|T_n|}{|T_n|_{\mathsf{e}}}. \tag{2.26}$$

For full binary trees with $|T_n|\xrightarrow{\mathrm{p}}\infty$, we simply have $\kappa=2$.

Proof. We have by (2.24) and the assumption

$$q(T_n;t\mid T_n)=|t|_{\mathsf{e}}\frac{\mathbb{P}(T_n^*=t\mid T_n)}{\mathbb{P}(T_n^*=\bullet\mid T_n)}\xrightarrow{\mathrm{p}}|t|_{\mathsf{e}}\frac{\mathbb{P}(T_\infty^*=t)}{\mathbb{P}(T_\infty^*=\bullet)}, \tag{2.27}$$

which can be written as (2.25)–(2.26).

For full binary trees with $|T_n|\xrightarrow{\mathrm{p}}\infty$, (2.1) yields $\kappa=2$. ☐

Results for extended fringe trees thus follow rather trivially from results for fringe trees T_n^*, but we find it interesting to state also such results explicitly below.

3. Patricia tries

We fix a full binary tree t and consider $N_t(\widehat{\Upsilon}_n)$, the number of fringe trees in the Patricia trie $\widehat{\Upsilon}_n$ that equal t. Let $m:=|t|_{\mathsf{e}}$, the number of leaves in t. The case $m=1$ is trivial with $t=\bullet$ and $N_t(\widehat{\Upsilon}_n)=n$, so we assume $m\geqslant 2$.

Asymptotic normality of $N_t(\widehat{\Upsilon}_n)$ follows by (2.12) and a straightforward application of results for tries in [28], see below for details, although some work is required to calculate the asymptotic mean and variance. Note that the argument below is very similar to the one in [28, Section 4.2] for the number of all fringe trees in Υ_n with a given number of leaves.

We first introduce some notation. Let

$$\pi_t:=\mathbb{P}(\Upsilon_m\in\check{\mathfrak{T}}_t), \tag{3.1}$$

(this will be calculated later) and

$$H:=-p\log p-q\log q, \tag{3.2}$$

the entropy of the bits in the random strings $\Xi^{(i)}$. Further, for $\lambda>0$, let $\widetilde{\Upsilon}_\lambda$ be the random trie constructed from a random number $N_\lambda\in\mathrm{Po}(\lambda)$ random strings; thus $\widetilde{\Upsilon}_\lambda$ has N_λ leaves. The results in [28] use heavily some functions defined in general (assuming $\varphi(\bullet)=0$ as in our case) by [28, (3.16)–(3.18)]:

$$f_{\mathsf{E}}(\lambda):=\mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda), \tag{3.3}$$

$$f_{\mathsf{V}}(\lambda) := 2\operatorname{Cov}\bigl(\varphi(\widetilde{\Upsilon}_\lambda), \Phi(\widetilde{\Upsilon}_\lambda)\bigr) - \operatorname{Var}\varphi(\widetilde{\Upsilon}_\lambda), \tag{3.4}$$

$$f_{\mathsf{C}}(\lambda) := \operatorname{Cov}\bigl(\varphi(\widetilde{\Upsilon}_\lambda), N_\lambda\bigr), \tag{3.5}$$

and their Mellin transforms defined by (for $\mathsf{X} = \mathsf{E}, \mathsf{V}, \mathsf{C}$ and suitable s)

$$f^*_{\mathsf{X}}(s) := \int_0^\infty f_{\mathsf{X}}(\lambda)\lambda^{s-1}\,\mathrm{d}\lambda. \tag{3.6}$$

For $\mathsf{X} = \mathsf{E}, \mathsf{V}, \mathsf{C}$ define also the function $\psi_{\mathsf{X}}(x)$, $x \in \mathbb{R}$ by [28, (3.13)–(3.14)]:

(i) In the aperiodic case ($\log p/\log q \notin \mathbb{Q}$), ψ_{X} is constant: for all x,

$$\psi_{\mathsf{X}}(x) := f^*_{\mathsf{X}}(-1). \tag{3.7}$$

(ii) In the periodic case, ψ_{X} is a continuous d-periodic function given by the Fourier series, recalling $d = d_p$ defined in (2.4),

$$\psi_{\mathsf{X}}(x) = \sum_{k=-\infty}^{\infty} f^*_{\mathsf{X}}\Bigl(-1 - \frac{2\pi k}{d}\mathrm{i}\Bigr)e^{2\pi\mathrm{i}kx/d}. \tag{3.8}$$

(The Fourier series converges absolutely in our case by (3.12)–(3.14) below, or by [28, Theorem 3.1] and the formulas for $f_X(\lambda)$ below.)

Theorem 3.1. *Let t be a full binary tree with $|t|_{\mathsf{e}} = m > 1$. Then $N_t(\widehat{\Upsilon}_n)$ is asymptotically normal as $n \to \infty$:*

$$\frac{N_t(\widehat{\Upsilon}_n) - \mathbb{E}\,N_t(\widehat{\Upsilon}_n)}{\sqrt{\operatorname{Var} N_t(\widehat{\Upsilon}_n)}} \xrightarrow{\mathrm{d}} N(0,1), \tag{3.9}$$

with convergence of all moments. We have $\operatorname{Var} N_t(\widehat{\Upsilon}_n) = \Theta(n)$ *as $n \to \infty$, and*

$$\mathbb{E}\,N_t(\widehat{\Upsilon}_n) = nH^{-1}\psi_{\mathsf{E}}(\log n) + o(n), \tag{3.10}$$

$$\operatorname{Var} N_t(\widehat{\Upsilon}_n) = n\bigl(H^{-1}\psi_{\mathsf{V}}(\log n) - H^{-2}\psi_{\mathsf{C}}(\log n)^2 + o(1)\bigr), \tag{3.11}$$

where ψ_{X} is given by (3.7)–(3.8) *with (for* $\operatorname{Re} s > -m$*)*

$$f^*_{\mathsf{E}}(s) = \pi_t\frac{\Gamma(m+s)}{m!}, \tag{3.12}$$

$$\begin{aligned} f^*_{\mathsf{V}}(s) = {} & \frac{\pi_t}{m!}\Gamma(m+s) - \frac{\pi_t^2}{m!^2}2^{-2m-s}\Gamma(2m+s) \\ & - 2\frac{\pi_t^2}{m!^2}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(2m+s+k)}{k!}\cdot\frac{p^{m+k}+q^{m+k}}{1-(p^{m+k}+q^{m+k})}, \end{aligned} \tag{3.13}$$

$$f^*_{\mathsf{C}}(s) = -sf^*_{\mathsf{E}}(s) = -\pi_t\frac{s\Gamma(m+s)}{m!}. \tag{3.14}$$

In the aperiodic case, (3.10)–(3.11) *simplify to*

$$\mathbb{E}\,N_t(\widehat{\Upsilon}_n)/n \to H^{-1}\frac{\pi_t}{m(m-1)}, \tag{3.15}$$

$$\operatorname{Var} N_t(\widehat{\Upsilon}_n)/n \to H^{-1}f^*_{\mathsf{V}}(-1) - \Bigl(H^{-1}\frac{\pi_t}{m(m-1)}\Bigr)^2. \tag{3.16}$$

Proof. By (2.12), we have $N_t(\widehat{\Upsilon}_n) = \Phi(\Upsilon_n)$; we apply [28, Theorem 3.9] to $\Phi(\Upsilon_n)$. We first verify the technical condition there that we can write $\varphi = \varphi_+ - \varphi_-$ with bounded $\varphi_\pm$ such that the corresponding additive functionals $\Phi_\pm$ are increasing, in the sense that $\Phi_\pm(T_1) \leqslant \Phi_\pm(T_2)$ if T_1 is a subtree of T_2. As in [28, Section 4.3], we let $\varphi_{>m}(T) := \mathbf{1}\{|T|_{\mathsf{e}} > m\}$, and it is easily seen that $\varphi_+ := \varphi + \varphi_{>m}$ and $\varphi_- := \varphi_{>m}$ satisfy the condition. Hence [28, Theorem 3.9] applies. Furthermore, $\varphi(\Upsilon_n) =$ for $n > m$, and it is easy to see that there exists n such that $\operatorname{Var}\Phi(\Upsilon_n) > 0$ (i.e., $\Phi(\Upsilon_n)$ is not constant); we may for example take $n = m+1$ if $m \geqslant 3$ and $n = 4$ for $m = 2$. Hence also [28, Lemma 3.14] applies, which shows that $\operatorname{Var} N_t(\widehat{\Upsilon}_n) = \operatorname{Var}\Phi(\Upsilon_n) = \Omega(n)$ as $n \to \infty$. Hence, (3.9) (with convergence of moments) follows by [28, Theorem 3.9(iv)]. Furthermore, the moment convergence in [28, Theorem 3.9(ii)] implies (3.11). In particular, $\operatorname{Var}\Phi(\Upsilon_n) = O(n)$, and thus $\operatorname{Var}\Phi(\Upsilon_n) = \Theta(n)$. The asymtotics (3.10) follows from [28, Theorem 3.9(v)].

It remains to find the Mellin transforms f_{X}^* (which yield $\psi_{\mathsf{X}}(x)$ by (3.7)–(3.8)). First, since $\Upsilon_n \in \check{\mathfrak{T}}_t$ is possible only when $n = |t|_{\mathsf{e}} = m$, it follows by (2.10) and (3.1) that

$$f_{\mathsf{E}}(\lambda) := \mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda) = \mathbb{P}(\widetilde{\Upsilon}_\lambda \in \check{\mathfrak{T}}_t) = \mathbb{P}(N_\lambda = m)\pi_t = \operatorname{Po}(\lambda; m)\pi_t = \pi_t \frac{\lambda^m}{m!} e^{-\lambda}. \tag{3.17}$$

This, or simpler [28, Lemma 3.16], yields the Mellin transform

$$f_{\mathsf{E}}^*(s) := \int_0^\infty f_{\mathsf{E}}(\lambda)\lambda^{s-1}\,\mathrm{d}\lambda = \frac{\Gamma(m+s)}{m!}\pi_t, \tag{3.18}$$

verifying (3.12). By [28, Lemma 3.6], we have $f_{\mathsf{C}}^*(s) = -s f_{\mathsf{E}}^*(s)$, showing (3.14). (We also obtain $f_{\mathsf{C}}(\lambda) = \lambda f_{\mathsf{E}}'(\lambda) = (m-\lambda) f_{\mathsf{E}}(\lambda)$.)

