

On one-orbit cyclic subspace codes of $\mathcal{G}_q(n, 3)$

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Abstract—Subspace codes have recently been used for error correction in random network coding. In this work, we focus on one-orbit cyclic subspace codes. If S is an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} , then the one-orbit cyclic subspace code defined by S is

$$\text{Orb}(S) = \{\alpha S : \alpha \in \mathbb{F}_{q^n}^*\},$$

where $\alpha S = \{\alpha s : s \in S\}$ for any $\alpha \in \mathbb{F}_{q^n}^*$.

Few classification results of subspace codes are known, therefore it is quite natural to initiate a classification of cyclic subspace codes, especially in the light of the recent classification of the isometries for cyclic subspace codes. We consider three-dimensional one-orbit cyclic subspace codes, which are divided into three families: the first one containing only $\text{Orb}(\mathbb{F}_{q^3})$; the second one containing the optimum-distance codes; and the third one whose elements are codes with minimum distance 2. We study inequivalent codes in the latter two families.

I. INTRODUCTION

Let k be a non-negative integer with $k \leq n$. The set of all k -dimensional \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} , viewed as an \mathbb{F}_q -vector space, forms a **Grassmannian space** over \mathbb{F}_q , which is denoted by $\mathcal{G}_q(n, k)$. A **constant dimension subspace code** is a subset C of $\mathcal{G}_q(n, k)$ endowed with the metric defined as follows

$$d(U, V) = \dim_{\mathbb{F}_q}(U) + \dim_{\mathbb{F}_q}(V) - 2 \dim_{\mathbb{F}_q}(U \cap V),$$

where $U, V \in C$. This metric is also known as **subspace metric**. As usual, we define the **minimum distance** of C as

$$d(C) = \min\{d(U, V) : U, V \in C, U \neq V\}.$$

Subspace codes have been recently used for the error correction in random network coding, see [15]. The first class of subspace codes studied was the one introduced in [7], known as **cyclic subspace codes**. A subspace code $C \subseteq \mathcal{G}_q(n, k)$ is said to be **cyclic** if $\alpha V \in C$ for every $\alpha \in \mathbb{F}_{q^n}^*$ and every $V \in C$. If C coincides with $\text{Orb}(S)$, for some subspace S of \mathbb{F}_{q^n} , we say that C is a **one-orbit** cyclic subspace code and S is said to be a **representative** of the orbit.

Let S be an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} of dimension k and let $d = \gcd(n, k)$. Then $|\text{Orb}(S)| = \frac{q^n - 1}{q^d - 1}$ if and only if \mathbb{F}_{q^d} is the maximum subfield of \mathbb{F}_{q^n} of linearity of S ; see [22, Theorem 1]. Therefore, every orbit of a subspace $V \in \mathcal{G}_q(n, k)$ defines a cyclic subspace code of size $(q^n - 1)/(q^t - 1)$, for some $t \mid n$. Let S be a strictly \mathbb{F}_q -linear subspace of dimension k , i.e. \mathbb{F}_q is the maximum field on which S is linear. Then the cyclic subspace code defined by S has size $(q^n - 1)/(q - 1)$. Also, in this case, the maximum value for the minimum distance is at most $2k$ and it is exactly $2k$ if and only if the orbit of S

is a k -spread of \mathbb{F}_{q^n} , i.e. $\text{Orb}(\mathbb{F}_{q^k})$ and $k = 1$ because of the linearity assumption.

In [26] the authors conjectured the existence of a cyclic code of size $\frac{q^n - 1}{q - 1}$ in $\mathcal{G}_q(n, k)$ and minimum distance $2k - 2$ for every pair of positive integers n, k such that $1 < k \leq n/2$. These codes are also known as **optimum-distance codes**.

In [3] the authors used subspace polynomials to generate cyclic subspace codes with size $\frac{q^n - 1}{q - 1}$ and minimum distance $2k - 2$, proving that the conjecture is true for any given k and infinitely many values of n . This was improved in [22]. Finally, the conjecture was solved in [24] for most of the cases, by making use of Sidon spaces originally introduced in [1], in relation with the linear analogue of Vosper’s Theorem (see also [2], [14]). An \mathbb{F}_q -subspace S of \mathbb{F}_{q^n} is called a **Sidon space** if S satisfies the following property: for all nonzero $a, b, c, d \in S$ such that $ab = cd$ then $\{a\mathbb{F}_q, b\mathbb{F}_q\} = \{c\mathbb{F}_q, d\mathbb{F}_q\}$, where $e\mathbb{F}_q = \{e\lambda : \lambda \in \mathbb{F}_q\}$. So, the study of cyclic subspace codes with size $\frac{q^n - 1}{q - 1}$ and minimum distance $2k - 2$ is equivalent to the study of Sidon spaces. Indeed, the following theorem explains the condition $k \leq n/2$ in the aforementioned conjecture. To this aim we need the following notation: if S is an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} then $S^2 = \langle ab : a, b \in S \rangle_{\mathbb{F}_q}$, i.e. S^2 is the \mathbb{F}_q -subspace spanned by the products of pairs of elements of S .

Theorem I.1. [1, Theorem 18] *Let $S \in \mathcal{G}_q(n, k)$ be a Sidon space of dimension $k \geq 3$, then $\dim_{\mathbb{F}_q}(S^2) \geq 2k$.*

Also, Sidon spaces with the smallest dimension of its square span were used in designing a multivariate public-key cryptosystem (see [23]).

Apart from some classification results on subspace codes, see for instance [12], [18], [19], very few classification results are known for cyclic subspace codes. In light of the classification of the isometries for cyclic subspace codes recently proved by Gluesing-Luerssen and Lehmann in [9] (see also [25]), it is quite natural to initiate a classification of cyclic subspace codes.

In this paper we will consider 3-dimensional one-orbit cyclic subspace codes and we will give some classification results. The possible 3-dimensional one-orbit cyclic subspace codes $\text{Orb}(S)$ are of the following types:

- i) $|\text{Orb}(S)| = (q^n - 1)/(q^3 - 1)$ and $d(\text{Orb}(S)) = 6$;
- ii) $|\text{Orb}(S)| = (q^n - 1)/(q - 1)$ and $d(\text{Orb}(S)) = 4$;
- iii) $|\text{Orb}(S)| = (q^n - 1)/(q - 1)$ and $d(\text{Orb}(S)) = 2$.

