MAHLER EQUATIONS FOR ZECKENDORF NUMERATION

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Abstract. We define generalised equations of Z-Mahler type, based on the Zeckendorf numeration system. We show that if a sequence over a commutative ring is Z-regular, then it is the sequence of coefficients of a series which is a solution of a Z-Mahler equation. Conversely, if the Z-Mahler equation is *isolating*, then its solutions define Z-regular sequences. This is a generalisation of results of Becker and Dumas. We provide an example to show that there exist non-isolating Z-Mahler equations whose solutions do not define Z-regular sequences. Our proof yields a new construction of weighted automata that generate classical q-regular sequences.

1. Introduction

Christol's theorem states that a series $f(x) = \sum_n f_n x^n$ with coefficients in the finite field $\mathbb{F}_q$ is algebraic over the field of rational functions $\mathbb{F}_q(x)$ if and only if the sequence $(f_n)_{n\geqslant 0}$ is q-automatic, i.e., f_n is a finite-state function of the base-q expansion of n [Chr79, CKMFR80]. In this article we extend Christol's theorem to sequences that are finite-state functions of the Zeckendorf numeration.

Christol's theorem is firmly anchored in algebra, and its beauty lies in the fact that it connects algebraicity of series over finite fields to automata theory. However, it falls short of being a complete characterisation of automatic sequences, as it only characterises q-automaticity when $q = p^k$ is a prime power. One way of generalising Christol's theorem is by replacing, on the one hand, algebraic equations with *Mahler equations*, and on the other, automaticity with *regularity*, as done by Becker and Dumas [Bec94, Dum93]. This larger context allows us to move from the setting of finite fields to that of commutative rings.

We can trace the passage from a polynomial to a Mahler equation as follows. Assuming that $q = p^k$, if (f_n) is q-automatic, then Christol's theorem tells us that $\sum_n f_n x^n$ is the root of a polynomial over $\mathbb{F}_q(x)$. It follows that $\sum_n f_n x^n$ must also be a root of an *Ore* polynomial, $P(x, y) = \sum_{i=0}^{d} A_i(x) y^{q^i}$.

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Now more generally let R be any commutative ring and let $q \geqslant 2$ be any natural number. Define the linear operator $\Phi : R[\![x]\!] \to R[\![x]\!]$ as $\Phi(f(x)) = f(x^q)$. Let $A_i(x) \in R[x]$ be polynomials. The equation

$$P(x, y) = \sum_{i=0}^{d} A_i(x)\Phi^i(y) = 0 \tag{1}$$

is called a *q-Mahler equation*, and if $f \in R[\![x]\!]$ satisfies $P(x, f(x)) = 0$, then it is called *q-Mahler*. If q is a power of a prime, and $R = \mathbb{F}_q$, then, using the fact that elements of $\mathbb{F}_q$ are left invariant by the Frobenius operator $x \mapsto x^q$, a q-Mahler functional equation is nothing but an Ore polynomial, and so the notions of being algebraic and q-Mahler coincide. In other words, the notion of a series being a solution of a q-Mahler equation is a generalisation of a series being algebraic.

Regular sequences, introduced by Allouche and Shallit in [AS92], are a generalisation of automatic sequences to infinite rings. Just as automatic sequences are generated by deterministic automata, regular sequences are generated by *weighted* automata. By a weighted automaton, we mean a nondeterministic automaton where each transition is labeled from an alphabet B and in addition carries a weight from a ring R; see Section 2.2 for a precise definition. Thus, a word w on B is assigned a weight by the automaton, namely, the sum of weights of all possible paths with w as label. Furthermore, as automatic sequences are characterised as having a finite *kernel*, so regular sequences are characterised as having a finitely generated kernel, and in fact Allouche and Shallit defined regular sequences this way. The equivalence of these two definitions follows from a classical result about weighted automata [BR11, Proposition 5.1] and [AS92, Theorem 2.2].

Becker [Bec94] and Dumas [Dum93] generalise Christol's theorem as follows: a q-regular series is q-Mahler, and conversely, if a Mahler equation is *isolating*, i.e., $A_0(x) \equiv 1$ in (1) then its roots are q-Mahler. They also give examples of non-isolating Mahler equations whose solutions are not q-regular. Finding a complete characterisation of regular sequences is still open in general, although it has been solved in [BCCD19] for $R = \mathbb{C}$. See also [CS18] for a general exposition of regular sequences, as well as recent developments.

Our first contribution, which is Theorem 15, is a direct construction of a weighted automaton computing the coefficients of the solution of an isolating q-Mahler equation, which in addition avoids the use of the *Cartier* operators. The construction is first given in the classical setting of the base-q numeration system. It can be used to obtain the result of Becker and Dumas, or indeed one direction of Christol's theorem in Corollary 16. Given an isolating Mahler equation P and $f_0 \in R$, we can define a weighted automaton $\mathcal{A} = \mathcal{A}(P, f_0)$; see Section 2.4.3. We show the following.

Theorem 1. *Let P be an isolating q-Mahler equation over the commutative ring R. Let $f(x) = \sum_{n\geqslant 0} f_n x^n$ satisfy $P(x, f(x)) = 0$. Then the weighted automaton $\mathcal{A} = \mathcal{A}_{P,f_0}$ generates f.*

Furthermore we bound the cardinality of the state set of $\mathcal{A}$ as a function of P's *exponent* and *height*; see Theorem 15.

The base-q numeration system is a fundamental underpinning of Christol's theorem. The main result of this paper is to extend Christol's theorem to the setting of the Zeckendorf numeration system, where each integer is written as a sum of Fibonacci numbers. We define (f_n) to be *Z-regular* if there exists a weighted automaton that computes f_n by reading the canonical Zeckendorf expansion of n. The main obstacle to extending the work of Allouche and Shallit, Becker and Dumas, i.e., to characterising Z-regular sequences as solutions of some generalised Mahler equation, is that the Zeckendorf numeration does not yield a ring. In particular, when moving from base-q to Zeckendorf numeration, the natural replacement of the map $n \mapsto qn$ is not linear, and this latter property is crucial in the construction of our automata in Theorem 15.

While the Zeckendorf analogue of $n \mapsto qn$ is not linear, we show, by using work of Frougny [Fro92], that its non-linearity can be calculated by a deterministic automaton. From this, we introduce a linear operator $\Phi : R[\![x]\!] \to R[\![x]\!]$ on series which plays the role of the Frobenius operator $x \mapsto x^q$ for q-Mahler equations, and with it we define Z-Mahler equations, analogously as in (1). Our main result is

Theorem 2. *Let R be a commutative ring, let $P(x, y) \in R[x, y]$ be an isolating Z-Mahler equation. If $f = \sum_{n\geqslant 0} f_n x^n$ is a solution of $P(x, f(x)) = 0$, then there is a weighted Z-automaton $\mathcal{A}$ which generates f.*

Furthermore in Theorem 30, we bound the state set of $\mathcal{A}$ as a function of P and the golden ratio φ, which is intimately connected to the Zeckendorf numeration. Conversely in Corollary 34 we see that Z-regular series are solutions of a Z-Mahler equation. Finally, as in the classical case, we show in Section 3.4.4, that the restriction to isolating equations is needed for our construction.

We remark that our results can be generalised to numeration systems generated by recurrences whose characteristic polynomial is the minimal polynomial of a Pisot number. In particular one can use general results of Frougny and Solomyak [Fro92, FS92], namely that *normalisation* of expansions can be realised by automata.

This paper arose out of a desire to understand whether there was a connection between two descriptions of certain substitutional fixed points as projections of a more regular structure in higher dimension. The first result concerns the realisation of Pisot substitutional tilings as a *model set*. A model set is a projection of a pseudo-lattice in $\mathbb{R}^n \times G$ into $\mathbb{R}^n$, via a window in a locally compact Abelian group G. A *Meyer* set is a finite set of translates of a model set. The importance of these sets is shown in Y. Meyer's work [Mey72, Thm V Chapter VII, Thm IV, page 48] and their relevance to substitutions was made clear by Lee and Solomyak [LS12], who proved that a substitutional tiling shift has pure discrete spectrum if and only if the discrete set of control points for the tiling are a Meyer set. The second result is Furstenberg's theorem [Fur67], which tells us that algebraic functions in $\mathbb{F}_q[\![x]\!]$ are projections, onto one dimension, of Laurent series expansions of rational functions in $\mathbb{F}_q(x, y)$. Combining Furstenberg's theorem with Christol's theorem [Chr79, CKMFR80], we conclude that these projections are codings of length-q substitutional fixed points.

