# Generalized cyclotomic polynomials associated with regular systems of divisors and arbitrary sets of positive integers 

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#### Abstract

We introduce and study the generalized cyclotomic polynomials $\Phi_{A, S, n}(x)$ associated with a regular system $A$ of divisors and an arbitrary set $S$ of positive integers. We show that all of these polynomials have integer coefficients, they can be expressed as the product of certain classical cyclotomic polynomials $\Phi_{d}(x)$ with $d \mid n$, and enjoy many other properties which are similar to the classical and unitary cases. We also point out some related Menon-type identities. One of them seems to be new even for the cyclotomic polynomials $\Phi_{n}(x)$.


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## 1 Introduction

The cyclotomic polynomials $\Phi_{n}(x)$ are defined by

$$
\Phi_{n}(x)=\prod_{\substack{j=1 \\(j, n)=1}}^{n}\left(x-\zeta_{n}^{j}\right),
$$

where $\zeta_{n}:=e^{2 \pi i / n}, n \in \mathbb{N}:=\{1,2, \ldots\}$.
We recall that a divisor $d$ of $n(d, n \in \mathbb{N})$ is a unitary divisor if $(d, n / d)=1$, notation $d \| n$. The unitary cyclotomic polynomials $\Phi_{n}^{*}(x)$ are defined using unitary divisors, as follows:

$$
\Phi_{n}^{*}(x)=\prod_{\substack{j=1 \\(j, n)_{*}=1}}^{n}\left(x-\zeta_{n}^{j}\right)
$$

where $(j, n)_{*}=\max \{d \in \mathbb{N}: d \mid j, d \| n\}$. These polynomials have properties similar to the classical cyclotomic polynomials. For example, for every $n \in \mathbb{N}$,

$$
\begin{equation*}
x^{n}-1=\prod_{d \| n} \Phi_{d}^{*}(x) \tag{1.1}
\end{equation*}
$$

and

$$
\Phi_{n}^{*}(x)=\prod_{d \| n}\left(x^{d}-1\right)^{\mu^{*}(n / d)}
$$

where $\mu^{*}(n)=(-1)^{\omega(n)}$ is the unitary Möbius function, $\omega(n)$ denoting the number of distinct prime divisors of $n$. Also, the unitary cyclotomic polynomials are in the class of inclusionexclusion polynomials. See Jones et al. [7], Moree and the author [12] for a detailed discussion of these and some related properties. Also see Bachman [1].

However, not all polynomials $\Phi_{n}^{*}(x)$ are irreducible over the rationals. More exactly, for every $n \in \mathbb{N}$ one has the irreducible factorization

$$
\begin{equation*}
\Phi_{n}^{*}(x)=\prod_{\substack{d \mid n \\ \kappa(d)=\kappa(n)}} \Phi_{d}(x) \tag{1.2}
\end{equation*}
$$

where $\kappa(n)=\prod_{p \mid n} p$ is the square-free kernel of $n$. See [12, Th. 2]. It follows from (1.2) that $\Phi_{n}^{*}(x)$ have integer coefficients.

The polynomials

$$
\begin{equation*}
Q_{n}(x)=\prod_{\substack{j=1 \\(j, n) \text { a square }}}^{n}\left(x-\zeta_{n}^{j}\right) \tag{1.3}
\end{equation*}
$$

have been discussed by Sivaramakrishnan [18, Sect. X.5]. For every $n \in \mathbb{N}$ we have the identities

$$
\begin{align*}
& x^{n}-1=\prod_{\substack{d \mid n \\
n / d \text { squarefree }}} Q_{d}(x)  \tag{1.4}\\
& Q_{n}(x)=\prod_{d \mid n}\left(x^{d}-1\right)^{\lambda(n / d)},
\end{align*}
$$

where $\lambda(n)=(-1)^{\Omega(n)}$ is the Liouville function, $\Omega(n)$ denoting the number of distinct prime power divisors of $n$. At the same time,

$$
Q_{n}(x)=\prod_{d^{2} \mid n} \Phi_{n / d^{2}}(x)
$$

hence $Q_{n}(x)$ have integer coefficients as well.
The inverse cyclotomic polynomials are defined by

$$
\Psi_{n}(x)=\prod_{\substack{j=1 \\(j, n)>1}}^{n}\left(x-\zeta_{n}^{j}\right)
$$

see Moree [11]. One has

$$
\Psi_{n}(x)=\frac{x^{n}-1}{\Phi_{n}(x)}=\prod_{\substack{d \mid n \\ d<n}} \Phi_{d}(x) .
$$

It is the goal of the present paper to investigate a common generalization of all of the above polynomials. Let $A=(A(n))_{n \in \mathbb{N}}$ be a regular system of divisors. See Section 2.1 for its definition and some basic properties. In particular, $D=(D(n))_{n \in \mathbb{N}}$ and $U=(U(n))_{n \in \mathbb{N}}$ are regular, where $D(n)$ is the set of all divisors of $n$ and $U(n)$ is the set of unitary divisors of $n$. Furthermore, let $S$ be an arbitrary nonempty subset of $\mathbb{N}$. We define the cyclotomic polynomials $\Phi_{A, S, n}(x)$ by

$$
\begin{equation*}
\Phi_{A, S, n}(x)=\prod_{\substack{j=1 \\(j, n)_{A} \in S}}^{n}\left(x-\zeta_{n}^{j}\right), \tag{1.5}
\end{equation*}
$$

where

$$
\begin{equation*}
(j, n)_{A}=\max \{d \in \mathbb{N}: d \mid j, d \in A(n)\} . \tag{1.6}
\end{equation*}
$$

If $S=\{1\}$ and $A$ is a regular system of divisors, then (1.5) reduces to the polynomials

$$
\begin{equation*}
\Phi_{A, n}(x):=\Phi_{A,\{1\}, n}(x)=\prod_{\substack{j=1 \\(j, n)_{A}=1}}^{n}\left(x-\zeta_{n}^{j}\right), \tag{1.7}
\end{equation*}
$$

introduced by Nageswara Rao [13]. Here (1.7) recovers $\Phi_{n}(x)$ and $\Phi_{n}^{*}(x)$ if $A(n)=D(n)$ and $A(n)=U(n)$, respectively. If $A(n)=D(n)$ and $S$ is arbitrary, then (1.5) are the polynomials investigated by the author [19], which recover the polynomials $Q_{n}(x)$ and $\Psi_{n}(x)$ if $S$ is the set of squares and $S=\mathbb{N} \backslash\{1\}$, respectively.

Note that generalizations of arithmetic functions, in particular Euler's function and Ramanujan sums, associated to regular systems of divisors and arbitrary sets are known in the literature, see Section 2.2. However, the corresponding cyclotomic polynomials have not been considered, as far as we know. Further generalizations can also be studied. Given a regular system $A$ of divisors and $k \in \mathbb{N}$ one can consider the system $A_{k}=\left(A_{k}(n)\right)_{n \in \mathbb{N}}$ with $A_{k}(n)=\left\{d \in \mathbb{N}: d^{k} \in A\left(n^{k}\right)\right\}$ and the cyclotomics defined with respect to $A_{k}$. See Section 2.1 for some more details. However, we confine ourselves with the previous generalizations.

