

# MANIFOLD WITH INFINITELY MANY FIBRATIONS OVER THE SPHERE

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ABSTRACT. We show that the manifold  $X = S^2 \times S^3$  has infinitely many structures of a fiber bundle over the base  $B = S^2$ . In fact for every lens space  $L(p, 1)$  there is a fibration  $L(p, 1) \rightarrow X \rightarrow B$ .

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## 1. INTRODUCTION

Let  $X, F, B$  be smooth manifold. Consider smooth fibrations  $F \rightarrow X \rightarrow B$ . If  $X = B = S^1$ , then there is infinitely many different fibers  $F_k$  such that we have a fibration  $F_k \rightarrow X \rightarrow B$ . This suggests that also in higher dimensions we can find manifolds  $X$ , which have infinitely many fibrations over  $S^1$ . And indeed Tollefson in [10] found some Seifert manifolds having infinitely many different fiber bundle structures over  $S^1$ . W.P. Thurston in [9] showed that if a hyperbolic 3-manifold with  $b_1 > 1$  fibers over  $S^1$ , then it fibers in infinitely many ways.

Later Hilden, Lozano, Montesinos-Amilibia in the paper - [7] considered a certain family of hyperbolic manifolds, obtained as branched covers of the 3-torus. They showed explicitly that each manifold in this family has infinitely many different fibrations over  $S^1$ .

Using Tollesfson example one can find diferent examples of this type, if  $N$  is the Tollesfson example and  $M$  any manifold then  $N \times M$  has infinitely many fibrations over  $S^1 \times M$  (see [1]). However all examples of this type have non simply connected base  $B$ . To the best knowledge of the authors there were no examples of a manifold with infinitely many different fibration over a simply connected base  $B$ .

There is natural question how many different fibration of this type can exists if the base  $B$  is simply connected. Is it possible that the number of different fibration is infinite? Here using ideas from [4], [5] and [6] we show the following:

**Theorem 3.3.** *The manifold  $X = S^2 \times S^3$  has infinitely many structures of a fiber bundle over the base  $B = S^2$ . In fact for every lens space  $L(p, 1)$  there is a fibration  $L(p, 1) \rightarrow X \rightarrow B$ ,  $p = 1, 2, \dots$*

## 2. PRELIMINARIES

We start with two definitions.

**Definition 2.1.** *Let  $X \subset \mathbb{CP}^n$  be an algebraic variety. We assume  $\mathbb{CP}^n$  to be a hyperplane at infinity of  $\mathbb{CP}^{n+1}$ . Then by an algebraic cone  $\overline{C(X)} \subset \mathbb{CP}^{n+1}$  with base  $X$  we mean the set*

$$\overline{C(X)} = \bigcup_{x \in X} \overline{O, x},$$

where  $O$  is the center of coordinates in  $\mathbb{C}^{n+1} \subset \mathbb{CP}^{n+1}$ , and  $\overline{O, x}$  means the projective line which goes through  $O$  and  $x$ . By an affine cone  $C(X)$  we mean  $\overline{C(X)} \setminus X$ . By the link of  $C(X)$  we mean the set  $L = \{x \in C(X) : \|x\| = 1\}$ .

**Definition 2.2.** *The three-dimensional lens spaces  $L(p; q)$  are quotients of the sphere  $S^3$  by  $\mathbb{Z}/p$ -actions. More precisely, let  $p$  and  $q$  be coprime integers and consider  $S^3$  as the unit sphere in  $\mathbb{C}^2$ . Then the  $\mathbb{Z}/p$ -action on  $S^3$  generated by the homeomorphism*

$$(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$$

*is free. The resulting quotient space is called the lens space  $L(p; q)$ .*

It is well known that

$$H_k(L(p, q), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, 3 \\ \mathbb{Z}/p\mathbb{Z}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

## 3. MAIN RESULT

Let  $W_k$  denotes the Veronese embedding of degree  $k$  of  $\mathbb{CP}^1$  into  $\mathbb{CP}^k$  given by  $\psi([z_0, z_1]) = [z_0^k : z_0^{k-1}z_1 : \dots, z_0z_1^{k-1} : z_1^k]$ . Let  $n \geq 2$  and consider the varieties  $X_k = \phi(W_k \times \mathbb{CP}^1) \subset \mathbb{CP}^{n_k}$ , where  $\phi : \mathbb{CP}^k \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^{n_k}$ ,  $\phi([z_0 : z_1 : \dots : z_k], [w_0, w_1]) = [z_0w_0 : z_0w_1 : z_1w_0 : z_1w_1, \dots : z_kw_0 : z_kw_1]$  is the Segre embedding. Consider the affine cone  $C(X_k)$ . Let  $L_k$  be the link of this cone. We have

**Theorem 3.1.** *All manifolds  $L_k$  are diffeomorphic to  $S^2 \times S^3 = X$ .*

*Proof.* This result follows from a more general result [4], however for the convenience of the reader we give here a direct proof of this fact (which is slightly simpler than this in [4]). By construction,  $X_k$  is the union of projective lines  $X_k = \bigcup_{a \in W_k} \phi(\{a\} \times \mathbb{P}^1)$ . This means that the projective cone  $\overline{C(X_k)}$  is the union of planes which have the line  $\phi(\{a\} \times \mathbb{P}^1)$  at infinity and go through the point  $O = (0, \dots, 0)$ . Thus the link  $L_k$ , of this cone is a union of 3-spheres  $S^3$ . In fact using the Ehresmann Theorem, it is easy to observe that this link is a sphere bundles over  $W_k \cong S^2$  with the projection being the composition of the projection  $p : \mathbb{CP}^{n_k+1} \setminus \{0\} \rightarrow \mathbb{CP}^{n_k}$  and the projection  $q : W_k \times \mathbb{CP}^1 \rightarrow W_k$ . From the homotopy sequence

