# THE ORDER OF THE (123, 132)-AVOIDING STACK SORT 

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#### Abstract

Let $s$ be West's deterministic stack-sorting map. A well-known result (West) is that any length $n$ permutation can be sorted with $n-1$ iterations of $s$. In 2020, Defant introduced the notion of highly-sorted permutations-permutations in $s^{t}\left(S_{n}\right)$ for $t \lesssim n-1$. In 2023, Choi and Choi extended this notion to generalized stack-sorting maps $s_{\sigma}$, where we relax the condition of becoming sorted to the analogous condition of becoming periodic with respect to $s_{\sigma}$. In this work, we introduce the notion of minimally-sorted permutations $\mathfrak{M}_{n}$ as an antithesis to Defant's highly-sorted permutations, and show that $\operatorname{ord}_{s_{123,132}}\left(S_{n}\right)=$ $2\left\lfloor\frac{n-1}{2}\right\rfloor$, strengthening Berlow's 2021 classification of periodic points.


## 1. Introduction

Knuth's Art of Computer Programming [17] first introduced the stack-sorting machine, in which an input sequence is sorted with a single external stack structure. The elements of the sequence are passed left-to-right through the machine, with two possible operations at every state: push, moving the next input element onto the stack, and pop, removing the top element from the stack and appending it to the output.

In 1990, West [21] introduced a deterministic version of Knuth's stack-sorting machine as the stack-sorting map $s$, insisting that the stack must always increase from top to bottom and employ a right-greedy process: the push operation is prioritized. Since then, various studies have been motivated by Knuth's original machine and West's deterministic $s$, including pop-stack-sorting [1, 2, 14, 18, 19], stack-sorting Coxeter groups [14, 15], sigma-tau machines [3, 4, 5], and stack-sorting of set-partitions [16, 23].


Figure 1. West's deterministic stack-sorting map $s$ on $\pi=2143$.
In his dissertation, West [21] proved that $s^{n-1}\left(S_{n}\right)$ contains only the identity permutation, justifying repeated applications of $s$ as a correct and terminating sorting algorithm. A natural direction of study, then, is the characterization of $t$-stack-sortable permutationspermutations $\pi$ such that $s^{t}(\pi)$ is sorted-for general $t \leq n-1$. Knuth [17] answered the question for $t=1$, showing that $\pi$ is 1 -stack-sortable if and only if $\pi$ avoids subsequences of the pattern 231, enumerating the number of such permutations of length $n$ to be $\frac{1}{n+1}\binom{2 n}{n}$,
the $n^{\text {th }}$ Catalan number. In 1990, West 21] characterized the 2 -stack-sortable permutations, proving that $\pi$ is 2 -stack-sortable if and only if $\pi$ avoids subsequences of the pattern 2341 and the barred pattern $3 \overline{5} 241$. He also conjectured that the number of such permutations of length $n$ is $\frac{2}{(n+1)(2 n+1)}\binom{3 n}{n}$, which was proven by Zeilberger [24] two years later. West [21, 22] then searched for a polynomial $P(n)$ such that 3 -stack-sortable permutations could be enumerated by $\frac{1}{P(n)}\binom{4 n}{n}$, but was unsuccessful for $\operatorname{deg}(P(n))<7$. In 2012, Úlfarsson [20] characterized 3 -stack-sortable permutations with "decorated patterns," but only in 2021, did Defant [11] discover a polynomial-time algorithm to enumerate 3 -stack-sortable permutations.

In 2020, Defant [13] first considered $t$-stack-sortable permutations to be duals of the $t$ sorted permutations [12]-permutations in the image of $s^{t}\left(S_{n}\right)$, a generalization of BousquetMélou's definition [6] of sorted. Defant then defined a permutation $\pi \in S_{n}$ to be highly-sorted if $\pi$ is $t$-sorted for some $t$ close to $n$, proving that a $t$-sorted permutation can contain at most $\left\lfloor\frac{n-t}{2}\right\rfloor$ descents 【13].