To find the more complicated f_{V}, note first that if $\varphi(T) = 1$, i.e., $T \in \check{\mathfrak{T}}_t$, then no fringe tree T^v except $T^o = T$ (where o is the root) belongs to $\check{\mathfrak{T}}_t$. Hence, if $\varphi(T) = 1$, then $\Phi(T) = 1$, and consequently,

$$\operatorname{Cov}\bigl(\varphi(\widetilde{\Upsilon}_\lambda), \Phi(\widetilde{\Upsilon}_\lambda)\bigr) = \mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda) - \mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda)\,\mathbb{E}\,\Phi(\widetilde{\Upsilon}_\lambda). \tag{3.19}$$

Similarly, $\operatorname{Var}\varphi(\widetilde{\Upsilon}_\lambda) = \mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda) - (\mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda))^2$, and thus (3.4) yields, using (3.19),

$$f_{\mathsf{V}}(\lambda) = \mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda) - 2\,\mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda)\,\mathbb{E}\,\Phi(\widetilde{\Upsilon}_\lambda) + (\mathbb{E}\,\varphi(\widetilde{\Upsilon}_\lambda))^2. \tag{3.20}$$

Let $\mathcal{A}^* := \bigcup_{n=0}^\infty \{0,1\}^n$ be the set of finite strings of $\{0,1\}$. If $\alpha = \alpha_1 \cdots \alpha_n \in \mathcal{A}^*$, define $P(\alpha)$ as the probability that the random string $\Xi^{(1)}$ begins with α, i.e.,

$$P(\alpha) := \prod_{i=1}^{|\alpha|} q^{1-\alpha_i} p^{\alpha_i}, \tag{3.21}$$

where $|\alpha| \geqslant 0$ is the length of α. We may regard the strings $\alpha \in \mathcal{A}^*$ as the nodes in the infinite binary tree, and, using again $\varphi(\bullet) = 0$, it follows (see [28, (2.25)]) that, using also (3.17),

$$\mathbb{E}\,\Phi(\widetilde{\Upsilon}_\lambda) = \sum_{\alpha\in\mathcal{A}^*} \mathbb{E}\,\varphi\bigl(\widetilde{\Upsilon}_{P(\alpha)\lambda}\bigr) = \sum_{\alpha\in\mathcal{A}^*} f_{\mathsf{E}}\bigl(P(\alpha)\lambda\bigr) = \pi_t \sum_{\alpha\in\mathcal{A}^*} \operatorname{Po}\bigl(P(\alpha)\lambda; m\bigr). \tag{3.22}$$

Let $\sum_\alpha'$ denote the sum over all $\alpha \in \mathcal{A}^*$ with $|\alpha| \geqslant 1$, i.e., over all strings α except the empty string. Then (3.20) and (3.22) yield, using (3.17) again and (2.3),

$$\begin{aligned} f_{\mathsf{V}}(\lambda) &= \pi_t \operatorname{Po}(\lambda; m) - 2\pi_t^2 \operatorname{Po}(\lambda; m) \sum_{\alpha\in\mathcal{A}^*} \operatorname{Po}\bigl(P(\alpha)\lambda; m\bigr) + \pi_t^2 \operatorname{Po}(\lambda; m)^2 \\ &= \pi_t \frac{\lambda^m}{m!} e^{-\lambda} - 2\frac{\pi_t^2}{m!^2} \sum_{\alpha\in\mathcal{A}^*} P(\alpha)^m \lambda^{2m} e^{-(1+P(\alpha))\lambda} + \pi_t^2 \frac{\lambda^{2m}}{m!^2} e^{-2\lambda} \end{aligned}$$

$$= \pi_t \frac{\lambda^m}{m!} e^{-\lambda} - 2\frac{\pi_t^2}{m!^2} \sum_{\alpha}{}' P(\alpha)^m \lambda^{2m} e^{-(1+P(\alpha))\lambda} - \pi_t^2 \frac{\lambda^{2m}}{m!^2} e^{-2\lambda}. \tag{3.23}$$

The Mellin transform is thus, for $\operatorname{Re} s > -m$, by a simple calculation,

$$f_{\mathsf{V}}^*(s) = \frac{\pi_t}{m!}\Gamma(m+s) - 2\frac{\pi_t^2}{m!^2}\sum_{\alpha}{}' P(\alpha)^m (1+P(\alpha))^{-2m-s}\Gamma(2m+s)$$
$$- \frac{\pi_t^2}{m!^2} 2^{-2m-s}\Gamma(2m+s). \tag{3.24}$$

By a binomial expansion we have, with absolutely convergent sums, e.g. by (3.26) below,

$$\sum_{\alpha}{}' P(\alpha)^m (1+P(\alpha))^{-2m-s} = \sum_{\alpha}{}' P(\alpha)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(2m+s+k)}{\Gamma(2m+s)} P(\alpha)^k$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(2m+s+k)}{\Gamma(2m+s)} \sum_{\alpha}{}' P(\alpha)^{m+k}. \tag{3.25}$$

For any exponent b and $\ell \geqslant 1$ we have from the definition (3.21), letting j be the number of 1s in α,

$$\sum_{\alpha:|\alpha|=\ell} P(\alpha)^b = \sum_{j=0}^{\ell} \binom{\ell}{j} \left(p^j q^{\ell-j}\right)^b = (p^b + q^b)^{\ell}. \tag{3.26}$$

Hence, summing over $\ell \geqslant 1$, we obtain from (3.25)

$$\sum_{\alpha}{}' P(\alpha)^m (1+P(\alpha))^{-2m-s} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(2m+s+k)}{\Gamma(2m+s)} \frac{p^{m+k}+q^{m+k}}{1-(p^{m+k}+q^{m+k})}. \tag{3.27}$$

Finally, we obtain (3.13) from (3.24) and (3.27).

In the aperiodic case, (3.7) holds, and thus (3.10)–(3.11) yield (3.15)–(3.16), using (3.12) and (3.14). ∎

We have postponed calculating π_t, which appears in the formulas above. We begin with an elementary (and certainly known) calculation of $\mathbb{P}(\Upsilon_m = t)$. Recall that a trie belongs to set set $\mathfrak{T}'$ of trees, and that $|\Upsilon_m|_{\mathsf{e}} = m$. Hence, it suffices to consider the case $t \in \mathfrak{T}'$ and $|t|_{\mathsf{e}} = m$, since otherwise the probability is 0.

Lemma 3.2. *Let t be a binary tree with $t \in \mathfrak{T}'$ and $|t|_{\mathsf{e}} = m \geqslant 1$. The probability that the random trie Υ_m equals t is given by*

$$\mathbb{P}(\Upsilon_m = t) = m!\, q^{\mathsf{LPL}(t)} p^{\mathsf{RPL}(t)}. \tag{3.28}$$

Proof. Each of the m leaves x_i of t corresponds to a finite string $A(x_i)$ of $\{0,1\}$ which encodes the path from the root to x_i by 0 for each left child and 1 for each right child. Since $t \in \mathfrak{T}'$, the random trie Υ_m constructed from the strings $\Xi^{(1)}, \dots, \Xi^{(m)}$ equals t if and only if the strings $A(x_1), \dots, A(x_m)$ are initial segments of $\Xi^{(1)}, \dots, \Xi^{(m)}$, in some order. The string $A(x_i)$ contains $d_{\mathsf{L}}(x_i)$ 0s and $d_{\mathsf{R}}(x_i)$ 1s, and thus the probability that $\Xi^{(j)}$ begins with $A(x_i)$ is $q^{d_{\mathsf{L}}(x_i)} p^{d_{\mathsf{R}}(x_i)}$. Since there are $m!$ possible permutations of the strings, we obtain

$$\mathbb{P}(\Upsilon_m = t) = m! \prod_{i=1}^{m} q^{d_{\mathsf{L}}(x_i)} p^{d_{\mathsf{R}}(x_i)} = m!\, q^{\mathsf{LPL}(t)} p^{\mathsf{RPL}(t)}, \tag{3.29}$$

showing (3.28). ☐

For a full binary tree t, recall from Section 2.4 that $\breve{\mathfrak{T}}_t$ is the set of binary trees obtained by subdividing the edges of t. Let

$$\nu_k(t) := \bigl|\{v \in t : |t^v|_{\mathsf{e}} = k\}\bigr|, \qquad k \geqslant 1. \tag{3.30}$$

Lemma 3.3. *Let t be a full binary tree with $|t|_{\mathsf{e}} = m \geqslant 1$. Then,*

$$\pi_t := \mathbb{P}(\Upsilon_m \in \breve{\mathfrak{T}}_t) = m!\, q^{\mathsf{LPL}(t)} p^{\mathsf{RPL}(t)} \prod_{k=2}^{m-1} \bigl(1 - (q^{w(e)} + p^{w(e)})\bigr)^{-\nu_k(t)}. \tag{3.31}$$

Proof. By Lemma 3.2,

$$\mathbb{P}(\Upsilon_m \in \breve{\mathfrak{T}}_t) = \mathbb{P}(\Upsilon_m \in \breve{\mathfrak{T}}_t \cap \mathfrak{T}') = \sum_{T \in \breve{\mathfrak{T}}_t \cap \mathfrak{T}'} \mathbb{P}(\Upsilon_m = T) = m! \sum_{T \in \breve{\mathfrak{T}}_t \cap \mathfrak{T}'} q^{\mathsf{LPL}(T)} p^{\mathsf{RPL}(T)}. \tag{3.32}$$

