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ORBITS AND INVARIANTS FOR COISOTROPY REPRESENTATIONS

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ABSTRACT. For a subgroup H of a reductive group G, let $\mathfrak m = \mathfrak h^\perp \subset \mathfrak g^*$ be the cotangent space of $\{H\} \in G/H$. The linear action $(H : \mathfrak m)$ is the *coisotropy representation*. It is known that the complexity and rank of G/H (denoted c and r, respectively) are encoded in properties of $(H : \mathfrak m)$. We complement existing results on c, r, and $(H : \mathfrak m)$, especially for quasiaffine varieties G/H. For instance, if the algebra of invariants $\Bbbk[\mathfrak m]^H$ is finitely generated, then $\mathfrak N_H(\mathfrak m) \subset \mathfrak m \cap \mathfrak N_G(\mathfrak g^*)$. Moreover, if G/H is affine, then $\mathfrak N_H(\mathfrak m) = \mathfrak m \cap \mathfrak N_G(\mathfrak g^*)$ if and only if $c = 0$. We also prove that the variety $\mathfrak m \cap \mathfrak N_G(\mathfrak g^*)$ is pure, of dimension $\dim \mathfrak m - r$. Two other topics considered are (i) a relationship between varieties G/H of complexity at most 1 and the homological dimension of the algebra $\Bbbk[\mathfrak m]^H$ and (ii) the Poisson structure of $\Bbbk[\mathfrak m]^H$ and Poisson-commutative subalgebras $\mathcal A \subset \Bbbk[\mathfrak m]^H$ such that $\operatorname{trdeg} \mathcal A$ is maximal.

INTRODUCTION

In this article, we study invariant-theoretic properties for the coisotropy representation of a homogeneous space of a reductive group G. The ground field $\Bbbk$ is algebraically closed and $\operatorname{char} \Bbbk = 0$. All groups and varieties are assumed to be algebraic, and all algebraic groups are affine. If Q is a group and X is a variety, then the notation $(Q : X)$ means that Q acts regularly on X. We also say that X is a *Q-variety*. Lie algebras of algebraic groups are denoted by the corresponding small gothic letters, e.g., $\mathfrak q = \mathsf{Lie}\, Q$.

Throughout, G is a connected reductive group and $\mathfrak g = \mathsf{Lie}\, G$. We also consider a Borel subgroup $B \subset G$, the maximal unipotent subgroup $U = (B, B)$, and a maximal torus $T \subset B$. This yields a bunch of related objects: roots, weights, simple roots, etc. For a reductive subgroup $H \subset G$, we denote by B_H, U_H, and T_H analogous subgroups of H.

For a subgroup $H \subset G$, let $c = c_G(G/H)$ and $r = r_G(G/H)$ be the *complexity* and *rank* of the G-variety G/H, respectively. Then $r \leqslant \operatorname{rk} G$ (see Section 1 for details.) These two integers are important for invariant theory and theory of equivariant embeddings of G/H. Let $\mathfrak m = \mathfrak h^\perp \subset \mathfrak g^*$ be the cotangent space of $\{H\} \in G/H$. The linear action $(H : \mathfrak m)$ is called the *coisotropy representation* of H (or G/H). It is shown in [12] that

• the integers c and r are closely related to properties of $(H : \mathfrak m)$. If G/H is quasiaffine, then $\dim \mathfrak m - \max_{x\in\mathfrak m} \dim H{\cdot}x = 2c + r$, the stabiliser H^x is reductive for generic $x \in \mathfrak m$, and $\operatorname{rk} G - \operatorname{rk} H^x = r$.

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• If $c = 0$ and $\mathfrak{c} \subset \mathfrak{m}$ is a *Cartan subspace*, then $\overline{H{\cdot}\mathfrak{c}} = \mathfrak{m}$ and there is a finite group $W \subset GL(\mathfrak{c})$ such that $\Bbbk[\mathfrak{c}]^W \simeq \Bbbk[\mathfrak{m}]^H$. Here the morphism $\pi : \mathfrak{m} \to \mathfrak{m}/\!\!/H := \mathsf{Spec}\,(\Bbbk[\mathfrak{m}]^H)$ is equidimensional, and if H is connected, then $\Bbbk[\mathfrak{m}]^H$ is a polynomial ring.

More generally, if $\Bbbk[\mathfrak{m}]^H$ is finitely generated, then π is also well-defined and the fibre $\pi^{-1}(\pi(0)) =: \mathfrak{N}_H(\mathfrak{m})$ is the *nullcone* (in $\mathfrak{m}$ with respect to H). The nullcone is a fibre of π of maximal dimension and if $c > 0$, then π is not necessarily equidimensional. It is convenient to consider the *defect* of equidimensionality (= *defect* of $\mathfrak{N}_H(\mathfrak{m})$)

$$\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = \dim \mathfrak{N}_H(\mathfrak{m}) - (\dim \mathfrak{m} - \dim \mathfrak{m}/\!\!/H).$$

If H is reductive, then $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) \leqslant c$ [15, Prop. 3.6].

In Section 2, we present new results related to the nullcones $\mathfrak{N}_H(\mathfrak{m})$ and $\mathfrak{N}_G(\mathfrak{g}^*)$, and the generalised Cartan subspace $\mathfrak{c} \subset \mathfrak{m}$.

– For any $x \in \mathfrak{m}$, we show that $\dim G{\cdot}x \geqslant 2\dim H{\cdot}x$, and the equality occurs if and only if $\mathfrak{g}{\cdot}x \cap \mathfrak{m} = \mathfrak{h}{\cdot}x$. Another general property is that $\dim\bigl(\mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)\bigr) = \dim \mathfrak{m} - r$ and all irreducible components of $\mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)$ have this dimension (Theorem 2.3).

– *If G/H is quasiaffine*, then one can define a generalised Cartan subspace $\mathfrak{c} \subset \mathfrak{m}$ (see Section 1) and we prove that $\operatorname{codim}_{\mathfrak{m}}\overline{H{\cdot}\mathfrak{c}} = c$.

– *If G/H is quasiaffine and $\Bbbk[\mathfrak{m}]^H$ is finitely generated*, then $\mathfrak{N}_H(\mathfrak{m}) \subset \mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)$. Moreover, if $c = 0$, then $\mathfrak{N}_H(\mathfrak{m}) = \mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)$ (Theorem 2.4).

– *If G/H is affine*, then $\mathfrak{N}_H(\mathfrak{m}) = \mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)$ if and only if G/H is spherical (Prop. 2.5). We also prove that if $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = c$, then $\mathfrak{h}$ contains a regular semisimple element of $\mathfrak{g}$ (Theorem 2.7). In particular, this applies to the affine homogeneous spaces with $c = 0$.

In Section 3, affine homogeneous spaces of the form $\tilde{\mathcal{O}} = (G \times H)/\Delta_H$ are studied. Here $H \subset G$ is reductive and Δ_H is the diagonal in $H \times H \subset G \times H = \tilde{G}$. Then $\tilde{\mathcal{O}} \simeq G$ and the isotropy representation of $\Delta_H \simeq H$ is identified with the H-module $\mathfrak{g}$. In this case, $\tilde{\mathcal{O}}$ has a group structure, $r_{\tilde{G}}(\tilde{\mathcal{O}}) = \operatorname{rk}\tilde{G}$ is maximal, and $\dim \mathfrak{N}_H(\mathfrak{g})$ can be computed via a result of R. Richardson [23]. Here we present some complements to results of Section 2.

In Section 4, we consider coisotropy representations $(H : \mathfrak{m})$ with $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) \leqslant 1$. If $c = 0$, then π is equidimensional and $\mathfrak{m}/\!\!/H \simeq \mathbb{A}^n$. This can be regarded as an illustration to the Popov conjecture [21]. In [15], we stated a related conjecture that if H is connected reductive and $c = 1$, then $\mathfrak{m}/\!\!/H$ is either an affine space or a hypersurface. We verify this in two cases:

(a) for the homogeneous spaces G/H with simple G;

(b) for the homogeneous spaces $\tilde{\mathcal{O}} = (G \times H)/\Delta_H$ with $c_{\tilde{G}}(\tilde{\mathcal{O}}) = 1$, where G is simple.

In both cases, classifications of such pairs (G, H) are known (see [13] for (a) and [2] for (b)), and we perform a case-by-case verification.

In Section 5, H is reductive and the natural Poisson bracket $\{\ ,\ \}$ on the affine variety $\mathfrak{m}/\!\!/H$ is considered. Let $\mathcal{Z}$ be the Poisson centre of $(\Bbbk[\mathfrak{m}]^H, \{\ ,\ \})$. Then there is the natural morphism $\boldsymbol{f} : \mathfrak{m} \to \mathsf{Spec}\,\mathcal{Z}$. Using results of F. Knop [5], we prove that $\boldsymbol{f}$ is equidimensional and $\boldsymbol{f}^{-1}(0) = \mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)$.

It is shown that if a subalgebra $\mathcal{A} \subset \Bbbk[\mathfrak{m}]^H$ is Poisson-commutative, then $\operatorname{trdeg}\mathcal{A} \leqslant c+r$. We conjecture that there is always such a subalgebra with $\operatorname{trdeg}\mathcal{A} = c + r$. Some partial results towards this conjecture are described.

Our basic reference for Invariant Theory is [31].

Data availability and conflict of interest statement.

This article has no associated data. There is no conflict of interest.

1. Generalities on group actions and coisotropy representations

Let $\Bbbk[X]$ denote the algebra of regular functions on a variety X. If X is irreducible, then $\Bbbk(X)$ is the field of rational functions on X. If X is acted upon by Q, then $\Bbbk[X]^Q$ and $\Bbbk(X)^Q$ are the subalgebra and subfield of invariant functions, respectively. The identity component of Q is denoted by Q^o. For $x \in X$, let Q^x denote the *stabiliser* of x in Q. Then $\mathfrak{q}^x = \mathsf{Lie}\,Q^x$. A stabiliser Q^x is said to be *generic*, if there is a dense open subset $\Omega \subset X$ such that Q^y is Q-conjugate to Q^x for all $y \in \Omega$. We say that a property (**P**) holds for *almost all points of* X, if there is a dense open subset $X_0 \subset X$ such that (**P**) holds for all $x \in X_0$.

1.1. **Complexity and rank.** Let X be an irreducible G-variety. Then

– the *complexity* of X is $c_G(X) = \dim X - \max_{x\in X} \dim B{\cdot}x$,
– the *rank* of X is $r_G(X) = \max_{x\in X} \dim B{\cdot}x - \max_{x\in X} \dim U{\cdot}x$.

By the Rosenlicht theorem (see e.g. [31, § 2.3]), we also have

$$c_G(X) = \operatorname{trdeg}\ k(X)^B \text{ and } c_G(X) + r_G(X) = \operatorname{trdeg}\ k(X)^U.$$

An alternate approach to the rank uses the weights of B-semi-invariants in $\Bbbk(X)$. For quasiaffine varieties, this boils down to the following. Write $\mathfrak{X}_+ = \mathfrak{X}_+(G)$ for the set of dominant weights of G with respect to (B, T). Let V_λ denote a simple G-module with highest weight $\lambda \in \mathfrak{X}_+$. Let $\Bbbk[X] = \bigoplus_{\lambda\in\mathfrak{X}_+} \Bbbk[X]_{(\lambda)}$ be the sum of G-isotypic components, where $\Bbbk[X]_{(0)} = \Bbbk[X]^G$. Then

$$\Gamma_X = \{\lambda \in \mathfrak{X}_+ \mid \Bbbk[X]_{(\lambda)} \neq 0\}$$

is the *rank monoid* of X and $r_G(X) = \dim_{\mathbb{Q}}(\mathbb{Q}\Gamma_X)$. Clearly $r \leqslant \operatorname{rk}\mathfrak{g}$. If $c_G(X) = 0$, then X is said to be a *spherical* G-variety. If $\Bbbk[X]^G = \Bbbk$, then $\dim \Bbbk[X]_{(\lambda)} < \infty$ for all λ and $m_\lambda(X) := \dim \Bbbk[X]_{(\lambda)}/ \dim \mathsf{V}_\lambda$ is the *multiplicity* of V_λ in $\Bbbk[X]$ (= the multiplicity of λ in Γ_X).

1.2. **The coisotropy representation.** Let H be an algebraic subgroup of G with $\mathsf{Lie}\, H = \mathfrak{h}$. Then $\mathfrak{g}/\mathfrak{h} \simeq \mathsf{T}_{\{H\}}(G/H)$ is an H-module and the linear action $(H : \mathfrak{g}/\mathfrak{h})$ is the *isotropy representation* of H. Set $\mathfrak{m} = \mathfrak{h}^{\perp} = \{\xi \in \mathfrak{g}^* \mid \xi|_{\mathfrak{h}} = 0\}$. Then $\mathfrak{m} \simeq (\mathfrak{g}/\mathfrak{h})^*$ as H-module and the linear action $(H : \mathfrak{m})$ is the *coisotropy representation* of H. If necessary, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using a non-degenerate G-invariant bilinear form Ψ on $\mathfrak{g}$ and regard $\mathfrak{m}$ as subspace of $\mathfrak{g}$.