For Case i), we only have $\text{Orb}(\mathbb{F}_{q^3})$. The codes of the Family ii) are the optimum-distance codes and are those for which S

is a Sidon space; whereas the codes of Family iii) correspond to those subspaces for which there exists at least one $\alpha \in \mathbb{F}_{q^n}^*$ such that $\dim_{\mathbb{F}_q}(S \cap \alpha S) = 2$. So, the problem is to determine inequivalent codes in both Family ii) and iii). To this aim, we first introduce in Section II some new invariants which can be used to distinguish inequivalent classes of codes, based on the square-span of a subspace and the span of a subspace over a larger field. In Section III we give a classification result based on the dimension of the square-span of a representative of the code and we study the equivalence problem for the codes in Family iii). In the last section (cf. Section IV) we investigate the equivalence problem for the codes in Family ii) under the assumption that a representative is contained in the sum of two multiplicative cosets of \mathbb{F}_{q^3} . Some of the technical proofs are in the Appendix to keep this paper short.

II. EQUIVALENCE AND INVARIANTS

The study of the equivalence for subspace codes was initiated by Trautmann in [25] and the case of cyclic subspace codes has been investigated in [9] by Gluesing-Luerssen and Lehmann. Therefore, motivated by [9, Theorem 6.2 (a)], we say that two cyclic subspace codes $\text{Orb}(S_1)$ and $\text{Orb}(S_2)$ in \mathbb{F}_{q^n} are **linearly equivalent** if there exists $i \in \{0, \dots, n-1\}$ such that $\text{Orb}(S_1) = \text{Orb}(S_2^{q^i})$, where $S_2^{q^i} = \{v^{q^i} : v \in S_2\}$. This happens if and only if $S_1 = \alpha S_2^{q^i}$, for some $\alpha \in \mathbb{F}_{q^n}^*$. We can replace the action of the q -Frobenius maps $x \in \mathbb{F}_{q^n} \mapsto x^{q^i} \in \mathbb{F}_{q^n}$ with any automorphism σ in $\text{Aut}(\mathbb{F}_{q^n})$, since this will still preserve the metric properties of the codes. Following [28], we consider an extension of this definition, where we will denote by $S^\sigma = \{\sigma(s) : s \in S\}$ with $\sigma \in \text{Aut}(\mathbb{F}_{q^n})$.

Definition II.1. Let S_1 and S_2 be two \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} . Then we say that S_1 and S_2 are **semilinearly equivalent** (or simply **equivalent**) if the associated codes $\text{Orb}(S_1)$ and $\text{Orb}(S_2)$ are semilinearly equivalent, that is there exist $\sigma \in \text{Aut}(\mathbb{F}_{q^n})$ and $\alpha \in \mathbb{F}_{q^n}^*$ such that $S_1 = \alpha S_2^\sigma$. In this case, we will also say that they are equivalent under the action of the pair $(\alpha, \sigma) \in \mathbb{F}_{q^n}^* \rtimes \text{Aut}(\mathbb{F}_{q^n})$.

In the following we describe some invariants that can be used to establish whether or not two one-orbit cyclic subspace codes are equivalent.

The first invariant that we introduce regards the dimension of the square-span of a subspace. Indeed, the following is easy to check.

Proposition II.2. Let S_1, S_2 be two \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} and suppose that S_1 and S_2 are equivalent under the action of (α, σ) , then S_1^2 and S_2^2 are equivalent under the action of (α^2, σ) . In particular, $\dim_{\mathbb{F}_q}(S_1^2) = \dim_{\mathbb{F}_q}(S_2^2)$.

Definition II.3. For any divisor t of n and for any \mathbb{F}_q -subspace S of \mathbb{F}_{q^n} , we denote by

$$\delta_t(S) := \dim_{\mathbb{F}_{q^t}}(\langle S \rangle_{\mathbb{F}_{q^t}})$$

and

$$w_t(S) := \max\{\dim_{\mathbb{F}_{q^t}}(S') : S' \in \mathcal{S}'\}$$

where

$$\mathcal{S}' = \{S' : S' \text{ is } \mathbb{F}_q\text{-subspace of } S \text{ and } \mathbb{F}_{q^t}\text{-subspace of } \mathbb{F}_{q^n}\}.$$

Note that the integer $\delta_t(S)$ has been introduced in [9, Definition 4.5]. Clearly, if $\dim_{\mathbb{F}_q}(S) > 0$, then

$$1 \leq \delta_t(S) \leq \min\left\{\frac{n}{t}, \dim_{\mathbb{F}_q}(S)\right\}$$

and

$$0 \leq w_t(S) \leq \frac{\dim_{\mathbb{F}_q}(S)}{t}.$$

It is easy to see that these two integers are invariant under semilinear equivalence.

Proposition II.4. Let S_1, S_2 be two \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} and suppose that S_1 and S_2 are equivalent under the action of (α, σ) . If $t \mid n$, then $\langle S_1 \rangle_{\mathbb{F}_{q^t}}$ and $\langle S_2 \rangle_{\mathbb{F}_{q^t}}$ are equivalent under the action of (α, σ) . In particular, $\delta_t(S_1) = \delta_t(S_2)$ and $w_t(S_1) = w_t(S_2)$.

III. A CLASSIFICATION RESULT

We now present a classification of three dimensional \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} , based on the equivalence given in Definition II.1, yielding a classification of three dimensional one-orbit cyclic subspace codes, by making use of the following lemma (an extension of [1, Lemma 4]).

Lemma III.1. [20, Lemma 3.1] Let S be an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} of dimension $k \geq 2$ and let $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$. Let $t = \dim_{\mathbb{F}_q}(\mathbb{F}_q(\lambda))$, where $\mathbb{F}_q(\lambda)$ denotes the field extension of \mathbb{F}_q generated by λ .

- (a) If $\dim_{\mathbb{F}_q}(S \cap \lambda S) = k$, then S is an $\mathbb{F}_q(\lambda)$ -subspace.
- (b) Suppose that $\dim_{\mathbb{F}_q}(S \cap \lambda S) = k-1$ and $t \geq k$. Then $S = \mu \langle 1, \lambda, \dots, \lambda^{k-1} \rangle_{\mathbb{F}_q}$, for some $\mu \in \mathbb{F}_{q^n}^*$ and $t \neq k$.
- (c) Suppose that $\dim_{\mathbb{F}_q}(S \cap \lambda S) = k-1$ and $t \leq k-1$. Write $k = t\ell + m$ with $m < t$, then $m > 0$ and $S = \overline{S} \oplus \mu \langle 1, \lambda, \dots, \lambda^{m-1} \rangle_{\mathbb{F}_q}$, where \overline{S} is an \mathbb{F}_{q^t} -subspace of dimension ℓ , $\mu \in \mathbb{F}_{q^n}^*$ and $\mu \mathbb{F}_{q^t} \cap \overline{S} = \{0\}$. In particular, t is a proper divisor of n .