We show that all polynomials $\Phi_{A, S, n}(x)$ have integer coefficients and they can be expressed as the product of certain polynomials $\Phi_{d}(x)$ with $d \mid n$. We also point out some other properties and identities concerning the polynomials $\Phi_{A, S, n}(x)$ and $\Phi_{A, n}(x)$. One of them seems to be new even for the classical cyclotomic polynomials. Namely, let $n \in \mathbb{N}$ and let $\chi$ be an arbitrary Dirichlet character $(\bmod n)$ with conductor $d(d \mid n)$. Then for real $x>1$ (or formally),

$$
\begin{equation*}
\prod_{j=1}^{n}\left(x^{(j-1, n)}-1\right)^{\operatorname{Re}(\chi(j))}=\prod_{\delta \mid n / d} \Phi_{d \delta}(x)^{\varphi(n) / \varphi(d \delta)}, \tag{1.8}
\end{equation*}
$$

where $\varphi$ is Euler's function. If $\chi$ is a primitive character $(\bmod n)$, then $d=n$, and $(1.8)$ gives

$$
\begin{equation*}
\Phi_{n}(x)=\prod_{j=1}^{n}\left(x^{(j-1, n)}-1\right)^{\operatorname{Re}(\chi(j))} . \tag{1.9}
\end{equation*}
$$

Note that the products in (1.8) and (1.9) are, in fact, over $j$ with $(j, n)=1$, since $\chi(j)=0$ for $(j, n)>1$. These are Menon-type identities, involving $(j-1, n)$, where $j$ runs over a reduced residue system $(\bmod n)$. The original Menon identity reads

$$
\begin{equation*}
\sum_{\substack{j=1 \\(j, n)=1}}^{n}(j-1, n)=\tau(n) \varphi(n) \quad(n \in \mathbb{N}) \tag{1.10}
\end{equation*}
$$

where $\tau(n)$ is the number of divisors of $n$. See the survey by the author [24].
We also deduce generalizations of the Möller-Endo identities and the Grytczuk-Tropak recursion formula concerning the coefficients of the discussed cyclotomic polynomials. The main results are included in Section 3 and their proofs are given in Section 4.

For an overview of properties of the classical cyclotomic polynomials, in particular those discussed and generalized in the present paper, we refer to Gallot et al. [4], Herrera-Poyatos and Moree [6], Sanna [15].

## 2 Preliminaries

### 2.1 Regular systems of divisors

Let $A(n)$ be a subset of the set of positive divisors of $n$ for every $n \in \mathbb{N}$. The $A$-convolution of the functions $f, g: \mathbb{N} \rightarrow \mathbb{C}$ is defined by

$$
\begin{equation*}
\left(f *_{A} g\right)(n)=\sum_{d \in A(n)} f(d) g(n / d) \quad(n \in \mathbb{N}) . \tag{2.1}
\end{equation*}
$$

Let $\mathcal{A}$ denote the set of arithmetic functions $f: \mathbb{N} \rightarrow \mathbb{C}$. The convolution (2.1) and the system $A=(A(n))_{n \in \mathbb{N}}$ of divisors are called regular, cf. Narkiewicz [14], if the following conditions hold true:
(a) $\left(\mathcal{A},+, *_{A}\right)$ is a commutative ring with unity,
(b) the $A$-convolution of multiplicative functions is multiplicative,
(c) the constant 1 function has an inverse $\mu_{A}$ (generalized Möbius function) with respect to $*_{A}$ and $\mu_{A}\left(p^{a}\right) \in\{-1,0\}$ for every prime power $p^{a}(a \geq 1)$.

It can be shown that the system $A=(A(n))_{n \in \mathbb{N}}$ is regular if and only if
(i) $A(m n)=\{d e: d \in A(m), e \in A(n)\}$ for every $m, n \in \mathbb{N}$ with $(m, n)=1$,
(ii) for every prime power $p^{a}(a \geq 1)$ there exists a divisor $t=t_{A}\left(p^{a}\right)$ of $a$, called the type of $p^{a}$ with respect to $A$, such that

$$
A\left(p^{i t}\right)=\left\{1, p^{t}, p^{2 t}, \ldots, p^{i t}\right\}
$$

for every $i \in\{0,1, \ldots, a / t\}$.
Given a regular system $A$, an integer $n>1$ is called primitive with respect to $A$ ( $A$-primitive) if $A(n)=\{1, n\}$. By (i) if $n$ is primitive, then $n=p^{a}$ for some prime $p$ and $a \geq 1$. Furthermore, a prime power $p^{a}(a \geq 1)$ is primitive if and only if $t_{A}\left(p^{a}\right)=a$. It turns out that

$$
\mu_{A}\left(p^{a}\right)= \begin{cases}-1, & \text { if } p^{a} \text { is primitive } \\ 0, & \text { otherwise }\end{cases}
$$

Examples of regular systems of divisors are $A=D$, where $D(n)$ is the set of all positive divisors of $n$ and $A=U$, where $U(n)$ is the set of unitary divisors of $n$. For every prime power $p^{a}(a \geq 1)$ one has $t_{D}\left(p^{a}\right)=1, D\left(p^{a}\right)=\left\{1, p, p^{2}, \ldots, p^{a}\right\}$, the primitive integers with respect to $D$ are the primes, and $t_{U}\left(p^{a}\right)=a, U\left(p^{a}\right)=\left\{1, p^{a}\right\}$, the primitive integers with respect to $U$ being the prime powers $p^{a}(a \geq 1)$. Here $*_{D}$ and $*_{U}$ are the Dirichlet convolution and the unitary convolution, respectively.

To give another example of a regular system of divisors let, for every prime $p$ and $a \in \mathbb{N}$, $t_{A}\left(p^{a}\right)=2$ if $a$ is even, and $t_{A}\left(p^{a}\right)=a$ if $a$ is odd. That is, $A\left(p^{a}\right)=\left\{1, p^{2}, p^{4}, \ldots, p^{a}\right\}$ if $a$ is even, and $A\left(p^{a}\right)=\left\{1, p^{a}\right\}$ if $a$ is odd.

Let $A=(A(n))_{n \in \mathbb{N}}$ be a regular system of divisors and $k \in \mathbb{N}$. Define $A_{k}=\left(A_{k}(n)\right)_{n \in \mathbb{N}}$, where $A_{k}(n)=\left\{d \in \mathbb{N}: d^{k} \in A\left(n^{k}\right)\right\}$. Then the system $A_{k}$ is also regular, see Sita Ramaiah [17, Th. 3.1]. Let $(j, n)_{A, k}=\max \left\{d^{k}: d^{k} \mid j, d^{k} \in A(n)\right\}$. Then

$$
\begin{equation*}
\Phi_{A, k, S, n}(x)=\prod_{\substack{1 \leq j \leq n^{k} \\\left(\left(j, n^{k}\right) A, k\right)^{1 / k} \in S}}\left(x-e^{2 \pi i j / n^{k}}\right) \tag{2.2}
\end{equation*}
$$

is the polynomial corresponding to the generalized Ramanujan sums, investigated in the literature. See Haukkanen [5, Sect. 5]. We will not consider (2.2) in what follows.

For properties of regular convolutions and related arithmetical functions we refer to Narkiewicz [14], McCarthy [8, 9], Sita Ramaiah [17].

### 2.2 Generalized arithmetic functions

If $A$ is a regular system of divisors, then the corresponding generalized Euler function $\varphi_{A}(n)$ is defined as

$$
\varphi_{A}(n)=\sum_{\substack{j=1 \\(j, n)_{A}=1}}^{n} 1
$$

where $(j, n)_{A}$ is given by (1.6). In the proofs we will use the property that

$$
\begin{equation*}
d \in A\left((j, n)_{A}\right) \text { holds if and only if } d \mid j \text { and } d \in A(n) . \tag{2.3}
\end{equation*}
$$

We note that

$$
\begin{equation*}
\varphi_{A}(n)=\sum_{d \in A(n)} d \mu_{A}(n / d)=n \prod_{p^{a} \| n}\left(1-\frac{1}{p^{t}}\right), \tag{2.4}
\end{equation*}
$$

where $t=t_{A}\left(p^{a}\right)$ is the type of $p^{a}$. Hence $\varphi_{A}$ is multiplicative. Here $\varphi_{D}(n)=\varphi(n)$ is the classical Euler function, and $\varphi_{U}(n)=\varphi^{*}(n)$ is its unitary analogue.