Recall that, for a reductive G, G/H is affine if and only if H is reductive. Another equivalent condition is that the form Ψ is non-degenerate on $\mathfrak{h}$. In this case, $\mathfrak{m} \simeq \mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{m}$ as H-module. Then the linear action $(H : \mathfrak{m})$ will be referred to as the *isotropy representation* of H.

We always assume that the G-action on G/H has a finite kernel. This is tantamount to saying that H contains no infinite normal subgroups of G. This condition is always satisfied, if G is simple. If G/H is spherical, then H is said to be a *spherical* subgroup of G. For simplicity, we write c and r for the complexity and rank of the homogeneous space G/H.

Theorem 1.1 ([12], [16, Ch. 2]). *If G/H is quasiaffine, then*

1. *There is a generic stabiliser for $(H : \mathfrak{m})$, say S, which is reductive;*
2. $\dim G + \dim S - 2\dim H = \dim \mathfrak{m} - \max_{x\in\mathfrak{m}} \dim H{\cdot}x = 2c + r$;
3. $\mathrm{rk}\, G - \mathrm{rk}\, S = r$.

The theory developed in [12] (and presented with more details in [16]) contains much more results. We mention those that will be needed later. Let us assume that B and T are fixed. Then the choice of H (up to conjugacy in G) and S (up to conjugacy in H) is at our disposal. It was proved that H and S can be chosen such that

($\mathcal{P}_1$) $Z_G(t)' \subset S \subset Z_G(t)$ for some $t \in \mathfrak{t} = \mathsf{Lie}\, T$. Hence $T \subset N_G(S)$ and S/S^o is abelian;
($\mathcal{P}_2$) $\mathfrak{b} \cap \mathfrak{s}$ is a Borel subalgebra of $\mathfrak{s}$ and $B \cap S$ is a generic stabiliser for $(B : G/H)$;
($\mathcal{P}_3$) $\mathfrak{u} \cap \mathfrak{s}$ is the nilradical of $\mathfrak{b} \cap \mathfrak{s}$ and $U \cap S$ is a generic stabiliser for $(U : G/H)$;
($\mathcal{P}_4$) $\mathfrak{t} \cap \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$;
($\mathcal{P}_5$) $B{\cdot}S = P$ is a parabolic subgroup and $P \cap H = S$.

Whenever the algebra $\Bbbk[\mathfrak{m}]^H$ is finitely generated, we consider the following objects:

- the *categorical quotient* $\mathfrak{m}/\!\!/H := \mathsf{Spec}\,(\Bbbk[\mathfrak{m}]^H)$;
- the *quotient morphism* $\pi = \pi_{H,\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}/\!\!/H$ induced by the inclusion $\Bbbk[\mathfrak{m}]^H \hookrightarrow \Bbbk[\mathfrak{m}]$;
- the *nullcone* $\mathfrak{N}_H(\mathfrak{m}) := \pi^{-1}(\pi(0)) \subset \mathfrak{m}$.

Example 1.1. Let $\sigma \in \mathsf{Aut}(\mathfrak{g})$ be an involution and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the sum of ± 1-eigenspaces of σ. If G_0 is the connected subgroup of G with $\mathsf{Lie}\, G_0 = \mathfrak{g}_0$, then G/G_0 is affine and $c_G(G/G_0) = 0$. We say that G/G_0 is a *symmetric variety* and G_0 is a *symmetric subgroup* of G. The isotropy representation $(G_0 : \mathfrak{g}_1)$ has thoroughly been studied by Kostant–Rallis [4].

For instance, they proved that $\Bbbk[\mathfrak g_1]^{G_0}$ is a polynomial algebra, $\dim \mathfrak g_1/\!\!/G_0 = r_G(G/G_0)$, and $\pi_{G_0,\mathfrak g_1} : \mathfrak g_1 \to \mathfrak g_1/\!\!/G_0$ is *equidimensional*, i.e., all fibres have the same dimension.

For any quasiaffine G/H, we introduced in [12] a certain subspace $\mathfrak c \subset \mathfrak m$, which is useful in the study of the linear action $(H : \mathfrak m)$. Let us recall the general construction of $\mathfrak c$. The definitions of the complexity and rank of G/H imply that

$$\dim G/H - \max_{x\in G/H} \dim B{\cdot}x = c \quad \& \quad \dim G/H - \max_{x\in G/H} \dim U{\cdot}x = c + r.$$

Without loss of generality, we may assume that $x = \{H\}$ is generic in both senses and properties $(\mathcal P_1)$-$(\mathcal P_5)$ are satisfied. Then $\operatorname{codim}_{\mathfrak g}(\mathfrak b + \mathfrak h) = c$, $\operatorname{codim}_{\mathfrak g}(\mathfrak u + \mathfrak h) = c + r$, and we set

$$\mathfrak c = (\mathfrak b + \mathfrak h)^\perp = \mathfrak u^\perp \cap \mathfrak m \quad \& \quad \tilde{\mathfrak c} = (\mathfrak u + \mathfrak h)^\perp = \mathfrak b^\perp \cap \mathfrak m.$$

Then $\dim \mathfrak c = c + r$ and $\dim \tilde{\mathfrak c} = c$. It follows that $\dim \mathfrak c \leqslant \dim \mathfrak m - \max_{x\in\mathfrak m} \dim H{\cdot}x = 2c + r$, and the equality occurs if and only if $c = 0$. Upon identification of $\mathfrak g$ and $\mathfrak g^*$, we have $\mathfrak c = \mathfrak b \cap \mathfrak m$ and $\tilde{\mathfrak c} = \mathfrak u \cap \mathfrak m$.

Consider the projection $p_{\mathfrak t} : \mathfrak b = \mathfrak t \oplus \mathfrak u \to \mathfrak t$ and set $\mathfrak t_1 = p_{\mathfrak t}(\mathfrak c) \subset \mathfrak t$. Then $\dim \mathfrak t_1 = r$. If $c = 0$, then $\tilde{\mathfrak c} = \{0\}$ and $p_{\mathfrak t}$ maps $\mathfrak c$ isomorphically to $\mathfrak t_1$. In this case, $\mathfrak c$ contains no nilpotent elements of $\mathfrak g$. Moreover, the following holds.

Theorem 1.2 ([12, Section 3.2])**.** *For $c = 0$, the subspace $\mathfrak c \subset \mathfrak m$ has the following properties:*

1. *the H-saturation of $\mathfrak c$ is dense in $\mathfrak m$, i.e., $\overline{H{\cdot}\mathfrak c} = \mathfrak m$;*
2. *almost all elements of $\mathfrak c$ have the same stabiliser in H, which is just S;*
3. *there is a finite group $W \subset GL(\mathfrak c)$ such that the restriction homomorphism $\Bbbk[\mathfrak m] \to \Bbbk[\mathfrak c]$ induces an isomorphism $\Bbbk[\mathfrak m]^H \xrightarrow{\sim} \Bbbk[\mathfrak c]^W$.*

It follows from (1) and (2) that, for almost all $x \in \mathfrak c$, the stabiliser H^x is generic, while (3) implies that $\Bbbk[\mathfrak m]^H$ is finitely generated and $\pi : \mathfrak m \to \mathfrak m/\!\!/H$ is *equidimensional*. The common dimension of fibres equals $\dim \mathfrak m - \dim \mathfrak m/\!\!/H = \dim \mathfrak m - r$. Furthermore, if H is connected, then W is a reflection group and $\Bbbk[\mathfrak m]^H$ is a polynomial ring, i.e., $\mathfrak m/\!\!/H \simeq \mathbb A^r$ is an affine space, see [12, Cor. 5].

If $c = 0$, then $\mathfrak c$ resembles a Cartan subspace for the isotropy representation a symmetric variety G/G_0. For this reason, a subspace $\mathfrak c$ satisfying properties of Theorem 1.2 was christened in [12] a *Cartan subspace* (of $\mathfrak m$).

Thus, Theorem 1.2 shows that many good properties of the symmetric variety G/G_0 and $(G_0 : \mathfrak g_1)$ are retained for quasiaffine spherical homogeneous spaces.

For an arbitrary quasiaffine G/H, we shall say that $\mathfrak c = \mathfrak u^\perp \cap \mathfrak m$, as above, is a *generalised Cartan subspace* of $\mathfrak m$. If G/H is not spherical, then $\mathfrak c$ does not satisfy properties (1) and (3) of Theorem 1.2 (cf. also Theorem 2.2 below).

Remarks. 1) In [12], the generalised Cartan subspace is denoted by $\mathfrak{z}$. Here we follow the notation of [16, Chapter 2].

2) Above results on the complexity, rank, and coisotropy representations are also obtained by Knop [5] via different methods. Our approach in [12, 16] is based on the study of 'doubled actions' (which is not discussed here), while Knop considers the cotangent bundles and moment map.

The nullcone $\mathfrak{N}_H(\mathfrak{m})$ is a fibre of π of maximal dimension [31, § 5.2]. The action $(H:\mathfrak{m})$ is said to be *equidimensional*, if the quotient morphism $\pi:\mathfrak{m}\to\mathfrak{m}/\!\!/H$ is equidimensional. The *defect* of equidimensionality of π (= of $\mathfrak{N}_H(\mathfrak{m})$) is introduced in [15, Section 3] as the difference between $\dim\mathfrak{N}_H(\mathfrak{m})$ and dimension of generic fibres of π, i.e.,

$$\operatorname{def}\mathfrak{N}_H(\mathfrak{m})=\dim\mathfrak{N}_H(\mathfrak{m})-(\dim\mathfrak{m}-\dim\mathfrak{m}/\!\!/H).$$

Therefore, π is equidimensional if and only if $\operatorname{def}\mathfrak{N}_H(\mathfrak{m})=0$.

If H is reductive, then $\mathfrak{N}_H(\mathfrak{m})=\{x\in\mathfrak{m}\mid\overline{H{\cdot}x}\ni 0\}$ and the representation $(H:\mathfrak{m})$ is orthogonal. The latter implies that the action $(H:\mathfrak{m})$ is stable [9] and therefore $\dim\mathfrak{m}/\!\!/H=2c+r$. Then $\dim\mathfrak{N}_H(\mathfrak{m})\geqslant\dim\mathfrak{m}-(2c+r)$. On the other hand, there is an upper bound on dimension of the nullcone for the self-dual representations of reductive groups [25, Prop. 2.10]. In our case, this shows that

$$\dim\mathfrak{N}_H(\mathfrak{m})\leqslant\dim U_H+\frac{1}{2}(\dim\mathfrak{m}-\dim\mathfrak{m}^{T_H})=\frac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{g}^{T_H}). \tag{1.1}$$

Theorem 1.3 ([15, Proposition 3.6])**.** *If H is reductive, then* $\operatorname{def}\mathfrak{N}_H(\mathfrak{m})\leqslant c$.

Outline of the proof. 1^o. If $r=\operatorname{rk}\mathfrak{g}$, then S is finite and $\dim G/H=\dim\mathfrak{m}=\dim B+c$. Hence $\dim\mathfrak{N}_H(\mathfrak{m})\geqslant\dim\mathfrak{m}-\dim\mathfrak{m}/\!\!/H=\dim H=\dim U-c$. On the other hand,

$$\dim\mathfrak{N}_H(\mathfrak{m})\leqslant\tfrac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{g}^{T_H})\leqslant\dim U. \tag{1.2}$$

2^o. If $r<\operatorname{rk}\mathfrak{g}$, then $\mathrm{Lie}\, S\neq 0$ and one can use the Luna–Richardson theorem [10, Theorem 4.2]. This provides a (rather technical) reduction to part 1^o. □

Remark 1.1. The relations $\dim H=\dim U-c$ and $\dim\mathfrak{m}=\dim B+c$ hold only if S is finite. In general, one has $\dim H=\dim U+\dim B_S-c$ and $\dim\mathfrak{m}=\dim B-\dim B_S+c$.

1.3. **Homogeneous spaces of complexity at most 1.** The Luna–Vust theory of equivariant embeddings of homogeneous spaces (1983) implies that a reasonably complete theory can be developed for homogeneous spaces of complexity $\leqslant 1$. For a modern account of that theory and related topics, we refer to [28].

As is already mentioned, if $\mathfrak{h}$ is a fixed point subalgebra for an involution of $\mathfrak{g}$, then $c_G(G/H)=0$. All connected spherical reductive subgroups H of **simple** algebraic groups G have been found by M. Krämer [8]. Then M. Brion and I. Mikityuk (independently) found all connected spherical reductive subgroups of the **semisimple** algebraic groups, see e.g. tables in [30, Ch. I, § 3.6].