Let the *weight* $w(e)$ of an edge e in a tree T be the number of leaves x such that e is on the path from the root to x. If we insert $\ell \geqslant 0$ new nodes in a given edge e, of which j get a left child and $k = \ell - j$ a right child, then $\mathsf{LPL}(T)$ and $\mathsf{RPL}(T)$ will increase by $jw(e)$ and $kw(e)$, respectively, and the term $q^{\mathsf{LPL}(T)} p^{\mathsf{RPL}(T)}$ will thus be multiplied by $q^{jw(e)} p^{kw(e)}$. Summing over all j and k with a given $\ell = j + k$ we obtain the factor, since the insertion may be done with $\binom{\ell}{j}$ orders of left and right,

$$\sum_{j=0}^{\ell} \binom{\ell}{j} q^{jw(e)} p^{(\ell-j)w(e)} = \bigl(q^{w(e)} + p^{w(e)}\bigr)^{\ell}. \tag{3.33}$$

And summing over all $\ell \geqslant 0$ gives the factor

$$\sum_{\ell=0}^{\infty} \bigl(q^{w(e)} + p^{w(e)}\bigr)^{\ell} = \frac{1}{1 - (q^{w(e)} + p^{w(e)})}. \tag{3.34}$$

The edges of weight 1 are precisely the edges that lead to a leaf. Hence, we are not allowed to insert any new nodes in any edge with weight 1, since that would yield a tree not in $\mathfrak{T}'$, but we may insert arbitrarily many in all other edges. Thus, it follows from (3.32) that $\mathbb{P}(\Upsilon_m \in \breve{\mathfrak{T}}_t)$ is obtained by multiplying $\mathbb{P}(\Upsilon_m = t)$ in (3.28) by the product of the factors in (3.34) for all edges e with $w(e) \geqslant 2$. Moreover, if $e = uv$ where v is a child of u, then

$$w(e) = |\mathcal{L}(T) \cap T^v|, \tag{3.35}$$

the number of leaves in the fringe tree T^v, and for $k \leqslant m - 1$, the number of edges with weight k thus equals $\nu_k(T)$. (This fails for $k = m$, since in a full binary tree T with $|T|_{\mathsf{e}} = m$, all edges have weight $\leqslant m - 1$ while $\nu_m(T) = 1$.) Consequently, (3.31) follows. ☐

Remark 3.4. We do not claim that the error term $o(n)$ in (3.10) is $o(\sqrt{n})$ so that $\mathbb{E}\, N_t(\widehat{\Upsilon}_n)$ may be replaced by $H^{-1}\psi_{\mathsf{E}}(\log n)$ in (3.9). It is known that in the corresponding results for the size of a random trie, this is in general *not* true, see [14] and [28, Appendix C]. We leave it as an open problem whether the same may happen here too. △

Remark 3.5. Theorem 3.1 extends to multivariate limits for several full binary trees t_i by the Cramér–Wold device, i.e., by considering linear combinations of different $N_{t_i}(\widehat{\Upsilon}_n)$, cf. [28, Remark 3.10]. In particular, for any set U of full binary trees with the same number of leaves $|t|_{\mathsf{e}} = m > 1$ for $t \in U$ (and thus the same size $|t| = 2m+1$), all results in Theorem 3.1 hold also for $N_U(\widehat{\Upsilon}_n) := \sum_{t\in U} N_t(\widehat{\Upsilon}_n)$ if we just replace π_t by $\pi_U := \sum_{t\in U} \pi_t$. (By the same proof.) For example, this yields the asymptotic distribution of the number $N_m(\widehat{\Upsilon}_n)$ of all fringe trees with m leaves; we then replace π_t by

$$\pi_m := \mathbb{P}\Big(\Upsilon_m \in \bigcup_{|t|_{\mathsf{e}}=m} \check{\mathfrak{T}}_t\Big) = 1 - p^m - q^m, \tag{3.36}$$

since $\bigcup_{|t|_{\mathsf{e}}=m} \check{\mathfrak{T}}_t$ (where t is a full binary tree) is the set of all tries with m leaves and a root of outdegree 2, and the probability that the root of Υ_m has outdegree 1 is $p^m + q^m$. In general, with trees t_i of different sizes, asymptotic covariances can be found by calculations similar to the ones above for the variance; we omit the details. △

The asymptotic normality of $N_t(\widehat{\Upsilon}_n)$ yields corresponding results for the distributions of fringe trees and extended fringe trees. We note first a simple corollary.

Corollary 3.6. *Let t be a full binary tree with $|t|_{\mathsf{e}} = m > 1$. Then*

$$N_t(\widehat{\Upsilon}_n)/n = \mathbb{E}\, N_t(\widehat{\Upsilon}_n)/n + o_{\mathrm{p}}(1) = H^{-1}\psi_{\mathsf{E}}(\log n) + o_{\mathrm{p}}(1), \tag{3.37}$$

with the periodic function $\psi_{\mathsf{E}}(t)$ as in Theorem 3.1. In particular, in the aperiodic case,

$$N_t(\widehat{\Upsilon}_n)/n \xrightarrow{\mathrm{p}} \frac{\pi_t}{m(m-1)H}. \tag{3.38}$$

Proof. The first equality in (3.37) follows from $\operatorname{Var} N_t(\widehat{\Upsilon}_n) = O(n)$ by Chebyshev's inequality, and the second is (3.10).

In the aperiodic case, we use (3.15) instead of (3.10) and obtain (3.38). □

The size of the random trie Υ_n shows oscillations in the periodic case, see e.g. [29; 31; 14; 17; 25; 18; 24; 28]. However, the Patricia trie $\widehat{\Upsilon}_n$ is a full binary tree with n leaves, and thus has a fixed size $|\widehat{\Upsilon}_n| = 2n-1$. (This is special to the binary case considered here.) Hence we obtain from (1.3) and Corollary 3.6 immediately the following for the random fringe tree. We state results both conditioned on the tree $\widehat{\Upsilon}_n$ and unconditioned (i.e., results of quenched and annealed type, repectively.)

Corollary 3.7. *Let t be a full binary tree with $|t|_{\mathsf{e}} = m > 1$. Then*

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t \mid \widehat{\Upsilon}_n) = \frac{1}{2H}\psi_{\mathsf{E}}(\log n) + o_{\mathrm{p}}(1), \tag{3.39}$$

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t) = \frac{1}{2H}\psi_{\mathsf{E}}(\log n) + o(1), \tag{3.40}$$

with the periodic function $\psi_{\mathsf{E}}(t)$ as in Theorem 3.1. In particular, in the aperiodic case,

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t \mid \widehat{\Upsilon}_n) \xrightarrow{\mathrm{p}} \frac{\pi_t}{2m(m-1)H}, \tag{3.41}$$

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t) \longrightarrow \frac{\pi_t}{2m(m-1)H}. \tag{3.42}$$

Furthermore, if $|t| = 1$*, i.e.,* $t = \bullet$*, then, for any* p*,*

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t \mid \widehat{\Upsilon}_n) = \mathbb{P}(\widehat{\Upsilon}_n^* = t) = \frac{n}{2n-1} \to \frac{1}{2}. \tag{3.43}$$

Proof. The quenched versions (3.39) and (3.41) are the same as (3.37) and (3.38) by (1.3) conditioned on $\widehat{\Upsilon}_n$, recalling $|\widehat{\Upsilon}_n| = 2n-1$. The annealed versions follow by taking expectations; note that the error term $o_{\mathrm{p}}(1)$ in (3.39) is bounded so dominated convergence applies and shows that its expectation is $o(1)$.

Finally, (3.43) is trivial (but included for completeness). □

For the random extended fringe tree $\widehat{\Upsilon}_n^{**}$, we obtain similarly, for the probabilities $q(\widehat{\Upsilon}_n; t)$ defined in Section 2.5.

Corollary 3.8. *Let* t *be a full binary tree with* $|t|_{\mathsf{e}} > 1$*. Then*

$$q(\widehat{\Upsilon}_n; t \mid \widehat{\Upsilon}_n) = |t|_{\mathsf{e}} H^{-1} \psi_{\mathsf{E}}(\log n) + o_{\mathrm{p}}(1), \tag{3.44}$$

$$q(\widehat{\Upsilon}_n; t) = |t|_{\mathsf{e}} H^{-1} \psi_{\mathsf{E}}(\log n) + o(1), \tag{3.45}$$

with the periodic function $\psi_{\mathsf{E}}(t)$ *as in Theorem 3.1. In particular, in the aperiodic case,*

$$q(\widehat{\Upsilon}_n; t \mid \widehat{\Upsilon}_n) \xrightarrow{\mathrm{p}} \frac{\pi_t}{(|t|_{\mathsf{e}} - 1)H}, \tag{3.46}$$

$$q(\widehat{\Upsilon}_n; t) \longrightarrow \frac{\pi_t}{(|t|_{\mathsf{e}} - 1)H}. \tag{3.47}$$

For $|t|_{\mathsf{e}} = 1$*,* $q(\widehat{\Upsilon}_n) = 1$ *by definition.*

Proof. Follows as Corollary 3.7 from Corollary 3.6, now using (2.19)–(2.20) and recalling $|\widehat{\Upsilon}_n|_{\mathsf{e}} = n$. □

Remark 3.9. We have here only stated first order results for the distributions of fringe trees and extended fringe trees. We similarly obtain from Theorem 3.1 also asymptotic normality of these distributions in the quenched version, meaning asymptotic normality of the conditional probabilities above. △