We are now ready to prove our first classification result, which is mainly based on the first invariant introduced in Section II.

Theorem III.2. Let S be an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} such that $\dim_{\mathbb{F}_q}(S) = 3$. Then

- 1) $\dim_{\mathbb{F}_q}(S^2) = 3$ if and only if S is a multiplicative coset of \mathbb{F}_{q^3} , i.e. $3 \mid n$ and $S = \mu \mathbb{F}_{q^3}$ for some $\mu \in \mathbb{F}_{q^n}^*$;
- 2) $\dim_{\mathbb{F}_q}(S^2) = 4$ if and only if one of the following holds:
 - 2.1) $S = \mu \langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$ for some $\mu, \lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$, $4 \mid n$ and $\delta_4(S) = 1$.
 - 2.2) $S = \omega \mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q}$ for some $\mu, \omega \in \mathbb{F}_{q^n}$ such that $\mu \notin \omega \mathbb{F}_{q^2}$, $4 \mid n$, $\delta_4(S) = 1$ and $w_2(S) = 1$.
- 3) $\dim_{\mathbb{F}_q}(S^2) = 5$ if and only if one of the following holds:
 - 3.1) $S = \mu \langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$ for some $\mu, \lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ such that $\dim_{\mathbb{F}_q}(\mathbb{F}_q(\lambda)) > 4$; also, if $2 \mid n$ then $\delta_2(S) = 3$ and $w_2(S) = 0$.

3.2) $S = \omega\mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q}$ for some $\mu, \omega \in \mathbb{F}_{q^n}$ such that $\mu \notin \omega\mathbb{F}_{q^4}$, $\delta_2(S) = 2$ and $w_2(S) = 1$.

4) $\dim_{\mathbb{F}_q}(S^2) = 6$ if and only if S is a Sidon space.

Proof. Since $\dim_{\mathbb{F}_q}(S) = 3$ then $3 \leq \dim_{\mathbb{F}_q}(S^2) \leq 6$, so we have four cases to analyze and we split the analysis according to the dimension of the square-span of S .

Case 1) Suppose that $\dim_{\mathbb{F}_q}(S^2) = 3 = \dim_{\mathbb{F}_q}(S)$ then, without loss of generality we may assume that $1 \in S$. Then $S^2 = S$ and so for any $\lambda \in S \setminus \mathbb{F}_q$, we get $S = \lambda S$, i.e. S is $\mathbb{F}_q(\lambda)$ -subspace of \mathbb{F}_{q^n} and since $\dim_{\mathbb{F}_q}(S) = 3$ then $\mathbb{F}_q(\lambda) = \mathbb{F}_{q^3}$ and $3 \mid n$.

Case 2) Suppose that $\dim_{\mathbb{F}_q}(S^2) = 4$, then S is not a Sidon space by Theorem I.1, i.e. there exists $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ such that

$$\dim_{\mathbb{F}_q}(S \cap \lambda S) > 1,$$

and hence $2 \leq \dim_{\mathbb{F}_q}(S \cap \lambda S) \leq 3$. If $\dim_{\mathbb{F}_q}(S \cap \lambda S) = 3 = \dim_{\mathbb{F}_q}(S)$ then $S = \lambda S$, which implies $\mathbb{F}_q(\lambda) = \mathbb{F}_{q^3}$ and $\dim_{\mathbb{F}_q}(S^2) = 3$, a contradiction to $\dim_{\mathbb{F}_q}(S^2) = 4$. Therefore $\dim_{\mathbb{F}_q}(S \cap \lambda S) = 2 = \dim_{\mathbb{F}_q}(S) - 1$. Lemma III.1 implies that one of the following cases occurs

- 2.1) $S = \mu\langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$ for some $\mu \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ and $\mathbb{F}_q(\lambda) \neq \mathbb{F}_{q^2}$;
- 2.2) $S = \omega\mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q}$ for some $\mu \in \mathbb{F}_{q^n} \setminus \omega\mathbb{F}_{q^2}$ and $\mathbb{F}_q(\lambda) = \mathbb{F}_{q^2}$.

If $S = \mu\langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$, then $S^2 = \mu^2\langle 1, \lambda, \lambda^2, \lambda^3, \lambda^4 \rangle_{\mathbb{F}_q}$ and since $\dim_{\mathbb{F}_q}(S^2) = 4$, the elements $1, \lambda, \lambda^2, \lambda^3, \lambda^4$ are \mathbb{F}_q -linearly dependent, i.e. λ is a root of a non-zero polynomial of degree less than or equal to 4 over \mathbb{F}_q . In particular, if the minimal polynomial of λ over \mathbb{F}_q has degree strictly less than 4, then this would give a contradiction to $\dim_{\mathbb{F}_q}(S^2) = 4$. Therefore the minimal polynomial of λ over \mathbb{F}_q has degree 4, which implies that $\mathbb{F}_q(\lambda) = \mathbb{F}_{q^4}$, so $4 \mid n$ and $S \subseteq \mu\mathbb{F}_{q^4}$, i.e. $\delta_4(S) = 1$.

If $S = \omega\mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q}$, then $S^2 = \omega^2\mathbb{F}_{q^2} + \omega\mu\mathbb{F}_{q^2} + \langle \mu^2 \rangle_{\mathbb{F}_q}$. Let observe that if $\omega^2\mathbb{F}_{q^2} = \omega\mu\mathbb{F}_{q^2}$ then $\frac{\mu}{\omega} \in \mathbb{F}_{q^2}$, i.e. $\mu \in \omega\mathbb{F}_{q^2}$, a contradiction. Therefore $\mu^2 \in \omega^2\mathbb{F}_{q^2} + \omega\mu\mathbb{F}_{q^2}$, i.e. there exist $\alpha, \beta \in \mathbb{F}_{q^2}$ such that $\mu^2 = \alpha\omega^2 + \beta\omega\mu$. This implies that $\frac{\omega}{\mu}$ is a root of the polynomial $\alpha x^2 + \beta x - 1 = 0$ whose coefficients are in \mathbb{F}_{q^2} . Then $\frac{\omega}{\mu} \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. So, we have that there exists $\rho \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$ such that $\mu = \rho\omega$. Then

$$S = \omega\mathbb{F}_{q^2} + \omega\langle \rho \rangle_{\mathbb{F}_q} = \omega(\mathbb{F}_{q^2} + \langle \rho \rangle_{\mathbb{F}_q}) \subset \omega\mathbb{F}_{q^4}$$

and so $4 \mid n$.