The classical stack-sorting map $s$ has since been generalized to $s_{\sigma}[7]$ for permutations $\sigma$, where instead of insisting that the stack increases, we insist that the stack avoids top-tobottom subsequences of the pattern $\sigma$. In 2021, Berlow [5] introduced the family of maps $s_{T}$, where the stack must avoid top-to-bottom subsequences of every pattern in set $T$ (see Figure (2). In 2023, Choi and Choi [8] generalized Defant's notion of highly-sorted permutations, defining $\pi$ to be highly-sorted with respect to $s_{\sigma}$ if $\pi$ is in the image of $s_{\sigma}^{t}$ for some $t$ close to $\operatorname{ord}_{s_{\sigma}}\left(S_{n}\right)$, where $\operatorname{ord}_{s_{\sigma}}(P)$ is the smallest integer $k$ such that every element in $s_{\sigma}^{k}(P)$ is periodic under $s_{\sigma}$. We straightforwardly extend this definition to generalized maps $s_{T}$.


Figure 2. The generalized stack-sorting map $s_{123,132}$ on $\pi=52431$.
Recently, Choi, Gan, Li, and Zhu [9] studied set partitions that require the maximum number of sorts through an $a b a$-avoiding stack. Similarly, we define a permutation $\pi$ to be minimally-sorted with respect to $s_{T}$ if $\operatorname{ord}_{s_{T}}\left(S_{n}\right)=\operatorname{ord}_{s_{T}}(\{\pi\})$, antithetical to Defant's notion of highly-sorted permutations. At the end of this work, we present two conjectures on $\mathfrak{M}_{n}$, the minimally-sorted permutations with respect to $s_{123,132}$.

In 2021, Berlow [5] studied the periodic points of $s_{123,132}$. She defined a permutation $\pi$ of length $n$ to be half-decreasing if the subsequence $\pi_{n-1} \pi_{n-3} \cdots \pi_{(3-(n \bmod 2))}$ is the identity of length $\left\lfloor\frac{n-1}{2}\right\rfloor$. In particular, being order-isomorphic to the identity is not sufficient.
Theorem 1.1 (Berlow [5]). A permutation $\pi$ is periodic under $s_{123,132}$ if and only if $\pi$ is half-decreasing.

Our main result is that we find the exact value of $\operatorname{ord}_{s_{123,132}}\left(S_{n}\right)$, extending Berlow's work on periodic permutations. An analogous result for $s_{321,312}$ follows directly from Theorem 1.2,
Theorem 1.2. For all positive integers $n$, we have $\operatorname{ord}_{s_{123,132}}\left(S_{n}\right)=2\left\lfloor\frac{n-1}{2}\right\rfloor$.

## 2. Preliminaries

We say that $a \in A$ is periodic under $f: A \rightarrow B$ if there exists a positive integer $k$ such that $f^{k}(a)=a$. For some ordered set $S$, we use $S_{i}$ to denote the $i$ th element of $S$.

Let $[n]$ denote $\{1,2, \cdots, n\}$ for positive integers $n$. A permutation, written $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$, is an ordering of distinct positive integers with length len $(\pi)=n$. We say that $\pi_{1}, \pi_{2}, \cdots, \pi_{n}$ are the elements of $\pi$, and use $\pi_{[i: j]}$ to denote the subpermutation $\pi_{i}, \pi_{i+1}, \cdots, \pi_{j}$. We define $\operatorname{ind}_{\pi}(i)$, the index of $i$ in $\pi$, to be $j$, where $\pi_{j}=i$. Let $S_{n}$ be the set of permutations with elements $[n$ ]. The reduction of a permutation $\pi$ (equivalently, the standardization [13]), is the unique permutation $\operatorname{red}(\pi) \in S_{n}$ such that $\operatorname{red}(\pi)_{i}=j$ for $1 \leq i \leq n$, where $\pi_{i}$ is the $j$ th smallest number in $\left\{\pi_{1}, \pi_{2}, \cdots, \pi_{n}\right\}$. Two permutations $\pi$ and $\sigma$ are order-isomorphic if $\operatorname{red}(\pi)=\operatorname{red}(\sigma)$, and we write $\pi \cong \sigma$. For instance, $\pi=57816$ and $\sigma=48917$ are orderisomorphic, since $\operatorname{red}(\pi)=\operatorname{red}(\sigma)=24513$. Given permutations $\pi$ and $\sigma$, we say that $\pi$ contains the pattern $\sigma$ if there exists a sequence of positive integers $a_{1}<a_{2}<\cdots<a_{k}$ such that $\pi^{\prime}=\pi_{a_{1}} \pi_{a_{2}} \cdots \pi_{a_{k}} \cong \sigma$. Otherwise, we say that $\pi$ avoids $\sigma$ (equivalently, is $\sigma$-avoiding). For instance, $\pi=24513$ contains $\sigma=132$ since $\pi_{1} \pi_{3} \pi_{5}=253 \cong \sigma$, but avoids $\tau=321$. We use $\pi \cdot \tau$ to denote the concatenation of $\pi$ and $\tau$, and let $\operatorname{rev}(\pi)$ denote the reverse of $\pi$, namely $\pi_{n} \pi_{n-1} \cdots \pi_{1}$.