The study of quasiaffine homogeneous spaces of complexity 1 was initiated in [13], where a classification of the pairs (G, H) such that G simple, H is connected reductive, and $c_G(G/H) = 1$ is also obtained. (See also [16, Chapter 3].)

2. New results for coisotropy representations

For the symmetric varieties (see Example 1.1), one has $\dim G{\cdot}x = 2\dim G_0{\cdot}x$ for **any** $x \in \mathfrak{g}_1$ [4, Prop. 5]. For quasiaffine spherical G/H, this equality holds generically, i.e., there is a dense open subset $\Omega \subset \mathfrak{m}$ such that $\dim G{\cdot}x = 2\dim H{\cdot}x$ for all $x \in \Omega$ [12, Theorem 5]. Then $H{\cdot}x$ is a Lagrangian subvariety of the symplectic variety $G{\cdot}x \subset \mathfrak{g}^*$ for all $x \in \Omega$. The following observation is a slight extension of [12, Proposition 1].

Lemma 2.1. (i) *For **any** algebraic subgroup $H \subset G$ and $x \in \mathfrak{m} = \mathfrak{h}^\perp$, one has*

$$\dim G{\cdot}x = \dim H{\cdot}x + \dim([\mathfrak{g}, x] \cap \mathfrak{m}) \geqslant 2\dim H{\cdot}x.$$

(ii) $\dim G{\cdot}x = 2\dim H{\cdot}x \Longleftrightarrow [\mathfrak{g}, x] \cap \mathfrak{m} = [\mathfrak{h}, x]$.

Proof. We have $([\mathfrak{g}, x] \cap \mathfrak{m})^\perp = ([\mathfrak{g}, x])^\perp + \mathfrak{m}^\perp = \mathfrak{g}^x + \mathfrak{h}$. Hence

$$\dim([\mathfrak{g}, x] \cap \mathfrak{m}) = \dim\mathfrak{g} - \dim\mathfrak{g}^x - \dim\mathfrak{h} + \dim(\mathfrak{g}^x \cap \mathfrak{h}) = \dim G{\cdot}x - \dim H{\cdot}x.$$

It is also clear that $[\mathfrak{g}, x] \cap \mathfrak{m} \supset [\mathfrak{h}, x]$. □

In the setting of symmetric varieties (Example 1.1), the relation $[\mathfrak{g}, x] \cap \mathfrak{g}_1 = [\mathfrak{g}_0, x]$ for all $x \in \mathfrak{g}_1$ readily follows from the presence of involution σ.

Recall that $\mathfrak{c} = \mathfrak{m} \cap \mathfrak{u}^\perp = \mathfrak{m} \cap \mathfrak{b}$ is a generalised Cartan subspace, $\tilde{\mathfrak{c}} = \mathfrak{c} \cap \mathfrak{u}$, and $\mathfrak{t}_1 = p_{\mathfrak{t}}(\mathfrak{c}) \subset \mathfrak{t}$. The following is a generalisation of Theorem 1.2(1).

Theorem 2.2. *Let G/H be a quasiaffine homogeneous space and $\mathfrak{c} \subset \mathfrak{m}$ a generalised Cartan subspace. Then $\operatorname{codim}_{\mathfrak{m}} \overline{H{\cdot}\mathfrak{c}} = c_G(G/H) = c$.*

Proof. 1. We assume that properties $(\mathcal{P}_1)$-$(\mathcal{P}_5)$ are satisfied for S, B, and H. Then $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{t}_1$ is a Levi subalgebra of $\mathfrak{p} = \mathrm{Lie}\, P$. Moreover, by [12, Lemma 3], one has $\mathfrak{g}^y = \mathfrak{l}$ for almost all $y \in \mathfrak{t}_1$. Let $\mathfrak{n}$ denote the nilradical of $\mathfrak{p}$ and $\mathfrak{n}_-$ the opposite nilradical, i.e., $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}_-$. Then $\mathfrak{c} = \mathfrak{m} \cap \mathfrak{b} = \mathfrak{m} \cap (\mathfrak{n} \oplus \mathfrak{t}_1)$ and $\tilde{\mathfrak{c}} = \mathfrak{m} \cap \mathfrak{u} = \mathfrak{m} \cap \mathfrak{n}$.

2. By part 1, we have $\mathfrak{c}^\perp = \mathfrak{m}^\perp + (\mathfrak{n} \oplus \mathfrak{t}_1)^\perp = \mathfrak{h} + (\mathfrak{s} \oplus \mathfrak{n}) = \mathfrak{h} + \mathfrak{n}$. For almost all $y \in \mathfrak{c}$, $\mathfrak{g}^y$ is a Levi subalgebra of $\mathfrak{p}$ (see Step 1 in [16, Theorem 2.2.6]). Therefore $([\mathfrak{g}, y] \cap \mathfrak{c})^\perp = \mathfrak{g}^y + \mathfrak{c}^\perp = \mathfrak{h} + (\mathfrak{g}^y + \mathfrak{n}) = \mathfrak{h} + \mathfrak{p}$. Hence

$$[\mathfrak{g}, y] \cap \mathfrak{c} = (\mathfrak{h} + \mathfrak{p})^\perp = \mathfrak{m} \cap \mathfrak{n} = \tilde{\mathfrak{c}} \ \text{ and } \ \dim([\mathfrak{g}, y] \cap \mathfrak{c}) = c. \tag{2.1}$$

3. Let us prove that $[\mathfrak{h}, y] \cap \mathfrak{c} = \{0\}$ for almost all $y \in \mathfrak{c}$. (For $c = 0$, this follows from (2.1), and this was already used in [12, 16] for proving Theorem 1.2(1).) By part 2, $[\mathfrak{g}, y] \cap \mathfrak{c} = \tilde{\mathfrak{c}} \subset \mathfrak{n}$. Therefore, it suffices to prove that $[\mathfrak{h}, y] \cap \mathfrak{n} = \{0\}$. Actually, we shall show that $[\mathfrak{h}, y] \cap \mathfrak{p} = \{0\}$.

Since $\mathfrak{h}\cap\mathfrak{p}=\mathfrak{s}$, we can write $\mathfrak{h}=\mathfrak{s}\oplus\hat{\mathfrak{h}}$, where $\hat{\mathfrak{h}}\cap\mathfrak{p}=\{0\}$. Then every nonzero element of $\hat{\mathfrak{h}}$ has a nonzero component in $\mathfrak{n}_-$ w.r.t. the sum $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{n}_-$. Write $y=y'+y''$, where $y'\in\mathfrak{t}_1$ and $y''\in\mathfrak{n}$, i.e., $p_{\mathfrak{t}}(y)=y'$. Let us say that $y'\in\mathfrak{t}_1$ is generic, if $\mathfrak{g}^{y'}=\mathfrak{l}$. Take a nonzero $x=x_{\mathfrak{s}}+\hat{x}\in\mathfrak{h}$, where $x_{\mathfrak{s}}\in\mathfrak{s}$ and $\hat{x}\in\hat{\mathfrak{h}}$. Write $\hat{x}=x_{\mathfrak{p}}+x_-$, where $x_{\mathfrak{p}}\in\mathfrak{p}$ and $x_-\in\mathfrak{n}_-$. If $\hat{x}\ne 0$, then $x_-\ne 0$ as well. We have

$$[x,y]=[\hat{x},y]=[x_{\mathfrak{p}},y]+[x_-,y]$$

and here $[x_{\mathfrak{p}},y]\in\mathfrak{p}$. Since y' is a generic element of $\mathfrak{t}_1$, there is $n\in N=\exp(\mathfrak{n})$ such that $n{\cdot}y=y'$. Then $n{\cdot}[x_-,y]=[n{\cdot}x_-,y']$. Again, since y' is generic, the last bracket has a nonzero component in $\mathfrak{n}_-$. Therefore, the same holds for $[x_-,y]=n^{-1}{\cdot}[n{\cdot}x_-,y']$.

Thus, we proved that $[\mathfrak{h},y]\cap\mathfrak{p}=\{0\}$ for almost all $y\in\mathfrak{c}$ and thereby $[\mathfrak{h},y]\cap\mathfrak{c}=\{0\}$.

4. Since $[\mathfrak{h},y]\cap\mathfrak{c}=\{0\}$ for almost all $y\in\mathfrak{c}$, the intersection $H{\cdot}y\cap\mathfrak{c}$ is finite. Hence

$$\begin{aligned}\dim\overline{H{\cdot}\mathfrak{c}}=\dim H{\cdot}y+\dim\mathfrak{c}&=\dim H-\dim S+(c+r)\\ &=\dim\mathfrak{m}-(2c+r)+(c+r)=\dim\mathfrak{m}-c. \qquad\square\end{aligned}$$

Theorem 2.3. *For **any** homogeneous space G/H, we have*

(1) $\dim\big(\mathfrak{m}\cap\mathfrak{N}_G(\mathfrak{g}^*)\big)=\dim\mathfrak{m}-r$ and all irreducible components of $\mathfrak{m}\cap\mathfrak{N}_G(\mathfrak{g}^*)$ have this dimension.

(2) The intersection $\mathfrak{m}\cap\mathfrak{N}_G(\mathfrak{g}^*)$ is proper if and only if $r=\mathrm{rk}\,\mathfrak{g}$.

Proof. (1) To use some results of F. Knop [5], we need more notation. Let $T^*_{\mathcal{O}}\simeq G\times_H\mathfrak{m}$ be the cotangent bundle of $\mathcal{O}=G/H$ and $\widetilde{\Phi}:T^*_{\mathcal{O}}\to\mathfrak{g}^*$ the associated moment map. Let $\widetilde{M}_{\mathcal{O}}=\overline{G{\cdot}\mathfrak{m}}$ denote the closure of the image of $\widetilde{\Phi}$ in $\mathfrak{g}^*$. Finally, $M_{\mathcal{O}}$ is the spectrum of the integral closure of $\Bbbk[\widetilde{M}_{\mathcal{O}}]$ in $\Bbbk[T^*_{\mathcal{O}}]$. This yields the commutative diagram of morphisms

(2.2)
$$\begin{tikzcd}
T^*_{\mathcal{O}} \arrow[r, "\Phi"] \arrow[rr, bend left, "\widetilde{\Phi}"] \arrow[dr, "\psi"'] \arrow[drr, bend right, "\tilde{\psi}"'] & M_{\mathcal{O}} \arrow[r, "\tilde{\tau}"] \arrow[d, "\pi_M"] & \widetilde{M}_{\mathcal{O}} \arrow[r, hook, "i"] \arrow[d, "\pi_{\tilde{M}}"] & \mathfrak{g}^* \arrow[d, "\pi_{\mathfrak{g}^*}"] \\
& M_{\mathcal{O}}/\!\!/G \arrow[r, "\tau"] & \widetilde{M}_{\mathcal{O}}/\!\!/G \arrow[r, hook] & \mathfrak{g}^*/\!\!/G
\end{tikzcd}$$

where the vertical arrows are quotient morphisms and $\tilde{\Phi}=\tilde{\tau}\circ\Phi$ see [5, Sect. 6]. By construction, $\widetilde{\tau}$ is finite and onto. Then so is τ. It is proved in [5] that

- $M_{\mathcal{O}}/\!\!/G$ is an affine space, see Satz 6.6(b);
- ψ is equidimensional and onto, see Satz 6.6(c);
- $\dim M_{\mathcal{O}}/\!\!/G=r$, see Satz 7.1.

Then $\tilde\psi = \tau\circ\psi$ is equidimensional and onto, too. Hence, for $\bar 0 = \pi_{\tilde M}(0)$, we obtain $\dim\tilde\psi^{-1}(\bar 0) = \dim T^*_{\mathcal O} - r = 2\dim\mathfrak m - r$. Note that $\pi^{-1}_{\tilde M}(\bar 0) = \overline{G{\cdot}\mathfrak m}\cap\mathfrak N_G(\mathfrak g^*)$. Hence

$$\tilde\psi^{-1}(\bar 0) = \tilde\Phi^{-1}(\overline{G{\cdot}\mathfrak m}\cap\mathfrak N_G(\mathfrak g^*)) = G\times_H(\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)).$$

Therefore, $\dim(\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)) = 2\dim\mathfrak m - r - \dim G/H = \dim\mathfrak m - r$ and all irreducible components have this dimension, as required.

(2) By definition, the intersection of $\mathfrak m$ and $\mathfrak N_G(\mathfrak g^*)$ in $\mathfrak g^*$ is proper if and only if

$$\dim\bigl(\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)\bigr) = \dim\mathfrak m + \dim\mathfrak N_G(\mathfrak g^*) - \dim\mathfrak g = \dim\mathfrak m - \operatorname{rk}\mathfrak g. \qquad \square$$

If H is reductive, then $\mathfrak N_H(\mathfrak m) = \{m\in\mathfrak m \mid \overline{H{\cdot}m}\ni 0\}$. Therefore, $\mathfrak N_H(\mathfrak m)\subset\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)$. As we prove below, this inclusion actually holds in a more general situation.