Remark 3.10. Corollaries 3.7 and 3.8 show that in the periodic case there is oscillation and no limit distribution, although suitable subsequences converge in distribution. It is well known that for some related functionals for tries, the oscillations are numerically very small; this is true here too when m is small, but not for large m. Consider the symmetric case $p = q = \frac{1}{2}$; then $d_p = \log 2$. (In other periodic cases, d_p is smaller and the oscillations are substantially smaller than in the symmetric case, but they still become large for large m.) In the Fourier series (3.8) for f_{E}, we have by (3.12) the constant term $f_{\mathsf{E}}^*(-1) = \pi_t/m(m-1)$, and if we normalize by this term, for the term $k = 1$ we have

$$\frac{f_{\mathsf{E}}^*(-1 - \frac{2\pi}{\log 2}\mathrm{i})}{f_{\mathsf{E}}^*(-1)} = \frac{\Gamma(m - 1 - \frac{2\pi}{\log 2}\mathrm{i})}{\Gamma(m-1)}. \tag{3.48}$$

For $m = 2$, this ratio has absolute value $|\Gamma(1 - \frac{2\pi}{\log 2}\mathrm{i})| \doteq 4.9 \cdot 10^{-6}$, and higher Fourier coefficients are much smaller. Hence, the oscillations are in this case hardly of practical importance. However, the absolute value of the ratio increases as m increases: for $m = 3$ it is $\doteq 4.5 \cdot 10^{-5}$, for $m = 4$ it is $\doteq 2.1 \cdot 10^{-4}$ and for $m = 100$ it is $\doteq 0.66$; in fact, the absolute value of the ratio converges to 1 as $m \to \infty$ (see [32, 5.11.12]), and the same holds for every Fourier coefficient. Hence we cannot always ignore the oscillations.

More precisely, still taking $p = q = \frac{1}{2}$, the normalized $(\log 2)$-periodic function $\psi_0(x) := \psi_{\mathsf{E}}(x)/f^*_{\mathsf{E}}(-1)$ is non-negative by (3.10) and has by (3.8) and (3.18) Fourier coefficients

$$\widehat{\psi}_0(k) = \frac{f^*_{\mathsf{E}}(-1 - k\frac{2\pi}{\log 2}\mathrm{i})}{f^*_{\mathsf{E}}(-1)} = \frac{\Gamma(m - 1 - \frac{2\pi k}{\log 2}\mathrm{i})}{\Gamma(m-1)}. \tag{3.49}$$

Let $\{x\} := x - \lfloor x \rfloor$ denote the fractional part of a real number x, and suppose that $m \to \infty$ along a subsequence such that $\{\lg m\} = \{(\log m)/d\} \to u \in [0, 1]$. (Recall that $d = \log 2$.) It then follows from (3.49) and [32, 5.11.12] that, for any $k \in \mathbb{Z}$,

$$\widehat{\psi}_0(k) \sim m^{-(2\pi k/\log 2)\mathrm{i}} = e^{-2\pi k(\lg m)\mathrm{i}} \to e^{-2\pi k u\mathrm{i}} = \widehat{\delta_{du}}(k), \tag{3.50}$$

where δ_{du} is a point mass at du. Hence, by the continuity theorem, the function $\psi_0(x)$ converges weakly (as a measure on the circle $\mathbb{R}/d\mathbb{Z}$) to δ_{du}, which roughly means that for large m, $\psi_0(x)$ is concentrated at x close to $du \approx d\{\lg m\}$. Hence, still roughly, $\psi_E(\log n)$ in (3.10) is large when $\{\lg n\} = \{\log n/d\} \approx \{\lg m\}$, but small otherwise. This should not be surprising. For a large n, in the first generations of the construction of the trie from n strings in Section 2.2, by the law of large numbers almost exatly half of the strings are passed to the left child and half to the right child. Consequently, there will be many fringe trees in Υ_n, and thus in $\widehat{\Upsilon}_n$, of size $\approx 2^{-j}n$ for integers j, but few for intermediate sizes, until we get down to small sizes m. Thus, for a fixed large m, we expect many fringe trees of size m when n/m is close to a power of 2, which is the same as $\{\lg n\} \approx \{\lg m\}$. △

Remark 3.11. If we consider the ratio between the probabilities for two given trees t of the same size, then there are no oscillations even in the periodic case, since the oscillations in Corollaries 3.7 and 3.8 for the two trees cancel by (3.8) and (3.12). △

4. Compressed binary search trees

In this section we study fringe trees of the compressed BST $\widehat{\mathcal{B}}_n$. Let t be a full binary tree, and let as in Section 2.4 $\check{\mathfrak{T}}_t$ be the set of all binary trees that can be obtained by inserting additional nodes of outdegree 1 in the edges. As said in the introduction, Aldous [1] shows that the random fringe trees $\mathcal{B}^*_n$ converge in distribution to some limiting random fringe tree $\mathcal{B}^*_\infty$ as $n \to \infty$, and Devroye [9, 10] show asymptotic normality of the subtree counts $N_t(\mathcal{B}_n)$. Furthermore, [1] shows that the distribution of the limiting random fringe tree $\mathcal{B}^*_\infty$ equals the mixture $\sum_{k=1}^\infty \frac{2}{(k+1)(k+2)}\mathrm{Law}(\mathcal{B}_k)$, i.e., for any set $\mathfrak{S} \subseteq \mathfrak{T}$,

$$\mathbb{P}\big(\mathcal{B}^*_\infty \in \mathfrak{S}\big) = \sum_{k=1}^\infty \frac{2}{(k+1)(k+2)}\,\mathbb{P}\big(\mathcal{B}_k \in \mathfrak{S}\big). \tag{4.1}$$

We use these results and (2.12) to obtain similar results for fringe trees of the compressed BST $\widehat{\mathcal{B}}_n$.

Theorem 4.1. *Let t be a full binary tree. Then $N_t(\widehat{\mathcal{B}}_n)$ is asymptotically normal: there exist constants $\beta_t > 0$ and $\gamma_t > 0$ such that, as $n \to \infty$,*

$$\frac{N_t(\widehat{\mathcal{B}}_n) - n\beta_t}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0, \gamma_t^2), \tag{4.2}$$

with convergence of mean and variance. In particular,

$$N_t(\widehat{\mathcal{B}}_n)/n \xrightarrow{\text{p}} \beta_t. \tag{4.3}$$

Furthermore, there exists a limiting fringe tree distribution given by a random full binary tree $\widehat{\mathcal{B}}_\infty^*$ such that for every $t \in \widehat{\mathfrak{T}}$,

$$\mathbb{P}(\widehat{\mathcal{B}}_\infty^* = t) = \frac{3}{2}\beta_t, \tag{4.4}$$

and (quenched version)

$$\mathbb{P}(\widehat{\mathcal{B}}_n^* = t \mid \widehat{\mathcal{B}}_n) = \frac{N_t(\widehat{\mathcal{B}}_n)}{|\widehat{\mathcal{B}}_n|} \xrightarrow{\text{p}} \mathbb{P}(\widehat{\mathcal{B}}_\infty^* = t) \tag{4.5}$$

and (annealed version)

$$\mathbb{P}(\widehat{\mathcal{B}}_n^* = t) = \mathbb{E}\,\frac{N_t(\widehat{\mathcal{B}}_n)}{|\widehat{\mathcal{B}}_n|} \to \mathbb{P}(\widehat{\mathcal{B}}_\infty^* = t). \tag{4.6}$$

We conjecture that also all higher moments converge in (4.2), but we have not pursued this and leave it as an open problem.

Proof. We use again (2.12); thus $N_t(\widehat{\mathcal{B}}_n) = \Phi(\mathcal{B}_n)$ where Φ is defined by (2.10)–(2.11). The asymptotic normality (4.2) (with convergence of mean and variance) then follows from [20, Corollary 1.15]; see also [10] for similar results. The convergence in probability (4.3) is an immediate consequence.

Furthermore, [20, Corollary 1.15 and (1.24)] yield

$$\beta_t = \sum_{k=1}^{\infty} \frac{2}{(k+1)(k+2)}\,\mathbb{E}\,\varphi(\mathcal{B}_k) = \sum_{k=1}^{\infty} \frac{2}{(k+1)(k+2)}\,\mathbb{P}(\mathcal{B}_k \in \check{\mathfrak{T}}_t). \tag{4.7}$$

Alternatively, (4.3) and dominated convergence yield, together with (2.12),

$$\begin{aligned}\beta_t &= \lim_{n\to\infty} \frac{\mathbb{E}\,N_t(\widehat{\mathcal{B}}_n)}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{v\in\mathcal{B}_n} \mathbf{1}\{\mathcal{B}_n^v \in \check{\mathfrak{T}}_t\} = \lim_{n\to\infty} \mathbb{P}\big(\mathcal{B}_n^* \in \check{\mathfrak{T}}_t\big)\\ &= \mathbb{P}\big(\mathcal{B}_\infty^* \in \check{\mathfrak{T}}_t\big),\end{aligned} \tag{4.8}$$

which agrees with (4.7) by (4.1). The union $\bigcup_{t\in\widehat{\mathfrak{T}}} \check{\mathfrak{T}}_t$ of the sets $\check{\mathfrak{T}}_t$ over all full binary trees t is the set $\mathfrak{T}^{\{0,2\}}$ of all binary trees where the root has degree 2 or 0. Hence, (4.8) implies

$$\sum_{t\in\widehat{\mathfrak{T}}} \beta_t = \sum_{t\in\widehat{\mathfrak{T}}} \mathbb{P}\big(\mathcal{B}_\infty^* \in \check{\mathfrak{T}}_t\big) = \mathbb{P}\big(\mathcal{B}_\infty^* \in \mathfrak{T}^{\{0,2\}}\big) = \lim_{n\to\infty} \mathbb{P}\big(\mathcal{B}_n^* \in \mathfrak{T}^{\{0,2\}}\big). \tag{4.9}$$

If $v \in \mathcal{B}_n$, then the fringe tree $\mathcal{B}_n^v \in \mathfrak{T}^{\{0,2\}} \iff$ the degree $d(v) \in \{0,2\} \iff v \in \widehat{\mathcal{B}}_n$. Hence, (4.9) yields, using (2.7) and dominated convergence,

$$\sum_{t\in\widehat{\mathfrak{T}}} \beta_t = \lim_{n\to\infty} \mathbb{E}\,\frac{|\{v \in \mathcal{B}_n : v \in \widehat{\mathcal{B}}_n\}|}{n} = \lim_{n\to\infty} \mathbb{E}\,\frac{|\widehat{\mathcal{B}}_n|}{n} = \frac{2}{3}. \tag{4.10}$$

Consequently, $\sum_{t\in\widehat{\mathfrak{T}}} \frac{3}{2}\beta_t = 1$, so (4.4) defines a probability distribution on full binary trees.