Next, an element $\pi_{i}$ of $\pi \in S_{n}$ is small if $\pi_{j} \leq\left\lfloor\frac{n-1}{2}\right\rfloor$. An element $\pi_{i}$ is a left-to-right minimum (equivalently, ltr-min) of $\pi$ if $\pi_{i}=\min \left(\pi_{[1: i]}\right)$. Additionally, we say that $\pi_{i}$ is a valley if $\pi_{i}$ is a ltr-min, $\pi_{i+1}$ (if $i+1 \leq n$ ) is not a ltr-min, and $\pi_{i+2}$ (if $i+2 \leq n$ ) is a ltr-min. A consecutive subsequence of elements $\pi_{[i: i+j]}$ is a valley-block $\bar{v}$ if $\pi_{i+j}$ is a valley and $\operatorname{red}\left(\pi_{[1: i+j]}\right)_{[i: i+j]}=j+1, j, \cdots, 1$. We say that the valley-boundary of $\pi \in S_{n}$, denoted $\mathfrak{B}(\pi)$, is the smallest index $i$ such that $\pi_{[i: n]}=\overline{v_{1}} \pi_{a_{1}} \overline{v_{2}} \pi_{a_{2}} \cdots \overline{v_{j}} \pi_{a_{j}}$ for valleys $\overline{v_{1}}, \cdots, \overline{v_{j}}$ and elements $\pi_{a_{1}}, \cdots, \pi_{a_{j}}$, and set $\mathfrak{B}(\pi)=n$ if no such index exists. The valley-region of $\pi$ is $\pi_{[\mathfrak{B}(\pi): n]}$. For instance, given $\pi=(11,12,7,5,8,4,3,6,2,9,1,10)$, the elements $1,2,3$, and 5 are valleys and the sets $(7,5),(4,3),(2),(1)$ form 4 valley-blocks in $\pi$. Finally, $\mathfrak{B}(\pi)=3$, since $\pi_{[3: n]}=\overline{7,5}, 8, \overline{4,3}, 6, \overline{2}, 9, \overline{1}, 10$.

We conclude by noting that permutation indices will be considered modulo $n$ for the duration of this paper. In particular, let $\pi_{i}:=\pi_{j}$, where $j$ is the unique element of $[n]$ such that $i \equiv j(\bmod n)$.

## 3. Proof of the Main Result

We preface this section with two propositions, immediate from the preliminaries.
Proposition 3.1. Given $\sigma, \tau \in S_{3}$, it holds that $\left(s_{\sigma, \tau}(\pi)\right)_{n}=\pi_{1}$ for all $\pi \in S_{n}$ and $n \geq 1$.
Proposition 3.2. Let $\overline{v_{1}}, \cdots, \overline{v_{i}}$ be the valley-blocks of $\pi$ from left to right, and let len $\left(v_{j}\right)=$ $l_{j}$ for all $j$. Then, the permutation $\overline{v_{1}} \cdot \overline{v_{2}} \cdots \overline{v_{i}}$ is the reverse of the identity of length $\sum l_{i}$.

We now begin the proof of Theorem 1.2 with several auxiliary lemmas that demonstrate the monovariant movement of valley-blocks under $s_{123,132}$.