Solutions of equations of Mahler type have provided a useful source of transcendental numbers. Indeed, Mahler started this line of investigation by showing that if α is algebraic with $0 < |\alpha| < 1$, and if $f(z)$ is the solution of the 2-Mahler equation $\Phi(f(x)) = f(x) - x$, then $f(\alpha)$ is transcendental [Mah82]. This approach has been greatly generalised and is now known as Mahler's method, see the book [Nis96] by Nishioka devoted to the foundations of this area, as well as the more recent survey article [Ada19] by Adamczewski. It would be interesting to investigate whether analogous results exist for Z-Mahler equations.

In Section 2 we set up notation, and define weighted automata, regular sequences, and Mahler equations. For the case of base-q numeration, $q \in \mathbb{N}$, we describe the correspondence between q-regular sequences and q-Mahler equations. In Section 2.4.3, we define a universal weighted automaton, and in Theorem 15 we show that, by varying the weights, we can generate any solution of any isolating q-Mahler equation. In Corollary 16 we recover Christol's theorem in the special case where the ring equals $\mathbb{F}_q$. We then focus on the Zeckendorf-numeration. In Section 3.2 we define an operator $n \mapsto \phi(n)$ and show in Corollary 23 that the non-linearity of ϕ can be calculated. In Section 3.4 we define Z-Mahler equations. By merging the ideas behind the weighted automata generating solutions of q-Mahler equations and the automaton computing the linearity defect of ϕ, we show in Theorem 30 that any solution of an isolating Z-Mahler equation is Z-regular. In the standard Section 3.4.2, we show that Z-regular sequences define solutions of Z-Mahler equations. In Section 3.4.3 we give a result analogous to that of Dumas for q-numeration, slighting relaxing the notion of Z-isolating which guarantees Z-regularity. Finally in Section 3.4.4 we give an example of a non-isolating Z-Mahler equation which has non-regular solutions. We end by describing some open problems.

2. Mahler equations and weighted automata

2.1. **Basic notation.** We will work with numeration systems $U = (u_n)_{n\geqslant 0}$ with $u_0 = 1$, and where each natural number can be represented using strings of symbols from a digit set B, i.e., for each n there is a k and $b_k, \dots, b_0$ from B with $n = \sum_{i=0}^{k} b_i u_i$. For the numeration systems we consider, every natural number has a *canonical* representation, which is the greatest representation for the lexicographic ordering, and we use $(n)_U$ to denote this canonical representation. Also, given a word on the digit set, we use $[w]_U$ to denote the natural number that has w as a (possibly non-canonical) representation in that system. Despite these definitions, we are permitted to sometimes pad $(n)_U$ with leading zeros, eg, when we have to compare the expansions of several integers at a time.

In this article we only work with base-q numerations and the Zeckendorf numeration. However we note that our work generalises to numerations where addition is not unreasonable, such as systems where U is defined using a linear recurrence, whose characteristic polynomial is the minimal polynomial of a Pisot number.

In Section 2, we work with base-q numerations, with $U = (q^n)_{n\geqslant 0}$, whose digit set is $\{0, 1, \dots, q-1\}$, and the canonical representation $(n)_q$ is that for which the most significant digit is non-zero. In Section 3, we work with the Zeckendorf numeration, $Z = (F_n)_{n\geqslant 0}$ defined by the Fibonacci numbers, where the digit set is $\{0, 1\}$ and where the canonical representation $(n)_Z$ has no consecutive occurrences of the digit 1; see [AS03, Section 3.8] for a summary.

2.2. **Automata.** In this section we recall the notion of a weighted automaton. There are many variants of automata. The main result (Theorem 30) of this paper is phrased using weighted automata, but we also use other kinds automata, such as deterministic automata with an output function or even classical automata. The inputs of automata are words, that is, sequences of symbols from an alphabet, but each numeration system U associates with each integer n a word, its canonical representation $(n)_U$, that can be fed as input to an automaton. This allows automata to deal with integers instead of words.

We assume the reader to be familiar with the basic notions concerning automata; otherwise we refer the reader to [Sak09] for a complete introduction. An *automaton* $\mathcal{B}$ is a tuple $\langle S, B, \Delta, I, F\rangle$ where S is the state set, B is the input alphabet, Δ is the transition relation and I and F are the sets of initial and final states. A transition $(p, b, q) \in \Delta$ is a labelled directed edge between states and is written $p \xrightarrow{b} q$. A word $w = b_1 \cdots b_n$ is *accepted* if there is a sequence $q_0 \xrightarrow{b_1} q_1 \cdots q_{n-1} \xrightarrow{b_n} q_n$ of consecutive transitions such that $q_0 \in I$ and $q_n \in F$. The automaton is *deterministic* if $I = \{s_0\}$ is a singleton, and if $p \xrightarrow{b} q$ and $p \xrightarrow{b} q'$ in Δ implies $q = q'$. In that case, the relation Δ is a function $\Delta : S \times B \to S$; it can be extended to a function from $S \times B^*$ to S by setting $\Delta(q, \varepsilon) = q$ and $\Delta(s, b_1 \cdots b_n) := \Delta(\Delta(s, b_1 \cdots b_{n-1}), b_n)$. If the set F of final states is replaced by an *output function* $\tau : S \to A$, where A is the *output alphabet*, then the automaton defines the function from B^* to A which maps the word w to $\tau(\Delta(s_0, w))$. Given a numeration system U, a sequence $(a_n)_{n\geqslant 0}$ is called *U-automatic* if there is a deterministic automaton $\langle S, B, \Delta, \{s_0\}, \tau\rangle$ such that $a_n = \tau(\Delta(s_0, (n)_U))$ for each $n \geqslant 0$. For the numeration systems that we study in this article, this definition is equivalent to requiring the existence of an automaton such that $a_n = \tau(\Delta(s_0, w))$ whenever $n = [w]_U$, for each $n \geqslant 0$. If $(n)_U$ is read starting with the most significant digit, we say that $(a_n)_{n\geqslant 0}$ is obtained in *direct reading*, otherwise we say that it is obtained in *reverse reading*. If $(a_n)_{n\geqslant 0}$ is U-automatic, then we will also say that $f(x) := \sum_{n\geqslant 0} a_n x^n$ is *U-automatic*.

If U is the classical base-q numeration, U-automatic sequences are called q-automatic, and these sequences have been extensively studied [AS03]. In [Sha88] Shallit studied more general U-automatic sequences; see also work by Allouche [All92] and Rigo [Rig00].

Let R be a commutative ring. A *weighted automaton* $\mathcal{A}$ with weights in R is a tuple $\langle S, B, \Delta, I, F\rangle$, where S is a finite state set, B is an alphabet, $I : S \to R$ and $F : S \to R$ are the functions that assign to each state an initial and a final weight and $\Delta : S \times B \times S \to R$ is a function that assigns to each transition, i.e., to each labelled edge, a weight. A transition (s, b, s') such that $\Delta(s, b, s') = r \in R$ is written $s \xrightarrow{b:r} s'$. A *path* γ in $\mathcal{A}$ is a finite sequence $s_0 \xrightarrow{b_1:r_1} s_1, s_1 \xrightarrow{b_2:r_2} s_2, \cdots, s_{n-1} \xrightarrow{b_n:r_n} s_n$ of consecutive transitions. The label of such a path is the word $w = b_1 \cdots b_n$ and the path is written $s_0 \xrightarrow{w:r} s_n$ where $r = r_1 \cdots r_n$. This notation is consistent with the notation $s \xrightarrow{b:r} s'$ for transitions since a transition can be viewed as a path of length 1. The *weight* $\mathrm{weight}_{\mathcal{A}}(\gamma)$ of the path γ is the product $I(s_0) r F(s_n) = I(s_0) r_1 \cdots r_n F(s_n)$. Furthermore, the weight of a word $w \in B^*$ is the sum of the weights of all paths with label w and it is denoted $\mathrm{weight}_{\mathcal{A}}(w)$, i.e.,

$$\mathrm{weight}_{\mathcal{A}}(w) = \sum_{\gamma = s_0 \xrightarrow{w:r} s_n} \mathrm{weight}_{\mathcal{A}}(\gamma).$$

If the input alphabet is $B = \{0, 1, \dots, q-1\}$, we say that $\mathcal{A}$ is a weighted *q-automaton*.