$$0 = \pi_2(S^3) \rightarrow \pi_2(L_k) \rightarrow \pi_2(S^2) = \mathbb{Z} \rightarrow \pi_1(S^3) = 0 \rightarrow \pi_1(L_k) \rightarrow \pi_1(S^2) = 0,$$

we get  $\pi_1(L_k) = 0$  and by the Hurewicz theorem  $H_2(L_k, \mathbb{Z}) = \pi_2(L_k) = \mathbb{Z}$ . Note that  $H_2(S^2 \times S^3) = \mathbb{Z}$ . Since  $w_1(S^2), w_1(S^3)$  and  $w_2(S^2), w_2(S^3)$  vanish by the Whitney Product Theorem the Stiefel-Whitney class  $w_2(S^2 \times S^3) = \sum_{i=0}^2 w_i(S^2) \times w_{2-i}(S^3) = 0$ . Hence by the Smale-Barden classification (see [2]) it is enough to prove that  $w_2(L_k) = 0$ .

Let  $\pi : L_k \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 = B$  be the Hopf fibration. Let  $E = \ker d\pi$ . Then  $E$  is a one dimensional (real) vector subbundle of  $TL_k$ . Let us introduce on  $L_k$  a Riemannian metric  $g$  and let  $F = E^\perp$ . Then  $d\pi : F \rightarrow TB$  induces the isomorphism of fibers:  $d\pi : F_x \rightarrow TB_{\pi(x)}$ . We will show that  $F$  is isomorphic to  $\pi^*TB$ . For a  $X \in T_{\pi(x)}B$  let  $X_x^* \in F_x \subset T_xL_k$  be a vector such that  $d\pi(X_x^*) = X$ . The isomorphism is  $\Phi : \pi^*TB \rightarrow F$  given on a fiber  $\pi^*TB_x$  as  $\Phi(X) = X_x^*$ . Hence  $TL_k = E \oplus \pi^*(TB)$  and  $w_2(L_k) = w_2(F) + w_1(F) \cup \pi^*w_1(B) + \pi^*w_2(B)$ .

Since  $w_1(S^2)$  and  $w_2(S^2)$  vanish, by the Whitney Product Theorem, it follows that  $w_1(B) = \sum_{i=0}^1 w_i(S^2) \times w_{1-i}(S^2) = 0$ ,  $w_2(B) = w_2(S^2 \times S^2) = \sum_{i=0}^2 w_i(S^2) \times w_{2-i}(S^2) = 0$ , which implies  $w_2(L_k) = 0$ .  $\square$

**Theorem 3.2.** *Let  $W \subset \mathbb{CP}^n$  be a smooth rational curve of degree  $d$ . Let  $C(W)$  be an affine cone with base  $W$ . Then the link  $L$  of this cone at 0 is diffeomorphic to the lens space  $L(d, 1)$ .*

*Proof.* Note that  $L$  is a principal circle bundle over the sphere. By [3] (see Theorem 2.1, Theorem 2.2 and Theorem 2.3) the group of such bundles is cyclic and generated by  $S^3 = L(1, 1)$ . Moreover,  $pS^3 = S^3/G_p$ , where  $G_p = \mathbb{Z}/(p)$ . Hence  $L$  is diffeomorphic to some  $pS^3$ , which is the lens space  $L(p, 1)$ . By [5], Thm. 3.5 we have  $H^2(L, \mathbb{Z}) = \mathbb{Z}/d$ . On the other hand by [8], point 11, the space  $L$  is a rational homology 3-sphere, in particular  $H_1(L, \mathbb{Z})$  is a torsion group. Since the torsion part of  $H_1(L, \mathbb{Z})$  coincides with the torsion part of  $H^2(L, \mathbb{Z})$  we have  $H_1(L, \mathbb{Z}) = \mathbb{Z}/(d)$ . Hence  $p = d$  and  $L = L(d, 1)$ .  $\square$

**Theorem 3.3.** *The manifold  $X = S^2 \times S^3$  has infinitely many structures of a fiber bundle over the base  $B = S^2$ . In fact for every lens space  $L(p, 1)$  there is a fibration  $L(p, 1) \rightarrow X \rightarrow B$ ,  $p = 1, 2, \dots$*

*Proof.* Let  $W_k$  denote the Veronese embedding of degree  $k$  of  $\mathbb{CP}^1$  into  $\mathbb{CP}^k$ . Let  $n \geq 2$  and consider the varieties  $X_k = \phi(W_k \times \mathbb{CP}^1) \subset \mathbb{CP}^{n_k}$ , where  $\phi$  is the Segre embedding. Consider the affine cone  $C(X_k)$ . Let  $L_k$  be the link of this cone. By construction,  $X_k$  is the union of projective rational curves  $X_k = \bigcup_{a \in \mathbb{CP}^1} W_k \times \{a\}$ . This means that the affine cone  $C(X_k)$  is the union of cones  $\bigcup_{a \in \mathbb{CP}^1} C(W_k \times \{a\})$ . Thus by Theorem 3.2 the link  $L_k$  of this cone is a union of lens spaces  $L(k, 1)$ . In fact using the Ehresmann Theorem, it is easy to observe that this link is a bundle over  $\mathbb{CP}^1 \cong S^2$  with the projection being the composition of the projection  $p : \mathbb{CP}^{n_k+1} \setminus \{0\} \rightarrow \mathbb{CP}^{n_k}$  and the projection  $q : W_k \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ . Consequently  $L_k$  is a  $L(k, 1)$  bundle over  $S^2$ . By Theorem 3.1 this finishes the proof.  $\square$

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