Case 3) Suppose that $\dim_{\mathbb{F}_q}(S^2) = 5$, then S is not a Sidon space by Theorem I.1, and so arguing as before, there exists $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ such that $\dim_{\mathbb{F}_q}(S \cap \lambda S) = 2 = \dim_{\mathbb{F}_q}(S) - 1$. Then Lemma III.1 and $\dim_{\mathbb{F}_q}(S^2) = 5$ implies that S has one of the following forms:

- 3.1) $S = \mu\langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$ for some $\mu \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ and $\dim_{\mathbb{F}_q}(\mathbb{F}_q(\lambda)) > 4$;
- 3.2) $S = \omega\mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q}$ for some $\mu \in \mathbb{F}_{q^n} \setminus \omega\mathbb{F}_{q^2}$ and $\mathbb{F}_q(\lambda) = \mathbb{F}_{q^2}$ and $\mu^2 \notin \omega^2\mathbb{F}_{q^2} + \omega\mu\mathbb{F}_{q^2}$.

In particular, in Case 3.2), $\mu^2 \notin \omega^2\mathbb{F}_{q^2} + \omega\mu\mathbb{F}_{q^2}$ because otherwise $\frac{\mu}{\omega}$ would be root of a non zero polynomial of degree

2 over \mathbb{F}_{q^2} , hence $\frac{\mu}{\omega} \in \mathbb{F}_{q^4}$. In this case $\omega S \subseteq \mathbb{F}_{q^4}$ and so $\dim_{\mathbb{F}_q}(S^2) \leq 4$, a contradiction.

Moreover, in Case 3.1), since $\lambda \notin \mathbb{F}_{q^2}$, if $\delta_2(S) = 2$, then there exist $a, b \in \mathbb{F}_{q^2}$ such that $\lambda^2 = a + b\lambda$, i.e. $\mathbb{F}_{q^2}(\lambda) = \mathbb{F}_{q^4}$, a contradiction. Hence $\delta_2(S) = 3$. If there exists $\xi \in \mathbb{F}_{q^n}$ such that $\xi\mathbb{F}_{q^2} \subseteq S$, then $S \subseteq \xi\mathbb{F}_{q^2} \oplus \mu\mathbb{F}_{q^2}$, where $\mu \in S \setminus \xi\mathbb{F}_{q^2}$, and since $\delta_2(S) = 3$, we have a contradiction. Thus $w_2(S) = 0$.

In Case 3.2) it is clear that $\delta_2(S) = 2$ and $w_2(S) = 1$.

Case 4) If $\dim_{\mathbb{F}_q}(S^2) = 6$, since $\dim_{\mathbb{F}_q}(S) = 3$, then S^2 has its maximum possible dimension and by [24, Lemma 20] it follows that S is a Sidon space. Conversely, if S is a Sidon space, then Theorem I.1 implies the assertion. \square

Cases 2.1 and 2.2 give rise to equivalent examples. To see this, we need the following well-known lemma (which follows by [17, Theorem 2.24]).

Lemma III.3. *Let H_1, H_2 be two \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} such that $\dim_{\mathbb{F}_q}(H_1) = \dim_{\mathbb{F}_q}(H_2) = n - 1$, then there exists $\xi \in \mathbb{F}_{q^n}^*$ such that $H_2 = \xi H_1$.*

Proposition III.4. *Let S_1 and S_2 be two three-dimensional \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} such that*

$$S_1 = \mu\langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q} \subseteq \mu\mathbb{F}_{q^4} \text{ and } S_2 = \omega\mathbb{F}_{q^2} + \langle \eta \rangle_{\mathbb{F}_q} \subseteq \omega\mathbb{F}_{q^4},$$

for some $\mu, \lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ and $\eta \in \mathbb{F}_{q^n} \setminus \omega\mathbb{F}_{q^2}$. Then S_1 and S_2 are equivalent.

Proof. By Lemma III.3 there exists $\xi \in \mathbb{F}_{q^n}^*$ such that $\mu^{-1}S_1 = \xi(\omega^{-1}S_2)$, that is $S_1 = \xi\mu\omega^{-1}S_2$, i.e. S_1 and S_2 are equivalent. \square

Remark III.5. *Proposition III.4 shows that all the subspaces in Case 2) are equivalent to a subspace of Case 2.1), i.e. they all admit a polynomial basis. Moreover, by Theorem III.2 and Proposition II.4 we get that three dimensional subspaces in \mathbb{F}_{q^n} as in Cases 3.1) and 3.2) are not equivalent under the action of semilinear equivalence.*

We now discuss the equivalence among the codes as in Case 3.1) and later those of in Case 3.2).

Theorem III.6. *Let S, T be two \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} of dimension 3 such that*

$$S = \langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q} \text{ and } T = \langle 1, \mu, \mu^2 \rangle_{\mathbb{F}_q}$$

for some $\lambda, \mu \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ such that $\dim_{\mathbb{F}_q}(\mathbb{F}_q(\lambda)) > 4$ and $\dim_{\mathbb{F}_q}(\mathbb{F}_q(\mu)) > 4$. Then S and T are equivalent under the action of $(\xi, \sigma) \in \mathbb{F}_{q^n}^ \rtimes \text{Aut}(\mathbb{F}_{q^n})$ if and only if $\lambda^\sigma = \frac{\alpha_0 + \alpha_1 \mu}{\beta_0 + \beta_1 \mu}$ with $(\alpha_1, \beta_1) \neq (0, 0)$.*

Theorem III.7. *Let S, T be two \mathbb{F}_q -subspaces of \mathbb{F}_{q^n} of dimension 3 such that*

$$S = \mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q} \text{ and } T = \mathbb{F}_{q^2} + \langle \eta \rangle_{\mathbb{F}_q}$$

for some $\mu, \eta \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^2}$. Then S and T are equivalent under the action of $(\xi, \sigma) \in \mathbb{F}_{q^n}^ \rtimes \text{Aut}(\mathbb{F}_{q^n})$ if and only if $\xi \in \mathbb{F}_{q^2}$ and $\eta = a + \xi\mu^\sigma b$ where $a \in \mathbb{F}_{q^2}$ and $b \in \mathbb{F}_q^*$.*

In terms of codes we obtain the following result, as a corollary of the results of this section.

Corollary III.8. *Let C be a one-orbit cyclic subspace code with dimension three in \mathbb{F}_{q^n} . Then C is equivalent to $\text{Orb}(S)$ where S satisfies one of the following conditions*

- I) $S = \mathbb{F}_{q^3}$, $d(C) = 6$;
- II) $\dim_{\mathbb{F}_q}(S^2) = 4$, $S = \langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$ for some $\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, $d(C) = 2$, $\delta_4(S) = 1$ and $w_2(S) = 1$;
- III) $\dim_{\mathbb{F}_q}(S^2) = 5$, $S = \langle 1, \lambda, \lambda^2 \rangle_{\mathbb{F}_q}$ for some $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^4}$, $d(C) = 2$, if n is even $\delta_2(S) = 3$ and $w_2(S) = 0$;
- IV) $\dim_{\mathbb{F}_q}(S^2) = 5$, $S = \mathbb{F}_{q^2} + \langle \mu \rangle_{\mathbb{F}_q}$ for some $\mu \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^4}$, $d(C) = 2$, $\delta_2(S) = 2$ and $w_2(S) = 1$;
- V) S is a Sidon space, $d(C) = 4$.