In our figures, non-zero initial and final weights are given over small incoming and outgoing arrows. Missing transitions implicitly have zero weight. If a state has zero initial weight, it will not have an initial arrow, likewise for states with final weight zero.

Recall the two-element Boolean semiring $\mathbb{B} = \{0, 1\}$, where the sum and the product are max and min respectively. A *non-deterministic automaton* $\langle S, B, \Delta, F \rangle$ is a weighted automaton where $R = \mathbb{B}$.

Let $\mathcal{A}$ be a q-weighted automaton and let U be a numeration system with digit set $\{0, 1, \dots, q-1\}$. Define the sequence $(a_n)_{n \geqslant 0}$ by $a_n := \mathrm{weight}_{\mathcal{A}}((n)_U)$, where $(n)_U$ is read starting with the most significant digit. Then we say that the sequence $(a_n)_{n \geqslant 0}$ and the generating function $f(x) := \sum_{n \geqslant 0} a_n x^n$ are *U-regular*, generated by $\mathcal{A}$. The notion of a q-regular sequence was introduced by Allouche and Shallit, that it is equivalent to this definition is shown in [AS92, Theorem 2.2]. Rigo studied automatic sequences for abstract numeration systems, using automata with output. While he did not define U-regularity, it is quite natural to extend regularity to other numeration systems.

Example 3. In Figure 1 we give a weighted automaton with weights in $\mathbb{F}_2$ that generates the Thue-Morse sequence with the base-2 numeration. As the state s is the only state with a nontrivial initial weight, and the state t is the only state with a nontrivial final weight, then, to compute the n-th term, one sums in $\mathbb{F}_2$ the weights of all paths with label $(n)_2$ from the state s to the state t. Note that there are as many such paths as the number of occurrences of the digit 1 in $(n)_2$. Since each such path has weight 1, the sum in $\mathbb{F}_2$ is the number of 1's in $(n)_2$ mod 2.

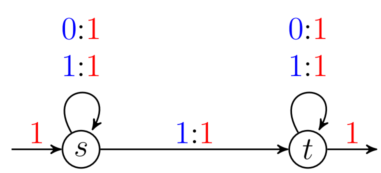


FIGURE 1. A weighted automaton that generates the Thue-Morse sequence (a_n), where $a_n = \mathrm{weight}_{\mathcal{A}}((n)_2)$. The weight of an edge is given in red, and the blue numbers are the digits we read in $(n)_2$.

2.2.1. *Matrix representations of weighted automata.* Let $\mathcal{A}$ be a weighted automaton and let n be its number of states. Then $\mathcal{A}$ can also be represented by a triple $\langle I, \mu, F \rangle$ where I is a row vector over R of dimension $1 \times n$, μ is a morphism from B^* into the set of $n \times n$-matrices over R, with the usual matrix multiplication, and F is a column vector of dimension $n \times 1$ over R. The vector I is the vector of initial weights, the vector F is the vector of final weights and, for each symbol b, $\mu(b)$ is the matrix whose (p, q)-entry is the weight $r \in R$ of the transition $p \xrightarrow{b:r} q$. Note that with this matricial notation, $\mathrm{weight}_{\mathcal{A}}(w) = I\mu(w)F$.

Example 4. The weighted automaton pictured in Figure 1 is represented by $\langle I, \mu, F\rangle$ where $I = (1, 0)$, $F = (0, 1)^t$ and the morphism μ from $\{0, 1\}^*$ to the set of 2×2-matrices over $\mathbb{F}_2$ is given by

$$\mu(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The following result can be proved as in [AS92, Theorem 2.10]. It will be used in Section 3.4.4 to show that the solution of a non-isolating Z-Mahler equation is not Z-regular.

Lemma 5. *If the sequence $(a_n)_{n\geqslant 0}$ is complex valued and U-regular, then there is a positive constant c such that $a_n = O(n^c)$.*

2.2.2. *From weighted automata to automatic sequences.* It is known that any weighted automaton with weights in a finite ring defines an automatic sequence; see for example [AS03, Sec. 4.3]. Nevertheless, for completeness we include a proof, which is a slight generalization of the power set construction, and which yields a deterministic automaton from a non-deterministic one.

Proposition 6. *Let $\mathcal{A}$ be a weighted q-automaton with weights in a finite commutative ring R. Then, for each of direct and reverse reading, there exists a q-deterministic automaton $\mathcal{B}$ with initial state s_0 and output $\tau : S \to R$ such that for each word w, the equality* $\mathrm{weight}_{\mathcal{A}}(w) = \tau(\Delta(s_0, w))$ *holds.*

Proof. Let n be the number of states of $\mathcal{A}$, and let $\langle I, \mu, F\rangle$ be a matrix representation of dimension n of the weighted automaton $\mathcal{A}$ as given in Section 2.2.1. The weight of a word $w \in B^*$ is by definition $I\mu(w)F$.

Consider the automaton $\mathcal{B}$ defined as follows. Its state set S is the finite set of all row vectors of dimension $1 \times n$ over R. Its initial state is the vector I. Set $\Delta(q, b) := q\mu(b)$ for each row vector q and each digit b. Finally define the function $\tau : S \mapsto R$ as $\tau(q) = qF$ for each $q \in S$. It is now routine to check that the automaton $\mathcal{B}$ defines, in direct reading, the same automatic sequence as the weighted automaton.

A reverse deterministic automaton is obtained similarly by taking S to be the set of column vectors of dimensions $n \times 1$ over R, by taking F as initial state and, setting $\Delta(q, b) := \mu(b)q$. □

Corollary 7. *Let $\mathcal{A}$ be a weighted q-automaton with weights in a finite commutative ring R, and let U be a numeration system with digit set $\{0, 1, \cdots, q - 1\}$. Then the U-regular sequence generated by $\mathcal{A}$ is U-automatic.*

2.3. **Robustness of weighted automata.** As we saw in Proposition 6, there is a direct link from sequences generated by weighted automata using a numeration system U to U-regular sequences. The purpose of this section is to show that the class of U-regular sequences is closed under the Cauchy product as soon as addition in U can be realized by an automaton. Obtaining this result is made easier by the use of weighted automata. Looking ahead, it is particularly interesting when Christol's theorem does not hold as for the Zeckendorf numeration system.

An *unambiguous automaton* is a weighted automaton over the ring $\mathbb{Z}$ of integers such that each weight, including initial and final weights, is either 0 or 1, and such that the weight of each word is either 0 or 1. The first condition implies that the weight of each path is in $\{0, 1\}$. The second condition implies that for each word w, there is at most one path labelled by w having a positive weight. If there is such a path, the word is said to be *accepted*.

Let u and v be two words over alphabets A and B respectively such that $|u| = |v|$. We denote by $u \otimes v$ the word w over the alphabet $A \times B$ such that $|w| = |u| = |v|$ and $w_i = (u_i, v_i)$. An unambiguous automaton *realizes addition* in a numeration system if it accepts all words of the form $(m)_U \otimes (n)_U \otimes (m+n)_U$ for non-negative integers m and n, where the expansions $(m)_U$ and $(n)_U$ have been possibly padded with leading zeros to have the same length as $(m+n)_U$, i.e., we momentarily relax our notation and use $(m)_U$ to refer to any representation of m. For example, the automaton pictured in Figure 2 realizes addition in base 2.

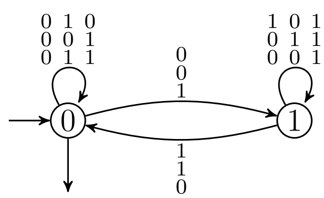


FIGURE 2. A automaton recognising addition base-2. A string over $\{0, 1\}^3$, whose letters here are written as column vectors $\substack{x\\y\\z}$, is accepted in direct reading if and only if it equals $(m)_2 \otimes (n)_2 \otimes (m+n)_2$.

Recall that the *Cauchy product*, or *convolution* of two series $f(x) = \sum_{n\geqslant 0} f_n x^n$ and $g(x) = \sum_{n\geqslant 0} g_n x^n$ is the series $h(x) = \sum_{n\geqslant 0} h_n x^n$ where $h_n = \sum_{i+j=n} f_i g_j$ for each $n \geqslant 0$.