Now let $S \subseteq \mathbb{N}$ be an arbitrary (nonempty) set and let $\varrho_{S}$ be the characteristic function of $S$. Given a regular system of divisors $A$, define the Möbius-type function $\mu_{A, S}$ by

$$
\begin{equation*}
\sum_{d \in A(n)} \mu_{A, S}(d)=\varrho_{S}(n) \quad(n \in \mathbb{N}) \tag{2.5}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\mu_{A, S}(n)=\sum_{d \in A(n)} \mu_{A}(d) \varrho_{S}(n / d) \tag{2.6}
\end{equation*}
$$

We say that $S$ is multiplicative if its characteristic function $\varrho_{S}$ is not identical zero and multiplicative (hence $1 \in S)$. In this case $\mu_{A, S}$ is multiplicative and $\mu_{A, S}\left(p^{a}\right)=\varrho\left(p^{a}\right)-\varrho\left(p^{a-1}\right) \in$ $\{-1,0,1\}$ for every prime power $p^{a}(a \geq 1)$. If $S$ is not multiplicative, then $\mu_{A, S}$ can be unbounded. For example, if $S=\mathbb{P}$ is the set of primes and $A=D$, then $\mu_{D, \mathbb{P}}\left(p_{1} \cdots p_{k}\right)=$ $(-1)^{k-1} k$ for distinct primes $p_{1}, \ldots, p_{k}$.

The Euler-type function $\varphi_{A, S}(n)$ is defined by

$$
\varphi_{A, S}(n)=\sum_{\substack{j=1 \\(j, n)_{A} \in S}}^{n} 1
$$

which reduces to $\varphi_{A}(n)$ if $S=\{1\}$. More generally, the corresponding Ramanujan sums $c_{A, S, n}(k)$ are given by

$$
c_{A, S, n}(k)=\sum_{\substack{j=1 \\(j, n)_{A} \in S}}^{n} \zeta_{n}^{j k}
$$

and they share properties similar to the classical Ramanujan sums $c_{n}(k)$. For example, for every regular $A$, every subset $S$ and $k, n \in \mathbb{N}$,

$$
\begin{equation*}
c_{A, S, n}(k)=\sum_{d \mid(k, n)_{A}} d \mu_{A, S}(n / d) \tag{2.7}
\end{equation*}
$$

holds, therefore all values of $c_{A, S, n}(k)$ are real integers. In particular, $c_{A, S, n}(1)=\mu_{A, S}(n)$, and

$$
\begin{equation*}
c_{A, S, n}(0)=\varphi_{A, S}(n)=\sum_{d \in A(n)} d \mu_{A, S}(n / d) \tag{2.8}
\end{equation*}
$$

If $S$ is multiplicative, then $c_{A, S, n}(k)$ and $\varphi_{A, S}(n)$ are multiplicative in $n$. If $S=\{1\}$, then one has the Hölder-type identity

$$
\begin{equation*}
c_{A, n}(k):=c_{A,\{1\}, n}(k)=\frac{\varphi_{A}(n) \mu_{A}\left(n /(k, n)_{A}\right)}{\varphi_{A}\left(n /(k, n)_{A}\right)} \tag{2.9}
\end{equation*}
$$

see McCarthy [9, p. 170].
In the case $A=D$ these functions were introduced by Cohen [2]. Also see the author [20]. For arbitrary $A$ and $S$ see Haukkanen [5], the author and Haukkanen [25, 26].

## 3 Results

### 3.1 Basic properties

Consider the cyclotomic polynomials $\Phi_{A, S, n}(x)$ defined by (1.5). It follows from the definition that these are monic polynomials, and the degree of $\Phi_{A, S, n}(x)$ is $\varphi_{A, S}(n)$.

Theorem 3.1. For every regular system $A$, every subset $S$ and $n \in \mathbb{N}$ we have

$$
\begin{align*}
\Phi_{A, S, n}(x) & =\prod_{d \in A(n)}\left(x^{d}-1\right)^{\mu_{A, S}(n / d)}  \tag{3.1}\\
& =(-1)^{\varrho_{S}(n)} \prod_{d \in A(n)}\left(1-x^{d}\right)^{\mu_{A, S}(n / d)}  \tag{3.2}\\
& =\prod_{\substack{d \in A(n) \\
n / d \in S}} \Phi_{A, d}(x) . \tag{3.3}
\end{align*}
$$

Furthermore,

$$
\begin{align*}
\prod_{d \in A(n)} \Phi_{A, S, d}(x) & =\prod_{\substack{d \in A(n) \\
n / d \in S}}\left(x^{d}-1\right)  \tag{3.4}\\
\prod_{d \in A(n)} \Phi_{A, d}(x) & =x^{n}-1 \tag{3.5}
\end{align*}
$$

If $A=D$ or $A=U$, respectively $S=\{1\}, S=\left\{m^{2}: m \in \mathbb{N}\right\}$ or $S=\mathbb{N} \backslash\{1\}$, then we recover properties of the classical cyclotomic polynomials and its analogues mentioned in the Introduction.

Corollary 3.2. For every $A, S$ and $n \in \mathbb{N}$,

$$
\Phi_{A, S, n}(x)=(-1)^{\varrho_{S}(n)} x^{\varphi_{A, S}(n)} \Phi_{A, S, n}(1 / x),
$$

hence the polynomial $\Phi_{A, S, n}(x)$ is palindromic or antipalindromic, according to $n \in S$ or $n \notin S$.
One may wonder what will be the generalized identity corresponding to (1.1) and (1.4). The answer is included in the next corollary.

Corollary 3.3. Assume that $1 \in S$. Then we have

$$
\begin{equation*}
x^{n}-1=\prod_{d \in A(n)} \Phi_{A, S, d}(x)^{h_{A, S}(n / d)}, \tag{3.6}
\end{equation*}
$$

where $h_{A, S}$ is the inverse with respect to $A$-convolution of the function $\mu_{A, S}$.
Here the values of $h_{A, S}$ can be computed for special choices of $A$ and $S$. To give another example, let $A=U$ and $S=\left\{m^{2}: m \in \mathbb{N}\right\}$. Then it turns out that $h_{U, S}$ is the characteristic function of the exponentially odd integers, i.e., integers with all exponents odd in their prime power factorization. Therefore, considering the unitary analogue of $Q_{n}(x)$, given by (1.3), namely

$$
Q_{n}^{*}(x)=\prod_{\substack{j=1 \\(j, n)_{*} \text { a square }}}^{n}\left(x-\zeta_{n}^{j}\right)
$$

we have

$$
x^{n}-1=\prod_{\substack{d \| \mid n \\ n / d \text { exponentially odd }}} Q_{d}^{*}(x) .
$$

For a regular system $A=(A(n))_{n \in \mathbb{N}}$ of divisors define the $A$-kernel function $\kappa_{A}$ by $\kappa_{A}(n)=$ $\prod_{p^{a} \| n} p^{t}$, where $t=t_{A}\left(p^{a}\right)$ is the type of $p^{a}$. Here $\kappa_{D}(n)=\kappa(n)=\prod_{p \mid n} p$ is the square-free kernel of $n$, and $\kappa_{U}(n)=n(n \in \mathbb{N})$.