Remark III.9. *By using Corollary III.8 together with Lemma III.3 and Theorems III.2, III.6 and III.7, we can get information about the number of inequivalent codes. The subspaces of Family II) of Corollary III.8, up to equivalence, generate only one orbit. Whereas, when n is odd, it can be proved that the number t of inequivalent orbits of those having minimum distance 2 is bounded as follows*

$$\frac{q^{n-1} - 1}{nh(q^2 - 1)} \leq t \leq \frac{q^{n-1} - 1}{q^2 - 1},$$

where $q = p^h$, p prime, $h \in \mathbb{N}$. When $h = 1$, n is a prime number such that $n > p + 1$, then t reaches the above lower bound. For small values of q and n this number has been computed also in [9], [10]. In the next section, we will investigate the equivalence issue for subspaces of Family V).

IV. OPTIMUM-DISTANCE CODES

In this section we will deal with three-dimensional one-orbit cyclic subspace codes of size $\frac{q^n-1}{q-1}$ having minimum distance 4, under the assumption that $\delta_3(S) = 2$. We will restrict our study to this family, since these are rare objects with respect to those having $\delta_3(S) = 3$. We start by proving that these subspaces/codes admit an interesting polynomial description via linearized polynomials. Recall that a **q -polynomial/linearized polynomial** over \mathbb{F}_{q^n} is a polynomial of the form $\sum_{i=0}^t a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$, for some $t \in \mathbb{N}_0$, and denote by $\mathcal{L}_{n,q}$ the set of q -polynomials over \mathbb{F}_{q^n} .

In the following we prove that a subspace S with $\delta_3(S) = 2$ can be represented via a linearized polynomial.

Proposition IV.1. *Let S be an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} of dimension 3 over \mathbb{F}_q , with $n = 3s$ and $s \geq 2$, such that $\delta_3(S) = 2$. Then there exists $\xi \in \mathbb{F}_{q^n}$ such that $S \cap \xi\mathbb{F}_{q^3} = \{0\}$ and $\xi\mathbb{F}_{q^3} \subseteq \langle S \rangle_{\mathbb{F}_{q^3}}$.*

Remark IV.2. *By the previous proposition, if $\delta_3(S) = 2$ then there exist $\xi, \rho \in \mathbb{F}_{q^n}$ such that $\xi\mathbb{F}_{q^3} \cap S = \{0\}$, $\xi\mathbb{F}_{q^3} \subseteq \langle S \rangle_{\mathbb{F}_{q^3}}$ and $\xi\mathbb{F}_{q^3} + \rho\mathbb{F}_{q^3} = \langle S \rangle_{\mathbb{F}_{q^3}}$. This means that, if $\delta_3(S) = 2$, without loss of generality, then we may assume that $S \subseteq \xi\mathbb{F}_{q^3} + \rho\mathbb{F}_{q^3}$ with $\xi, \rho \in \mathbb{F}_{q^n}$ such that $S \cap \xi\mathbb{F}_{q^3} = \{0\}$ and $\frac{\lambda}{\rho} \notin \mathbb{F}_{q^3}$.*

Proposition IV.3. *Let S be an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} , with $n = 3s$ and $s \geq 2$, such that $\dim_{\mathbb{F}_q}(S) = 3$ and $\langle S \rangle_{\mathbb{F}_{q^3}} = \lambda\mathbb{F}_{q^3} + \rho\mathbb{F}_{q^3}$ with $\frac{\lambda}{\rho} \notin \mathbb{F}_{q^3}$ and $S \cap \lambda\mathbb{F}_{q^3} = \{0\}$. Then there exists a q -polynomial $f \in \mathcal{L}_{3,q}$ such that $S = \{\rho u + \lambda f(u) : u \in \mathbb{F}_{q^3}\}$.*

By the above proposition we have that

$$\rho^{-1}S = \{u + \rho^{-1}\lambda f(u) : u \in \mathbb{F}_{q^3}\}.$$

Therefore, up to multiplication by a scalar in \mathbb{F}_{q^n} , we have that we may represent an \mathbb{F}_q -subspace S of \mathbb{F}_{q^n} , with $n = 3s$ and $s \geq 2$, such that $\dim_{\mathbb{F}_q}(S) = 3$ and $\delta_3(S) = 2$ by a linearized polynomial over \mathbb{F}_{q^3} defined as in Proposition IV.3, i.e.

$$S = \{u + \gamma f(u) : u \in \mathbb{F}_{q^3}\} = V_{f,\gamma}.$$

So, up to equivalence, we may assume that the subspaces we want to study are of the form $V_{f,\gamma} \subseteq \mathbb{F}_{q^n}$. The equivalence among subspaces of the form $V_{f,\gamma}$ has been studied in [5], in a more general setting.

Theorem IV.4. [5, Theorem 6.2] *Let k and n be two positive integers such that $k \mid n$. Let U, W be two m -dimensional \mathbb{F}_q -subspaces of \mathbb{F}_{q^k} . Consider*

$$V_{U,\gamma} = \{u + u'\gamma : (u, u') \in U\}$$

and

$$V_{W,\xi} = \{w + w'\xi : (w, w') \in W\},$$

where $\gamma, \xi \in \mathbb{F}_{q^n}$ are such that $\{1, \gamma\}$ and $\{1, \xi\}$ are \mathbb{F}_{q^k} -linearly independent and $\delta_k(V_{U,\gamma}) = \delta_k(V_{W,\xi}) = 2$. Then $V_{U,\gamma}$ and $V_{W,\xi}$ are equivalent under the action of $(\lambda, \sigma) \in \mathbb{F}_{q^n}^* \rtimes \text{Aut}(\mathbb{F}_{q^n})$ if and only if there exists $A = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \in \text{GL}(2, \mathbb{F}_{q^k})$ such that $\xi = \frac{a+b\gamma\sigma}{c+d\gamma\sigma}$, $\lambda = \frac{1}{c+d\gamma\sigma}$ and $U^\sigma = \{wA : w \in W\} = W \cdot A$.