Theorem 8. *Suppose that there is an unambiguous automaton realizing addition in the numeration system U. If the sequences $f_1 = (f_{1,n})_{n\geqslant 0}$ and $f_2 = (f_{2,n})_{n\geqslant 0}$ are generated by weighted automata in U, then the Cauchy product of f_1 and f_2 is also generated by a weighted automaton in U.*

Proof. Let B be the digit alphabet of the numeration system U. Suppose that the unambiguous automaton $\mathcal{A} = (Q, B^3, \Delta, I, F)$ realizes addition in U. Let the sequences f_1 and f_2 be generated by the weighted automata $\mathcal{B}_1 = (S_1, B, \Delta_1, I_1, F_1)$ and $\mathcal{B}_2 = (S_2, B, \Delta_2, I_2, F_2)$ respectively. We construct a new weighted automaton $\mathcal{B}$ whose state set is $Q \times S_1 \times S_2$. The initial (respectively final) weight of a state (q, s_1, s_2) of $\mathcal{B}$ is $I_1(s_1)I_2(s_2)$ (respectively $F_1(s_1)F_2(s_2)$) if q is initial (respectively final) in $\mathcal{A}$, and 0 otherwise. Its transition set Δ

is given by

$$\Delta = \left\{ (q, s_1, s_2) \xrightarrow{b:\alpha_1\alpha_2} (q', s_1', s_2') : \exists b_1, b_2 \in B \quad \begin{array}{lll} q & \xrightarrow{b_1,b_2,b} q' & \text{in } \mathcal{A} \\ s_1 & \xrightarrow{b_1:\alpha_1} s_1' & \text{in } \mathcal{B}_1 \\ s_2 & \xrightarrow{b_2:\alpha_2} s_2' & \text{in } \mathcal{B}_2 \end{array} \right\}.$$

A path $(q, s_1, s_2) \xrightarrow{w:\alpha} (q', s_1', s_2')$ in $\mathcal{B}$ corresponds first to a choice of a path $s \xrightarrow{w,u_1,u_2} s'$ in $\mathcal{A}$ giving two words u_1 and u_2 such that $[w]_U = [u_1]_U + [u_2]_U$, and second to a choice of two paths $s_1 \xrightarrow{u_1:\alpha_1} s_1'$ and $s_2 \xrightarrow{u_2:\alpha_2} s_2'$ in $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\alpha = \alpha_1\alpha_2$. From this remark and the choices of initial and final weights in $\mathcal{B}$, it follows that the weight of a word w is the sum, over all decompositions $[w]_U = [u_1]_U + [u_2]_U$ of the products $\alpha_1\alpha_2$ of the weights α_1 and α_2 of u_1 and u_2. Note that the commutativity of the ring is here essential. $\square$

2.4. q-Mahler equations and weighted automata.

2.4.1. *From weighted automata to q-Mahler equations.* Given a weighted q-automaton generating $(a_n)_{n\geqslant 0}$, there is a standard technique to obtain a Mahler equation for which $\sum_{n\geqslant 0} a_n x^n$ is a solution, similar to the techniques described in the proof of Christol's theorem in [AS03, Theorem 12.2.5]. We illustrate with a simple example.

Example 9. *Consider again the Thue-Morse sequence $(b_n)_{n\geqslant 0}$, whose weighted automaton is given in Figure 1. We interpret the state t to be the formal power series $t(x) = \sum_n b_n x^n$ in $\mathbb{F}_2[\![x]\!]$, where $b_n = \mathrm{weight}_{\mathcal{A}}((n)_2) \in \mathbb{F}_2$. Similarly the state s corresponds to the series $s(x) = \sum_{n\geqslant 0} a_n x^n$ in $\mathbb{F}_2[\![x]\!]$, whose coefficients would be generated by the automaton if s were the only final state with weight 1. As we read $(n)_2$ starting with the most significant digit, we have,*

$$\begin{aligned} a_{2n} = a_n \quad &\textit{and} \quad a_{2n+1} = a_n. \\ b_{2n} = b_n \quad &\textit{and} \quad b_{2n+1} = b_n \oplus a_n \end{aligned} \tag{2}$$

where the symbol $\oplus$ denotes the sum in $\mathbb{F}_2$.

Since $\mathbb{F}_2[\![x]\!]$ has characteristic 2, then (2) implies that

$$t(x) = (1+x)t(x^2) + xs(x^2) \text{ and } s(x) = (1+x)s(x^2). \tag{3}$$

Again using that $\mathbb{F}_2[\![x]\!]$ has characteristic 2, we have

$$t(x^2) = (1+x)^2 t(x^4) + x^2 s(x^4) \text{ and } s(x^2) = (1+x^2)s(x^4). \tag{4}$$

Now substituting (4) in (3), we obtain

$$\begin{aligned} xt(x) &= (x + x^2 + x^3 + x^4)t(x^4) + (x^2 + x^3)s(x^4) \\ &= (1+x)^3 t(x^4) + x^2(1+x)s(x^4) + (1+x^4)t(x^4) \\ &= (1+x)t(x^2) + (1+x^4)t(x^4), \end{aligned}$$

i.e., the Thue-Morse power series is a solution of the 2-Mahler equation $P(x,y) = xy + (1+x)\Phi(y) + (1+x^4)\Phi^2(y) = 0$.

2.4.2. *Reformulating isolating q-Mahler equations.* Given an isolating q-Mahler equation $P(x,y) = y - \sum_{i=1}^{d} A_i(x)\Phi^i(y)$, we write

$$A_i(x) = \sum_{j=0}^{h} \alpha_{i,j} x^j \qquad \text{for } 1 \leqslant i \leqslant d, \tag{5}$$

where $\alpha_{i,j} \in R$ for each i, j. Let $f(x) = \sum_{n \geqslant 0} f_n x^n$ satisfy $P(x, f(x)) = 0$. To find the coefficients f_n of f, we obtain from (5) that

$$f_n = \sum_{\substack{kq^i + j = n \\ 1 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant h}} \alpha_{i,j} f_k = \sum_{kq^i + j = n} \alpha_{i,j} f_k \,, \tag{6}$$

where the last equality follows if we set $\alpha_{i,j} = 0$ for i, j outside the prescribed bounds. Therefore we can drop the superfluous constraints on the indices.

Note that if we set $n = 0$ in (6), we get the following equation satisfied by the coefficient $f_0 = f(0)$.

$$f_0 = \left(\sum_{i=0}^{h} \alpha_{i,0} \right) f_0. \tag{7}$$

Therefore, if it were to be the case that $f_0 = f(0)$ for some solution f of $P(x, f(x)) = 0$, Equality (7) must hold. In this case we say that f_0 is *P-compatible*. Conversely, if f_0 is P-compatible, then, again using (6), there is a unique series f such that $P(x, f(x)) = 0$ and $f(0) = f_0$. Clearly, if $\sum_{i=0}^{h} \alpha_{i,0} = 1$, then any $f_0 \in R$ is P-compatible. Also, if R is an integral domain, then $\left(\sum_{i=0}^{h} \alpha_{i,0}\right) f_0 = f_0$ is equivalent to either $\sum_{i=0}^{h} \alpha_{i,0} = 1$ or $f_0 = 0$.

In (6) we have reduced solving a functional equation to a linear problem, to which weighted automata are well suited. The identity (6) motivates the definition of the automaton associated to a Mahler equation that we will use in Section 2.4.3, and it is also what makes the proof of Proposition 14 work.

2.4.3. *From isolating q-Mahler equations to weighted automata.* In this section, we define a weighted automaton which directly computes the coefficients of the solution of a q-Mahler equation. Note that we will avoid the need for Cartier operators Λ_i, $0 \leqslant i \leqslant q-1$ ([AS03, Definition 12.2.1]); this is relevant, because when we move to more general numeration systems in Section 3, the property $\Lambda_0(fg^p) = \Lambda_0(f)g$ that Cartier operators enjoy does not hold there. This automaton is given in two steps. First we describe a *universal* weighted automaton with an infinite number of states. The structure of this universal automaton is fixed and does not depend on the ring R; furthermore it can accommodate any q-Mahler equation P. Given such an equation, which has finitely many non-zero coefficients, only finitely many states of this universal automaton are needed to compute the solution, and the weights of its transitions are given by the coefficients occurring in P.