Combining (4.3) and (2.7) we obtain

$$\frac{N_t(\widehat{\mathcal{B}}_n)}{|\widehat{\mathcal{B}}_n|} \xrightarrow{\mathrm{p}} \frac{3}{2}\beta_t. \tag{4.11}$$

The quenched and annealed convergence in distribution (4.5) and (4.6) then follow from (1.2) and (1.3) (and dominated convergence again).

It remains to show that $\beta_t > 0$ and $\gamma_t > 0$. For β_t, this is immediate from (4.7). For γ_t, let $m := |t|_{\mathsf{e}}$ and note first that we may assume $m \geqslant 2$, since for $t = \bullet$, we have $N_\bullet(\widehat{\mathcal{B}}_n) = |\widehat{\mathcal{B}}_n|_{\mathsf{e}} = |\mathcal{B}_n|_{\mathsf{e}}$, and thus $\gamma_\bullet^2 = 2/45$ by (2.8). Fix a suitable $k \geqslant m$; we may take $k = 4$ if $m = 2$ and $k = m$ if $m \geqslant 3$. Assume $n \geqslant 2k - 1$. In the construction of the binary search tree $\mathcal{B}_n$ in Section 2.3, stop at every node that receives exactly $2k - 1$ items. At each such node, peek into the future to see whether the fringe tree at that node will be a full binary tree (with k leaves), or it will contain some node of outdegree 1; in the latter case, continue the recursive constraction at this node too, but in the first case, just mark the node and leave it. The result is a subtree of $\mathcal{B}_n$ that we denote by $\mathcal{B}'_n$; it has a number N'_k of marked nodes with $2k - 1$ items each, and we recover $\mathcal{B}_n$ by replacing each marked node by a random full binary tree with k leaves (more precisely, a copy of $\mathcal{B}_{2k-1}$ conditioned on being a full binary tree); denote these trees by $T_1, \dots, T_{N'_k}$. Every fringe tree $\mathcal{B}_n^v$ that belongs to $\check{\mathfrak{T}}_t$, and thus has m leaves, either lies completely in $\mathcal{B}'_n$ or in one of the N'_k trees T_i; furthermore, any fringe tree of T_i that belongs to $\check{\mathfrak{T}}_t$ has to be a copy of t. Thus we have

$$N_t(\widehat{\mathcal{B}}_n) = N'' + \sum_{i=1}^{N'_k} N_t(T_i) \tag{4.12}$$

where N'' is determined by $\mathcal{B}'_n$. Condition on $\mathcal{B}'_n$ (which also determines N'_k). Then the trees T_i are (conditionally) independent and identically distributed, and, by our choice of k, $0 < \mathbb{P}(N_t(T_i) = 1) < 1$ so $c := \operatorname{Var} N_t(T_i) > 0$. Hence (4.12) implies that the conditional variance

$$\operatorname{Var}\bigl[N_t(\widehat{\mathcal{B}}_n) \mid \mathcal{B}'_n\bigr] = \sum_{i=1}^{N'_k} \operatorname{Var} N_t(T_i) = cN'_k. \tag{4.13}$$

The number N'_k equals the number of fringe trees of $\mathcal{B}_n$ that are full binary trees of size $2k - 1$; we let $\widehat{\mathfrak{T}}_k$ be the set of all such full binary trees. Thus

$$N'_k/n = \mathbb{P}\bigl(\mathcal{B}_n^* \in \widehat{\mathfrak{T}}_k \mid \mathcal{B}_n\bigr). \tag{4.14}$$

Hence we obtain from the known convergence $\mathcal{B}_n^* \xrightarrow{\mathrm{d}} \mathcal{B}_\infty^*$

$$\mathbb{E}\, N'_k/n = \mathbb{E}\, \mathbb{P}\bigl(\mathcal{B}_n^* \in \widehat{\mathfrak{T}}_k \mid \mathcal{B}_n\bigr) = \mathbb{P}\bigl(\mathcal{B}_n^* \in \widehat{\mathfrak{T}}_k\bigr) \to \mathbb{P}\bigl(\mathcal{B}_\infty^* \in \widehat{\mathfrak{T}}_k\bigr) =: c', \tag{4.15}$$

where it follows from (4.1) that $c' > 0$. Recall also the law of total variance: for any square-integrable random variable X and random variable (or σ-field) Y

$$\operatorname{Var} X = \mathbb{E}\, \operatorname{Var}[X \mid Y] + \operatorname{Var} \mathbb{E}\,[X \mid Y] \geqslant \mathbb{E}\, \operatorname{Var}[X \mid Y]. \tag{4.16}$$

Consequently, (4.13) and (4.15) yield

$$\operatorname{Var}\bigl[N_t(\widehat{\mathcal{B}}_n)\bigr] \geqslant \mathbb{E}\, \operatorname{Var}\bigl[N_t(\widehat{\mathcal{B}}_n) \mid \mathcal{B}'_n\bigr] = c\,\mathbb{E}\, N'_k = cc'n + o(n). \tag{4.17}$$

The convergence of variance in (4.2) thus yields $\gamma_t^2 \geqslant cc' > 0$. $\square$

Remark 4.2. Theorem 4.1 extends to multivariate limits for several full binary trees by the Cramér–Wold device; we omit the details. △

Remark 4.3. The random BST can be constructed by a simple continuous-time branching process, which yields a simple description of the limiting random fringe tree $\mathcal{B}^*_\infty$ as this branching process stopped at an exponentially distributed random time [1] (see also [21, Example 6.2]). In principle, this leads to a description of the compressed BST and its limiting fringe tree $\widehat{\mathcal{B}}^*_\infty$, but this becomes more complicated and we have not been able to use it, for example to compute β_t and the probabilities in (4.4); we therefore do not give the details. The rather complicated explicit values of β_t given in Appendix A for small t also suggest that no really simple description of $\widehat{\mathcal{B}}^*_\infty$ exists. △

For the random extended fringe tree $\widehat{\mathcal{B}}^{**}_n$ we obtain for the probabilities $q(\widehat{\mathcal{B}}_n;t)$ defined in Section 2.5:

Corollary 4.4. *Let t be a full binary tree. Then*

$$q(\widehat{\mathcal{B}}_n;t \mid \widehat{\mathcal{B}}_n) \xrightarrow{\mathrm{p}} q(\widehat{\mathcal{B}}^{**}_\infty;t) = 3|t|_{\mathsf{e}}\beta_t, \tag{4.18}$$

$$q(\widehat{\mathcal{B}}_n;t) \longrightarrow q(\widehat{\mathcal{B}}^{**}_\infty;t). \tag{4.19}$$

Proof. Immediate from (4.4)–(4.5) and Lemma 2.2. □

The numbers β_t are given by (4.7), but since $\check{\mathfrak{T}}_t$ is an infinite set, this formula is of limited use for explicit calculations. In the remainder of the section, we give one way to find β_t and thus the limiting fringe distribution $\widehat{\mathcal{B}}^*_\infty$ more explicitly.

Problem 4.5. It seems possible that similar but more complicated arguments might make it possible to compute also the asymptotic variance γ_t^2, but we have not pursued this, and we leave it as an open problem.

4.1. **Computing β_t.** We define a generating function for binary trees and sets of binary trees as follows. For a binary tree T, let

$$p_T := \mathbb{P}(\mathcal{B}_{|T|} = T), \tag{4.20}$$

$$F_T(x) := p_T x^{|T|}. \tag{4.21}$$

For any set $\mathfrak{T}_0$ of binary trees, let

$$F_{\mathfrak{T}_0}(x) := \sum_{T\in\mathfrak{T}_0} F_T(x). \tag{4.22}$$

These generating functions will help us to compute fringe tree probabilities by the following simple formula for the BST.

Lemma 4.6. *If $\mathfrak{T}_0$ is a set of binary trees, then*

$$\mathbb{P}(\mathcal{B}^*_\infty \in \mathfrak{T}_0) = 2\int_0^1 F_{\mathfrak{T}_0}(x)(1-x)\,\mathrm{d}x. \tag{4.23}$$

Proof. By (4.22) and linearity, it suffices to consider the case when $\mathfrak{T}_0 = \{T\}$ for a single binary tree T. Let $k := |T|$. Then, by (4.1),

$$\mathbb{P}(\mathcal{B}^*_\infty = T) = \frac{2}{(k+1)(k+2)}\mathbb{P}(\mathcal{B}_k = T) = 2p_T\int_0^1 x^k(1-x)\,\mathrm{d}x$$

$$= 2\int_0^1 F_T(x)(1-x)\,\mathrm{d}x, \tag{4.24}$$

which shows (4.23). ☐

Let t be a full binary tree, and recall that $\breve{\mathfrak{T}}_t^+$ is the set of all binary trees that contract to t, and $\breve{\mathfrak{T}}_t$ the subset of all such trees where the root has degree 2 or 0. In other words, $\breve{\mathfrak{T}}_t$ is the set of all binary trees that can be obtained from t by replacing any edge by a path, and $\breve{\mathfrak{T}}_t^+$ is the set of all binary trees that can be obtained from these by adding a path of length $\ell \geqslant 0$ to the root. Define the corresponding generating functions

$$G_t(x) := F_{\breve{\mathfrak{T}}_t^+}(x), \tag{4.25}$$

$$H_t(x) := F_{\breve{\mathfrak{T}}_t}(x). \tag{4.26}$$

We state a series of lemmas to help us compute these generating functions.