Lemma 3.3. For any $\pi \in S_{n}$ and ltr-min $\pi_{i}$ with $i>1$, let $j \leq n$ be the largest index such that $\pi_{i}=\min \left(\pi_{[1: j]}\right)$. It holds that $s_{123,132}(\pi)_{j-1}=\pi_{i}$.

Proof. Since $\pi_{i}$ is a ltr-min, just before $\pi_{i}$ enters the stack, $\pi_{1}$ must be the only element in the stack. After the elements $\pi_{[i+1: j]}$ have all entered the stack, $\pi_{i}$ and $\pi_{1}$ necessarily remain in the stack since $\pi_{i+1}, \cdots, \pi_{j}>\pi_{i}$. Additionally, since $\pi_{j+1}<\pi_{i}$, just before $\pi_{j+1}$ enters the
stack, $\pi_{j}$ must exit the stack. At this moment, the $j-1$ elements $\pi_{2}, \pi_{3}, \cdots, \pi_{j}$ have been the only elements to exit the stack, with $\pi_{i}$ being the last, so $s_{123,132}(\pi)_{j-1}=\pi_{i}$.
Lemma 3.4. Given a valley-block $\bar{v}=\pi_{[i: i+j]}$ of $\pi$, we have $s_{123,132}(\pi)_{i+j}=\pi_{i+j}$ and $s_{123,132}(\pi)_{k-1}=\pi_{k}$ for $i \leq k<i+j$.
Proof. Just before $\pi_{i}$ enters the stack, $\pi_{1}$ must be the only element in the stack. Since $\bar{v}$ consists of the $j+1$ smallest elements of $\pi_{[1: i+j]}$ in descending-order, just before any element of $\bar{v}$ enters the stack, the previous element must exit. Hence, $k-2$ elements exit before $\pi_{k}$ for $i \leq k<i+j$, and thus $s_{123,132}(\pi)_{k-1}=\pi_{k}$. Finally, by Lemma 3.3, $\pi_{i+j}$ is a fixed point.