Theorem 2.4. *Suppose that G/H be quasi-affine and $\Bbbk[\mathfrak m]^H$ is finitely generated. Then*

(i) $\mathfrak N_H(\mathfrak m)\subset\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)$;
(ii) *if $c=0$, then $\mathfrak N_H(\mathfrak m) = \mathfrak m\cap\mathfrak N_G(\mathfrak g^*)$.*

Proof. (i) Consider a version of the commutative diagram (2.2) :

(2.3)
$$\begin{tikzcd}
\mathfrak N_H(\mathfrak m) \arrow[r, hook] & \mathfrak m \arrow[r, hook, "j"] \arrow[d, "\pi_{\mathfrak m}"] & G\times_H\mathfrak m \arrow[r, "\tilde\Phi"] \arrow[d] \arrow[dr, "\tilde\psi"] & \overline{G{\cdot}\mathfrak m} \arrow[r, hook] \arrow[d, "\pi_{\tilde M}"] & \mathfrak g^* \\
& \mathfrak m/\!\!/H \arrow[r, "\sim"] & (G\times_H\mathfrak m)/\!\!/G \arrow[r, "f"] & \overline{G{\cdot}\mathfrak m}/\!\!/G &
\end{tikzcd}$$

Here $\overline{G{\cdot}\mathfrak m} = \widetilde{M}_{\mathcal O}$ and j is the embedding of $\mathfrak m$ as fibre of $\{H\}\in G/H$. As above, all vertical arrows are quotient morphisms. Since $\Bbbk[\mathfrak m]^H$ is finitely generated, we get two new objects in the lower row. Using the path through $\pi_{\mathfrak m}$, we see that $\mathfrak N_H(\mathfrak m)$ maps into $\bar 0 = \pi_{\tilde M}(0)\in\overline{G{\cdot}\mathfrak m}/\!\!/G$. On the other hand, using the path through j and $\tilde\psi$, we see that $j(\mathfrak N_H(\mathfrak m))\subset\tilde\psi^{-1}(\bar 0) = G\times_H\bigl(\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)\bigr)$, i.e., $\mathfrak N_H(\mathfrak m)\subset\mathfrak m\cap\mathfrak N_G(\mathfrak g^*)$.

(ii) Here we use a fragment of the previous diagram:

$$\begin{tikzcd}
\mathfrak m \arrow[r, hook, "\tilde\Phi\circ j"] \arrow[d, "\pi_{\mathfrak m}"] & \overline{G{\cdot}\mathfrak m} \arrow[d, "\pi_{\tilde M}"] \\
\mathfrak m/\!\!/H \arrow[r, "f"] & \overline{G{\cdot}\mathfrak m}/\!\!/G
\end{tikzcd}$$

If $c=0$, then $\dim\mathfrak m/\!\!/H = \dim\overline{G{\cdot}\mathfrak m}/\!\!/G = r$. Since $\tilde\psi$ is equidimensional, the same is true for f. The affine varieties $\mathfrak m/\!\!/H$ and $\overline{G{\cdot}\mathfrak m}/\!\!/G$ are conical, hence f is finite and $\pi_{\mathfrak m}(0) = f^{-1}(\bar 0)$. Therefore, $\mathfrak N_H(\mathfrak m) = \overline{G{\cdot}\mathfrak m}\cap\mathfrak N_G(\mathfrak g^*)\cap\mathfrak m = \mathfrak N_G(\mathfrak g^*)\cap\mathfrak m$. $\square$

Remark 2.1. (a) There is an alternate proof of Theorem 2.4(ii) that exploits properties of the Cartan subspace $\mathfrak c = \mathfrak b\cap\mathfrak m$. In the spherical case, the projection $p_{\mathfrak t}:\mathfrak b\to\mathfrak t$ maps $\mathfrak c$

isomorphically onto $\mathfrak{t}_1$ and one proves that $\overline{G{\cdot}\mathfrak{t}_1} = \overline{G{\cdot}\mathfrak{c}} = \overline{G{\cdot}\mathfrak{m}}$. Then the finiteness of the morphism $f : \mathfrak{m}/\!\!/H \to \overline{G{\cdot}\mathfrak{m}}/\!\!/G$ is obtained without using the equidimensional map $\tilde{\psi}$.

(b) For the symmetric varieties, the equality $\mathfrak{N}_{G_0}(\mathfrak{g}_1) = \mathfrak{g}_1 \cap \mathfrak{N}_G(\mathfrak{g})$ is proved in [4].

For the affine homogeneous spaces, one can strengthen Theorem 2.4 as follows.

Proposition 2.5. *If G/H is affine, then*

(i) $\dim\mathfrak{m} - 2c - r \leqslant \dim\mathfrak{N}_H(\mathfrak{m}) \leqslant \dim\mathfrak{m} - c - r$;

(ii) $\mathfrak{N}_H(\mathfrak{m}) = \mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)$ *if and only if* G/H *is spherical.*

Proof. (i) The first inequality means that $\dim\mathfrak{N}_H(\mathfrak{m}) \geqslant \dim\mathfrak{m} - \dim\mathfrak{m}/\!\!/H$. By Theorem 1.3, if G/H is affine, then

$$\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = \dim\mathfrak{N}_H(\mathfrak{m}) - (\dim\mathfrak{m} - (2c + r)) \leqslant c.$$

Hence the second inequality.

(ii) It follows from part (i) and Theorem 2.3 that $\dim\mathfrak{N}_H(\mathfrak{m}) < \dim\big(\mathfrak{m} \cap \mathfrak{N}_G(\mathfrak{g}^*)\big)$ unless $c = 0$. □

Remark 2.2. The equality in the first place in Proposition 2.5(i) is equivalent to that π is equidimensional, while the equality in the second place means that $\dim\mathfrak{N}_H(\mathfrak{m}) = c$, i.e., it is maximal possible. Hence, for $c = 0$, both properties hold (as we already know). For $c = 1$ exactly one property takes place, i.e., either π is equidimensional, or $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = 1$.

Let us say that a reductive subgroup $H \subset G$ is *s-regular*, if $\mathfrak{h}$ contains a regular semisimple element of $\mathfrak{g}$.

Proposition 2.6. *Suppose that G/H is affine and S^o is a torus. If $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = c$, then H is s-regular.*

Proof. If S^o is a torus, then $\dim H = \dim U + \dim S - c$ (see Remark 1.1) and $\dim\mathfrak{m} - \dim\mathfrak{m}/\!\!/H = \dim H - \dim S$. Then

$$\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = \dim\mathfrak{N}_H(\mathfrak{m}) - \dim H + \dim S = \dim\mathfrak{N}_H(\mathfrak{m}) - \dim U + c.$$

If $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = c$, then $\dim\mathfrak{N}_H(\mathfrak{m}) = \dim U$. By Eq. (1.2), this means that $\dim\mathfrak{g}^{T_H} = \operatorname{rk}\mathfrak{g}$, i.e., T_H contains a regular semisimple element of G. □

In particular, Proposition 2.6 asserts that if $c = 0$ and S^o is a torus, then H is s-regular. However, it is easily seen that if a symmetric subgroup G_0 does not contain infinite normal subgroups of G (e.g. if G is simple), then it is s-regular regardless of the structure of S. This suggests that the condition on S in Proposition 2.6 is superfluous.

Theorem 2.7. *If G/H is affine and $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = c$, then H is s-regular.*

Proof. By Proposition 2.6, it suffices to handle the case in which $r < \operatorname{rk}\mathfrak{g}$, i.e., $\operatorname{rk} S > 0$.

Let $\mathcal{N}_A(B)$ (resp. $\mathcal{Z}_A(B)$) denote the *normaliser* (resp. *centraliser*) of the group B in A. Since S is contained between a Levi subgroup of G and its commutant, there is a connected reductive group $K \subset G$ such that $\mathcal{N}_G(S)^o = S^o{\cdot}K$ and $S^o \cap K$ is finite. Then $\operatorname{rk} K = \operatorname{rk} G - \operatorname{rk} S = r$. Since $\mathcal{N}_G(S)^o = \mathcal{Z}_G(S)^o{\cdot}S^o$, one also has $\mathcal{Z}_G(S)^o = K{\cdot}Z(S^o)$, where $Z(S^o)$ is the centre of S^o. For Lie algebras, this means that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{k} \oplus \mathfrak{z}(\mathfrak{s})$.

As $S \subset H$, we have $N_H(S)^o = S^o{\cdot}(K \cap H)^o$ and $K \cap H$ is reductive. The linear action $(K \cap H : \mathfrak{m}^S)$ is the coisotropy representation of the affine homogeneous space $K/K \cap H$, and it follows from the construction that it's generic stabiliser is finite.

By the Luna–Richardson theorem [10, Theorem 4.2], the restriction homomorphism $\Bbbk[\mathfrak{m}] \to \Bbbk[\mathfrak{m}^S]$, $f \mapsto f|_{\mathfrak{m}^S}$, induces an isomorphism $\Bbbk[\mathfrak{m}]^H \xrightarrow{\sim} \Bbbk[\mathfrak{m}^S]^{N_H(S)}$. Hence $\Bbbk[\mathfrak{m}^S]^{N_H(S)^0} = \Bbbk[\mathfrak{m}^S]^{K\cap H}$ is a finite $\Bbbk[\mathfrak{m}]^H$-module and $\mathfrak{N}_H(\mathfrak{m}) \cap \mathfrak{m}^S = \mathfrak{N}_{K\cap H}(\mathfrak{m}^S)$. It is also known that

- $c_K(K/K \cap H) = c_G(G/H) = c$, see [14, 1.9] and
- $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) \leqslant \operatorname{def}\mathfrak{N}_{K\cap H}(\mathfrak{m}^S)$ [15, Lemma 3.4].

It then follows from Theorem 1.3 that $\operatorname{def}\mathfrak{N}_{K\cap H}(\mathfrak{m}^S) = c$. Therefore, Proposition 2.6 applies to $K/K \cap H$ and we conclude that $\mathfrak{k} \cap \mathfrak{h}$ contains regular semisimple elements of $\mathfrak{k}$. Let $\mathfrak{t}_S$ be a Cartan subalgebra of $\mathfrak{s}$. (Then $\mathfrak{t}_S \subset \mathfrak{h}$.) Let $\mathfrak{t}_K$ be a Cartan subalgebra of $\mathfrak{k}$ such that $\mathfrak{t}_K \cap \mathfrak{h}$ contains a regular semisimple element of $\mathfrak{k}$. Since K and S^o commute and their intersection is finite, $\mathfrak{t}_S \oplus \mathfrak{t}_K =: \tilde{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $x \in \mathfrak{t}_S$ be a sufficiently general semisimple element of $\mathfrak{s}$. Then $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_S) \supset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{k} \oplus \mathfrak{z}(\mathfrak{s})$. Consequently, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_S) = \mathfrak{t}_S + \tilde{\mathfrak{k}}$ for a reductive Lie algebra $\tilde{\mathfrak{k}} \supset [\mathfrak{k}, \mathfrak{k}]$. However, since $\mathfrak{s}$ contains a commutant of a Levi subalgebra of $\mathfrak{g}$, it is not hard to prove that $\tilde{\mathfrak{k}} = [\mathfrak{k}, \mathfrak{k}]$. In other words,

$$[\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_S), \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_S)] = [\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}), \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})] = [\mathfrak{k}, \mathfrak{k}].$$

Therefore, if $x \in \mathfrak{t}_S$ is sufficiently general and $y \in \mathfrak{t}_K \cap \mathfrak{h}$ is regular in $\mathfrak{k}$, then $x + y \in \mathfrak{h}$ is a regular semisimple element of $\mathfrak{g}$. □

Corollary 2.8. *If G/H is an affine spherical homogeneous space, then H is s-regular.*

Remark 2.3. (1) If G/H is quasiaffine but not affine, then the condition $c = 0$ does not guarantee that H is s-regular. For instance, take $H = U$.

(2) It can happen that $c = 1$, S is finite, and H is s-regular, but $\operatorname{def}\mathfrak{N}_H(\mathfrak{m}) = 0 < 1$. For instance, take $G = SO_{2n+1}$ and $H = SL_n$.