Corollary 3.4. For every regular $A$ and $n \in \mathbb{N}$,

$$
\Phi_{A, n}(x)=\Phi_{\kappa_{A}(n)}\left(x^{n / \kappa_{A}(n)}\right) .
$$

For a regular system $A=(A(n))_{n \in \mathbb{N}}$ of divisors the $A$-core function $\gamma_{A}$ is defined by $\gamma_{A}(n)=$ $n \kappa(n) / \kappa_{A}(n)=\prod_{p^{a} \| n} p^{a-t+1}$, where $t=t_{A}\left(p^{a}\right)$ is the type of $p^{a}$. See McCarthy [9, p. 166]. Note that $\gamma_{D}(n)=n(n \in \mathbb{N})$ and $\gamma_{U}(n)=\kappa(n)=\prod_{p \mid n} p$.

The following result is a generalization of identity (1.2).
Theorem 3.5. For every regular system $A$ of divisors and every $n \in \mathbb{N}$,

$$
\begin{equation*}
\Phi_{A, n}(x)=\prod_{\substack{d\left|n \\ \gamma_{A}(n)\right| d}} \Phi_{d}(x) \tag{3.7}
\end{equation*}
$$

As a corollary we deduce the following more general identity.
Corollary 3.6. For every regular $A$, every subset $S$ and $n \in \mathbb{N}$ we have

$$
\Phi_{A, S, n}(x)=\prod_{\substack{d \in A(n) \\ n / d \in S \\ \gamma_{A}(d) \mid e}} \prod_{e} \Phi_{e}(x)
$$

therefore all the polynomials $\Phi_{A, S, n}(x)$ have integer coefficients.
Theorem 3.5 is a special case of the following general result.
Theorem 3.7. Let $A$ be a regular system of divisors, and let the functions $g$ and $g_{A}$ be defined for every $n \in \mathbb{N}$ by

$$
\sum_{d \mid n} g(d)=\sum_{d \in A(n)} g_{A}(d) .
$$

Then

$$
g_{A}(n)=\sum_{\substack{d\left|n \\ \gamma_{A}(n)\right| d}} g(d) .
$$

Another application of Theorem 3.7 is to the Ramanujan sums $c_{A, n}(k)=c_{A,\{1\}, n}(k)$, namely

$$
\begin{equation*}
c_{A, n}(k)=\sum_{\substack{d\left|n \\ \gamma_{A}(n)\right| d}} c_{d}(k), \tag{3.8}
\end{equation*}
$$

in particular,

$$
\begin{aligned}
& \varphi_{A}(n)=\sum_{\substack{d\left|n \\
\gamma_{A}(n)\right| d}} \varphi(d), \\
& \mu_{A}(n)=\sum_{\substack{d\left|n \\
\gamma_{A}(n)\right| d}} \mu(d) .
\end{aligned}
$$

The last three identities are given and proved by McCarthy [8, Th. 2, Cor. 2.1, Cor. 2.2] by different arguments, namely by using properties of finite Fourier representations of $n$-even functions. Also see McCarthy [9, pp. 165-167]. Our proof is direct and short. See Cohen [3, Lemma 3.1] for a different approach in the case $A=U$.

### 3.2 Other identities

Theorem 3.8. Let $A$ be a regular system of divisors and let $n \in \mathbb{N}$. Then for $x>1$ (or formally),

$$
\begin{equation*}
\Phi_{A, n}(x)=\prod_{j=1}^{n}\left(x^{(j, n)_{A}}-1\right)^{\cos (2 \pi j / n)} \tag{3.9}
\end{equation*}
$$

For $A=D$ identity (3.9) was proved by Schramm [16] and for $A=U$ by Moree and the author [12, Th. 5]. In fact, (3.9) is a special case of the following general result and its corollary concerning the discrete Fourier transform (DFT) of functions involving the quantity $(j, n)_{A}$.
Theorem 3.9. Let $A$ be a regular system of divisors, $f: \mathbb{N} \rightarrow \mathbb{C}$ be an arbitrary arithmetic function and $n, k \in \mathbb{N}$. Then

$$
\begin{equation*}
\sum_{j=1}^{n} f\left((j, n)_{A}\right) \zeta_{n}^{j k}=\sum_{d \in(k, n)_{A}} d\left(\mu_{A} *_{A} f\right)(n / d) \tag{3.10}
\end{equation*}
$$

Corollary 3.10. If $f$ is a real valued function, then

$$
\begin{align*}
& \sum_{j=1}^{n} f\left((j, n)_{A}\right) \cos (2 \pi j k / n)=\sum_{d \in(k, n)_{A}} d\left(\mu_{A} *_{A} f\right)(n / d),  \tag{3.11}\\
& \sum_{j=1}^{n} f\left((j, n)_{A}\right) \sin (2 \pi j k / n)=0 .
\end{align*}
$$

Now we present a Menon-type identity, involving $(j-1, n)_{A}$, where $j$ runs over an $A$-reduced residue system $(\bmod n)$, that is, $(j, n)_{A}=1$.

Theorem 3.11. Let $A$ be a regular system of divisors and let $n \in \mathbb{N}$. Then

$$
\begin{equation*}
\prod_{\substack{j=1 \\(j, n)_{A}=1}}^{n}\left(x^{(j-1, n)_{A}}-1\right)=\prod_{d \in A(n)} \Phi_{A, d}(x)^{\varphi_{A}(n) / \varphi_{A}(d)} \tag{3.12}
\end{equation*}
$$

If $A=D$, then (3.12) was given by the author [23], [24, Eq. (45)]. Here (3.12) can be deduced from the following general result, known in the literature, even in a more general form, see [17, Th. 9.1]. However, for the sake of completeness we also give a direct and short proof of it.

Theorem 3.12. Let $A$ be a regular system of divisors and $f: \mathbb{N} \rightarrow \mathbb{C}$ an arbitrary arithmetic function. Then for every $n \in \mathbb{N}$,

$$
\begin{equation*}
\sum_{\substack{j=1 \\(j, n)_{A}=1}}^{n} f\left((j-1, n)_{A}\right)=\varphi_{A}(n) \sum_{d \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(d)}{\varphi_{A}(d)} . \tag{3.13}
\end{equation*}
$$

Note that if $A=D$ and $f(n)=n(n \in \mathbb{N})$, then identity (3.13) recovers (1.10).
Another Menon-type identity is the following.
Theorem 3.13. Let $A$ be a regular system of divisors, let $n \in \mathbb{N}$ and $\chi$ be a Dirichlet character $(\bmod n)$ with conductor $d(d \mid n)$. Then for real $x>1$ (or formally),

$$
\begin{equation*}
\prod_{j=1}^{n}\left(x^{(j-1, n)_{A}}-1\right)^{\operatorname{Re}(\chi(j))}=\prod_{d \delta \in A(n)} \Phi_{A, d \delta}(x)^{\varphi(n) / \varphi(d \delta)} \tag{3.14}
\end{equation*}
$$

If $\chi$ is a primitive character $(\bmod n)$, then

$$
\begin{equation*}
\Phi_{A, n}(x)=\prod_{j=1}^{n}\left(x^{(j-1, n)_{A}}-1\right)^{\operatorname{Re}(\chi(j))} . \tag{3.15}
\end{equation*}
$$

If $A=D$, then (3.14) and (3.15) recover identities (1.8) and (1.9), respectively. See the author [21] for some related identities in the case $A=D$.