Next, we show that $s_{123,132}$ preserves the elements in the valley-region of $\pi$.
Lemma 3.5. Suppose $\pi_{[:: j]}$ and $\pi_{[j+2: k]}$ are two valley-blocks of $\pi$. Then, $s_{123,132}(\pi)_{j-1}=\pi_{j+1}$.
Proof. Right before $\pi_{j}$ enters the stack, the only element remaining must be $\pi_{1}$. Now, since $\pi_{j+1}>\pi_{j}$, the stack will read $\pi_{j+1} \pi_{j} \pi_{1}$ top to bottom just after $\pi_{j+1}$ enters. Finally, since $\pi_{j+2}$ is also a ltr-min, just before it enters, $\pi_{j+1}$ and $\pi_{j}$ must have left the stack. Hence, every element in $\pi_{[1: j]}$ exits the stack before $\pi_{j+1}$ except $\pi_{1}$ and $\pi_{j}$, yielding $s(\pi)_{j-1}=\pi_{j+1}$.
Lemma 3.6. If $\pi_{i}$ is in the valley-region of $\pi$, then $\pi_{i}$ is also in the valley-region of $s_{123,132}(\pi)$.
Proof. Let $\pi_{[\mathfrak{B}(\pi): n]}=\overline{v_{1}} \pi_{a_{1}} \overline{v_{2}} \cdots \overline{v_{j}} \pi_{a_{j}}$, the valley-region of $\pi$, and let len $\left(\overline{v_{i}}\right)=l_{i}$ for $1 \leq$ $i \leq j$. Then, by Lemma 3.3 and Lemma [3.4, we have that $s_{123,132}(\pi)$ ends with the suffix $\left(v_{1\left[1: l_{1}-1\right]}\right) \cdot \pi_{b_{1}} \cdot\left(v_{1\left[l_{1}\right]} \cdot v_{2\left[1: l_{2}-1\right]}\right) \cdot \pi_{b_{2}} \cdot\left(v_{2\left[l_{2}\right]} \cdot v_{3\left[1: l_{3}-1\right]}\right) \cdots\left(v_{j-1\left[l_{j-1}\right]} \cdot v_{j[1]}\right) \cdot \pi_{b_{j-1}} \cdot\left(v_{j\left[l_{j}\right]}\right) \cdot \pi_{b_{j}}$ for some elements $\pi_{b_{1}}, \pi_{b_{2}}, \cdots, \pi_{b_{j}}$. By Proposition [3.2, this suffix is of the form $\overline{w_{1}} \pi_{b_{c_{1}}} \overline{w_{2}} \pi_{b_{c_{2}}} \cdots \overline{w_{k}} \pi_{b_{c_{k}}}$, where $\pi_{b_{c_{1}}}, \cdots, \pi_{b_{c_{k}}}$ are the elements of $\left\{\pi_{b_{1}}, \cdots, \pi_{b_{j}}\right\}$ that are not ltr-mins. Hence, this suffix is fully contained in the valley-region of $s_{123,132}(\pi)$. However, it also contains all the elements in valley-blocks in $\pi_{[\mathfrak{B}(\pi): n]}$, and all the elements in between valley-blocks in $\pi_{[\mathfrak{B}(\pi): n]}$ by Lemma 3.5, which fully encompass all of elements in the valley-block, finishing the proof.
Lemma 3.7. Let $\pi_{i}=\min \left(\pi_{[1: \mathfrak{B}(\pi)-1]}\right)$. If $\pi_{i}$ is small, then $\pi_{i}$ is in the valley-region of $s_{123,132}(\pi)$.
Proof. If $i=1$, then the claim follows from Proposition 3.1. Otherwise, just before $\pi_{i}$ enters the stack, $\pi_{1}$ must be the only element remaining in the stack, since $\pi_{i}$ is a ltr-minimum. Then, after $\pi_{i+1}, \cdots, \pi_{\mathfrak{B}(\pi)-1}$ have all entered the stack, $\pi_{i}$ will remain in the stack. However, when $\pi_{\mathfrak{B}(\pi)}$ enters the stack, $\pi_{i}$ will necessarily leave, since $\pi_{\mathfrak{B}}$ is part of a valley-block to the right of $\pi_{i}$, so $\pi_{\mathfrak{B}(\pi)}<\pi_{i}$. Thus, since every other element in $\pi_{1}, \cdots, \pi_{\mathfrak{B}(\pi)-1}$ was popped out before $\pi_{i}$, except for $\pi_{1}$, we have $s(\pi)_{\mathfrak{B}(\pi)-2}=\pi_{i}$. However, since $\pi_{i}=\min \left(\pi_{1}, \cdots, \pi_{\mathfrak{B}(\pi)-1}\right)$, the proof of Lemma 3.6 shows that $\pi_{i}$ is in the valley-region of $s_{123,132}(\pi)$.

By Lemma 3.6, elements never leave the valley-region, and by Lemma 3.7, a small element is always added to the valley-region every iteration, implying the following result.
Corollary 3.8. For any $\pi \in s_{123,132}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(S_{n}\right)$, it holds that $i \geq \mathfrak{B}(\pi)$ for all small elements $\pi_{i}$.
Corollary 3.8 gives a characterization of the $\left\lfloor\frac{n-1}{2}\right\rfloor$-sorted permutations under a $s_{123,132}$ map. We continue by showing that these permutations become periodic with at most $\left\lfloor\frac{n-1}{2}\right\rfloor$ further passes.
Lemma 3.9. For $\pi \in S_{n}$ and small element $i$, if $\pi_{n-2 i+2}=i$ and $i$ is in the valley-region of $\pi$, then $s_{123,132}(\pi)_{n-2 i+1}=i$.

Proof. Suppose for the sake of contradiction that $\pi_{n-2 i+2}$ is directly in between two valleyblocks, so that $\pi_{[j: n-2 i+1]}$ is a valley-block for some $j \leq n-2 i$. By definition, $\pi_{n-2 i+1}$ is a valley, and by Lemma 3.3, $s^{k}(\pi)_{n-2 i+1}=\pi_{n-2 i+1}$ for all $k$. But this contradicts Theorem 1.1, since we have $s^{k}(\pi)_{n-2 i+1} \neq \pi_{n-2 i+2}=i$. Now, suppose that $\pi_{n-2 i+2}$ is itself a valley. This similarly contradicts Theorem 1.1, since we have $s^{k}(\pi)_{n-2 i+2}=\pi_{n-2 i+2}$ for all $k$ by Lemma 3.3,

Since $\pi_{n-2 i+2}$ is in the valley-region of $\pi$, the only remaining possibility is that $\pi_{n-2 i+2}$ is part of a valley-block but not a valley. Hence, by Lemma 3.6, we have $s_{123,132}(\pi)_{n-2 i+1}=i$, as desired.