Let $(\alpha_{i,j})_{i \geqslant 1, j \geqslant 0}$ and f_0 be elements of the commutative ring R. Define the state set S

$$S := \{ s_{i,j} : 0 \leqslant i < \infty \text{ and } 0 \leqslant j < \infty \}.$$

Let $B = \{0, 1, \dots, q-1\}$ be the input alphabet, and define the transition set Δ

$$\Delta := \left\{ s_{i,j} \xrightarrow{b:1} s_{i+1,qj+b} : b \in B, 0 \leqslant i, j \right\}$$
$$\cup \left\{ s_{i,j} \xrightarrow{b:\alpha_{i+1,qj+b-k}} s_{0,k} : b \in B, 0 \leqslant i, j, k \text{ and } 0 \leqslant qj + b - k \right\}.$$

We set the initial and final weights I and F as

$$I(s_{i,j}) := \begin{cases} f_0 \text{ if } j = 0 \\ 0 \text{ otherwise,} \end{cases} \qquad \text{and} \quad F(s_{i,j}) := \begin{cases} 1 \text{ if } i = j = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Then we call the automaton $\mathcal{A} := \langle S, B, \Delta, I, F \rangle$ the *universal weighted q-automaton associated to $(\alpha_{i,j})_{i \geqslant 1, j \geqslant 0}$*. If needed, we can specify that $\mathcal{A}$ depends on the choice of initial condition f_0.

We think of this weighted automaton as universal because only the edge weights depend on $(\alpha_{i,j})_{i \geqslant 1, j \geqslant 0}$. Given a finite set of coefficients $(\alpha_{i,j})_{1 \leqslant i \leqslant d, 0 \leqslant j \leqslant h}$, in particular, those associated to a q-Mahler equation with coefficients as in (5), we extend them to $(\alpha_{i,j})_{i \geqslant 1, j \geqslant 0}$ by setting $\alpha_{i,j} = 0$ if $i > d$ or $j > h$. The following lemma shows that if there are finitely many non-zero coefficients $\alpha_{i,j}$, only finitely many states of the universal automaton are useful. Useful means here that they occur in a path with non-zero weight. Note that such a path must end in $s_{0,0}$ since this state is the only one with non-zero final weight. Lemma 10 also provides an explicit upper bound of the number of useful states.

Lemma 10. *Let P be an isolating q-Mahler equation with exponent d and height h. If either $i \geqslant d$ or $j \geqslant \frac{h}{q-1}$, then the state $s_{i,j}$ does not occur in a path with non-zero weight.*

Proof. Let $s_{i,j} \xrightarrow{b:\alpha} s_{i',j'}$ be a transition in the universal weighted q-automaton associated to $(\alpha_{i,j})_{i \geqslant 1, j \geqslant 0}$ where $\alpha_{i,j} = 0$ if $j \geqslant h$ or $i \geqslant d$. We claim that if $\alpha \neq 0$ and $j \geqslant \frac{h}{q-1}$ then $j' \geqslant \frac{h}{q-1}$. If $s_{i,j} \xrightarrow{b:\alpha} s_{i',j'}$ is a transition of the form $s_{i,j} \xrightarrow{b:1} s_{i+1,qj+b}$, the claim is clear. If it is a transition of the form $s_{i,j} \xrightarrow{b:\alpha} s_{0,j'}$, with $\alpha = \alpha_{i+1,qj+b-j'}$, then, since α is assumed non-zero, we have $qj + b - j' \leqslant h$, and so

$$j' \geqslant qj + b - h \geqslant q\left(\frac{h}{q-1}\right) - h = \frac{h}{q-1}.$$

The statement of the lemma follows. □

In view of Lemma 10, and because $s_{0,0}$ is the only state with a non-zero final weight, we define the weighted automaton associated to a q-Mahler equation P and initial condition f_0 is defined as follows.

Let $q \geqslant 2$ be a natural number. Let $P(x, y) = y - \sum_{i=1}^{d} A_i(x) y^{q^i}$ be an isolating q-Mahler equation whose coefficients $A_i(x) \in R[x]$ are as in (5). Set $\tilde{h} := \lceil \frac{h}{q-1} \rceil - 1$. If we truncate the universal weighted q-automaton associated to $(\alpha_{i,j})_{i \geqslant 1, j \geqslant 0}$ and f_0 to the state and transition sets

$$S := \{s_{i,j} : 0 \leqslant i \leqslant d-1 \text{ and } 0 \leqslant j \leqslant \tilde{h}\}$$

and

$$\Delta := \left\{ s_{i,j} \xrightarrow{b:1} s_{i+1,qj+b} : b \in B \begin{array}{c} 0 \leqslant i \leqslant d-2 \\ 0 \leqslant qj+b \leqslant \tilde{h} \end{array} \right\}$$
$$\cup \left\{ s_{i,j} \xrightarrow{b:\alpha_{i+1,qj+b-k}} s_{0,k} : b \in B \begin{array}{c} 0 \leqslant i \leqslant d-1 \\ 0 \leqslant j,k \leqslant \tilde{h} \\ 0 \leqslant qj+b-k \leqslant h \end{array} \right\},$$

The automaton $\mathcal{A} := \langle S, B, \Delta, I, F \rangle$ is called the *weighted automaton associated to P and f_0*, $\mathcal{A} = \mathcal{A}(P, f_0)$.

In other words, the weighted automaton associated to P is a restriction of the universal q-automaton to a subgraph which will contain all paths in the universal automaton that have non-zero weight and end at $s_{0,0}$. Because of this, we will frequently relax some of the constraints on the indices. Note also that while this definition does not require f_0 to be P-compatible, the latter will be a condition for all our results.

Example 11. In Figure 3 we give the weighted automaton associated to the 2-Mahler equation $f(x) = A_1(x)f(x^2)$ where $A_1(x)$ is the polynomial of degree 3 $\alpha_{1,0}+\alpha_{1,1}x+\alpha_{1,2}x^2+\alpha_{1,3}x^3$. We will see in Proposition 14 that setting $\alpha_{1,0} = 1$ and $f_0 \in R$, this automaton generates the unique solution $f(x) = \sum_{n\geqslant 0} f_n x^n \in R[\![x]\!]$ of the equation $f(x) = A_1(x)f(x^2)$ such that $f(0) = f_0$.

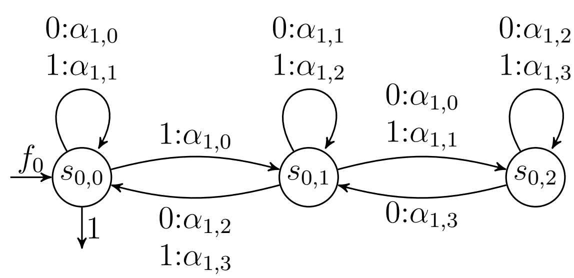


FIGURE 3. The weighted automaton for $f(x) = (\alpha_{1,0} + \alpha_{1,1}x + \alpha_{1,2}x^2 + \alpha_{1,3}x^3)f(x^2)$

Example 12. Figure 17 depicts the weighted automaton associated to a 2-Mahler equation of exponent 2 and height 3, that is, the equation

$$f(x) = A_1(x)f(x^2) + A_2(x)f(x^4)$$

where $A_1(x) = \alpha_{1,0} + \alpha_{1,1}x + \alpha_{1,2}x^2 + \alpha_{1,3}x^3$ and $A_2(x) = \alpha_{2,0} + \alpha_{2,1}x + \alpha_{2,2}x^2 + \alpha_{2,3}x^3$.