Theorem 3.13 can be deduced from the next general result and its corollary.
Theorem 3.14. Let $A$ be a regular system of divisors, $f: \mathbb{N} \rightarrow \mathbb{C}$ be an arbitrary arithmetic function, let $n \in \mathbb{N}$ and $\chi$ be a Dirichlet character $(\bmod n)$ with conductor $d(d \mid n)$. Then

$$
\begin{equation*}
\sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \chi(j)=\varphi(n) \sum_{d \delta \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(d \delta)}{\varphi(d \delta)} . \tag{3.16}
\end{equation*}
$$

If $\chi$ is a primitive character $(\bmod n)$, then

$$
\begin{equation*}
\sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \chi(j)=\left(\mu_{A} *_{A} f\right)(n) . \tag{3.17}
\end{equation*}
$$

Corollary 3.15. Let $f$ be a real valued function. If $\chi$ is a Dirichlet character $(\bmod n)$ with conductor $d(d \mid n)$, then

$$
\begin{aligned}
& \sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \operatorname{Re}(\chi(j))=\varphi(n) \sum_{d \delta \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(d \delta)}{\varphi(d \delta)}, \\
& \sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \operatorname{Im}(\chi(j))=0 .
\end{aligned}
$$

If $\chi$ is a primitive character $(\bmod n)$, then

$$
\sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \operatorname{Re}(\chi(j))=\left(\mu_{A} *_{A} f\right)(n) .
$$

It is possible to deduce some further related identities. For example, we have the next result, known in the case $A=D$, and proved by Moree and the author [12, Cor. 6] for $A=U$.

Theorem 3.16. For every regular system $A, n>1$ and $x \in \mathbb{C},|x|<1$ (or formally),

$$
\begin{equation*}
\Phi_{A, n}(x)=\exp \left(-\sum_{k=1}^{\infty} \frac{c_{A, n}(k)}{k} x^{k}\right) . \tag{3.18}
\end{equation*}
$$

### 3.3 Coefficients

Now consider the coefficients of the monic polynomials $\Phi_{A, S, n}(x)$ of degree $\varphi_{A, S}(n)$. It follows from identity (2.7) applied for $k=1$ that the coefficient of the term $x^{\varphi_{A, S}(n)-1}$ is $-c_{A, S, n}(1)=-\mu_{S, A}(n)$. In order to deduce formulas for the other coefficients as well, let

$$
\Phi_{A, S, n}(x)=\sum_{j=0}^{\varphi_{A, S}(n)} a_{A, S, n}(j) x^{j} .
$$

We have the following generalization of the Möller-Endo identities.
Theorem 3.17. For every $A, S, n, k$,

$$
\begin{equation*}
a_{A, S, n}(k)=(-1)^{\varrho_{S}(n)} \sum_{\substack{j_{1}, j_{2}, \ldots, j_{k} \geq 0 \\ j_{1}+2 j_{2}+\cdots+k j_{k}=k}} \prod_{d=1}^{k}(-1)^{j_{d}}\binom{\mu_{A, S}(n / d)}{j_{d}}, \tag{3.19}
\end{equation*}
$$

with the convention $\mu_{A, S}(t)=0$ if $t$ is not an integer.
This shows that $a_{A, S, n}(1)=-(-1)^{\varrho_{S}(n)} \mu_{A, S}(n)$. Hence, according to Corollary 3.2,

$$
a_{A, S, n}\left(\varphi_{A, S}(n)-1\right)=-\mu_{A, S}(n),
$$

as mentioned above. Also,

$$
(-1)^{\varrho_{S}(n)} a_{A, S, n}(2)=a_{A, S, n}\left(\varphi_{A, S}(n)-2\right)=\frac{\mu_{A, S}(n)\left(\mu_{A, S}(n)-1\right)}{2}-\mu_{A, S}(n / 2),
$$

and so on, similar to the classical case.
Now let $S=\{1\}$ and let $a_{A, n}(k):=a_{A,\{1\}, n}(k)$ denote the coefficients of $\Phi_{A, n}(x)$. The identity of Corollary 3.4 shows that to study these coefficients it is enough to consider the case when $n$ is replaced by $\kappa_{A}(n)$. As a generalization of the Grytczuk-Tropak recursion formula we prove the following result.

Theorem 3.18. Let $A$ be a regular system of divisors. If $n$ is a product of $A$-primitive integers, then for every $k$ with $1 \leq k \leq \varphi_{A}(n)$,

$$
\begin{equation*}
a_{A, n}(k)=-\frac{\mu_{A}(n)}{k} \sum_{j=1}^{k} a_{A, n}(k-j) \mu_{A}\left((j, n)_{A}\right) \varphi_{A}\left((j, n)_{A}\right), \tag{3.20}
\end{equation*}
$$

where $a_{A, n}(0)=1$.
Note that in the classical case $(A=D)(3.20)$ only holds for squarefree values of $n$ (fact omitted in some texts). However, in the unitary case $(A=U)(3.20)$ holds for every $n \in \mathbb{N}$.

## 4 Proofs

Proof of Theorem 3.1. More generally, let $f: \mathbb{N} \rightarrow \mathbb{C}$ be an arbitrary function and $F_{f}(n):=$ $\sum_{j=1}^{n} f(j / n)$. Then

$$
S_{f, A, S}(n):=\sum_{\substack{j=1 \\(j, n)_{A} \in S}}^{n} f(j / n)=\sum_{j=1}^{n} f(j / n) \varrho_{S}\left((j, n)_{A}\right)=\sum_{j=1}^{n} f(j / n) \sum_{d \in A\left((j, n)_{A}\right)} \mu_{A, S}(d),
$$

by (2.5). Using property (2.3) we deduce that

$$
S_{f, A, S}(n)=\sum_{d \in A(n)} \mu_{A, S}(d) \sum_{\substack{j=1 \\ d \mid j}}^{n} f(j / n)=\sum_{d \in A(n)} \mu_{A, S}(d) F_{f}(n / d) .
$$

Note that this is generalization of the the Hurwitz lemma, recovered for $A=D, S=\{1\}$. Now if (formally) $f(n)=\log \left(x-e^{2 \pi i n}\right)$, then $F_{f}(n)=\sum_{j=1}^{n} \log \left(x-e^{2 \pi i j / n}\right)=\log \left(x^{n}-1\right)$, and deduce that

$$
\begin{equation*}
\sum_{\substack{j=1 \\(j, n)_{A} \in S}}^{n} \log \left(x-e^{2 \pi i j / n}\right)=\sum_{d \in A(n)} \log \left(x^{d}-1\right) \mu_{A, S}(n / d), \tag{4.1}
\end{equation*}
$$

equivalent to (3.1). Now (3.2) follows by (3.1) and (2.5)
In terms of the $A$-convolution (4.1) shows that

$$
\begin{equation*}
\log \Phi_{A, S, \bullet}(x)=\log \left(x^{\bullet}-1\right) *_{A} \mu_{A, S}, \tag{4.2}
\end{equation*}
$$

that is, using (2.6),

$$
\begin{equation*}
\log \Phi_{A, S, \bullet}(x)=\log \left(x^{\bullet}-1\right) *_{A} \mu_{A} *_{A} \varrho_{S} \tag{4.3}
\end{equation*}
$$

If $S=\{1\}$, then (4.3) gives

$$
\begin{equation*}
\log \Phi_{A, \bullet}(x)=\log \left(x^{\bullet}-1\right) *_{A} \mu_{A}, \tag{4.4}
\end{equation*}
$$

and combining (4.3) and (4.4) we have

$$
\log \Phi_{A, S, \bullet}(x)=\log \Phi_{A, \bullet}(x) *_{A} \varrho_{S},
$$

giving (3.3).
From (4.3) we also have

$$
\log \Phi_{A, S, \bullet}(x) *_{A} \mathbf{1}=\log \left(x^{\bullet}-1\right) *_{A} \varrho_{S},
$$

where $\mathbf{1}(n)=1(n \in \mathbb{N})$. This shows the validity of (3.4), which reduces to (3.5) if $S=\{1\}$.
Proof of Corollary 3.2. This is a direct consequence of identities (3.1), (3.2) and (2.8).
Proof of Corollary 3.3. If $1 \in S$, then $\mu_{A, S}(1)=\mu_{A}(1) \varrho(1)=1 \neq 0$. Hence the function $\mu_{A, S}$ has an inverse with respect to $A$-convolution, we denote it by $h_{A, S}$. That is, $h_{A, S} *_{A} \mu_{A, S}=\varepsilon$, where $\varepsilon(n)=\lfloor 1 / n\rfloor(n \in \mathbb{N})$. From (4.2) we obtain that

