# BURNING HAMMING GRAPHS 

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#### Abstract

The Hamming graph $H(n, q)$ is defined on the vertex set $[q]^{n}$ and two vertices are adjacent if and only if they differ in precisely one coordinate. Alon [1] proved that the burning number of $H(n, 2)$ is $\left\lceil\frac{n}{2}\right\rceil+1$. In this note we show that the burning number of $H(n, q)$ is $\left(1-\frac{1}{q}\right) n+O(\sqrt{n \log n})$ for fixed $q \geq 2$ and $n \rightarrow \infty$.


Graph burning has been actively studied in recent years as a model for the spread of influence in a network, see e.g., [3]. One of the seminal results due to Alon determines the burning number of the $n$-dimensional cube. In this note we obtain the burning number of the Hamming graph asymptotically.

Let us define the burning number of a finite graph $G$ with the vertex set $V$. For vertices $x, y \in V$ let $d(x, y)$ denote the distance between $x$ and $y$. For a non-negative integer $k$, let $\Gamma_{k}(x)$ denote the $k$-neighbors of $x$, that is, the set of vertices $y \in V$ such that $d(x, y) \leq k$. We say that $x_{0}, x_{1}, \ldots, x_{b} \in V$ is a burning sequence of length $b+1$ if $\bigcup_{k=0}^{b} \Gamma_{b-k}\left(x_{k}\right)=V$. The burning number of $G$, denoted by $\beta(G)$, is defined to be the minimum length of the burning sequences.

A path with $n$ vertices has burning number $\lceil\sqrt{n}$, and it is conjectured that $\beta(G) \leq$ $\lceil\sqrt{n}\rceil$ for every $n$-vertex graph $G$. Although this is one of the major open problems concerning graph burning, it is also interesting to find graphs which have small burning number and small maximum degree. We will show that Hamming graphs satisfy these conditions.

Recall some basic facts about Hamming graphs. For positive integers $n, q$, let $[q]:=$ $\{0,1, \ldots, q-1\}$ and let $[q]^{n}$ denote the set of $n$-tuple of elements of $[q]$. For $x \in[q]^{n}$ we write $(x)_{i}$ for the $i$ th coordinate of $x$. The Hamming distance between $x, y \in[q]^{n}$ is defined to be $\#\left\{i:(x)_{i} \neq(y)_{i}\right\}$. The Hamming graph $H(n, q)$ has the vertex set $V=[q]^{n}$, and two vertices are adjacent if they have Hamming distance one. Thus $H(n, q)$ is $(q-1) n$-regular, and $\left|\Gamma_{k}(x)\right|=\sum_{i=0}^{k}(q-1)^{i}\binom{n}{i}$ is independent of the choice of $x \in V$. Note that $H(n, 2)$ is the $n$-dimensional cube.

Alon considered a problem of transmitting on the $n$-dimensional cube. His result is restated in terms of burning number as follows.

Theorem 1 (Alon [1]). $\beta(H(n, 2))=\left\lceil\frac{n}{2}\right\rceil+1=\left\lfloor\frac{n}{2}+\frac{3}{2}\right\rfloor$.

It is easy to see that $\beta(H(n, 2)) \leq b+1$, where $b:=\left\lceil\frac{n}{2}\right\rceil$. Indeed, by choosing $x_{0}=$ $(0, \ldots, 0)$ and $x_{1}=(1, \ldots, 1)$, we see that $\Gamma_{b}\left(x_{0}\right) \cup \Gamma_{b-1}\left(x_{1}\right)=[2]^{n}$. (Thus you can choose $x_{2}, \ldots, x_{b}$ arbitrarily.) On the other hand, it is non-trivial to see that $b+1$ is the correct lower bound. To this end, Alon used the Beck-Fiala Theorem [2] in discrepancy theory.

Now we show that $\beta(H(n, q))=\left(1-\frac{1}{q}\right) n+O(\sqrt{n \log n})$ for fixed $q \geq 2$ and $n \rightarrow \infty$.