Lemma 4.7. *If T is a path with $l \geqslant 1$ nodes, then*

$$F_T(x) = \frac{x^\ell}{\ell!}. \tag{4.27}$$

Proof. By the construction of the BST,

$$p_T = \mathbb{P}(\mathcal{B}_\ell = T) = \frac{1}{\ell}\cdot\frac{1}{\ell-1}\cdots\frac{1}{1} = \frac{1}{\ell!}, \tag{4.28}$$

since a given path T is obtained by exactly one choice of pivot each time. Hence, (4.27) follows by the definition (4.21). ☐

Lemma 4.8. *If $\mathfrak{T}_P$ is the set consisting of all paths with any number $\ell \geqslant 1$ of nodes, then*

$$F_{\mathfrak{T}_P}(x) = \tfrac{1}{2}\bigl(e^{2x}-1\bigr). \tag{4.29}$$

Proof. There are $2^{\ell-1}$ different paths with ℓ nodes. Thus Lemma 4.7 yields, letting P_ℓ denote any path with $|P_\ell| = \ell$,

$$F_{\mathfrak{T}_P}(x) = \sum_{\ell=1}^{\infty} 2^{\ell-1}F_{P_\ell}(x) = \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(2x)^\ell}{\ell!} = \frac{1}{2}\bigl(e^{2x}-1\bigr). \tag{4.30}$$

☐

Lemma 4.9. *Let T be a binary tree and let T_1 be a tree obtained by adding a path with $\ell \geqslant 1$ nodes to the root of T. Then*

$$F_{T_1}(x) = \int_0^x F_T(y)\frac{(x-y)^{\ell-1}}{(\ell-1)!}\,\mathrm{d}y. \tag{4.31}$$

Proof. Let $|T| = k$; then $|T_1| = k+\ell$. In analogy with (4.28), the construction of the BST yields

$$\begin{aligned} p_{T_1} &= \mathbb{P}(\mathcal{B}_{k+\ell} = T_1) = \frac{1}{k+\ell}\cdot\frac{1}{k+\ell-1}\cdots\frac{1}{k+1}\cdot\mathbb{P}(\mathcal{B}_k = T) \\ &= \frac{k!}{(k+\ell)!}p_T, \end{aligned} \tag{4.32}$$

since T_1 is obtained by exactly one choice of pivot each of the first ℓ times, and then by the same choices as for T. On the other hand, a standard Beta integral yields

$$\begin{aligned}\int_0^x F_T(y)(x-y)^{\ell-1}\,\mathrm{d}y &= p_T\int_0^x y^k(x-y)^{\ell-1}\,\mathrm{d}y = p_T x^{k+\ell}\int_0^1 z^k(1-z)^{\ell-1}\,\mathrm{d}z\\ &= p_T\frac{\Gamma(k+1)\Gamma(\ell)}{\Gamma(k+1+\ell)}x^{k+\ell} = p_T\frac{k!\,(\ell-1)!}{(k+\ell)!}x^{k+\ell}.\end{aligned} \tag{4.33}$$

By (4.32) and (4.21), this equals $(\ell-1)!\,x^{k+\ell}p_{T_1}$, and thus (4.31) follows by the definition (4.21). ☐

Lemma 4.10. *Let $\mathfrak{T}_0$ be a set of binary trees and let $\mathfrak{T}_1$ be the set of trees obtained by adding a path with any number $\ell \geqslant 1$ of nodes to the root of any tree $T \in \mathfrak{T}_0$. Then*

$$F_{\mathfrak{T}_1}(x) = 2\int_0^x F_{\mathfrak{T}_0}(y)e^{2(x-y)}\,\mathrm{d}y. \tag{4.34}$$

Proof. For each ℓ, there are 2^ℓ different paths of length ℓ that can be added (including the choice of edge from the end of the path to the former root). Hence, by Lemma 4.9, summing over all $T \in \mathfrak{T}_0$ and all possible paths,

$$F_{\mathfrak{T}_0}(x) = \sum_{T\in\mathfrak{T}_0}\sum_{\ell\geqslant 1}2^\ell\int_0^x F_T(y)\frac{(x-y)^{\ell-1}}{(\ell-1)!}\,\mathrm{d}y = 2\int_0^x F_{\mathfrak{T}_0}(y)e^{2(x-y)}\,\mathrm{d}y. \tag{4.35}$$

☐

Lemma 4.11. *Let T_L and T_R be binary trees and let T_1 be a tree obtained by taking a path with $\ell \geqslant 1$ nodes and adding T_L and T_R as the left and right subtrees of the last node in the path. Then*

$$F_{T_1}(x) = \int_0^x F_{T_\mathsf{L}}(y)F_{T_\mathsf{R}}(y)\frac{(x-y)^{\ell-1}}{(\ell-1)!}\,\mathrm{d}y. \tag{4.36}$$

Proof. This is similar to the proof of Lemma 4.9. Let $|T_\mathsf{L}| = k_\mathsf{L}$ and $|T_\mathsf{R}| = k_\mathsf{R}$; then $|T_1| = k_\mathsf{L}+k_\mathsf{R}+\ell$. The same argument as for (4.32) now yields

$$\begin{aligned}p_{T_1} &= \mathbb{P}(\mathcal{B}_{k_\mathsf{L}+k_\mathsf{R}+\ell} = T_1) = \frac{1}{k_\mathsf{L}+k_\mathsf{R}+\ell}\cdots\frac{1}{k_\mathsf{L}+k_\mathsf{R}+1}\cdot\mathbb{P}(\mathcal{B}_{k_\mathsf{L}} = T_\mathsf{L})\cdot\mathbb{P}(\mathcal{B}_{k_\mathsf{R}} = T_\mathsf{R})\\ &= \frac{(k_\mathsf{L}+k_\mathsf{R})!}{(k_\mathsf{L}+k_\mathsf{R}+\ell)!}p_{T_\mathsf{L}}p_{T_\mathsf{R}},\end{aligned} \tag{4.37}$$

since if the first ℓ pivots have been chosen correctly to form the desired path, with the last node having left and right subtrees of sizes k_L and k_R, respectively, then the shapes of those subtrees are independent. The rest of the proof is as for Lemma 4.9 with only notational changes. ☐

Lemma 4.12. *Let $\mathfrak{T}_\mathsf{L}$ and $\mathfrak{T}_\mathsf{R}$ be two sets of binary trees and let $\mathfrak{T}_1$ be the set of trees obtained by taking a path with any number $\ell \geqslant 1$ of nodes and adding two trees $T_\mathsf{L} \in \mathfrak{T}_\mathsf{L}$ and $T_\mathsf{R} \in \mathfrak{T}_\mathsf{R}$ as the left and right subtrees of the last node in the path. Then*

$$F_{\mathfrak{T}_1}(x) = \int_0^x F_{\mathfrak{T}_\mathsf{L}}(y)F_{\mathfrak{T}_\mathsf{R}}(y)e^{2(x-y)}\,\mathrm{d}y. \tag{4.38}$$

Proof. By Lemma 4.11 and summing over $T_\mathsf{L} \in \mathfrak{T}_\mathsf{L}$, $T_\mathsf{R} \in \mathfrak{T}_\mathsf{R}$, and $\ell \geqslant 1$, just as in the proof of Lemma 4.10; note that for a given ℓ, now there are $2^{\ell-1}$ paths. ☐

Finally, we obtain our formula for β_t, using the functions $G_t(x)$ for which we provide a recursion.

Theorem 4.13. *Let t be a full binary tree.*

(a) *The generating function $G_t(x)$ can be computed recursively as follows.*
  (i) *If $t = \bullet$, then*

$$G_\bullet(x) = \frac{1}{2}(e^{2x} - 1). \tag{4.39}$$

  (ii) *If $|t| > 1$, and the root of t has left and right subtrees t_L and t_R, then*

$$G_t(x) = \int_0^x G_{t_\mathsf{L}}(y) G_{t_\mathsf{R}}(y) e^{2(x-y)} \,\mathrm{d}y. \tag{4.40}$$

(b) *Then β_t is given by:*
  (i) *If $|t| = 1$, i.e., $t = \bullet$, then $\beta_t = 1/3$.*
  (ii) *If $|t| > 1$, and the root of t has left and right subtrees t_L and t_R, then*

$$\beta_t = \int_0^1 (1-x)^2 G_{t_\mathsf{L}}(x) G_{t_\mathsf{R}}(x) \,\mathrm{d}x. \tag{4.41}$$

Proof. (a): The recursion (4.39)–(4.40) is an immediate consequence of the definition (4.25) and Lemmas 4.8 and 4.12.

(b)(i): If $t = \bullet$, then $N_t(\widehat{\mathcal{B}}_n) = |\widehat{\mathcal{B}}_n|_\mathsf{e}$, and thus (2.6) and (4.3) show $\beta_t = 1/3$.