$$
\log \left(x^{\bullet}-1\right)=\log \Phi_{A, S, \bullet}(x) *_{A} h_{A, S},
$$

equivalent to (3.6).
Proof of Corollary 3.4. By the definition of the function $\mu_{A}$, if $d$ is not a product of $A$-primitive integers, then $\mu_{A}(d)=0$. Therefore, for every $n \in \mathbb{N}$, using (3.1),

$$
\begin{aligned}
\Phi_{A, n}(x) & =\prod_{d \in A(n)}\left(x^{n / d}-1\right)^{\mu_{A}(d)}=\prod_{\substack{d \in A(n) \\
d \in A\left(\kappa_{A}(n)\right)}}\left(x^{n / d}-1\right)^{\mu_{A}(d)} \\
& =\prod_{d \in A\left(\kappa_{A}(n)\right)}\left(\left(x^{n / \kappa(n)}\right)^{\kappa(n) / d}-1\right)^{\mu_{A}(d)}=\Phi_{A, \kappa(n)}\left(x^{n / \kappa(n)}\right)
\end{aligned}
$$

Proof of Theorem 3.5. Apply (formally) Theorem 3.7 in the case $g(n)=\log \Phi_{n}(x), g_{A}(n)=$ $\log \Phi_{A, n}(x)$, where $f(n)=\log \left(x^{n}-1\right)$ by taking into account (3.5). We deduce that

$$
\log \Phi_{A, n}(x)=\sum_{\substack{d\left|n \\ \gamma_{A}(n)\right| d}} \log \Phi_{d}(x)
$$

which gives (3.7).
Proof of Corollary 3.6. This is a direct consequence of identities (3.3) and (3.7).
Proof of Theorem 3.7. Let

$$
f(n)=\sum_{d \mid n} g(d)=\sum_{d \in A(n)} g_{A}(d) \quad(n \in \mathbb{N}) .
$$

Then we have

$$
g_{A}(n)=\sum_{d \in A(n)} f(d) \mu_{A}(n / d)=\sum_{d \in A(n)} \mu_{A}(n / d) \sum_{\delta \mid d} g(\delta)
$$

$$
=\sum_{\substack{\delta j m=n \\ \delta j \in A(n)}} g(\delta) \mu_{A}(m)=\sum_{\delta t=n} g(\delta) \sum_{\substack{j m=t \\ \delta j \in A(n)}} \mu_{A}(m)
$$

Since for every regular system $A, d \in A(n)$ holds if and only if $n / d \in A(n)$ we deduce that $\delta j \in A(n)$ if and only if $n /(\delta j)=m \in A$, and have

$$
g_{A}(n)=\sum_{\delta t=n} g(\delta) \sum_{\substack{j m=t \\ m \in A(n)}} \mu_{A}(m)=\sum_{\delta \mid n} g(\delta) \sum_{\substack{m \mid n / \delta \\ m \in A(n)}} \mu_{A}(m)
$$

We show that for every fixed $n$ and $\delta$ with $\delta \mid n$,

$$
\sum_{\substack{m \mid n / \delta \\ m \in A(n)}} \mu_{A}(m)= \begin{cases}1, & \text { if } \gamma_{A}(n) \mid \delta \\ 0, & \text { otherwise }\end{cases}
$$

which will finish the proof.
To do this, let $n=p_{1}^{a_{1}} \cdots p_{r}^{a_{r}}, \delta=p_{1}^{b_{1}} \cdots p_{r}^{b_{r}}$ with $0 \leq b_{i} \leq a_{i}(1 \leq i \leq r)$. Note that $m \in A(n)$ holds if and only if $m=p_{1}^{c_{1} t_{1}} \cdots p_{r}^{c_{r} t_{r}}$ with $0 \leq c_{i} \leq a_{i} / t_{i}$, where $t_{i}$ is the type of $p_{i}^{a_{i}}$ $(1 \leq i \leq r)$. Therefore, since the function $\mu_{A}$ is multiplicative,

$$
\begin{aligned}
& =\sum_{0 \leq c_{1} \leq\left(a_{1}-b_{1}\right) / t_{1}} \mu_{A}\left(p_{1}^{c_{1} t_{1}}\right) \cdots \sum_{0 \leq c_{r} \leq\left(a_{r}-b_{r}\right) / t_{r}} \mu_{A}\left(p_{r}^{c_{r} t_{r}}\right) \\
& =\sum_{0 \leq c_{1} \leq\left(a_{1}-b_{1}\right) / t_{1}} \mu\left(p_{1}^{c_{1}}\right) \cdots \sum_{0 \leq c_{r} \leq\left(a_{r}-b_{r}\right) / t_{r}} \mu\left(p_{r}^{c_{r}}\right),
\end{aligned}
$$

where $\mu$ is the classical Möbius function.
Here if $\left(a_{i}-b_{i}\right) / t_{i} \geq 1$ for some $i$, then $\sum_{c_{i}} \mu\left(p_{i}^{c_{1}}\right)=\mu(1)+\mu(p)=0$, and the product of the sums is also zero. Otherwise, $\left(a_{i}-b_{i}\right) / t_{i}<1$ for all $i$ holds if and only if $a_{i}-b_{i}<t_{i}$ for all $i$, equivalent to $a_{i}-t_{i}+1 \leq b_{i}$ for all $i$, that is, $\gamma_{A}(n) \mid \delta$. In this case for all $i$, $\sum_{c_{i}} \mu\left(p_{i}^{c_{i}}\right)=\mu(1)=1$, and the proof is ready.

Proof of Theorem 3.8. Apply identity (3.11) to the function $f(n)=\log \left(x^{n}-1\right)$, where $x>1$ is real (or formally), and take into account that $\mu_{A} *_{A} f=\Phi_{A, \bullet}(x)$ by (3.5) and Möbius inversion.

Proof of Theorem 3.9. We have by using that $f(n)=\sum_{d \in A(n)}\left(\mu_{A} *_{A} f\right)(d)(n \in \mathbb{N})$ and property (2.3),

$$
\sum_{j=1}^{n} f\left((j, n)_{A}\right) \zeta_{n}^{j k}=\sum_{j=1}^{n} \zeta_{n}^{j k} \sum_{d \in A\left((j, n)_{A}\right)}\left(\mu_{A} *_{A} f\right)(d)=\sum_{j=1}^{n} \zeta_{n}^{j k} \sum_{\substack{d \mid j \\ d \in A(n)}}\left(\mu_{A} *_{A} f\right)(d)
$$

$$
=\sum_{d \in A(n)}\left(\mu_{A} *_{A} f\right)(d) \sum_{\substack{j=1 \\ d \mid j}}^{n} \zeta_{n}^{j k}=\sum_{\substack{d \in A(n) \\ n / d \mid k}}\left(\mu_{A} *_{A} f\right)(d) \frac{n}{d}=\sum_{\substack{d \in A(n) \\ d \mid k}} d\left(\mu_{A} *_{A} f\right)(n / d),
$$

giving (3.10) by using (2.3) again.
Proof of Corollary 3.10. If $f$ is a real valued function, then the right hand side of (3.10) is real, and we deduced the given identities.