3. On a class of affine homogeneous spaces

For a reductive subgroup $H \subset G$, we set $\tilde{G} = G \times H$ and $\tilde{H} = \Delta_H = H \times H \subset \tilde{G}$. Then $\tilde{\mathcal{O}} = \tilde{G}/\tilde{H}$ is an affine homogeneous space of the reductive group $\tilde{G}$, which is isomorphic to G. The transitive $\tilde{G}$-action on G is given by the formula $(g,h)\circ s = gsh^{-1}$, where $(g,h)\in\tilde{G}$ and $s\in G$. Then the space of the coisotropy representation of $\tilde{H}$ is

$$\tilde{\mathfrak{m}} = \tilde{\mathfrak{h}}^{\perp} = \{(\xi,\eta)\in\mathfrak{g}^*\times\mathfrak{h}^* \mid \xi|_{\mathfrak{h}} = -\eta\}.$$

We shall identify the $\tilde{H}$-module $\tilde{\mathfrak{m}}$ with the H-module $\mathfrak{g}^*$ via the projection to the first factor in $\tilde{\mathfrak{g}}^* = \mathfrak{g}^*\times\mathfrak{h}^*$. If we use the isomorphisms $\mathfrak{g}^*\simeq\mathfrak{g}$ and $\mathfrak{h}^*\simeq\mathfrak{h}$, and the sum $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}=\mathfrak{h}\oplus\mathfrak{h}^{\perp}$, then

$$\tilde{\mathfrak{m}} = (\mathfrak{m}\times\{0\})\oplus\{(x,-x)\mid x\in\mathfrak{h}\}\subset\mathfrak{g}\times\mathfrak{h}. \tag{3.1}$$

The multiplicities in the algebra $\Bbbk[\tilde{G}/\tilde{H}]$ are closely related to the multiplicities in the branching rule $G\downarrow H$. Let $\tilde{\lambda}=(\lambda,\mu)$ be a dominant weight of $\tilde{G}$, where $\lambda\in\mathfrak{X}_+(G)$ and $\mu\in\mathfrak{X}_+(H)$. Write $\tilde{\mathsf{V}}_{\tilde{\lambda}}=\mathsf{V}_\lambda\otimes\mathsf{W}_\mu$ for a simple $\tilde{G}$-module. Let $m_{\tilde{\lambda}}(\tilde{\mathcal{O}})$ denote the multiplicity of $\tilde{\mathsf{V}}_{\tilde{\lambda}}$ in $\Bbbk[\tilde{\mathcal{O}}]$, i.e., the multiplicity of $\tilde{\lambda}$ in the rank monoid $\tilde{\Gamma}=\Gamma_{\tilde{\mathcal{O}}}$. Then $m_{\tilde{\lambda}}(\tilde{\mathcal{O}})$ equals the multiplicity of W_μ in $\mathsf{V}^*_\lambda|_H$, see [2]. It was also shown therein that if $c_{\tilde{G}}(\tilde{O})\leqslant 1$, then one can explicitly describe all the multiplicities and the branching rule.

Set $\tilde{c}=c_{\tilde{G}}(\tilde{O})$, $\tilde{r}=r_{\tilde{G}}(\tilde{O})$. By [2, Section 3] one has $\tilde{c}=c_G(G/B_H)$ and

$$\tilde{r}=\operatorname{rk}\tilde{G}=\operatorname{rk}G+\operatorname{rk}H. \tag{3.2}$$

If H does not contain infinite normal subgroups of G (in particular, H is a proper subgroup of G), then there is a more practical formula

$$\tilde{c}=\dim U-\dim B_H. \tag{3.3}$$

Note that if $H=G$, then $\tilde{\mathcal{O}}=G\times G/\Delta_G$ is a spherical homogeneous space of $G\times G$, i.e., $\tilde{c}=0$. That is, the constraint on H is necessary for (3.3) to be valid. Whenever we consider the homogeneous space $\tilde{\mathcal{O}}=\tilde{G}/\tilde{H}$, it is also assumed that H does not contain infinite normal subgroups of G and thereby (3.3) holds.

We have here the quotient morphism $\tilde{\pi}:\tilde{\mathfrak{m}}\simeq\mathfrak{g}\to\mathfrak{g}/\!\!/H\simeq\tilde{\mathfrak{m}}/\!\!/\tilde{H}$ and the nullcone $\tilde{\pi}^{-1}(\tilde{\pi}(0))=\mathfrak{N}_{\tilde{H}}(\tilde{\mathfrak{m}})\simeq\mathfrak{N}_H(\mathfrak{g})$. It follows from (3.2) and (3.3) that

$$\dim\mathfrak{g}/\!\!/H = 2\tilde{c}+\tilde{r} = \dim\mathfrak{g}-\dim\mathfrak{h}=\dim\mathfrak{m}. \tag{3.4}$$

Proposition 3.1. *All irreducible components of the nullcone $\mathfrak{N}_H(\mathfrak{g})$ have the same dimension, which is equal to* $\dim U_H+\frac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{g}^{T_H})$.

Proof. Since the H-module $\mathfrak{g}$ contains $\mathfrak{h}$ as a summand, all roots of $\mathfrak{h}$ occur as H-weights in $\mathfrak{g}$. Moreover, $\mathfrak{g}$ is a self-dual H-module. Therefore, $\mathfrak{g}$ satisfies the conditions (C.1) and (C.3) considered by Richardson in [23], and his Theorem 7.3 applies here. □

The point of this result is that the upper bound on dimension of the nullcone given in [25], cf. Eq. (1.1), provides now the exact value. Properties of $\tilde{G}/\tilde{H}$ are better than those of arbitrary affine homogeneous spaces G/H, because $\tilde r_{\tilde G}(\tilde G/\tilde H) = \operatorname{rk}\tilde G$ and $\dim\mathfrak{N}_H(\mathfrak g)$ is known. Preceding formulae also show that

$$\operatorname{def}\mathfrak{N}_H(\mathfrak g) = \frac{1}{2}(\dim\mathfrak g - \dim\mathfrak g^{T_H}) - \dim B_H \leqslant \dim U - \dim B_H = \tilde c,$$

which illustrates to the easy part of Theorem 1.3. Since $\mathfrak g^{T_H}$ contains a Cartan subalgebra of $\mathfrak g$ and $\dim\mathfrak{N}_H(\mathfrak g) \geqslant \dim\mathfrak g - \dim\mathfrak g/\!\!/H = \dim H$, Proposition 3.1 implies that

$$\dim B_H \leqslant \tfrac{1}{2}(\dim\mathfrak g - \dim\mathfrak g^{T_H}) \leqslant \dim U. \tag{3.5}$$

Here the equality in the first place is equivalent to that $\dim\mathfrak{N}_H(\mathfrak g) = \dim H$, i.e., $\tilde\pi$ is equidimensional. Whereas the equality in the second place is equivalent to that $\mathfrak g^{T_H}$ is a Cartan subalgebra of $\mathfrak g$, i.e., H is s-regular. It is easily seen that H is s-regular in G if and only if $\tilde H$ is s-regular in $\tilde G$.

Comparing equations (3.3) and (3.5) shows that

- if $\tilde c = 0$, then H is s-regular and $\tilde\pi$ is equidimensional;
- for $\tilde c = 1$, exactly one of these two properties is satisfied.

However, for homogeneous spaces $\tilde G/\tilde H$ and the isotropy representation $(H : \mathfrak g)$, there is a more precise assertion for any $\tilde c > 0$.

Theorem 3.2. *Suppose that $\mathfrak g$ is simple, $\tilde c > 0$, and $\mathfrak h \neq 0$. Then the first inequality in (3.5) is always strict, i.e., $\tilde\pi$ cannot be equidimensional. In particular, if $\tilde c = 1$, then H is s-regular and* $\operatorname{def}\mathfrak{N}_H(\mathfrak g) = 1$.

Proof. Without loss of generality, we may assume that H is connected. Let $x \in \mathfrak h$ be a semisimple element such that $H^x = T_H$. The orbit $H{\cdot}x \subset \mathfrak g$ is closed and, since $\mathfrak g = \mathfrak h \oplus \mathfrak m$, the slice representation at x equals $(T_H : \mathfrak t_H \oplus \mathfrak m)$. Here $\mathfrak t_H$ is a trivial T_H-module. The property of being equidimensional is inheritable, see [31, §8.2]. Therefore, if $(H : \mathfrak g)$ is equidimensional, then so are $(T_H : \mathfrak t_H \oplus \mathfrak m)$ and $(T_H : \mathfrak m)$.

Assume that $\tilde\pi$ is equidimensional, i.e., $\dim B_H = \frac{1}{2}(\dim\mathfrak g - \dim\mathfrak g^{T_H})$. Then we have $\dim U = \frac{1}{2}(\dim\mathfrak g - \dim\mathfrak g^{T_H}) + \tilde c$ and hence

$$\dim\mathfrak g^{T_H} = \operatorname{rk}\mathfrak g + 2\tilde c. \tag{3.6}$$

Take a 1-parameter subgroup $\lambda : \Bbbk^\times \to T_H$ such that $\mathfrak m^{T_H} = \mathfrak m^{\lambda(\Bbbk^\times)}$. Then

$$\mathfrak m = \mathfrak m^+ \oplus \mathfrak m^{T_H} \oplus \mathfrak m^-,$$

where $\mathfrak m^+$ (resp. $\mathfrak m^-$) is the sum of weight spaces $\mathfrak m_\nu$ such that $(\lambda,\nu) > 0$ (resp. $(\lambda,\nu) < 0$). Since $\mathfrak m$ is a self-dual H-module, the T_H-weights in $\mathfrak m^+$ and $\mathfrak m^-$ are opposite to each other. As $\mathfrak g^{T_H} = \mathfrak t_H \oplus \mathfrak m^{T_H}$, it follows from (3.6) that $\dim\mathfrak m^{T_H} = \operatorname{rk}\mathfrak g - \operatorname{rk}\mathfrak h + 2\tilde c$. Then using (3.2) and (3.4), we obtain $\dim\mathfrak m = \operatorname{rk}\mathfrak g + \operatorname{rk}\mathfrak h + 2\tilde c$ and $\dim\mathfrak m^+ = \dim\mathfrak m^- = \operatorname{rk}\mathfrak h$.

The equidimensional representations of tori are described by Wehlau [32]. For the self-dual representations, his description implies that the nonzero weights in $\mathfrak{m}^+$ are linearly independent. Therefore, the nonzero T_H-weights in $\mathfrak{m}^+$ (and in $\mathfrak{m}^-$) are of multiplicity 1. As the same is true for the T_H-weights in $\mathfrak{h}$, we obtain the following conditions:

($\Diamond_1$) the multiplicity of any nonzero T_H-weight in $\mathfrak{g}$ is $\leqslant 2$;
($\Diamond_2$) the number of weights with multiplicity 2 is at most $2\mathrm{rk}\,\mathfrak{h} = \dim(\mathfrak{m}^+ \oplus \mathfrak{m}^-)$.

Let us prove that ($\Diamond_1$) and ($\Diamond_2$) cannot be satisfied if $\mathfrak{h} \ne \{0\}$. By (3.6), $\mathfrak{l} = \mathfrak{g}^{T_H}$ is not abelian. Without loss of generality, we may assume that $\mathfrak{l}$ is a standard Levi subalgebra w.r.t. $T \subset B$, i.e., $\mathfrak{l}$ is determined by the set of simple roots α such that $\alpha|_{\mathfrak{t}_H} = 0$.

(a) Assume that $[\mathfrak{l}, \mathfrak{l}]$ has a simple factor of rank $\geqslant 2$. Then there is a chain of simple roots $\alpha_1, \alpha_2, \beta$ in the Dynkin diagram of $\mathfrak{g}$ such that $\alpha_i|_{\mathfrak{t}_H} = 0$ $(i = 1, 2)$ and $\beta|_{\mathfrak{t}_H} \ne 0$. Then $\beta, \beta + \alpha_2, \beta + \alpha_2 + \alpha_1$ have the same (nonzero) restriction to $\mathfrak{t}_H$, which contradicts ($\Diamond_1$).

(b) Assume that $[\mathfrak{l}, \mathfrak{l}] \simeq k\mathbf{A}_1$ and $k \geqslant 2$. Take simple roots α_1, α_2 in $[\mathfrak{l}, \mathfrak{l}]$ such that the simple roots between them, say $\beta_1, \dots, \beta_r$, do not belong to $[\mathfrak{l}, \mathfrak{l}]$. If $\beta = \sum_{i=1}^r \beta_i$, then the roots $\beta, \beta + \alpha_1, \beta + \alpha_2$ yield a T_H-weight of multiplicity $\geqslant 3$, which again contradicts ($\Diamond_1$).

(c) Assume that $[\mathfrak{l}, \mathfrak{l}] \simeq \mathbf{A}_1$. Then $\tilde{c} = 1$, generic elements of $\mathfrak{t}_H$ are subregular in $\mathfrak{g}$, and there is a unique root $\alpha \in \Pi$ such that $\alpha|_{\mathfrak{t}_H} = 0$. Each pair of roots of the form $\{\mu, \mu + \alpha\}$ gives rise to a T_H-weight in $\mathfrak{g}$ of multiplicity 2.

• If there are roots of different length and α is short, then one can find a triple of roots $\mu, \mu + \alpha, \mu + 2\alpha$, which again provides a T_H-weight of multiplicity $\geqslant 3$.