(b)(ii): Lemma 4.11 with $\ell = 1$ shows, by summing over all $T_\mathsf{L} \in \check{\mathfrak{T}}^+_{t_\mathsf{L}}$ and $T_\mathsf{R} \in \check{\mathfrak{T}}^+_{t_\mathsf{R}}$, and recalling (4.26),

$$H_t(x) = F_{\check{\mathfrak{T}}_t}(x) = \int_0^x G_{t_\mathsf{L}}(y) G_{t_\mathsf{R}}(y) \,\mathrm{d}y. \tag{4.42}$$

Hence, (4.8) and Lemma 4.6 yield

$$\begin{aligned}\beta_t &= \mathbb{P}(\mathcal{B}^*_\infty \in \check{\mathfrak{T}}_t) = 2\int_0^1 H_t(x)(1-x)\,\mathrm{d}x = 2 \iint\limits_{0 \leqslant y \leqslant x \leqslant 1} (1-x) G_{t_\mathsf{L}}(y) G_{t_\mathsf{R}}(y) \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 (1-y)^2 G_{t_\mathsf{L}}(y) G_{t_\mathsf{R}}(y)\,\mathrm{d}y, \end{aligned} \tag{4.43}$$

which proves (4.41). (Alternatively, one can use integration by parts in (4.43).) ☐

Remark 4.14. Note that (4.40) and (4.41) hold for $t = \bullet$ too if we define $G_\varnothing := 1$.

Furthermore, (4.41), (4.40), and an integration by parts yield the alternative formula

$$\begin{aligned}\beta_t &= \int_0^1 (1-x)^2 e^{2x} \cdot G_{t_\mathsf{L}}(x) G_{t_\mathsf{R}}(x) e^{-2x} \,\mathrm{d}x = -\int_0^1 \Bigl((1-x)^2 e^{2x}\Bigr)' \cdot \Bigl(e^{-2x} G_t(x)\Bigr)\,\mathrm{d}x \\ &= \int_0^1 2x(1-x)e^{2x} \cdot e^{-2x} G_t(x)\,\mathrm{d}x = \int_0^1 2x(1-x) G_t(x)\,\mathrm{d}x. \end{aligned} \tag{4.44}$$

This is perhaps more elegant than (4.41), but (4.41) seems better for calculations. △

Appendix A. Examples

We give here some examples of exact and numerical values for the limits of $\mathbb{P}(T_n^* = t)$ and $q(T_n;t)$ that describe the distribution of random fringe trees and extended fringe trees when T_n is a random Patricia trie $\widehat{\Upsilon}_n$ or a compressed binary search tree $\widehat{\mathcal{B}}_n$. We consider only the smallest fringe trees t, with $|t|_{\mathsf{e}} \leqslant 4$. Since we consider only full binary trees T_n, we assume that t is a full binary tree; moreover, the case $|t|_{\mathsf{e}} = 1$ is trivial, since then for any sequence T_n of full binary trees

$$\mathbb{P}(T_n^* = \bullet \mid T_n) = \frac{|T_n|_{\mathsf{e}}}{|T_n|} = \frac{|T_n|_{\mathsf{e}}}{2|T_n|_{\mathsf{e}} - 1} \to \frac{1}{2} \tag{A.1}$$

provided $|T_n|_{\mathsf{e}} \to \infty$. Hence we consider the small trees in Figure 1 with the notations given there, and their mirror images which we may ignore since they give the same result, with p and q exchanged for $\widehat{\Upsilon}_n$. For simplicity, we state only the annealed versions of the results; note that the (stronger) quenched results too hold by the results above. Also for simplicity, for Patricia tries, we do not show the oscillations in that appear in the periodic case (in particular in the symmetric case $p = q = \frac{1}{2}$); we give only the constant terms coming from the constant term $f_{\mathsf{E}}^*(-1)$ in (3.8) and ignore the oscillating terms there (which is the same as averaging ψ_{E} over a period). We denote this by $\approx$ below; note that in the aperiodic case, $\approx$ thus means $\to$. We use freely notation from the previous sections.

Remark A.1. If we ignore orientations and regard the random trees as cladograms, see Remark 1.1, and for Patricia tries consider only the symmetric case $p = \frac{1}{2}$, then we do not have to consider mirror images, and also t_{4b} disappears; instead we should multiply the results below for t_3 by 2, and for t_{4a} by 4 (the numbers of possible orientations). △

A.1. **The uniform full binary tree.** For comparison, we give also the corresponding values for the uniform random full binary tree with n leaves, here denoted $\mathcal{U}_n$. (Note that this random tree is quite different from $\widehat{\Upsilon}_n$ and $\widehat{\mathcal{B}}_n$; for example, as is well known, typically $\mathcal{U}_n$ has height of order $\sqrt{n}$, while $\widehat{\Upsilon}_n$ and $\widehat{\mathcal{B}}_n$ have heights of order logn.)

The random full binary tree $\mathcal{U}_n$ can be regarded as a conditioned Galton–Watson tree with critical offspring distribution $\mathbb{P}(\xi = 0) = \mathbb{P}(\xi = 2) = \frac{1}{2}$, see e.g. [2], and thus it follows by Aldous [1, Lemma 9] (see also [27]) that the asymptotic fringe tree distribution is the corresponding (unconditioned) Galton–Watson tree, which in this case simply means that for every full binary tree t,

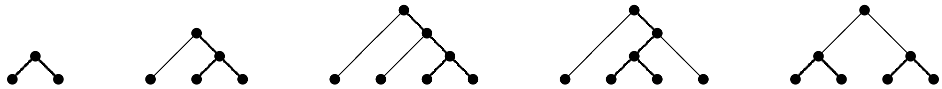
$$\mathbb{P}(\mathcal{U}_n^* = t) \to 2^{-|t|} = 2^{1-2|t|_{\mathsf{e}}}. \tag{A.2}$$

We have also the quenched version [26, Theorem 7.12] $\mathbb{P}(\mathcal{U}_n^* = t \mid \mathcal{U}_n) \to 2^{-|t|}$, and thus Lemma 2.2 yields

$$q(\mathcal{U}_n;t) \to |t|_{\mathsf{e}} 2^{1-|t|} = |t|_{\mathsf{e}} 2^{2-2|t|_{\mathsf{e}}}. \tag{A.3}$$

A.2. **The examples.** We consider the trees $t_2, \dots, t_{4c}$ in Figure 1 one by one; for each of them we consider the different random fringe trees studied above.

Example A.2. $\mathbf{t_2}$: For the Patricia trie $\widehat{\Upsilon}_n$, note first that we have $|t_2|_{\mathsf{e}} = 2$, $\mathsf{LPL}(t_2) = \mathsf{RPL}(t_2) = 1$, $\nu_1(t_2) = 2$, $\nu_2(t_2) = 1$, and $\nu_k(t_2) = 0$, $k > 2$. Hence,



t_2 t_3 t_{4a} t_{4b} t_{4c}

FIGURE 1. Some small full binary trees.

Lemma 3.3 yields

$$\pi_{t_2} = 2pq. \tag{A.4}$$

(Which perhaps is more easily seen directly.) Corollaries 3.7 and 3.8 then yield

$$\mathbb{P}(\widehat{\Upsilon}^*_n = t_2) \approx \frac{pq}{2H}, \tag{A.5}$$

$$q(\widehat{\Upsilon}_n; t_2) \approx \frac{2pq}{H}. \tag{A.6}$$

In particular, in the symmetric case $p = \frac{1}{2}$, when $H = \log 2$,

$$\mathbb{P}(\widehat{\Upsilon}^*_n = t_2) \approx \frac{1}{8\log 2} \doteq 0.1803, \tag{A.7}$$

$$q(\widehat{\Upsilon}_n; t_2) \approx \frac{1}{2\log 2} \doteq 0.7213. \tag{A.8}$$

For the compressed BST $\widehat{\mathcal{B}}_n$, Theorem 4.13 yields $G_{t_2}(x) = \frac{1}{8}e^{4x} - \frac{1}{2}xe^{2x} - \frac{1}{8}$ and

$$\beta_{t_2} = \tfrac{1}{128}e^4 - \tfrac{1}{8}e^2 + \tfrac{233}{384} \doteq 0.1097, \tag{A.9}$$

Hence, Theorem 4.1 and Lemma 2.2 yield

$$\mathbb{P}(\widehat{\mathcal{B}}^*_n = t_2) \to \tfrac{3}{2}\beta_{t_2} = \tfrac{3}{256}e^4 - \tfrac{3}{16}e^2 + \tfrac{233}{256} \doteq 0.1645. \tag{A.10}$$

$$q(\widehat{\mathcal{B}}_n; t_2) \to 6\beta_{t_2} = \tfrac{3}{64}e^4 - \tfrac{3}{4}e^2 + \tfrac{233}{64} \doteq 0.6581. \tag{A.11}$$

For the uniformly random full binary tree $\mathcal{U}_n$, (A.2)–(A.3) yield

$$\mathbb{P}(\mathcal{U}^*_n = t_2) \to \frac{1}{8} = 0.125, \tag{A.12}$$

$$q(\mathcal{U}_n; t_2) \to \frac{1}{2} = 0.5. \tag{A.13}$$

$\triangle$

Example A.3. $\mathbf{t_3}$: For $\widehat{\Upsilon}_n$, we have $|t_3|_{\mathsf{e}} = 3$, $\mathsf{LPL}(t_3) = 2$, $\mathsf{RPL}(t_3) = 3$, and $\nu_1(t_3) = 3$, $\nu_2(t_3) = 1$, $\nu_3(t_3) = 1$. Hence, Lemma 3.3 yields

$$\pi_{t_3} = \frac{6p^3q^2}{1-p^2-q^2} = \frac{6p^3q^2}{2pq} = 3p^2q. \tag{A.14}$$

Corollaries 3.7 and 3.8 then yield

$$\mathbb{P}(\widehat{\Upsilon}^*_n = t_3) \approx \frac{p^2q}{4H}, \tag{A.15}$$

$$q(\widehat{\Upsilon}_n; t_3) \approx \frac{3p^2q}{2H}. \tag{A.16}$$

In particular, in the symmetric case $p = \frac{1}{2}$, when $H = \log 2$,

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t_3) \approx \frac{1}{32\log 2} \doteq 0.0451, \tag{A.17}$$

$$q(\widehat{\Upsilon}_n; t_3) \approx \frac{3}{16\log 2} \doteq 0.2705. \tag{A.18}$$