Theorem 2. We have

$$
\beta(H(n, q)) \leq\left\lfloor\left(1-\frac{1}{q}\right) n+\frac{q+1}{2}\right\rfloor .
$$

Proof. Let $n=q k+r$ where $r \in[q]$, and let $s:=\left\lceil r-\frac{r}{q}+\frac{q-1}{2}+\frac{1}{2 q}\right\rceil$. Let $b+1=$ $(q-1) k+s$, and let $x_{i}=(i, \ldots, i)$ for $i \in[q]$. Note that $x \in \Gamma_{b}\left(x_{0}\right)$ if and only if $\#\left\{j:(x)_{j} \neq 0\right\} \leq b$, or equivalently, $\#\left\{j:(x)_{j}=0\right\} \geq n-b=k+r-s+1$. Similarly, $x \in \Gamma_{b-i}\left(x_{i}\right)$ if and only if $\#\left\{j:(x)_{j}=i\right\} \geq k+r-s+1+i$ for $i \in[q]$. We claim that $W:=\bigcup_{i=0}^{q-1} \Gamma_{b-i}\left(x_{i}\right)=[q]^{n}$. To the contrary, suppose that $y \in[q]^{n}$ is not covered by $W$. Then $\#\left\{j:(y)_{j}=i\right\} \leq k+r-s+i$ for all $i \in[q]$. By summing both sides from $i=0$ to $q-1$, we get $n=q k+r \leq(k+r-s) q+\binom{q}{2}$, or equivalently, $s \leq r-\frac{r}{q}+\frac{q-1}{2}$, a contradiction. Thus we have $\beta(H(n, q)) \leq b+1=(q-1) \frac{n-r}{q}+s=\left\lfloor\left(1-\frac{1}{q}\right) n+\frac{q+1}{2}\right\rfloor$, where we used $\lceil N /(2 q)\rceil=\lfloor(N+2 q-1) /(2 q)\rfloor$ for $N \in \mathbb{Z}$ in the last equality.
Theorem 3. Let $p=1-\frac{1}{q}$. Then we have

$$
\beta(H(n, q))>p n-\sqrt{2 p n \log n}
$$

Proof. Let $b=\lfloor p n-\sqrt{2 p n \log n}\rfloor$. We need to show that no matter how $x_{0}, \ldots, x_{b} \in V$ are chosen, we have $\left|\bigcup_{i=0}^{b} \Gamma_{b-i}\left(x_{i}\right)\right|<|V|$. Even more strongly, we claim that $(b+1)\left|\Gamma_{b}\left(x_{0}\right)\right|<$ $|V|$.

Let $X_{1}, \ldots, X_{n}$ be independent random variables with $\mathbb{P}\left[X_{i}=1\right]=p, \mathbb{P}\left[X_{i}=0\right]=1-p$. Let $X=\sum_{i=1}^{n} X_{i}$. Then we have $\mu:=\mathbb{E}[X]=p n$ and $\mathbb{P}[X=k]=\binom{n}{k} p^{k}(1-p)^{n-k}$. It follows from the Chernoff bound [4] (see also e.g., Corollary 23.7 in [5]), by setting $\epsilon=\sqrt{(2 \log n) /(p n)}$, that

$$
\mathbb{P}[X \leq p n-\sqrt{2 p n \log n}]=\mathbb{P}[X \leq(1-\epsilon) \mu] \leq \exp \left(-\frac{\mu \epsilon^{2}}{2}\right)=\frac{1}{n}
$$

Thus we have

$$
\begin{aligned}
q^{-n}\left|\Gamma_{b}\left(x_{0}\right)\right| & =q^{-n} \sum_{k=0}^{b}(q-1)^{k}\binom{n}{k}=\sum_{k=0}^{b}\binom{n}{k} p^{k}(1-p)^{n-k} \\
& =\mathbb{P}[X \leq b] \leq \mathbb{P}[X \leq p n-\sqrt{2 p n \log n}] \leq \frac{1}{n}
\end{aligned}
$$

and so

$$
(b+1)\left|\Gamma_{b}\left(x_{0}\right)\right| \leq(b+1) \frac{|V|}{n}<p|V|<|V|
$$

as desired.
Conjecture. $\beta(H(n, q))=\left(1-\frac{1}{q}\right) n+O(1)$.

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