• For α long, the number of pairs of **positive** roots $\{\mu, \mu + \alpha\}$ equals $\boldsymbol{h}^* - 2$, where $\boldsymbol{h}^*$ is the *dual Coxeter number* of $\mathfrak{g}$, see [17, Section 1]. Then the total number of such pairs equals $2(\boldsymbol{h}^* - 2)$ and ($\Diamond_2$) means that $2\mathrm{rk}\,\mathfrak{h} \geqslant 2(\boldsymbol{h}^* - 2)$. Since $\boldsymbol{h}^* - 1 \geqslant \mathrm{rk}\,\mathfrak{g}$, one must have

$$\boldsymbol{h}^* - 2 \leqslant \mathrm{rk}\,\mathfrak{h} < \mathrm{rk}\,\mathfrak{g} \leqslant \boldsymbol{h}^* - 1.$$

Hence $\mathrm{rk}\,\mathfrak{g} = \boldsymbol{h}^* - 1$ and $\mathrm{rk}\,\mathfrak{h} = \mathrm{rk}\,\mathfrak{g} - 1$. The equality $\mathrm{rk}\,\mathfrak{g} = \boldsymbol{h}^* - 1$ holds only for $\mathbf{A}_n$ and $\mathbf{C}_n$, and we look more carefully at these two series. Since $\tilde{c} = 1$ and $\mathrm{rk}\,\mathfrak{h} = \mathrm{rk}\,\mathfrak{g} - 1$, we obtain using (3.4) that

$$\dim \mathfrak{h} = \dim \mathfrak{g} - \mathrm{rk}\,\mathfrak{g} - \mathrm{rk}\,\mathfrak{h} - 2\tilde{c} = \dim \mathfrak{g} - 2\mathrm{rk}\,\mathfrak{g} - 1.$$

$\mathfrak{g} = \mathfrak{sl}_{n+1}$: Here $\dim \mathfrak{h} = n^2 - 1$ and hence $\mathfrak{h} = \mathfrak{sl}_n$ is the only possibility. But the subgroup $SL_n \subset SL_{n+1}$ is s-regular, if $n \geqslant 2$, i.e., if $\mathfrak{h} \ne 0$.
$\mathfrak{g} = \mathfrak{sp}_{2n}$: Here $\dim \mathfrak{h} = 2n^2 - n - 1 > \dim \mathfrak{sp}_{2n-2}$, and this case is also impossible.

Thus, the assumption that $\tilde{\pi}$ is equidimensional leads to a contradiction. □

Remark 3.1. This result is specific for homogeneous spaces of the form $(G \times H)/\Delta_H$. For arbitrary affine homogeneous spaces G/H, it can happen that $c = 1$, but $\pi : \mathfrak{m} \to \mathfrak{m}/\!\!/H$

is equidimensional and H is not s-regular. For instance, take $(G,H) = (Sp_{2n}, Sp_{2n-2})$ or (SO_{2n+1}, SO_{2n-1}) with $n \geqslant 2$.

3.1. **More on the nullcone for $\tilde{\pi}$.** For $\tilde{G} = G \times H$ and $\tilde{G}/\tilde{H} \simeq G$, we have

$$\mathfrak{N}_{\tilde{H}}(\tilde{\mathfrak{m}}) \subset \mathfrak{N}_{\tilde{G}}(\tilde{\mathfrak{g}}) \cap \tilde{\mathfrak{m}} \ \text{ and } \ \mathfrak{N}_{\tilde{G}}(\tilde{\mathfrak{g}}) = \mathfrak{N}_G(\mathfrak{g}) \times \mathfrak{N}_H(\mathfrak{h}) \subset \mathfrak{g} \times \mathfrak{h}.$$

Using (3.1), one readily verifies that under the isomorphism $\tilde{\mathfrak{m}} \simeq \mathfrak{g}$ the variety $\mathfrak{N}_{\tilde{G}}(\tilde{\mathfrak{g}}) \cap \tilde{\mathfrak{m}}$ is identified with $\mathfrak{N}_G(\mathfrak{g}) \cap \big(\mathfrak{N}_H(\mathfrak{h}) \times \mathfrak{m}\big) \subset \mathfrak{g}$. Since $\tilde{r} = \operatorname{rk} \tilde{\mathfrak{g}}$, translating Theorem 2.3, Theorem 2.4, and Proposition 2.5 into this setting, we obtain

Theorem 3.3. *If* $\tilde{c} = 0$, *then* $\mathfrak{N}_H(\mathfrak{g}) = \mathfrak{N}_G(\mathfrak{g}) \cap (\mathfrak{N}_H(\mathfrak{h}) \times \mathfrak{m})$. *For arbitrary* $\tilde{c} \geqslant 0$, *we have*

- $\dim \mathfrak{N}_H(\mathfrak{g}) \leqslant \dim \mathfrak{h} + \tilde{c} = \dim \tilde{U} = \dim U + \dim U_H$;
- $\dim(\mathfrak{N}_G(\mathfrak{g}) \cap (\mathfrak{N}_H(\mathfrak{h}) \times \mathfrak{m})) = \dim \mathfrak{h} + 2\tilde{c} = \dim \mathfrak{g} - \operatorname{rk} \mathfrak{g} - \operatorname{rk} \mathfrak{h}$;
- *the intersection* $\mathfrak{N}_G(\mathfrak{g}) \cap (\mathfrak{N}_H(\mathfrak{h}) \times \mathfrak{m})$ *is proper.*

Moreover, if $\dim \mathfrak{N}_H(\mathfrak{g}) = \dim \mathfrak{h} + \tilde{c}$, *i.e.,* $\operatorname{def} \mathfrak{N}_H(\mathfrak{g}) = \tilde{c}$, *then H is s-regular in G.*

3.2. **Homogeneous spaces $\tilde{G}/\tilde{H}$ of complexity $\leqslant 1$.**

The pairs (G,H) such that $c_{\tilde{G}}(\tilde{G}/\tilde{H}) = 0$ can be characterised by a number of equivalent properties. An extensive list of such properties is given and discussed in [18, Section 3.2].

In particular, (G,H) is a *strong Gelfand pair*, which means that any simple G-module V_λ is a multiplicity free H-module. A classification of strong Gelfand pairs (in the category of compact Lie groups) is obtained by Manfred Krämer in [7].

If G is simple, then the (very short) list of strong Gelfand pairs consists of two series:

$$(\mathfrak{sl}_n, \mathfrak{gl}_{n-1}),\ n \geqslant 2, \text{ and } (\mathfrak{so}_n, \mathfrak{so}_{n-1}),\ n \geqslant 5.$$

See also comments in [2, Section 4] and other details in [18, Section 3.2].

Remark 3.2. For a symmetric variety G/G_0 with simple G, it is proved in [3] that

$$\tilde{\pi} : \mathfrak{g} \to \mathfrak{g}/\!\!/G_0 \text{ is equidimensional} \iff (\mathfrak{g}, \mathfrak{g}_0) \text{ is either } (\mathfrak{sl}_n, \mathfrak{gl}_{n-1}) \text{ or } (\mathfrak{so}_n, \mathfrak{so}_{n-1}).$$

Since these two series gives rise to the only spherical homogeneous spaces $\tilde{G}/\tilde{H}$, our Theorem 3.2 generalises that result of [3].

The list of pairs (G,H) such that G is simple, H is connected, and $\tilde{c} = 1$ is obtained in [2, Section 4]. For the reader convenience, we recall it in Table 1.

TABLE 1. The pairs (G,H) with simple G and $\tilde{c} = 1$

№	1	2	3	4	5	6	7	8
G	SL_{n+1}	Sp_6	$Spin_7$	$\mathbf{G}_2$	SL_3	SL_3	Sp_4	SL_4
H	SL_n	$Sp_4 \times SL_2$	$\mathbf{G}_2$	SL_3	SO_3	T	$SL_2 \cdot \Bbbk^\times$	$(SL_2)^2 \cdot \Bbbk^\times$

For № 6, $T = \mathsf{T}_2$ is a maximal torus of SL_3, and № 7 represents actually two different pairs. Here H is a Levi subgroup of Sp_4, and there are two non-conjugate Levi subgroups corresponding to either the **long** or the **short** simple root of Sp_4. In Section 4, these two cases will be referred to as 7(l) and 7(s), respectively.

4. THE DEFECT OF THE NULL-CONE AND INVARIANTS

Let $G \to GL(\mathsf{V})$ be a linear representation of a connected reductive group G. In [21], V. Popov conjectured that if G is semisimple and $\pi : \mathsf{V} \to \mathsf{V}/\!\!/G$ is equidimensional, then $\mathsf{V}/\!\!/G$ is an affine space, i.e, $\Bbbk[\mathsf{V}]^G$ is a polynomial algebra. Afterwards, this conjecture was extended to arbitrary connected reductive groups. Using our terminology, the conjecture can be stated as follows:

If G is a connected reductive group and $\operatorname{def}\mathfrak{N}_G(\mathsf{V}) = 0$, then $\mathsf{V}/\!\!/G$ is an affine space.

There had been a good few classification work related to this conjecture. To the best of my knowledge, it is verified in the following cases:

- G is simple, V is irreducible (V.L. Popov, 1976 [21]);
- G is simple, V is reducible (O.M. Adamovich, 1980);
- G is semisimple, V is irreducible (P. Littelmann, 1989);
- G is a torus (E.B. Vinberg (oral lecture at MSU) 1983; D. Wehlau, 1992 [32]);
- G is a product of two simple factors, with some exceptions (D. Wehlau, 1993 [33]).

An interesting approach to an *a priori* proof of the Popov conjecture is presented in [29]. More information on this conjecture and other references can be found in [31, § 8.7].

Some time ago, I stated a similar conjecture on non-equidimensional representations. Let $\operatorname{ed}\mathsf{V}/\!\!/G$ denote the *embedding dimension* of $\mathsf{V}/\!\!/G$, i.e., the minimal number of generators of $\Bbbk[\mathsf{V}]^G$. Then $\operatorname{hd}\mathsf{V}/\!\!/G := \operatorname{ed}\mathsf{V}/\!\!/G - \dim\mathsf{V}/\!\!/G$ is the *homological dimension* of $\mathsf{V}/\!\!/G$, see [22].

Conjecture 4.1 ([15, Conj. 3.5])**.** *Suppose that G is connected, V is a self-dual G-module, and $\operatorname{def}\mathfrak{N}_G(\mathsf{V}) = 1$. Then $\mathsf{V}/\!\!/G$ is either an affine space or a hypersurface, i.e., $\operatorname{hd}\mathsf{V}/\!\!/G \leqslant 1$.*

The assumption on self-duality is essential here, see Example in [15, p. 94]. This conjecture is proved for tori [15, Prop. 3.10] and $G = SL_2$ [15, Example 3.12(1)].

The isotropy representation of a reductive subgroup $H \subset G$ is orthogonal and the complexity of G/H provides an upper bound on $\operatorname{def}\mathfrak{N}_H(\mathfrak{m})$, see Theorem 1.3. Therefore, Conjecture 4.1 can be specialised to the following

Conjecture 4.2. *If H is connected reductive and $c_G(G/H) = 1$, then $\operatorname{hd}\mathfrak{m}/\!\!/H \leqslant 1$.*

As a support to Conjecture 4.2, we prove below two theorems. Before stating these theorems, we describe our general approach. If H is simple and $H \subset GL(\mathsf{V})$, then we use various classification results on the structure of $\mathsf{V}/\!\!/H$:

- when $\mathsf{V}/\!\!/H \simeq \mathbb{A}^N$ [1, 24];
- when $\mathsf{V}/\!\!/H$ is a complete intersection, especially a hypersurface [26, 27];
- when $H = SL_2$ and $\mathrm{hd}\,(\mathsf{V}/\!\!/SL_2) \leqslant 3$ [22, Theorem 4].

If H is not simple, then we work with consecutive quotients, using factors of H. Suppose that $H = H_1 \times H_2$ is a product of reductive groups. Then one has the quotient morphisms

$$\mathfrak{m} \xrightarrow{\pi_1} \mathfrak{m}/\!\!/H_1 \xrightarrow{\pi_2} (\mathfrak{m}/\!\!/H_1)/\!\!/H_2 = \mathfrak{m}/\!\!/H. \tag{4.1}$$

In all cases below, we can choose H_1 such that $\mathrm{hd}\,(\mathfrak{m}/\!\!/H_1) \leqslant 2$, hence $\mathfrak{m}/\!\!/H_1$ is a complete intersection. (In most cases, we actually obtain $\mathrm{hd}\,(\mathfrak{m}/\!\!/H_1) \leqslant 1$.) This yields an embedding $\mathfrak{m}/\!\!/H_1 \hookrightarrow \mathsf{V}_2$ into an H_2-module V_2 such that $\mathrm{codim}_{\mathsf{V}_2}(\mathfrak{m}/\!\!/H_1) \leqslant 2$. Then using the equations of $\mathfrak{m}/\!\!/H_1$ in V_2, we describe $\mathfrak{m}/\!\!/H$ as a subvariety of $\mathsf{V}_2/\!\!/H_2$. This allows us to handle the second step in (4.1) and prove that $\mathrm{hd}\,(\mathfrak{m}/\!\!/H) \leqslant 1$.