Remark 13. The aim is to generate a sequence $(f_n)_{n\geqslant 0}$ by feeding $(n)_q$ into a q-weighted automaton. We claim that the automaton associated to P will generate the same sequence even if we allow leading zeros in $(n)_q$. It is sufficient to show that $I\mu(0) = I$. The only states with non-zero initial weight are $s_{i,0}$. The transitions between these states with $b = 0$

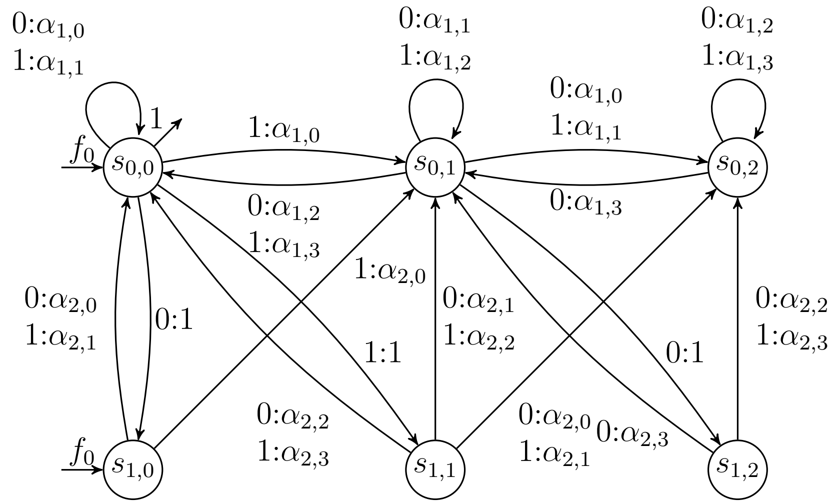


FIGURE 4. The automaton for a 2-Mahler equation of exponent 2 and height 3.

are $s_{i,0} \xrightarrow{0:\alpha_{i+1,0}} s_{0,0}$ and $s_{i,0} \xrightarrow{0:1} s_{i+1,0}$ for $0 \leqslant i \leqslant d-2$. Thus

$$I\mu(0) = (\overbrace{f_0, \dots f_0}^{d}|0 \dots 0)\left(\begin{array}{c|c} \underbrace{\begin{matrix} \alpha_{1,0} & 1 & 0 & \dots & 0 \\ \alpha_{2,0} & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ \alpha_{d-1,0} & 0 & 0 & \dots & 1 \\ \alpha_{d,0} & 0 & 0 & \dots & 0 \end{matrix}}_{M_2} & \mathbf{0} \end{array}\right) = I,$$

where $\mathbf{0}$ represents the appropriate dimension matrix all of whose entries are zero, and where we have used the property that $\left(\sum_{i=1}^{d} \alpha_{i,0}\right) f_0 = f_0$.

Given a state s in the automaton $\mathcal{A}(P, f_0)$, define

$$\text{weight}^*_{\mathcal{A},s}(w) = \sum_{s_0 \xrightarrow{w:r} s} I(s_0) r,$$

where the sum is taken over all possible paths that end at s. The following proposition states the main property of the automaton associated to a given Mahler equation, showing that it indeed computes the solution of the equation.

Proposition 14. *Let R be a commutative ring, let $P(x, y) \in R[x, y]$ be an isolating q-Mahler equation of exponent d and height h, and let $f = \sum_{n \geqslant 0} f_n x^n$ be a solution of $P(x, f(x)) = 0$. If $\mathcal{A}$ is the weighted automaton associated to $P(x, y)$ and $f_0 \in R$, if $w \in \{0, \dots, q-1\}^*$ and if i and j are integers with $0 \leqslant i \leqslant d-1$ and $0 \leqslant j \leqslant \lceil \frac{h}{q-1} \rceil - 1$, then*

$$\text{weight}^*_{\mathcal{A},s_{i,j}}(w) = \begin{cases} f_k & \text{if } kq^i + j = [w]_q \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction on the length of the word w. If w is the empty word ε, then $\mathrm{weight}^*_{\mathcal{A},s_{i,j}}(\varepsilon) = I(s_{i,j})$; this equals f_0 if $j = 0$ and equals 0 otherwise, so the claim is proved for $w = \varepsilon$ since $[\varepsilon]_q = 0$.

For $|w| > 0$, we consider first the easier case $i > 0$ and then the case $i = 0$.

Let $i > 0$. In this case there is a unique incoming transition $s_{i-1,j'} \xrightarrow{b:1} s_{i,j}$ to $s_{i,j}$ if $j \equiv b \mod q$ and $qj' + b = j$, and no incoming transition otherwise. In the latter case the statement follows trivially, so in what follows we assume that there is an incoming transition. Using the inductive hypothesis,

$$\begin{aligned}\mathrm{weight}^*_{\mathcal{A},s_{i,j}}(wb) &= \mathrm{weight}^*_{\mathcal{A},s_{i-1,j'}}(w)\\ &= f_k \text{ where } q^{i-1}k + j' = [w]_q\end{aligned}$$

By multiplying $q^{i-1}k+j' = [w]_q$ by q and then adding b, we obtain $q^ik+j = q[w]_q+b = [wb]_q$ and the statement of the proposition is proved in this case. Notice here that we use linearity of the map $m \mapsto qm$.

Next we consider the case $i = 0$. We have

$$\begin{aligned}\mathrm{weight}^*_{\mathcal{A},s_{0,j}}(wb) &= \sum_{i',j'} \alpha_{i'+1,qj'+b-j}\, \mathrm{weight}^*_{\mathcal{A},s_{i',j'}}(w)\\ &= \sum_{q^{i'}k+j'=[w]_q} \alpha_{i'+1,qj'+b-j} f_k,\end{aligned}$$

where the first equality follows by the definition of the allowed transitions in $\mathcal{A}$ and the second equality follows by the inductive hypothesis. Setting $i = i' + 1$ and $\ell = qj' + b - j$, the equation $q^{i'}k + j' = [w]_q$ becomes $q^ik + \ell = [wb]_q - j$, and we have

$$\begin{aligned}\mathrm{weight}^*_{\mathcal{A},s_{0,j}}(wb) &= \sum_{q^ik+\ell=[wb]_q-j} \alpha_{i,\ell} f_k\\ &= f_{[wb]_q-j},\end{aligned}$$

where the last equality follows from (6). □

Theorem 15. *Let R be a commutative ring, and let $P(x, y) \in R[x, y]$ be an isolating q-Mahler equation of exponent d and height h. Then the automaton $\mathcal{A}$ associated to P and f_0 satisfies*

$$\mathrm{weight}_{\mathcal{A}}(w) = f_{[w]_q},$$

where $f = \sum_{n\geqslant 0} f_n x^n$ is a solution of $P(x, f(x)) = 0$.

Consequently if R is finite, then there exists a deterministic automaton with at most $|R|^{\lceil \frac{h}{q-1}\rceil d}$ states that generates $f(x)$.

Proof. Given $P(x, y) = y - \sum_{i=1}^{d} A_i(x)y^{q^i}$, with A_i defined by Equation (5), let $\mathcal{A}$ be the weighted automaton associated to P; by definition it has at most $\lceil \frac{h}{q-1}\rceil d$ states. Let $[w]_q = n$. By Remark 13, $f_{[w]_q} = f_n$, so it suffices to check the statement for $w = (n)_q$.

By Proposition 14, we have $\mathrm{weight}^*_{\mathcal{A},s_{0,0}}(w) = f_n$. As $s_{0,0}$ is the only state with a non-zero final weight, and $\mathrm{weight}^*_{\mathcal{A},q}(w) = (I\mu(w))_q$, we have

$$\mathrm{weight}_{\mathcal{A}}(w) = I\mu(w)F = \mathrm{weight}^*_{\mathcal{A},s_{0,0}}(w) = f_n.$$

The second statement follows by determinising, either in reverse or in direct reading, the automaton $\mathcal{A}$, as in Proposition 6. □

For the case of automatic sequences over $R = \mathbb{F}_q$, with $q = p^j$, our approach gives a novel proof of one implication in Christol's theorem, and construction of a deterministic automaton generating (f_n), without the use of Cartier operators.