Lemma 3.10. For all positive integers $i \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ and $\pi \in S_{n}$, the permutation $\sigma_{n-1} \sigma_{n-3} \cdots \sigma_{n-2 i+1}$ is the identity of length $i$, where $\sigma=s_{123,132}^{i+\left\lfloor\frac{n-1}{2}\right\rfloor}(\pi)$.

Proof. We induct on $i$. The base case $i=1$ is immediate - in particular, $s_{123,132}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(\pi)_{n-1}=1$, which becomes a fixed element by Lemma 3.3, since otherwise $\mathfrak{B}(\pi)=n$ which contradicts Corollary 3.8.

Now suppose that for some $1<j \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, it holds that for all $\pi$ and $i<j$, the permutation $\sigma_{n-1} \sigma_{n-3} \cdots \sigma_{n-2 i+1}$ is the identity of length $i$, where $\sigma=s_{123,132}^{i+\left\lfloor\frac{n-1}{2}\right\rfloor}(\pi)$. First, we note that by Lemma 3.4 and Lemma3.5, if an element $\pi_{i}$ is in the valley-region of $\pi$, we have $s_{123,132}(\pi)_{x}=$ $\pi_{i}$ for some $x \in\{i-2, i-1, i\}$. Next, consider some $\pi \in S_{n}$, and let $\mathfrak{Z}=\left\{\operatorname{ind}_{s_{123,132}^{k}(\pi)}(j) \mid\right.$ $\left.\left\lfloor\frac{n-1}{2}\right\rfloor \leq k \leq\left\lfloor\frac{n-1}{2}\right\rfloor+j\right\}$. By Lemma [3.3, if $\mathfrak{Z}_{l}-\mathfrak{Z}_{l+1}=0$ for some $l \leq j$, we must have $s_{123,132}^{\left\lfloor\frac{n-1}{2}\right\rfloor+l}(\pi)_{n-2 j+1}=j$, or equivalently $\mathfrak{Z}_{l} \leq n-2 j+1$. Similarly, if $\mathfrak{Z}_{l}-\mathfrak{Z}_{l+1}=1$, we have by Lemma 3.4 and Lemma 3.5 that the element $j$ must be in a valley-block (but not a valley) of $s_{123,132}^{\left.\frac{\lfloor n-1}{2}\right\rfloor+l-1}(\pi)$, so by the inductive hypothesis, $\mathfrak{Z}_{l} \leq n-j-l+2$. Otherwise, $\mathfrak{Z}_{l}-\mathfrak{Z}_{l+1}=2$, so we conclude recursively that $\mathfrak{Z}_{j+1} \leq n-2 j+1$. But combining Lemma 3.3, Lemma 3.9, and the fact that $\mathfrak{Z}_{l}-\mathfrak{Z}_{l+1} \leq 2$ for all $l$, we derive $\mathfrak{Z}_{j+1}=n-2 j+1$, or equivalently $s_{123,132}^{\left\lfloor\frac{n-1}{2}\right\rfloor+j}(\pi)_{n-2 j+1}=j$. Hence, for all $\pi$ and $i<j+1$, the permutation $\sigma_{n-1} \sigma_{n-3} \cdots \sigma_{n-2 i+1}$ is the identity of length $i$, where $\sigma=s_{123,132}^{i+\left\lfloor\frac{n-1}{2}\right\rfloor}(\pi)$, completing the induction.

In particular, any $\pi \in s_{123,132}^{2\left\lfloor\frac{n-1}{2}\right\rfloor}\left(S_{n}\right)$ is half-decreasing, which implies the following by Theorem 1.1 .