To describe $\mathfrak{m}$ as H-module, we need some notation. The fundamental weights of H are denoted by $\{\varphi_i\}$ and ε stands for the basic character of one-dimensional torus $\Bbbk^\times = \mathsf{T}_1$. The fundamental weights for the second (resp. third) simple factor of H are marked with prime (resp. double prime). The unique fundamental weight of SL_2 is denoted by φ. Write $1\!\!1$ for the trivial one-dimensional representation.

As in [24, 25, 27], the simple H-module W_λ is identified with its highest weight λ, using the multiplicative notation for λ in terms of the fundamental weights. For instance, we write $\varphi_j\varphi_k + 3\varphi_i^2$ in place of $\mathsf{W}_{\varphi_j+\varphi_k} + 3\mathsf{W}_{2\varphi_i}$. Finally, λ^* is a dual H-module to λ.

Theorem 4.3. *If G is simple and $c_G(G/H) = 1$, then*

- *either $\mathfrak{m}/\!\!/H$ is an affine space and* $\mathrm{def}\,\mathfrak{N}_H(\mathfrak{m}) = 0$*;*
- *or $\mathfrak{m}/\!\!/H$ is a hypersurface and* $\mathrm{def}\,\mathfrak{N}_H(\mathfrak{m}) = 1$*.*

(Hence an a priori *conceivable case, where $\mathfrak{m}/\!\!/H \simeq \mathbb{A}^n$ and* $\mathrm{def}\,\mathfrak{N}_H(\mathfrak{m}) = 1$*, does not occur.)*

Proof. The list of such pairs (G, H) consists of 17 items, and we refer to their numbering in [13, Table 1] (see also Table 1 in [16]). The output is that $\Bbbk]\mathfrak{m}]^H$ is a polynomial algebra for № 1, 4–9, 13, 16,17. For the other cases, $\Bbbk]\mathfrak{m}]^H$ is a hypersurface.

Let us provide some details to our computations. If H is simple, then the pairs with a polynomial algebra $\Bbbk]\mathfrak{m}]^H$ can be picked from the list of "coregular representations" of H obtained by Schwarz [24] and Adamovich–Golovina [1]. This applies to № 4–8, 13, 16–17. Moreover, for all these cases, one also has $\mathrm{def}\,\mathfrak{N}_H(\mathfrak{m}) = 0$, see [25].

For № 1, we have $(G, H) = (SL_{2n}, SL_n \times SL_n)$ and $\mathfrak{m} = \varphi_1\varphi_1' + (\varphi_1\varphi_1')^* + 1\!\!1$. Here one can use the fact that $\hat{H} = (SL_n)^2{\cdot}\mathsf{T}_1$ is a symmetric subgroup of G, with isotropy representation $\hat{\mathfrak{m}} = \varphi_1\varphi_1'\varepsilon + (\varphi_1\varphi_1'\varepsilon)^*$, and hence $\Bbbk[\hat{\mathfrak{m}}]^{\hat{H}}$ is a polynomial algebra.

For № 9, we have $(G, H) = (\mathbf{C}_n, \mathbf{C}_{n-2} \times \mathbf{A}_1 \times \mathbf{A}_1)$, $n \geqslant 3$, and $\mathfrak{m} = \varphi_1\varphi' + \varphi_1\varphi'' + \varphi'\varphi''$. Here $\mathfrak{m}|_{\mathbf{C}_{n-2}} = 4\varphi_1 + 41\!\!1$. If $n \geqslant 4$, then $(4\varphi_1)/\!\!/\mathbf{C}_{n-2} \simeq \mathbb{A}^6$, and it is isomorphic to

$\varphi'\varphi'' + 2\mathbb{1}$ as $\mathbf{A}_1 \times \mathbf{A}_1$-module. Hence $\mathfrak{m}/\!\!/\mathbf{C}_{n-2} \simeq \varphi'\varphi'' + 2\mathbb{1} + \varphi'\varphi''$. It is easily seen that $(2\varphi'\varphi'')/\!\!/\mathbf{A}_1 \times \mathbf{A}_1 \simeq \mathbb{A}^3$ (it is also № 1 with $n = 2$). Therefore, $\mathfrak{m}/\!\!/H \simeq \mathbb{A}^5$. By [25], $(\mathbf{C}_{n-2}, 4\varphi_1)$ is equidimensional if and only if $n \geqslant 5$. Then both quotient morphisms

$$\mathfrak{m} \to \mathfrak{m}/\!\!/\mathbf{C}_{n-2} \to (\mathfrak{m}/\!\!/\mathbf{C}_{n-2})/\!\!/(\mathbf{A}_1 \times \mathbf{A}_1) = \mathfrak{m}/\!\!/H$$

are equidimensional. This already shows that $\mathfrak{m}/\!\!/H$ is an affine space for $n \geqslant 4$ and, moreover, $(H : \mathfrak{m})$ is equidimensional, if $n \geqslant 5$. Some other *ad hoc* methods allow us to handle the case with $n = 3$ and prove the equidimensionality for $n = 4$.

The other cases, where H is semisimple, are $Sp_4 \supset SL_2$ (№ 14) and $\mathbf{B}_5 \supset \mathbf{B}_3 \times \mathbf{A}_1$ (№ 12).

– In № 14, the SL_2-module $\mathfrak{m}$ equals φ^6 (binary forms of degree 6), and it is a classical fact from XIX century that $\varphi^6/\!\!/SL_2$ is a hypersurface, cf. also [22, Theorem 4].

– In № 12, $\mathfrak{m} = \varphi_3\varphi'^2 + \varphi_1$ as $\mathbf{B}_3 \times \mathbf{A}_1$-module. Then $\mathfrak{m} = 3\varphi_3 + \varphi_1$ as $\mathbf{B}_3$-module and $\mathfrak{m}/\!\!/\mathbf{B}_3 \simeq \mathbb{A}^{10}$. Using explicit multi-degrees of basic $\mathbf{B}_3$-invariants, see № 6 in [1, Table 3], one sees that $\mathfrak{m}/\!\!/\mathbf{B}_3 \simeq \varphi'^2 + \varphi'^4 + 2\mathbb{1}$ as $\mathbf{A}_1$-module, and therefore $(\mathfrak{m}/\!\!/\mathbf{B}_3)/\!\!/\mathbf{A}_1 = \mathfrak{m}/\!\!/H$ is a hypersurface.

Consider an item, where H is not semisimple. For № 11, we have $(G, H) = (\mathbf{B}_4, \mathbf{G}_2{\cdot}\mathsf{T}_1)$ and $\mathfrak{m} = \varphi_1\varepsilon + \varphi_1 + \varphi_1\varepsilon^{-1}$. Then $\mathfrak{m} = 3\varphi_1$ as $\mathbf{G}_2$-module and $(3\varphi_1)/\!\!/\mathbf{G}_2 \simeq \mathbb{A}^7$. Using explicit multi-degrees of basic $\mathbf{G}_2$-invariants, cf. № 1 in [1, Table 4], we obtain that the T_1-weights on $\mathbb{A}^7$ are $\varepsilon^2, \varepsilon, \varepsilon^{-1}, \varepsilon^{-2}, 1, 1, 1$. Hence $\mathbb{A}^7/\!\!/\mathsf{T}_1 = \mathfrak{m}/\!\!/H$ is a hypersurface. □

Remark 4.1. I would like to fix some misprints and omissions in [13, Table 1].

- In № 1, the group H has to be $SL_n \times SL_n$;
- the summand $\mathbb{1}$ has to be added to $\mathfrak{m}$ in № 3, 6. One also has $r = 4$ in № 3.
- for № 12, the right formula for $\mathfrak{m}$ is given above.

A similar approach works for affine homogeneous spaces $\tilde G/\tilde H$ with $\tilde c = 1$.

Theorem 4.4. *If G is simple and $\tilde c = c_{\tilde G}(\tilde G/\tilde H) = 1$, then $\mathfrak{g}/\!\!/H$ is a hypersurface.*

Proof. We check the assertion for all items in Table 1. The H-modules $\mathfrak{g}$ are given below. The underlined summands give rise to the adjoint representation of H.

1. $\mathfrak{sl}_{n+1} = \underline{\varphi_1\varphi_{n-1}} + \varphi_1 + \varphi_{n-1} + \mathbb{1}$ as SL_n-module.
2. $\mathfrak{sp}_6 = \underline{\varphi_1^2} + \varphi_1\varphi' + \underline{\varphi'^2}$ as $Sp_4 \times SL_2$-module.
3. $\mathfrak{so}_7 = \varphi_1 + \underline{\varphi_2}$ as $\mathbf{G}_2$-module.
4. $\mathbf{G}_2 = \underline{\varphi_1\varphi_2} + \varphi_1 + \varphi_2$ as SL_3-module.
5. $\mathfrak{sl}_3 = \underline{\varphi^2} + \varphi^4$ as SL_2-module (we use the isomorphism $\mathfrak{sl}_2 \simeq \mathfrak{so}_3$).
6. $\mathfrak{sl}_3 = (\varepsilon + \mu + \varepsilon\mu) + (\varepsilon + \mu + \varepsilon\mu)^* + \underline{2\mathbb{1}}$ as T_2-module.

7(s). $\mathfrak{sp}_4 = \varphi^2\varepsilon^2 + \underline{\varphi^2} + \varphi^2\varepsilon^{-2} + \underline{\mathbb{1}}$ as $SL_2{\cdot}\mathsf{T}_1$-module.

7(l). $\mathfrak{sp}_4 = \underline{\varphi^2} + \varphi\varepsilon + \varphi\varepsilon^{-1} + \varepsilon^2 + \varepsilon^{-2} + \underline{\mathbb{1}}$ as $SL_2{\cdot}\mathsf{T}_1$-module.

8. $\mathfrak{sl}_4 = \varphi\varphi'\varepsilon + \varphi\varphi'\varepsilon^{-1} + \underline{\varphi^2} + \underline{\varphi'^2} + \underline{1\!\!1}$ as $(SL_2 \times SL_2){\cdot}\mathsf{T}_1$-module.

▶ Items 1, 3–5 are representations admitting a *finite coregular extension* in the sense of Shmel'kin [26], and he proves that here $\mathfrak g/\!\!/H$ is an (explicitly described) hypersurface.

▶ Items 7(l, s) can be handled in a similar way, and we provide details for one of them.

– In the s-case, we have $\mathfrak{sp}_4 = 3\varphi^2 + 1\!\!1$ as SL_2-module, and $(3\varphi^2)/\!\!/SL_2$ is a hypersurface, see [22, Theorem 4]. We skip below the trivial H-module $1\!\!1$. It is not hard to write explicitly down the basic invariants for $(SL_2 : 3\varphi^2)$. Let F denote the basic invariant of degree 2 for the adjoint representation $(SL_2 : \varphi^2)$, i.e., $F(v) = (v, v)$ for $v \in \varphi^2$. If $(v_1, v_2, v_3) \in 3\varphi^2$, then the basic SL_2-invariants are: F_{ij}, $1 \leqslant i \leqslant j \leqslant 3$, and $\tilde F$, where $F_{ij}(v_1, v_2, v_3) = (v_i, v_j)$ and $\tilde F = \det[v_1, v_2, v_3]$. The basic relation is

$$\det\big((F_{ij})_{i,j=1}^3\big) = \tilde F^2, \tag{4.2}$$

where $\boldsymbol F = (F_{ij})_{i,j=1}^3$ is the symmetric 3 by 3 matrix. If $t{\cdot}(v_1, v_2, v_3) = (t^{-2}v_1, v_2, t^2v_3)$ for $t \in \mathsf{T}_1$, then $F_{13}, F_{22}, \tilde F$ are already T_1-invariants, but $t{\cdot}F_{11} = t^4F_{11}$, $t{\cdot}F_{12} = t^2F_{12}$, $t{\cdot}F_{23} = t^{-2}F_{23}$, and $t{\cdot}F_{33} = t^{-4}F_{33}$. Therefore, the other $SL_2{\cdot}\mathsf{T}_1$-invariants are:

$$x_1 = F_{11}F_{33},\ y_1 = F_{12}F_{23},\ z_1 = F_{11}F_{23}^2,\ z_2 = F_{33}F_{12}^2.$$

Thus, we get seven generators and yet another relation

$$x_1y_1^2 = z_1z_2. \tag{4.3}$$

Expressing the $\det\big(\boldsymbol F\big)$ via these $SL_2{\cdot}\mathsf{T}_1$-invariants, we rewrite (4.2) as

$$x_1F_{22} + 2y_1F_{13} + F_{13}^2F_{22} + z_1 + z_2 = \tilde F^2.$$

Therefore, either z_1 or z_2 can be excluded from the minimal generating system of $\Bbbk[\mathfrak m]^H$. Afterwards, (4.3) yields the relation for the remaining six invariants.