Corollary 16. *Let R be the finite field $\mathbb{F}_q$. If $\sum_n f_n x^n \in \mathbb{F}_q[\![x]\!]$ is algebraic over $\mathbb{F}_q(x)$, then there is a deterministic automaton that generates $(f_n)_{n\geqslant 0}$.*

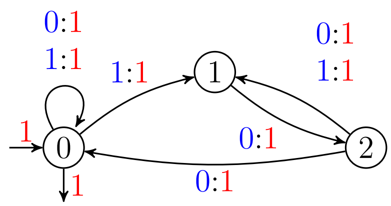
Proof. If $\sum_n f_n x^n \in \mathbb{F}_q[\![x]\!]$ is algebraic, then standard techniques imply that it is the root of an Ore polynomial. If f is the root of an Ore q-polynomial $P(x,y) = \sum_{i=0}^d A_i(x)y^{q^i}$, then the series $g := f/A_0$ is the root of the isolating Ore Polynomial $Q(x,y) = \sum_{i=0}^d B_i(x)y^{p^i}$ where $B_0 = 1$ and $B_i = A_i A_0^{q^i-2}$. By Theorem 15 we can construct a weighted automaton $\mathcal{B}$ that generates g. Then, we apply Theorem 8, which allows us to construct a weighted automaton $\mathcal{A}$ of the Cauchy product $f = A_0 g$. Now using Proposition 6, $\mathcal{A}$ can be determinised, to yield automata that can generate $(f_n)_{n\geqslant 0}$ in either reverse or direct reading. □

3. Mahler equations for Zeckendorf numeration

In this section we investigate links between generalized Mahler equations and weighted automata in the Zeckendorf numeration. Our principal result in this section, Thm 30, is a version of Becker's theorem, which says that a solution of an isolating Z-Mahler equation, as defined in (9), has coefficients computed by a weighted automaton reading integers in the Zeckendorf base. Conversely, any such weighted automaton generates a solution of a Z-Mahler equation, which is possibly non-isolating. The gap between the two results is similar to the one in [Bec94]. The notion of Z-Mahler equation that we introduce is based on a linear operator Φ, defined on power series, which is the analogue of the operator $f(x) \mapsto f(x^q)$ in base-q numeration. This operator Φ is defined thanks to a function $\phi : \mathbb{N} \to \mathbb{N}$ which plays the role of the function $n \mapsto qn$. The main obstacle to our extension of Becker's result is that ϕ is not linear.

3.1. **The Zeckendorf numeration.** Recall that the Fibonacci numbers satisfy the recurrence $F_n = F_{n-1}+F_{n-2}$ with initial conditions $F_{-2} = 0$ and $F_{-1} = 1$. The strictly increasing sequence $(F_n)_{n\geqslant 0}$ defines the *Zeckendorf* numeration system: every natural number has a unique expansion as $n = \sum_{i=0}^k b_i F_i$ where $b_i \in \{0,1\}$ for each i and $b_i b_{i+1} = 0$ for each i. We write $(n)_Z := b_k \cdots b_0$. Conversely, given any finite digit set $B \subset \mathbb{Z}$ and any word $w = w_k \dots w_0 \in B^+$, let $[w]_Z$ denote the natural number n such that $n = \sum_{i=0}^k w_i F_i$. Note that $(n)_Z$ is the canonical Zeckendorf expansion of n, but conversely, the map $w \mapsto [w]_Z$ can be applied to any word over a finite digit set $B \subset \mathbb{Z}$. Thorough descriptions of the properties of addition in this numeration system can be found in Frougny's exposition [Lot02, Chapter 7].

Example 17. *The following weighted automaton generates $(a_n)_{n\geqslant 0}$, where a_n equals the number of representations of n as a sum of distinct Fibonacci numbers. Note that each non-zero weight (in red in the figure) is equal to* 1*. This implies that the weight of a path is either* 0 *or* 1*. This weight is* 1 *if the path starts and ends at state* 0*. The number of such paths labeled by $(n)_Z$ is exactly a_n.*



3.2. **The function ϕ and its almost-linearity.** The following function is the analogue, in the Zeckendorf numeration, of the map $n \mapsto qn$ in base-q. Define $\phi : \mathbb{N} \to \mathbb{N}$ as $\phi(n) := [w0]_Z$, where $w = (n)_Z$. In particular $\phi(F_n) = F_{n+1}$. By definition,

$$\phi\left(\sum_{i=0}^{k} b_i F_i\right) := \sum_{i=0}^{k} b_i F_{i+1}$$

where $(n)_Z = b_k \cdots b_0$. Furthermore, the above equality holds for any word w over $\{0,1\}$ such that $[w]_Z = n$ even if there are consecutive digits b_i such that $b_i = b_{i+1} = 1$. The function ϕ and the map $w \mapsto [w]_Z$ are linked by the equation $[wb]_Z = \phi([w]_Z) + b$ for any word w over $\{0,1\}$ and any digit $b \in \{0,1\}$.

If $(m)_Z = b_k \cdots b_0$ and $(n)_Z = c_k \cdots c_0$ with possibly some leading zeros, m and n are said to have *disjoint support* if $b_i c_i = 0$ for $0 \leqslant i \leqslant k$. If $(m)_Z$ and $(n)_Z$ have disjoint support, then $\phi(m+n) = \phi(m) + \phi(n)$. However, the function ϕ is not linear, for example $\phi(2) = 3 \neq 4 = \phi(1) + \phi(1)$. Nevertheless, ϕ is *almost* linear: In Lemma 19 we show that $\phi(m+n) - \phi(m) - \phi(n)$ belongs to $\{-1, 0, 1\}$ for each $m, n \geqslant 0$. We write ϕ^2 for the function $\phi \circ \phi$. Note that if $(n)_Z = w$, then $(\phi^2(n))_Z = w00$ and $(\phi^2(n)+1)_Z = w01$. We give below the first few values of $\phi(n)$ and $\phi^2(n) + 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi(n)$	0	2	3	5	7	8	10	11	13	15	16	18	20	21
$\phi^2(n)+1$	1	4	6	9	12	14	17	19	22	25	27	30	33	35

The sets $A_1 = \{\phi(n) : n \geqslant 0\}$ and $A_2 = \{\phi^2(n) + 1 : n \geqslant 0\}$ form a partition of $\mathbb{N}$, as A_1 and A_2 are the sets of integers whose Zeckendorf expansion ends with 0 and 1 respectively. In fact, Lemma 18 below implies that A_1 and A_2 are translations of the Beatty sequences $B_1 = \{\lfloor \varphi n \rfloor : n \geqslant 1\}$ and $B_2 = \{\lfloor \varphi^2 n \rfloor : n \geqslant 1\}$ which are known to form a partition of $\mathbb{N} \setminus \{0\}$ as $\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1$.

The following lemma will allow us to bound the non-linearity of ϕ in Lemma 19.

Lemma 18. *For $n \geqslant 0$ an integer, we have $\phi(n) = \lfloor \varphi n + \varphi - 1 \rfloor$ and $\phi^2(n) = \lfloor \varphi^2 n + \varphi - 1 \rfloor$.*

The proof is an easy application of Binet's formula, which we include for completeness.

Proof. Let $\bar{\varphi} = -1/\varphi$ be the algebraic conjugate of φ; we prove the first equality. From $F_n = \frac{1}{\sqrt{5}}(\varphi^{n+2} - \bar{\varphi}^{n+2})$ it follows that $\varphi F_n - F_{n+1} = -\bar{\varphi}^{n+2}$.

Suppose that $n = \sum_{i=0}^{k} F_{n_i}$ for a sequence $n_0, \ldots, n_k$ of integers such that $n_{i+1} > n_i + 1$ for each integer $0 \leqslant i \leqslant k - 1$.

$$\varphi n - \phi(n) = \sum_{i=0}^{k} (\varphi F_{n_i} - F_{n_i+1}) = -\sum_{i=0}^{k} \bar{\varphi}^{n_i+2}$$

We now bound the summation $\sum_{i=0}^{k} \bar{\varphi}^{n_i+2}$. Since $\bar{\varphi}$ is negative, odd powers of $\bar{\varphi}$ are negative while even powers are positive, so

$$-\bar{\varphi}^2 = \frac{\bar{\varphi}^3}{1 - \bar{\varphi}^2} = \sum_{n \geqslant 1} \bar{\varphi}^{2n+1} < \sum_{i=0}^{k} \bar{\varphi}^{n_i+2} < \sum_{n \geqslant 1} \bar{\varphi}^{2n} = \frac{\bar{\varphi}^2}{1 - \bar{\varphi}^2} = -\bar{\varphi}.$$

It follows that

$$0 = \bar{\varphi} + \varphi - 1 < \varphi n + \varphi - 1 - \phi(n) < \bar{\varphi}^2 + \varphi - 1 = 1$$

Since $\phi(n)$ is an integer, this completes the proof of the first equality. The proof of the second one uses $\varphi^2 F_n - F_{n+2} = -\bar{\varphi}^{n+2}$ and follows the same lines. $\square$

We define the *linearity defect* δ of ϕ by $\delta(m, n) = \phi(m + n) - \phi(m) - \phi(n)$ for each non-negative integers m and n. It turns out that although ϕ is not linear, its linearity defect is small, as the following lemma shows.