Clearly, if $U_f = \{(u, f(u)) : u \in \mathbb{F}_{q^3}\}$, which is an \mathbb{F}_q -subspace of $\mathbb{F}_{q^3}^2$, then $V_{f,\gamma} = V_{U_f,\gamma}$.

We are now ready to provide a classification result for three dimensional subspaces S of \mathbb{F}_{q^n} such that $\delta_3(S) = 2$. We recall that $\text{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x) = x + x^q + x^{q^2} + \dots + x^{q^{k-1}}$ for $x \in \mathbb{F}_{q^k}$.

Theorem IV.5. *Let n be a multiple of 3 and let S be an \mathbb{F}_q -subspace of \mathbb{F}_{q^n} with dimension 3 and $\delta_3(S) = 2$. Then there exists $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$ such that S is equivalent either to $V_{x^q,\gamma}$ or to $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x),\gamma}$.*

Proof. Up to equivalence, because of the above results, we can assume that

$$S = V_{f,\gamma},$$

for some $f \in \mathcal{L}_{3,q}$ and $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. Let $U_f \subseteq \mathbb{F}_{q^3}^2$. By [16] and [8] (see also [6]), U_f is $\text{GL}(2, q^3)$ -equivalent either to U_{x^q} or to $U_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x)}$, i.e. there exist $A = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \in \text{GL}(2, q^3)$ and $\sigma \in \text{Aut}(\mathbb{F}_{q^3})$ such that $U_f^\sigma = U_{x^q}A$ or $U_f^\sigma = U_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x)}A$. By Theorem IV.4 S is equivalent either to $V_{x^q,\gamma'}$ or to $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x),\gamma'}$, where $\gamma' = \frac{a+b\gamma\sigma}{c+d\gamma\sigma}$. \square

We can characterize the values of γ for which $V_{x^q, \gamma}$ and $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$ are Sidon spaces. The former one has been already characterized in [24, Theorems 12 and 16] and [5, Theorem 4.5], which in our case reads as follows.

Theorem IV.6. *Let $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. If $n > 6$ then $V_{x^q, \gamma}$ is a Sidon space for any $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. If $n = 6$, then $V_{x^q, \gamma}$ is a Sidon space if and only if $N_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) = \gamma^{\frac{q^6-1}{q-1}} \neq 1$.*

For the case of the trace function we have the following characterization.

Theorem IV.7. *Let $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. If $n > 6$ then $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$ is a Sidon space for any $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. If $n = 6$, then $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$ is a Sidon space if and only if $\text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) \neq -2$.*

Proof. Note that $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$ is equivalent to $V_{x^q + x^{q^2}, \gamma}$, via the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. If $n > 6$ the assertion follows by [5, Proposition 4.8]. So, assume that $n = 6$ and let $V = V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$. Since $\dim_{\mathbb{F}_q}(V) = 3$ and $\dim_{\mathbb{F}_q}(\ker(\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x))) = 2$ we have that

$$V = \langle u_1, u_2, u_3 + \gamma \rangle_{\mathbb{F}_q},$$

for some u_1, u_2 and u_3 in \mathbb{F}_{q^3} such that $\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u_1) = \text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u_2) = 0$ and $\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u_3) = 1$. Let consider

$$V^2 = \langle u_1^2, u_1 u_2, u_1 u_3 + u_1 \gamma, u_2^2, u_2 u_3 + u_2 \gamma, u_3^2 + 2u_3 \gamma + \gamma^2 \rangle_{\mathbb{F}_q}.$$

Let observe that $u_1^2, u_1 u_2, u_2^2$ are \mathbb{F}_q -linearly independent because otherwise there would exist $\alpha, \beta \in \mathbb{F}_q$ such that

$$\frac{u_1^2}{u_2^2} = \alpha + \beta \frac{u_1}{u_2},$$

i.e. $\frac{u_1}{u_2}$ is a root of a polynomial of degree 2 over \mathbb{F}_q , and this is not possible since $\frac{u_1}{u_2} \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$. Hence $\mathbb{F}_{q^3} = \langle u_1^2, u_1 u_2, u_2^2 \rangle_{\mathbb{F}_q}$ and then

$$V^2 = \mathbb{F}_{q^3} + \gamma \langle u_1, u_2, 2u_3 + \gamma \rangle_{\mathbb{F}_q}.$$

Also $\dim_{\mathbb{F}_q}(\mathbb{F}_{q^3} + \langle u_1 \gamma \rangle_{\mathbb{F}_q}) = 4$ since $\gamma \in \mathbb{F}_{q^6} \setminus \mathbb{F}_{q^3}$. Moreover $\dim_{\mathbb{F}_q}(\mathbb{F}_{q^3} + \langle u_1 \gamma, u_2 \gamma \rangle_{\mathbb{F}_q}) = 5$. Indeed, if there exist $\alpha \in \mathbb{F}_q$ and $\beta \in \mathbb{F}_{q^3}$ such that $u_2 \gamma = \alpha u_1 \gamma + \beta$ then $(u_2 - \alpha u_1) \gamma = \beta$ and since $u_2 - \alpha u_1$ cannot be zero (because u_1, u_2 are \mathbb{F}_q -linearly independent) this would imply that $\gamma \in \mathbb{F}_{q^3}$, a contradiction. Finally, since $\gamma^2 = A + B\gamma$, where $A, B \in \mathbb{F}_{q^3}$ and $B = \text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^3}}(\gamma)$ and $A = -N_{\mathbb{F}_{q^6}/\mathbb{F}_{q^3}}(\gamma)$, we have

$$2u_3 \gamma + \gamma^2 = (2u_3 + B)\gamma + A$$

and hence $V^2 = \mathbb{F}_{q^3} + \gamma \langle u_1, u_2, 2u_3 + B \rangle_{\mathbb{F}_q}$. Note that $\dim_{\mathbb{F}_q}(\ker(\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x))) = 2$ and $\ker(\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x)) = \langle u_1, u_2 \rangle_{\mathbb{F}_q}$ then $\dim_{\mathbb{F}_q}(V^2) = 6$ if and only if $2u_3 + B \notin \ker(\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x))$, i.e. $\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(2u_3 + B) = 2\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u_3) + \text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(B) = 2 + \text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(B) \neq 0$, which happens if and only if $\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(B) = \text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_{q^3}}(\gamma)) = \text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) \neq -2$. \square

For more details on the subspace associated with the trace function see [4].

The above result also gives a classification of three dimensional Sidon spaces S with $\delta_3(S) = 2$.