– In the l-case, $\mathfrak{sp}_4 = \varphi^2 + 2\varphi + 3 1\!\!1$ as SL_2-module, and $(\varphi^2 + 2\varphi)/\!\!/SL_2$ is a hypersurface, too. The rest is similar to the s-case.

▶ № 6 is easy and left to the reader.

▶ № 8 is a slice representation for № 2. Therefore, using the monotonicity results for homological dimension of algebras of invariants [22, Theorem 2], it suffices to handle № 2.

▶ № 2 is the most difficult case, and we only give some hints. Here $\mathfrak{sp}_6 = \varphi_1^2 + 2\varphi_1 + 3 1\!\!1$ as Sp_4-module. The representation $(Sp_4 : \varphi_1^2{+}2\varphi_1)$ is a slice for $(SL_4 : \tilde\varphi_1^2{+}\tilde\varphi_2{+}2\tilde\varphi_1^* = \tilde V)$ (use $v \in \tilde\varphi_2$ such that $(SL_4)^v = Sp_4$) and $\mathrm{hd}\,(\tilde V/\!\!/SL_4) = 2$ [27, Table 9]. Therefore, $(\varphi_1^2{+}2\varphi_1)/\!\!/Sp_4$ is a complete intersection and $\mathrm{hd}\,(\varphi_1^2 + 2\varphi_1)/\!\!/Sp_4 \leqslant 2$ [22]. The subsequent argument is similar in spirit with that in case 7(s), but much more elaborated. We also need the fact that, for the truncated $Sp_4 \times SL_2$-module $\varphi_1^2 + \varphi_1\varphi' = \mathsf V \subset \mathfrak{sp}_6$, the quotient $\mathsf V/\!\!/(Sp_4 \times SL_2)$ is an affine space of dimension 5 [33]. □

5. Coisotropy representations and related Poisson structures

In this section, G/H is affine, and we think of $\mathfrak{m}$ as a subspace of $\mathfrak{g}^*$. The cotangent bundle $T^*_{G/H} = G\times_H \mathfrak{m}$ is a symplectic G-variety, hence the algebra $\Bbbk[T^*_{G/H}]$ is equipped with the associated Poisson bracket $\{\ ,\ \}$. This bracket restricts to the algebra of G-invariants $\Bbbk[T^*_{G/H}]^G \simeq \Bbbk[\mathfrak{m}]^H$, which makes $\mathfrak{m}/\!\!/H$ a Poisson variety. Recall that $\dim \mathfrak{m}/\!\!/H = 2c+r$.

The Poisson bracket $(\Bbbk[\mathfrak{m}]^H, \{\ ,\ \})$ has the following explicit description, see e.g. [30, Ch. II, § 1.8]. The algebra of regular functions $\Bbbk[\mathfrak{m}]$ is also the symmetric algebra of $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}^*$, hence $\Bbbk[\mathfrak{m}]^H = \mathcal{S}(\mathfrak{g}/\mathfrak{h})^H$. Let $f_1, f_2 \in \mathcal{S}(\mathfrak{g}/\mathfrak{h})^H$ and $\alpha \in (\mathfrak{g}/\mathfrak{h})^* = \mathfrak{m}\subset \mathfrak{g}^*$. Then

$$\{f_1,f_2\}(\alpha) = \langle \alpha, [d_\alpha f_1, d_\alpha f_2]\rangle. \tag{5.1}$$

The commutator on the right-hand side of this formula is understood as the commutator in $\mathfrak{g}$ of any representatives of the cosets $d_\alpha f_1, d_\alpha f_2 \in \mathfrak{g}/\mathfrak{h}$, because the result does not depend on the choice of these representatives in $\mathfrak{g}$ (if f_1 and f_2 are H-invariants!). Note that if $\mathfrak{h}=\mathfrak{g}^\sigma$ for an involution σ, then the right-hand side in (5.1) is identically zero. That is, for the symmetric variety G/H, the Poisson bracket vanishes on $\Bbbk[T^*_{G/H}]^G$. Let $\mathrm{rk}\,\{\ ,\ \}$ denote the *rank of the Poisson bracket*, i.e., the maximal dimension of symplectic leaves in $\mathfrak{m}/\!\!/H$. It follows from [30, Ch. II, § 3, Theorem 2] that in our case $\mathrm{rk}\,\{\ ,\ \} = 2c$.

Let $(\mathcal{P}, \{\ ,\ \}_{\mathcal{P}})$ be an affine Poisson variety. A subalgebra $\mathcal{A}$ of $\Bbbk[\mathcal{P}]$ is said to be *Poisson-commutative*, if $\{\mathcal{A},\mathcal{A}\}=0$. As is well-known, if $\mathcal{A}$ is Poisson-commutative, then

$$\mathrm{trdeg}\,\mathcal{A} \leqslant \dim\mathcal{P} - \frac{1}{2}\mathrm{rk}\{\ ,\ \}_{\mathcal{P}}.$$

Therefore, we arrive at the following conclusion.

Lemma 5.1. *If $\mathcal{A}$ is a Poisson-commutative subalgebra of $\Bbbk[\mathfrak{m}]^H$, then $\mathrm{trdeg}\,\mathcal{A}\leqslant c+r$.*

Conjecture 5.2. *For any affine homogeneous space G/H, there is a Poisson-commutative subalgebra $\mathcal{A}\subset\Bbbk[\mathfrak{m}]^H$ such that $\mathrm{trdeg}\,\mathcal{A} = c+r$.*

Let $\mathcal{Z}$ denote the Poisson centre of $(\Bbbk[\mathfrak{m}]^H, \{\ ,\ \})$. By [5, Section 7], $\mathcal{Z}$ is a polynomial ring and $\mathrm{trdeg}\,\mathcal{Z} = r$. Some stronger results can also be found in [6, Section 9].

Example 5.1. For $c=0$, one has $\mathcal{Z}=\Bbbk[\mathfrak{m}]^H$, and there is nothing to prove. For $c=1$, $\mathrm{trdeg}\,\Bbbk[\mathfrak{m}]^H = \mathrm{trdeg}\,\mathcal{Z}+2$ and one can take any $f\in\Bbbk[\mathfrak{m}]^H$ that is not algebraic over $\mathcal{Z}$. Then the subalgebra generated by $\mathcal{Z}$ and f is Poisson-commutative and its transcendence degree equals $r+1$, as required. Thus, Conjecture 5.2 is true, if $c\leqslant 1$.

By [5, Theorem 7.6], one has $\mathcal{Z}=\Bbbk[M_{\mathcal{O}}]^G$ and the morphism $T^*_{G/H}\to \mathsf{Spec}\,\mathcal{Z}$ is given by the map ψ in (2.2). Therefore, using commutative diagrams (2.2) and (2.3), we get the chain morphisms

$$\mathfrak{m} \xrightarrow{\pi_{\mathfrak{m}}} \mathfrak{m}/\!\!/H \longrightarrow \mathsf{Spec}\,\mathcal{Z} = M_{\mathcal{O}}/\!\!/G \xrightarrow{\tau} \overline{G{\cdot}\mathfrak{m}}/\!\!/G,$$

where τ is finite, the morphisms $\boldsymbol{f} : \mathfrak{m} \to \operatorname{Spec} \mathcal{Z}$ and $\tilde{\boldsymbol{f}} : \mathfrak{m} \to \overline{G{\cdot}\mathfrak{m}}/\!\!/G = \tilde{M}_{\mathcal{O}}/\!\!/G$ are equidimensional, and $\tilde{\boldsymbol{f}}^{-1}(\bar{0}) = \boldsymbol{f}^{-1}(\bar{0}) = \mathfrak{N}_G(\mathfrak{g}^*) \cap \mathfrak{m}$.

5.1. **The case of $\tilde{G}/\tilde{H}$.** For the homogeneous spaces of the form $\tilde{G}/\tilde{H} = (G \times H)/\Delta_H$, one can say more. Recall that $\tilde{r} = \operatorname{rk} \mathfrak{g} + \operatorname{rk} \mathfrak{h}$, $\tilde{c} = \dim U - \dim B_H$, and $\tilde{\mathfrak{m}} \simeq \mathfrak{g}^*$. Here Lemma 5.1 says that if $\mathcal{A} \subset \Bbbk[\mathfrak{g}^*]^H = \mathcal{S}(\mathfrak{g})^H$ is Poisson-commutative, then

$$\operatorname{trdeg} \mathcal{A} \leqslant \tilde{c} + \tilde{r} = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{h} + \operatorname{rk} \mathfrak{g} + \operatorname{rk} \mathfrak{h}). \tag{5.2}$$

In this setting, the existence of $\mathcal{A}$ such that $\operatorname{trdeg} \mathcal{A} = \tilde{c} + \tilde{r}$ has been proved for several classes of reductive subalgebras $\mathfrak{h}$:

- $\mathfrak{h} = \mathfrak{g}^\sigma$ is a symmetric subalgebra [19, Theorem 2.7];
- $\mathfrak{h} = \mathfrak{g}^\theta$, where ϑ is an automorphism of $\mathfrak{g}$ of finite order $\geqslant 3$ [20, Theorem 3.10]. Here one also needs the condition that a certain contraction of $\mathfrak{g}$ associated with ϑ, denoted $\mathfrak{g}_{(0)}$, has the same index as $\mathfrak{g}$. It should be noted, however, that this condition has been verified in many cases, and it is likely that this condition always holds.
- $\mathfrak{h}$ is the centraliser of a semisimple element of $\mathfrak{g}$ [11, Lemma 2.1], i.e., $\mathfrak{h}$ is a Levi subalgebra of $\mathfrak{g}$.

Note also that if $\mathfrak{h} \subset \mathfrak{g}$ has a non-trivial centre, then, for any $\mathfrak{r}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{r} \subset \mathfrak{h}$, we have $\dim \mathfrak{h} - \operatorname{rk} \mathfrak{h} = \dim \mathfrak{r} - \operatorname{rk} \mathfrak{r}$. Therefore, if a Poisson-commutative subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{h}}$ has the maximal transcendence degree, then it follows from (5.2) that $\mathcal{A}$ has the maximal transcendence degree as subalgebra of the larger Poisson algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{r}}$.

Remark. An advantage of this case is that $\Bbbk[\tilde{\mathfrak{m}}] = \mathcal{S}(\mathfrak{g})$ is a Poisson algebra. Therefore, one can construct 'large' Poisson-commutative subalgebras in $\mathcal{S}(\mathfrak{g})^H$ using compatible Poisson brackets and Mishchenko-Fomenko subalgebras of $\mathcal{S}(\mathfrak{g})$, see [11, 19, 20]. But for an arbitrary affine G/H, the algebra $\Bbbk[\mathfrak{m}]$ does not possess a natural Poisson structure.

5.2. **A more general setting.** Let $R \subset Q$ be arbitrary connected affine algebraic groups. Then $Q{\times}R/\Delta_R \simeq Q$ is an affine homogeneous space of $Q{\times}R$ and the coisotropy representation of $R \simeq \Delta_R$ is isomorphic to $(R : \mathfrak{q}^*)$. Here we are led to consider Poisson-commutative subalgebras of the Poisson algebra $\mathcal{S}(\mathfrak{q})^R = \mathcal{S}(\mathfrak{q})^{\mathfrak{r}}$.

Our luck is that this problem (without connection to coisotropy representations) has been considered in [11], where an upper bound on $\operatorname{trdeg} \mathcal{A}$ similar to (5.2) is given. The only difference is that the *rank* of a Lie algebra has to be replaced with the *index*. (Recall that $\operatorname{rk} \mathfrak{q} = \operatorname{ind} \mathfrak{q}$ whenever $\mathfrak{q}$ is reductive.) That is, if $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{r}}$ and $\{\mathcal{A}, \mathcal{A}\} = 0$, then

$$\operatorname{trdeg} \mathcal{A} \leqslant \frac{1}{2}(\dim \mathfrak{q} - \dim \mathfrak{r} + \operatorname{ind} \mathfrak{q} + \operatorname{ind} \mathfrak{r}), \tag{5.3}$$

see [11, Prop. 1.1]. It is also shown in [11] that if $\mathfrak{r} = \mathfrak{q}^\xi$ is the stabiliser of $\xi \in \mathfrak{q}^*$ under the coadjoint representation of $\mathfrak{q}$ and $\mathrm{ind}\,\mathfrak{q}^\xi = \mathrm{ind}\,\mathfrak{q}$, then this bound is achieved in many cases. In particular, if $\mathfrak{q}$ is reductive, then this happens for any ξ.

However, results of [5] do not apply in this setting, unless both groups Q and R are reductive.

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