Proof of Theorem 3.11. Apply Theorem 3.12 by taking (formally) $f(n)=\log \left(x^{n}-1\right)$. Then $\mu_{A} *_{A} f=\log \Phi_{A, \bullet}(x)$ by (3.5) and Möbius inversion. We deduce that

$$
\sum_{\substack{j=1 \\(j, n)_{A}=1}}^{n} \log \left(x^{(j-1, n)_{A}}-1\right)=\varphi_{A}(n) \sum_{d \in A(n)} \frac{\log \Phi_{A, d}(x)}{\varphi_{A}(d)},
$$

equivalent to the given identity.
Proof of Theorem 3.12. Let $M_{A, f}(n)$ denote the left hand side of (3.13). We have by property (2.3),

$$
M_{A, f}(n)=\sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \sum_{d \mid(j, n)_{A}} \mu_{A}(d)=\sum_{d \in A(n)} \mu_{A}(d) \sum_{\substack{j=1 \\ d \mid j}}^{n} f\left((j-1, n)_{A}\right) .
$$

By using that $f(n)=\sum_{d \in A(n)}\left(\mu_{A} *_{A} f\right)(d)(n \in \mathbb{N})$, we deduce

$$
\begin{aligned}
S_{A, f, d}(n):= & \sum_{\substack{j=1 \\
d \mid j}}^{n} f\left((j-1, n)_{A}\right)=\sum_{k=1}^{n / d} f\left((k d-1, n)_{A}\right)=\sum_{k=1}^{n / d} \sum_{e \in A\left((k d-1, n)_{A}\right)}\left(\mu_{A} *_{A} f\right)(e) \\
& =\sum_{k=1}^{n / d} \sum_{\substack{e \mid k d-1 \\
e \in A(n)}}\left(\mu_{A} *_{A} f\right)(e)=\sum_{e \in A(n)}\left(\mu_{A} *_{A} f\right)(e) \sum_{\substack{k=1 \\
k d \equiv 1(\bmod e)}}^{n / d} 1,
\end{aligned}
$$

where the inner sum is $n /(d e)$ if $(d, e)=1$ and 0 otherwise. This gives

$$
S_{A, f, d}(n)=\frac{n}{d} \sum_{\substack{e \in A(n) \\(e, d)=1}} \frac{\left(\mu_{A} *_{A} f\right)(e)}{e} .
$$

Thus

$$
M_{A, f}(n)=\sum_{d \in A(n)} \mu_{A}(d) S_{A, f, d}(n)=n \sum_{e \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(e)}{e} \sum_{\substack{d \in A(n) \\(d, e)=1}} \frac{\mu_{A}(d)}{d},
$$

and for every $e \in A(n)$,

$$
\begin{gathered}
\sum_{\substack{d \in A(n) \\
(d, e)=1}} \frac{\mu_{A}(d)}{d}=\prod_{\substack{p^{a} \| n \\
p \nmid e}}\left(1-\frac{1}{p^{t}}\right) \\
=\prod_{p^{a} \| n}\left(1-\frac{1}{p^{t}}\right) \prod_{\substack{p^{a} \| n \\
p \mid e}}\left(1-\frac{1}{p}\right)^{-1}=\frac{\varphi_{A}(n)}{n} \cdot \frac{e}{\varphi_{A}(e)},
\end{gathered}
$$

where $t=t_{A}\left(p^{a}\right)$, by using (2.4) and the fact that if $e \in A(n)$ and $e=\prod p^{b}, n=\prod p^{a}$, then $t_{A}\left(p^{b}\right)=t_{A}\left(p^{a}\right)$ for all prime powers in question, see [9, Cor. 4.2].

We obtain that

$$
M_{A, f}(n)=\varphi_{A}(n) \sum_{e \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(e)}{\varphi_{A}(e)},
$$

which is identity (3.13).
Proof of Theorem 3.13. Apply Theorem 3.14 to the function $f(n)=\log \left(x^{n}-1\right)$, where $\mu_{A} *_{A} f=$ $\log \Phi_{A, .}(x)$.

Proof of Theorem 3.14. We need the following known results, see, e.g., [10, Ch. 9].
If $\chi$ is a Dirichlet character $(\bmod n)$ with conductor $d$, then there is a unique primitive character $\chi^{*}(\bmod d)$ that induces $\chi$. That is,

$$
\chi(k)= \begin{cases}\chi^{*}(k), & \text { if }(k, n)=1  \tag{4.5}\\ 0, & \text { if }(k, n)>1\end{cases}
$$

Let $\chi$ be a primitive character $(\bmod n)$. Then for every $d \mid n, d<n$ and every $s \in \mathbb{Z}$,

$$
\begin{equation*}
\sum_{\substack{k=1 \\ k \equiv s(\bmod d)}}^{n} \chi(k)=0 . \tag{4.6}
\end{equation*}
$$

We have, according to (4.5),

$$
\begin{aligned}
S_{f}:=\sum_{j=1}^{n} f\left((j-1, n)_{A}\right) \chi(j)=\sum_{\substack{j=1 \\
(j, n)=1}}^{n} f\left((j-1, n)_{A}\right) \chi^{*}(j) \\
=\sum_{r=1}^{d} \sum_{\substack{j=1 \\
(j, n)=1 \\
j \equiv r(\bmod d)}}^{n} f\left((j-1, n)_{A}\right) \chi^{*}(j)=\sum_{r=1}^{d} \chi^{*}(r) \sum_{\substack{j=1 \\
(j, n)=1 \\
j \equiv r(\bmod d)}}^{n} f\left((j-1, n)_{A}\right) .
\end{aligned}
$$

Here, since $d \mid n$, if $(j, n)=1$ and $j \equiv r(\bmod d)$, then $(r, d)=(j, d)=1$. Therefore, the inner sum is empty in the case $(r, d)>1$.

Now assume that $(r, d)=1$. We have

$$
\begin{gathered}
T:=\sum_{\substack{j=1 \\
(j, n)=1 \\
j \equiv r(\bmod d)}}^{n} f\left((j-1, n)_{A}\right)=\sum_{\substack{j=1 \\
(j, n)=1 \\
j \equiv r(\bmod d)}}^{n} \sum_{\substack{e \in A\left((j-1, n)_{A}\right)}}\left(\mu_{A} *_{A} f\right)(e) \\
=\sum_{e \in A(n)}\left(\mu_{A} *_{A} f\right)(e) \sum_{\substack{j=1 \\
(j, n)=1 \\
j \equiv r \bmod d) \\
j \equiv 1(\bmod e)}}^{n} 1,
\end{gathered}
$$

by property (2.3). Here the inner sum is

$$
U:=\sum_{\substack{j=1 \\(j, n)=1 \\ j=r(\bmod d) \\ j \equiv 1(\bmod e)}}^{n} 1= \begin{cases}\frac{\varphi(n)(d, e)}{\varphi(d e)}=\frac{\varphi(n) \varphi((d, e))}{\varphi(d) \varphi(e)}, & \text { if }(d, e) \mid r-1, \\ 0, & \text { otherwise },\end{cases}
$$

see the author [22, Lemma 2.1]. This gives

$$
\begin{aligned}
& S_{f}=\sum_{r=1}^{d} \chi^{*}(r) \frac{\varphi(n)}{\varphi(d)} \sum_{\substack{e \in A(n) \\
(d, e) \mid r-1}} \frac{\left(\mu_{A} *_{A} f\right)(e)}{\varphi(e)} \varphi((d, e)) \\
= & \frac{\varphi(n)}{\varphi(d)} \sum_{e \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(e)}{\varphi(e)} \varphi((d, e)) \sum_{\substack{r=1 \\
r \equiv 1(\bmod (d, e))}}^{d} \chi^{*}(r),
\end{aligned}
$$

where by (4.6) the last sum is 0 unless $(d, e)=d$, that is, $d \mid e$. We deduce

$$
S_{f}=\frac{\varphi(n)}{\varphi(d)} \sum_{\substack{e \in A(n) \\ d \mid e}} \frac{\left(\mu_{A} *_{A} f\right)(e)}{\varphi(e)} \varphi(d)=\varphi(n) \sum_{d \delta \in A(n)} \frac{\left(\mu_{A} *_{A} f\right)(d \delta)}{\varphi(d \delta)},
$$

finishing the proof of (3.16).
If $\chi$ is a primitive character $(\bmod n)$, then its conductor is $d=n$. Therefore, (3.16) reduces to (3.17).