Corollary IV.8. *Let n be a multiple of three and let S be a Sidon space in \mathbb{F}_{q^n} with dimension 3 and $\delta_3(S) = 2$. If $n > 6$ then S is equivalent to one of the following:*

- $S_{x^q, \gamma}$;
- $S_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$,

for some $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. If $n = 6$ then S is equivalent to one of the following:

- $S_{x^q, \gamma}$, for some $\gamma \in \mathbb{F}_{q^6}$ such that $N_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) \neq 1$;
- $S_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$, for some $\gamma \in \mathbb{F}_{q^n}$ such that $\text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) \neq -2$.

We can now use Proposition A.1 to show that the two families found in Theorem IV.5, up to equivalence, are distinct.

Corollary IV.9. *Let n be a multiple of 3 and let $\gamma, \xi \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$. Then $V_{x^q, \gamma}$ and $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \xi}$ are inequivalent.*

In terms of codes we obtain a classification result for one-orbit cyclic subspace codes of dimension three when $3 \mid n$ and $\delta_3(S) = 2$.

Corollary IV.10. *Let $C = \text{Orb}(S)$ be a one-orbit cyclic subspace code with dimension three in \mathbb{F}_{q^n} . Suppose that $3 \mid n$ and $\delta_3(S) = 2$.*

- If $n > 6$ then $d(C) = 4$ and C is equivalent either to $\text{Orb}(V_{x^q, \gamma})$ or to $\text{Orb}(V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma})$, for some $\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^3}$.
- If $n = 6$ and $d(C) = 4$ then C is equivalent either to $\text{Orb}(V_{x^q, \gamma})$ with $N_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) \neq 1$ or to $\text{Orb}(V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma})$ with $\text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) \neq -2$.
- If $n = 6$ and $d(C) = 2$ then C is equivalent either to $\text{Orb}(V_{x^q, \gamma})$ with $N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\gamma) = 1$ or to $\text{Orb}(V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma})$ with $\text{Tr}_{\mathbb{F}_{q^6}/\mathbb{F}_q}(\gamma) = -2$.

In [7, Example 1] (see also [11, Example 3.14] and [27, Example 18]) Etzion and Vardy provide an explicit example of optimum-distance code when $n = 6$, $k = 3$ and $q = 2$. Under this assumption, the construction $V_{x^q, \gamma}$ of Roth, Raviv and Tamo [24] does not provide any example of optimum-distance codes. Therefore, by the above corollary, we know that [7, Example 1] arises from $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$, for some $\gamma \in \mathbb{F}_{2^6} \setminus \mathbb{F}_{2^3}$ with $\text{Tr}_{\mathbb{F}_{2^6}/\mathbb{F}_2}(\gamma) = 1$.

ACKNOWLEDGEMENTS

The research was supported by the project ‘‘COMBINE’’ of the University of Campania ‘‘Luigi Vanvitelli’’ and was partially supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM). This research was also supported by Bando Galileo 2024 – G24-216 and by the project ‘‘The combinatorics of minimal codes and security aspects’’, Bando Cassini.

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APPENDIX A
SOME PROOFS

Because of lack of space, here we prove some of the stated results.

Proof of Theorem III.6

Suppose that T and S are equivalent under the action of $(\xi, \sigma) \in \mathbb{F}_{q^n}^* \rtimes \text{Aut}(\mathbb{F}_{q^n})$, then $T = \xi S^\sigma$, i.e.

$$\langle 1, \mu \mu^2 \rangle_{\mathbb{F}_q} = \xi \langle 1, \lambda^\sigma, \lambda^{2\sigma} \rangle_{\mathbb{F}_q}.$$

Then

$$\begin{cases} \xi = p_0(\mu) \\ \xi \lambda^\sigma = p_1(\mu) \\ \xi \lambda^{2\sigma} = p_2(\mu) \end{cases} \quad (1)$$

where $p_i(x) \in \mathbb{F}_q[x]$ and $\deg_{\mathbb{F}_q}(p_i(x)) \leq 2$ for any $i \in \{0, 1, 2\}$. Without loss of generality, up to change ξ , we may assume that $\gcd(p_0(x), p_1(x), p_2(x)) = 1$. From (1) we get

$$\begin{cases} \lambda^\sigma = \frac{p_1(\mu)}{p_0(\mu)} \\ \lambda^{2\sigma} = \frac{p_2(\mu)}{p_0(\mu)}. \end{cases} \quad (2)$$

This implies that

$$p_1^2(\mu) = p_2(\mu)p_0(\mu), \quad (3)$$

and since $\dim_{\mathbb{F}_q}(\mathbb{F}_q(\mu)) \geq 5$, Equation (3) implies the following polynomial identity

$$p_1^2(x) = p_2(x)p_0(x). \quad (4)$$

Since $\lambda^\sigma \notin \mathbb{F}_q$, we have that $p_1(x)$ and $p_0(x)$ are not \mathbb{F}_q -proportional and the same holds for $p_1(x)$ and $p_2(x)$. Therefore, by Equation (4) we have that $p_1(x)$ is reducible over \mathbb{F}_q , i.e.

$$p_1(x) = t(x)s(x)$$

where $t(x), s(x) \in \mathbb{F}_q[x]$ and $\deg(t(x)) = \deg(s(x)) = 1$. Then, since $\gcd(p_0(x), p_1(x), p_2(x)) = 1$, by (4) if $t(x) \mid p_2(x)$ then $t^2(x) \mid p_2(x)$ and $s^2(x) \mid p_0(x)$. Thus $p_0(x) = \alpha s^2(x)$ and $p_2(x) = \beta t^2(x)$ where $\alpha, \beta \in \mathbb{F}_q$ and $\alpha\beta = 1$.

Then by (2) it follows that

$$\lambda^\sigma = \frac{t(\mu)s(\mu)}{\alpha s^2(\mu)} = \frac{t(\mu)}{\alpha s(\mu)} = \frac{\alpha_0 + \alpha_1\mu}{\beta_0 + \beta_1\mu}.$$

for some $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{F}_q$ with $(\alpha_1, \beta_1) \neq (0, 0)$. We obtain the same condition in the case in which $s(x) \mid p_2(x)$.