Corollary 3.11. For all positive integers $n$, we have $\operatorname{ord}_{s_{123,132}}\left(S_{n}\right) \leq 2\left\lfloor\frac{n-1}{2}\right\rfloor$.
Finally, we present a family of minimally-sorted permutations to show that precisely $2\left\lfloor\frac{n-1}{2}\right\rfloor$ iterations are required to sort all of $S_{n}$. Define

$$
\gamma_{n}=\left(\frac{n+1}{2}, 2,3, \cdots, \frac{n-1}{2}, \frac{n+3}{2}, \cdots, n-2,1, n-1, n\right)
$$

for odd $n \geq 5$ and $\gamma_{n}=\gamma_{n-1} \cdot n$ for even $n \geq 6$. It is immediate that $\operatorname{ord}_{s_{123,132}}([n])=2\left\lfloor\frac{n-1}{2}\right\rfloor$ for $n \leq 4$. Hence, we consider $n \geq 5$. Let $\delta_{n}$ denote the permutation $\operatorname{rev}\left(\left(\gamma_{n}\right)_{[2: n-3]}\right)$ when $n$ is odd and $\operatorname{rev}\left(\left(\gamma_{n}\right)_{[2: n-4]}\right)$ when $n$ is even.
Lemma 3.12. For positive integers $n \geq 5$ and $k \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, we have $s_{123,132}^{k}\left(\gamma_{n}\right)_{[1: n-2 k-2]}=$ $\left(\delta_{n}\right)_{[k: n-k-3]}$ for odd $n$ and $s_{123,132}^{k}\left(\gamma_{n}\right)_{[1: n-2 k-3]}=\left(\delta_{n}\right)_{[k: n-k-4]}$ for even $n$. Furthermore, $\zeta_{n-1} \zeta_{n-3} \cdots \zeta_{n-2 k+1}$ is the identity permutation of length $k$, where $\zeta=s_{123,132}^{k}\left(\gamma_{n}\right)$.

| $n$ | $\gamma_{n}$ |
| :---: | :---: |
| 5 | $(3,2,1,4,5)$ |
| 6 | $(3,2,1,4,5,6)$ |
| 7 | $(4,2,3,5,1,6,7)$ |
| 8 | $(4,2,3,5,1,6,7,8)$ |
| 9 | $(5,2,3,4,6,7,1,8,9)$ |

Table 1. The first few $\gamma_{n}$ for $n \geq 5$.
Proof. We induct on $k$. For brevity, we will prove the lemma for when $n$ is odd-the proof for even $n$ is directly analogous. For the base case $k=1$, we have $s_{123,132}\left(\gamma_{n}\right)_{n}=\left(\gamma_{n}\right)_{1}=\frac{n+1}{2}$ by Proposition [3.1. Since $\left(\gamma_{n}\right)_{[2: n-3]}$ is strictly increasing, these elements are popped out in reverse order just before 1 enters the stack. Hence, $s_{123,132}\left(\gamma_{n}\right)_{[1: n-4]}=\delta_{n}=\left(\delta_{n}\right)_{[1: n-4]}$. Finally, $s_{123,132}\left(\gamma_{n}\right)_{n-1}=1$ by Lemma 3.3, completing the base case.