Proof of Corollary 3.15. If $f$ is a real valued function, then the right hand side of (3.16) is real, and we deduced the given identities.

Proof of Theorem 3.16. It is known that for every $n>1$ and $|x|<1$,

$$
\begin{equation*}
\Phi_{n}(x)=\exp \left(-\sum_{k=1}^{\infty} \frac{c_{n}(k)}{k} x^{k}\right) \tag{4.7}
\end{equation*}
$$

see, e.g., Herrera-Poyatos and Moree [6, Eq. (1.2)]. By identities (3.7), (4.7) and (3.8) we obtain

$$
\Phi_{A, n}(x)=\prod_{\substack{d\left|n \\ \gamma_{A}(n)\right| d}} \Phi_{d}(x)=\exp \left(-\sum_{k=1}^{\infty} \frac{x^{k}}{k} \sum_{\substack{d\left|n \\ \gamma_{A}(n)\right| d}} c_{d}(k)\right)=\exp \left(-\sum_{k=1}^{\infty} \frac{c_{A, n}(k)}{k} x^{k}\right),
$$

completing the proof of (3.18).
Proof of Theorem 3.17. We adopt the simple approach concerning the classical Möller-Endo formulas due to Gallot et al. [4, Sect. 3]. Also see Herrera-Poyatos and Moree [6, Sect. 4].

From (3.2) we have

$$
\Phi_{A, S, n}(x)=(-1)^{\varrho_{S}(n)} \prod_{d=1}^{\infty}\left(1-x^{d}\right)^{\mu_{A, S}(n / d)}
$$

with the notation $\mu_{A, S}(t)=0$ if $t$ is not an integer. Writing for $|x|<1$ the Taylor series expansion of $\left(1-x^{d}\right)^{\mu_{A, S}(n / d)}$ we deduce

$$
\Phi_{A, S, n}(x)=(-1)^{\varrho_{S}(n)} \prod_{d=1}^{\infty} \sum_{j_{d}=0}^{\infty}(-1)^{j_{d}}\binom{\mu_{A, S}(n / d)}{j_{d}} x^{d j_{d}}
$$

and by identifying the coefficient of $x^{k}$ we obtain (3.19).
Proof of Theorem 3.18. Similar to the classical case, from Viète's and Newton's formulas we deduce the recursion formula

$$
a_{A, n}(k)=-\frac{1}{k} \sum_{j=1}^{k} a_{A, n}(k-j) c_{A, n}(j),
$$

where $a_{A, n}(0)=1$.
If $n=p^{t}$ is an $A$-primitive integer, then $A\left(p^{t}\right)=\left\{1, p^{t}\right\}$, and by the Hölder-type identity (2.9) we have

$$
\begin{aligned}
c_{A, p^{t}}(k) & = \begin{cases}p^{t}-1, & \text { if } p^{t} \mid k ; \\
-1, & \text { otherwise, }\end{cases} \\
& =\mu_{A}\left(p^{t}\right) \mu_{A}\left(\left(k, p^{t}\right)_{A}\right) \varphi_{A}\left(\left(k, p^{t}\right)_{A}\right) .
\end{aligned}
$$

Hence, by multiplicativity, if $n$ is a product of $A$-primitive integers, namely $n=p^{t_{1}} \cdots p^{t_{r}}$, then

$$
c_{A, n}(k)=\mu_{A}(n) \mu_{A}\left((k, n)_{A}\right) \varphi_{A}\left((k, n)_{A}\right),
$$

holds and we deduce that

$$
a_{A, n}(k)=-\frac{\mu_{A}(n)}{k} \sum_{j=1}^{k} a_{A, n}(k-j) \mu_{A}\left((j, n)_{A}\right) \varphi_{A}\left((j, n)_{A}\right),
$$

as stated.

## References

[1] G. Bachman, Coefficients of unitary cyclotomic polynomials of order three, Integers 22 (2022), Paper No. A80, 13 pp.
[2] E. Cohen, Arithmetical functions associated with arbitrary sets of integers, Acta Arith. 5 (1959), 407-415.
[3] E. Cohen, An elementary method in the asymptotic theory of numbers, Duke Math. J. 28 (1961), 183-192.
[4] Y. Gallot, P. Moree, and H. Hommersom, Value distribution of cyclotomic polynomial coefficients, Unif. Distrib. Theory 6 (2011), 177-206.
[5] P. Haukkanen, Some generalized totient functions, Math. Student 56 (1988), 65-74.
[6] A. Herrera-Poyatos and P. Moree, Coefficients and higher order derivatives of cyclotomic polynomials: Old and new, Expo. Math. 39 (2021), 309-343.
[7] G. Jones, P. I. Kester, L. Martirosyan, P. Moree, L. Tóth, B. B. White, and B. Zhang, Coefficients of (inverse) unitary cyclotomic polynomials, Kodai Math. J. 43 (2) (2020), 325-338.
[8] P. J. McCarthy, Regular arithmetical convolutions, Portugal. Math. 27 (1968), 1-13.
[9] P. J. McCarthy, Introduction to Arithmetical Functions, Springer, 1986.
[10] H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory I. Classical Theory, Cambridge University Press, 2007.
[11] P. Moree, Inverse cyclotomic polynomials, J. Number Theory 129 (2009), 667-680.
[12] P. Moree and L. Tóth, Unitary cyclotomic polynomials, Integers 20 (2020), Paper No. A65, 21 pp .
[13] K. Nageswara Rao, A generalization of the cyclotomic polynomial, Canad. Math. Bull. 19 (1976), 461-466.
[14] W. Narkiewicz, On a class of arithmetical convolutions, Colloq. Math. 10 (1963), 81-94.
[15] C. Sanna, A survey on coefficients of cyclotomic polynomials, Expo. Math. 40 (2022), 469494.
[16] W. Schramm, Eine alternative Produktdarstellung für die Kreisteilungspolynome, Elem. Math. 70 (2015), 137-143.
[17] V. Sita Ramaiah, Arithmetical sums in regular convolutions, J. Reine Angew. Math. 303/304 (1978), 265-283.
[18] R. Sivaramakrishnan, Classical Theory of Arithmetic Functions, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 126, Marcel Dekker, 1989.
[19] L. Tóth, A generalization of the cyclotomic polynomials, Gaz. Mat., Perfecţ. Metod. Metodol. Mat. Inf. (Bucharest) 11 (1990), no. 1, 26-29.
[20] L. Tóth, Remarks on generalized Ramanujan sums and even functions, Acta Math. Acad. Paedagog. Nyházi. (N. S.) 20 (2004), 233-238.
[21] L. Tóth, Menon-type identities concerning Dirichlet characters, Int. J. Number Theory 14 (2018), 1047-1054.
[22] L. Tóth, Short proof and generalization of a Menon-type identity by Li, Hu and Kim, Taiwanese J. Math. 23 (2019), 557-561.
[23] L. Tóth, Proposed problem 274, Eur. Math. Soc. Mag. (2023), no. 127, p. 54.
[24] L. Tóth, Proofs, generalizations and analogs of Menon's identity: a survey, Acta Univ. Sapientiae Math. 15 (2023), no.1, 142-197.
[25] L. Tóth and P. Haukkanen, A generalization of Euler's $\varphi$-function with respect to a set of polynomials, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 39 (1996), 69-83.
[26] L. Tóth and P. Haukkanen, On an operator involving regular convolutions, Mathematica 42 (65) (2000), 199-209.