Conversely, suppose that

$$\lambda^\sigma = \frac{\alpha_0 + \alpha_1\mu}{\beta_0 + \beta_1\mu} = \frac{t(\mu)}{s(\mu)}.$$

for some $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{F}_q$ with $(\alpha_1, \beta_1) \neq (0, 0)$. Let $\xi = s^2(\mu)$, then

$$\xi S^\sigma = s^2(\mu) \left\langle 1, \frac{t(\mu)}{s(\mu)}, \frac{t^2(\mu)}{s^2(\mu)} \right\rangle_{\mathbb{F}_q} \subseteq \langle 1, \mu, \mu^2 \rangle_{\mathbb{F}_q} = T, \quad (5)$$

and, since $\dim_{\mathbb{F}_q}(\xi S^\sigma) = \dim_{\mathbb{F}_q}(S) = \dim_{\mathbb{F}_q}(T) = 3$, we get

$$\xi S^\sigma = \xi \langle 1, \lambda^\sigma, \lambda^{2\sigma} \rangle_{\mathbb{F}_q} = \langle 1, \mu, \mu^2 \rangle_{\mathbb{F}_q} = T,$$

i.e. S and T are equivalent. \square

Proof of Theorem III.7

Suppose that T and S are equivalent under the action of (ξ, σ) , i.e. $T = \xi S^\sigma$. Then $\mathbb{F}_{q^2} + \xi \mathbb{F}_{q^2} \subseteq T$ and hence $\mathbb{F}_{q^2} = \xi \mathbb{F}_{q^2}$, i.e. $\xi \in \mathbb{F}_{q^2}$. Moreover we get

$$T = \mathbb{F}_{q^2} + \langle \eta \rangle_{\mathbb{F}_q} = \mathbb{F}_{q^2} + \xi \langle \mu^\sigma \rangle_{\mathbb{F}_q}$$

and so

$$\eta = a + \xi \mu^\sigma b$$

where $a \in \mathbb{F}_{q^2}$ and $b \in \mathbb{F}_q$.

Conversely, suppose that $\eta = a + \xi \mu^\sigma b$ with $a, \xi \in \mathbb{F}_{q^2}$ and $b \in \mathbb{F}_q^*$. Then

$$T = \mathbb{F}_{q^2} + \langle \eta \rangle_{\mathbb{F}_q} = \xi \mathbb{F}_{q^2} + \xi \langle \mu^\sigma \rangle_{\mathbb{F}_q} = \xi S^\sigma.$$

\square

Proof of Proposition IV.1

Suppose by contradiction that $S \cap \xi \mathbb{F}_{q^3} \neq \{0\}$ for every $\xi \in \mathbb{F}_{q^n}^*$ such that $\xi \mathbb{F}_{q^3} \subseteq \langle S \rangle_{\mathbb{F}_{q^3}}$. Then, since

$$\langle S \rangle_{\mathbb{F}_{q^3}} \setminus \{0\} = \bigcup_{\xi \mathbb{F}_{q^3} \subseteq \langle S \rangle_{\mathbb{F}_{q^3}}} \xi \mathbb{F}_{q^3}^*,$$

this implies that $S^* = S \setminus \{0\}$ contains at least as many elements as the multiplicative cosets of \mathbb{F}_{q^3} contained in $\langle S \rangle_{\mathbb{F}_{q^3}}$, i.e. since $\delta_2(S) = 2$, we have that

$$q^3 - 1 = |S^*| \geq \frac{q^6 - 1}{q^3 - 1} = q^3 + 1$$

which gives a contradiction. \square

Proof of Proposition IV.3

Let consider $x, y \in S$ such that $x \neq y$. Then there exist $u, u', v, v' \in \mathbb{F}_{q^3}$ such that

$$x = \lambda u + \rho u' \text{ and } y = \lambda v + \rho v'.$$

Let observe that $u' \neq v'$ because otherwise $x - y = \lambda(u - v) \in S \cap \lambda \mathbb{F}_{q^3}$ and this implies $x = y$, a contradiction. Therefore

$$\begin{aligned} f: \mathbb{F}_{q^3} &\rightarrow \mathbb{F}_{q^3} \\ u &\mapsto u' : \lambda u + \rho u' \in S \end{aligned}$$

is well defined. Also, since $S = \{\rho u + \lambda f(u) : u \in \mathbb{F}_{q^3}\}$ is an \mathbb{F}_q -subspace, f is an \mathbb{F}_q -linear map, i.e. $f \in \mathcal{L}_{3,q}$. \square

The following is a well-known result in linear sets theory (see e.g. [21, Proposition 2.2]), but we reformulate it in a more algebraic flavour including also a short proof. This will allow us to prove Corollary IV.9.

Proposition A.1. *Let $V_1 = V_{U_1, \gamma}$ and $V_2 = V_{U_2, \xi}$ be two k -dimensional equivalent \mathbb{F}_q -subspaces in \mathbb{F}_{q^n} . For every $i \in \{0, 1, \dots, k\}$ and $j \in \{1, 2\}$, define*

$$N_i(U_j) = |\{\langle v \rangle_{\mathbb{F}_{q^k}} : v \in \mathbb{F}_{q^k}^2 \setminus \{0\} \text{ and } \dim_{\mathbb{F}_q}(U_i \cap \langle v \rangle_{\mathbb{F}_{q^k}}) = i\}|.$$

Then $N_i(U_1) = N_i(U_2)$ for any i .

Proof. By Theorem IV.4 it follows that U_1 and U_2 are $\Gamma\text{L}(2, q^k)$ -equivalent, then there exist a matrix $A \in \text{GL}(2, q^k)$ and an automorphism $\rho \in \text{Aut}(\mathbb{F}_{q^k})$ such that

$$U_1^\rho A = U_2.$$

Let $v \in \mathbb{F}_{q^k}^2 \setminus \{(0, 0)\}$. Then $\dim_{\mathbb{F}_q}(U_1 \cap \langle v \rangle_{\mathbb{F}_{q^k}}) = i$ if and only if $i = \dim_{\mathbb{F}_q}(U_1^\rho A \cap \langle v^\rho A \rangle_{\mathbb{F}_{q^k}}) = \dim_{\mathbb{F}_q}(U_2 \cap \langle v^\rho A \rangle_{\mathbb{F}_{q^k}})$. This means that $N_i(U_1) = N_i(U_2)$. \square

Proof of Corollary IV.9

By contradiction, suppose that $V_{x^q, \gamma}$ and $V_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x), \gamma}$ are equivalent, then by Theorem IV.4 the \mathbb{F}_q -subspaces

$$U_{x^q} = \{(u, u^q) : u \in \mathbb{F}_{q^3}\}$$

and

$$U_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x)} = \{(u, \text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u)) : u \in \mathbb{F}_{q^3}\}$$

are $\Gamma\text{L}(2, q^3)$ -equivalent. Note that U_{x^q} is scattered, that is for any $v \in \mathbb{F}_{q^3}^2$

$$\dim_{\mathbb{F}_q}(U_{x^q} \cap \langle v \rangle_{\mathbb{F}_{q^3}}) \leq 1,$$

whereas

$$\dim_{\mathbb{F}_q}(U_{\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x)} \cap \langle (1, 0) \rangle_{\mathbb{F}_{q^3}}) = \dim_{\mathbb{F}_q}(\ker(\text{Tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(x))) = 2.$$

Therefore, Proposition A.1 yields a contradiction. \square