Next, suppose $s_{123,132}^{k}\left(\gamma_{n}\right)_{[1: n-2 k-2]}=\left(\delta_{n}\right)_{[k: n-k-3]}$ for some $k$ and $\zeta_{n-1} \zeta_{n-3} \cdots \zeta_{n-2 k+1}$ is the identity of length $k$ where $\zeta=s_{123,132}^{k}\left(\gamma_{n}\right)$. By Proposition 3.1, we have $s_{123,132}^{k+1}\left(\gamma_{n}\right)_{n}=$ $s_{123,132}^{k}\left(\gamma_{n}\right)_{1}$, and since $s_{123,132}^{k}\left(\gamma_{n}\right)_{[1: n-2 k-2]}$ is strictly decreasing, it follows that these elements will exit the stack in the same order, giving $s_{123,132}^{k+1}\left(\gamma_{n}\right)_{[1: n-2 k-4]}=\left(\delta_{n}\right)_{[k+1: n-k-4]}$ by the inductive hypothesis. Finally, by Lemma 3.3, we have $s_{123,132}^{k+1}\left(\gamma_{n}\right)_{n-2 k-1}=k+1$, completing the induction.
Lemma 3.13. For all positive integers $n$, we have $\operatorname{ord}_{s_{123,132}}\left(S_{n}\right) \geq 2\left\lfloor\frac{n-1}{2}\right\rfloor$.
Proof. It follows from Lemma 3.12 that $s_{123,132}^{\left\lfloor\frac{n-1}{2}\right\rfloor-1}\left(\gamma_{n}\right)_{1}=\left\lfloor\frac{n-1}{2}\right\rfloor$. By Proposition 3.1 and Lemma 3.5, we have ind $s_{s_{123,132}\left(\gamma_{n}\right)}\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right)=n-2\left(k-\left\lfloor\frac{n-1}{2}\right\rfloor\right)$ for $k \geq\left\lfloor\frac{n-1}{2}\right\rfloor$. Hence, $k=2\left\lfloor\frac{n-1}{2}\right\rfloor$ is the minimal $k$ such that $s_{123,132}^{k}\left(\gamma_{n}\right)$ is half-decreasing, giving us the desired bound.

Finally, we conclude that exactly $2\left\lfloor\frac{n-1}{2}\right\rfloor$ iterations are required to sort $S_{n}$.
Proof of Theorem 1.2. Corollary 3.11 and Lemma3.13 directly $\operatorname{imply}^{\operatorname{ord}}{ }_{s_{123,132}}\left(S_{n}\right)=2\left\lfloor\frac{n-1}{2}\right\rfloor$.

## 4. Future Directions

To study Defant's notion of highly-sorted permutations and our newly-introduced notion of minimally-sorted permutations, characterizing the periodic permutations under generalized stack-sorting maps is a prerequisite. We state a conjecture on the periodic points of other $s_{\sigma, \tau}$ stack-sorting maps for three pairs of $(\sigma, \tau)$, and restate a conjecture from Berlow.
Conjecture 4.1. For $(\sigma, \tau)=(123,213),(132,312),(231,321)$, the map $s_{\sigma, \tau}$ is a bijection from $S_{n}$ to itself, and all permutations are periodic.
Conjecture 4.2 (Berlow [5]). For $(\sigma, \tau)=(213,231),(132,213),(231,312)$, the only periodic points of $s_{\sigma, \tau}$ are the identity permutation and its inverse.

Recall that $\mathfrak{M}_{n}$ is the set of minimally-sorted permutations under $s_{123,132}$. We conjecture several properties of elements in $\mathfrak{M}_{n}$. However, these conditions are not sufficient for $n \geq 7$.

Conjecture 4.3. For $\pi \in \mathfrak{M}_{n}$, the following conditions hold true:

- $\pi_{1} \geq\left\lfloor\frac{n+1}{2}\right\rfloor$.
- For odd $n: \pi_{n-2}=1$ and $\pi_{n-1}, \pi_{n} \geq\left\lfloor\frac{n+1}{2}\right\rfloor$.
- For even $n: \pi_{n-3}=1$ and $\pi_{n-2}, \pi_{n-1}, \pi_{n} \geq\left\lfloor\frac{n+1}{2}\right\rfloor$.

Next, an enumerative conjecture on $\mathfrak{M}_{n}$, computationally verified for $n \leq 6$.
Conjecture 4.4. For all positive integers $n$, we have $\left|\mathfrak{M}_{2 n}\right|=(n+1)\left|\mathfrak{M}_{2 n-1}\right|$.
Finally, we conclude with an enumerative conjecture on $\operatorname{Sort}_{t, n}(123,132)$, the set of length $n$ permutations that are $t$-stack-sortable under $s_{123,132}$.
Conjecture 4.5. For any positive integer $t$ and $n \geq 2 t+1$, we have:

- $\left|\operatorname{Sort}_{t, n}(123,132)\right|=\frac{n+3}{2}\left|\operatorname{Sort}_{t, n-2}(123,132)\right|$ if $n$ is odd.
- $\left|\operatorname{Sort}_{t, n}(123,132)\right|=\frac{n+4}{2}\left|\operatorname{Sort}_{t, n-2}(123,132)\right|$ if $n$ is even.


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