For $\widehat{\mathcal{B}}_n$, Theorem 4.13 yields

$$\beta_{t_3} = \frac{1}{1728}e^6 - \frac{1}{256}e^4 - \frac{3}{64}e^2 + \frac{2447}{6912} \doteq 0.0279. \tag{A.19}$$

Hence, Theorem 4.1 and Lemma 2.2 yield

$$\mathbb{P}(\widehat{\mathcal{B}}_n^* = t_3) \to \frac{3}{2}\beta_{t_3} = \frac{1}{1152}e^6 - \frac{3}{512}e^4 - \frac{9}{128}e^2 + \frac{2447}{4608} \doteq 0.0418, \tag{A.20}$$

$$q(\widehat{\mathcal{B}}_n; t_3) \to 9\beta_{t_3} = \frac{1}{192}e^6 - \frac{9}{256}e^4 - \frac{27}{64}e^2 + \frac{2447}{768} \doteq 0.2507. \tag{A.21}$$

For $\mathcal{U}_n$, (A.2)–(A.3) yield

$$\mathbb{P}(\mathcal{U}_n^* = t_3) \to \frac{1}{32} = 0.03125, \tag{A.22}$$

$$q(\mathcal{U}_n; t_3) \to \frac{3}{16} = 0.1875. \tag{A.23}$$

$\triangle$

Example A.4. $\mathbf{t_{4a}}$: For $\widehat{\Upsilon}_n$, we have $|t_{4a}|_{\mathsf{e}} = 4$, $\mathsf{LPL}(t_{4a}) = 3$, $\mathsf{RPL}(t_{4a}) = 6$, and $\nu_1(t_{4a}) = 4$, $\nu_2(t_{4a}) = 1$, $\nu_3(t_{4a}) = 1$, $\nu_4(t_{4a}) = 1$. Hence, Lemma 3.3 yields

$$\pi_{t_{4a}} = \frac{24p^6q^3}{(1-p^2-q^2)(1-p^3-q^3)} = \frac{24p^6q^3}{2pq\cdot 3pq} = 4p^4q. \tag{A.24}$$

Corollaries 3.7 and 3.8 then yield

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t_{4a}) \approx \frac{p^4q}{6H}, \tag{A.25}$$

$$q(\widehat{\Upsilon}_n; t_{4a}) \approx \frac{4p^4q}{3H}. \tag{A.26}$$

In particular, in the symmetric case $p = \frac{1}{2}$, when $H = \log 2$,

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t_{4a}) \approx \frac{1}{192\log 2} \doteq 0.0075, \tag{A.27}$$

$$q(\widehat{\Upsilon}_n; t_{4a}) \approx \frac{1}{24\log 2} \doteq 0.0601. \tag{A.28}$$

For $\widehat{\mathcal{B}}_n$, Theorem 4.13 yields

$$\beta_{t_{4a}} = \frac{1}{32768}e^8 - \frac{1}{4608}e^6 - \frac{11}{512}e^2 + \frac{47503}{294912} \doteq 0.0057. \tag{A.29}$$

Hence, Theorem 4.1 and Lemma 2.2 yield

$$\mathbb{P}(\widehat{\mathcal{B}}_n^* = t_{4a}) \to \frac{3}{2}\beta_{t_{4a}} = \frac{3}{65536}e^8 - \frac{1}{3072}e^6 - \frac{33}{1024}e^2 + \frac{47503}{196608} \doteq 0.0086, \tag{A.30}$$

$$q(\widehat{\mathcal{B}}_n; t_{4a}) \to 12\beta_{t_{4a}} = \frac{3}{8192}e^8 - \frac{1}{384}e^6 - \frac{33}{128}e^2 + \frac{47503}{24576} \doteq 0.0690. \tag{A.31}$$

For $\mathcal{U}_n$, (A.2)–(A.3) yield

$$\mathbb{P}(\mathcal{U}_n^* = t_{4a}) \to \frac{1}{128} \doteq 0.0078, \tag{A.32}$$

$$q(\mathcal{U}_n; t_{4a}) \to \frac{1}{16} = 0.0625. \tag{A.33}$$

$\triangle$

Example A.5. $\mathbf{t_{4b}}$: For $\widehat{\Upsilon}_n$, we have $|t_{4b}|_{\mathsf{e}} = 4$, $\mathsf{LPL}(t_{4b}) = 4$, $\mathsf{RPL}(t_{4b}) = 5$, and $\nu_1(t_{4b}) = 4$, $\nu_2(t_{4b}) = 1$, $\nu_3(t_{4b}) = 1$, $\nu_4(t_{4b}) = 1$. Hence, Lemma 3.3 yields

$$\pi_{t_{4b}} = \frac{24p^5q^4}{(1-p^2-q^2)(1-p^3-q^3)} = \frac{24p^5q^4}{2pq \cdot 3pq} = 4p^3q^2. \tag{A.34}$$

Corollaries 3.7 and 3.8 then yield

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t_{4b}) \approx \frac{p^3q^2}{6H}, \tag{A.35}$$

$$q(\widehat{\Upsilon}_n; t_{4b}) \approx \frac{4p^3q^2}{3H}. \tag{A.36}$$

All other results are by symmetry the same as for t_{4b}. $\triangle$

Example A.6. $\mathbf{t_{4c}}$: For $\widehat{\Upsilon}_n$, we have $|t_{4c}|_{\mathsf{e}} = 4$, $\mathsf{LPL}(t_{4c}) = \mathsf{RPL}(t_{4c}) = 4$, and $\nu_1(t_{4c}) = 4$, $\nu_2(t_{4c}) = 2$, $\nu_3(t_{4c}) = 0$, $\nu_4(t_{4c}) = 1$. Hence, Lemma 3.3 yields

$$\pi_{t_{4c}} = \frac{24p^4q^4}{(1-p^2-q^2)^2} = \frac{24p^4q^4}{(2pq)^2} = 6p^2q^2. \tag{A.37}$$

Corollaries 3.7 and 3.8 then yield

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t_{4c}) \approx \frac{p^2q^2}{4H}, \tag{A.38}$$

$$q(\widehat{\Upsilon}_n; t_{4c}) \approx \frac{2p^2q^2}{H}. \tag{A.39}$$

In particular, in the symmetric case $p = \frac{1}{2}$, when $H = \log 2$,

$$\mathbb{P}(\widehat{\Upsilon}_n^* = t_{4c}) \approx \frac{1}{64 \log 2} \doteq 0.0225, \tag{A.40}$$

$$q(\widehat{\Upsilon}_n; t_{4c}) \approx \frac{1}{8 \log 2} \doteq 0.1803. \tag{A.41}$$

For $\widehat{\mathcal{B}}_n$, Theorem 4.13 yields

$$\beta_{t_{4c}} = \frac{1}{16384}e^8 - \frac{1}{1728}e^6 + \frac{1}{1024}e^4 - \frac{1}{64}e^2 + \frac{54973}{442368} \doteq 0.0106. \tag{A.42}$$

Hence, Theorem 4.1 and Lemma 2.2 yield

$$\mathbb{P}(\widehat{\mathcal{B}}_n^* = t_{4c}) \to \frac{3}{2}\beta_{t_{4c}} = \frac{3}{32768}e^8 - \frac{1}{1152}e^6 + \frac{3}{2048}e^4 - \frac{3}{128}e^2 + \frac{54973}{294912} \doteq 0.0159, \tag{A.43}$$

$$q(\widehat{\mathcal{B}}_n; t_{4c}) \to 12\beta_{t_{4c}} = \frac{3}{4096}e^8 - \frac{1}{144}e^6 + \frac{3}{256}e^4 - \frac{3}{16}e^2 + \frac{54973}{36864} \doteq 0.1273. \tag{A.44}$$

For $\mathcal{U}_n$, (A.2)–(A.3) yield (just as for t_{4a})

$$\mathbb{P}(\mathcal{U}_n^* = t_{4c}) \to \frac{1}{128} \doteq 0.0078, \tag{A.45}$$

$$q(\mathcal{U}_n; t_{4c}) \to \frac{1}{16} = 0.0625. \tag{A.46}$$

△

We summarize the numerical values above in Tables 1 and 2. In particular, note the large differences in the relative importance of t_{4a} and t_{4c} for the three random full binary trees considered here: the asymptotic ratio between the probabilities for t_{4c} and t_{4a} are 3 for symmetric Patricia tries (the oscillations cancel, see Remark 3.11), 1.846... for compressed BST, and 1 for uniform full binary trees.

	t_2	t_3	t_{4a}	t_{4c}
Patricia trie $\widehat{\Upsilon}_n$ ($p = q = \frac{1}{2}$)	0.1803	0.0451	0.0075	0.0225
Compressed BST $\widehat{\mathcal{B}}_n$	0.1645	0.0418	0.0086	0.0159
Uniform full binary tree $\mathcal{U}_n$	0.125	0.0312	0.0078	0.0078

TABLE 1. Limits or approximations of $\mathbb{P}(T_n^* = t)$ for three random full binary trees.

	t_2	t_3	t_{4a}	t_{4c}
Patricia trie $\widehat{\Upsilon}_n$ ($p = q = \frac{1}{2}$)	0.7213	0.2705	0.0601	0.1803
Compressed BST $\widehat{\mathcal{B}}_n$	0.6581	0.2507	0.0690	0.1273
Uniform full binary tree $\mathcal{U}_n$	0.5	0.1875	0.0625	0.0625

TABLE 2. Limits or approximations of $q(T_n; t)$ for three random full binary trees.

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Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

Email address: `svante.janson@math.uu.se`