Lemma 19. *For natural numbers m, n, we have $-1 \leqslant \delta(m, n) \leqslant 1$.*

Proof. We apply the relation $\phi(k) = \lfloor \varphi k + \varphi - 1 \rfloor$, obtained in Lemma 18, to $k = m + n$, $k = m$ and $k = n$:

$$\begin{gathered}
\varphi(m + n) + \varphi - 2 < \phi(m + n) < \varphi(m + n) + \varphi - 1, \\
1 - \varphi m - \varphi < -\phi(m) < 2 - \varphi m - \varphi \text{ and} \\
1 - \varphi n - \varphi < -\phi(n) < 2 - \varphi n - \varphi.
\end{gathered}$$

Adding these three relations gives $-\varphi < \phi(m+n) - \phi(m) - \phi(n) < 3 - \varphi$, and the statements follows as $1 < \varphi < 2$. $\square$

3.3. **Regularity of the linearity defect.** We will sometimes use $\bar{1}$ to denote -1, in particular when -1 is an element of an alphabet. Set $B := \{0, 1\}$ and $\bar{B} := \{\bar{1}, 0, 1\}$. Lemma 19 tells us that $\delta(m, n) \in \bar{B}$. Next, we obtain a more precise version of Lemma 19, as we will later need to use an automaton that, given $m > n$, computes $\delta(m - n, n)$.

The following theorem is proved in [Lot02, Proposition 7.3.11, Chapter 7]. Recall that a set of words K is *regular* if there is a deterministic automaton $\mathcal{B} = \langle S, B, \Delta, \{s_0\}, F \rangle$ such that $\Delta(s_0, w)$ is in an accepting state if and only if $w \in K$.

Theorem 20. *For any finite set $C \subset \mathbb{Z}$, the set $\{w \in C^* : [w]_Z = 0\}$ is regular.*

Let $u = u_m \cdots u_0$ and $v = v_n \cdots u_0$ be words in C^*. By padding with leading zeros, it can be assumed that $m = n$. We denote respectively by $u \boxplus v$ and $u \boxminus v$ the two words $(u_m + v_m) \cdots (u_0 + v_0)$ and $(u_m - v_m) \cdots (u_0 - v_0)$. Note that this definition is slightly ambiguous as padding u and v with more leading zeros yields a word with more leading zeros. However, it does not hurt as we are only interested in $[u \boxplus v]_Z$ and $[u \boxminus v]_Z$. Note that both operations are associative. It is easily verified that $[u \boxplus v]_Z = [u]_Z + [v]_Z$ and $[u \boxminus v]_Z = [u]_Z - [v]_Z$.

Note that equality $u0 \boxplus u0 = (u \boxplus v)0$ holds for words $u, v \in C^*$ and similarly for the operation $\boxminus$. The equality $[(m)_Z 0 \boxplus (n)_Z 0]_Z = \phi(m) + \phi(n)$ also holds for integers $m, n \geqslant 0$. Let $L = \{(m)_Z : m \geqslant 0\} \subset B^*$ be the set of canonical expansions.

Next we discuss the regularity of δ. For $b \in \bar{B}$, define

$$X_b := \{(m)_Z \boxminus (n)_Z : m \geqslant n \geqslant 0 \text{ and } \delta(m-n, n) = b\}.$$

Proposition 21. *The sets $X_{\bar{1}}$, X_0 and X_1 are pairwise disjoint and regular.*

The fact that the three sets $X_{\bar{1}}$, X_0 and X_1 are pairwise disjoint means that whenever $(m)_Z \boxminus (n)_Z = (m')_Z \boxminus (n')_Z$, then $\delta(m-n, n) = \delta(m'-n', n')$. We remark that using similar methods, we can also show that the sets $\{(m)_Z \boxplus (n)_Z : m, n \geqslant 0 \text{ and } \delta(m, n) = b\}$, for $b \in \bar{B}$, are also pairwise disjoint and regular. The proof of Proposition 21 is based on the following key lemma which gives a characterization of $\delta(m-n, n) = k$.

Lemma 22. *Let $b \in \bar{B}$. Then for $m \geqslant n \geqslant 0$*

$$\delta(m-n, n) = b \iff \exists w \in L \text{ such that } \begin{cases} [w \boxminus ((m)_Z \boxminus (n)_Z)]_Z = 0 \\ [w0 \boxminus ((m)_Z 0 \boxminus (n)_Z 0)]_Z = -b. \end{cases}$$

Proof. Suppose that $\delta(m-n, n) = b$ and let w be equal to $(m-n)_Z$. Then $[w \boxminus ((m)_Z \boxminus (n)_Z)]_Z = [w]_Z - m + n = 0$, and $[w0 \boxminus ((m)_Z 0 \boxminus (n)_Z 0)]_Z = [w0] - \phi(m) + \phi(n) = -\delta(m-n, n) = -b$.

Conversely, suppose that there exists $w \in L$ satisfying both required equalities. The first equality $[w \boxminus ((m)_Z \boxminus (n)_Z)]_Z = 0$ implies that $[w]_Z = m - n$ and $[w0]_Z = \phi(m-n)$. Combined with $w \in L$, it shows that $w = (m-n)_Z$. The second equality yields $[w0 \boxminus ((m)_Z 0 \boxminus (n)_Z 0)]_Z = [w0]_Z - \phi(m) + \phi(n) = -\delta(m-n, n)$. □

Now we come to the proof of Proposition 21.

Proof of Proposition 21. By Lemma 22, the value of $\delta(m-n, n)$ is determined by the word $(m)_Z \boxminus (n)_Z$. This shows that the three sets $X_{\bar{1}}$, X_0 and X_1 are pairwise disjoint. It remains to show that they are regular. We will prove that X_0 is regular, the proofs for $X_{\bar{1}}$ and X_1 being similar.

The construction of a non-deterministic automaton accepting X_0 is based on the statement of Lemma 22. Such an automaton reads $(m)_Z \boxminus (n)_Z$, and non-deterministically "guesses" the word $w \in L$ given by Lemma 22. In particular, it checks that both equalities $[w \boxminus ((m)_Z \boxminus (n)_Z)]_Z = 0$ and $[(w0 \boxminus ((m)_Z \boxminus (n)_Z))0] = 0$ hold. This means that each of its transitions reads a symbol $b' \in \bar{B}$ of $(m)_Z \boxminus (n)_Z$, and guesses a symbol $b \in B$ of w. Non-deterministic guessing is done by having two transitions reading a given symbol b', one making the guess $b = 0$ and another one making the guess $b = 1$. Note that $(m)_Z \boxminus (n)_Z$ has entries from $\bar{B}$ and w has entries from B, so $w \boxminus ((m)_Z \boxminus (n)_Z)$ has entries from $C := \{\bar{1}, 0, 1, 2\}$. In order to check both required equalities, the automaton simulates a deterministic automaton accepting $Y = \{u \in C^* : [u]_Z = 0\}$ for $C = \{\bar{1}, 0, 1, 2\}$. If the read symbol is b' and the guessed symbol for w is b, the automaton simulates the automaton for Y with entry $b - b'$. The automaton also stores in its states the last guessed b to check that there are no consecutive digits 1s in w.

We give below a more formal description of a non-deterministic automaton $\mathcal{B}$ accepting X_0. Let $C = \{\bar{1}, 0, 1, 2\}$ and let $\mathcal{C} = \langle Q, C, \Delta_C, \{q_0\}, F\rangle$ be a deterministic automaton accepting $\{u \in C^* : [u]_Z = 0\}$, whose existence is guaranteed by Theorem 20. The input alphabet of $\mathcal{B}$ is $\bar{B}$ since $\mathcal{B}$ is fed with words of the form $(m)_Z \boxminus (n)_Z$ for $m, n \geqslant 0$. The state set of $\mathcal{B}$ is the set $Q \times \{0, 1\}$ and its unique initial state is $(q_0, 0)$. Its transition set Δ is defined as follows.

$$\begin{aligned} \Delta := &\{(p,0) \xrightarrow{b'} (q,0) : \Delta_C(p, -b') = q\} \\ &\cup \{(p,1) \xrightarrow{b'} (q,0) : \Delta_C(p, -b') = q\} \\ &\cup \{(p,0) \xrightarrow{b'} (q,1) : \Delta_C(p, 1 - b') = q\} \end{aligned}$$

Each transition reads a symbol b' of the input word $(m)_Z \boxminus (n)_Z$ and guesses a symbol $b \in B$ for the word w (see Lemma 22). The corresponding symbol of $w \boxminus ((m)_Z \boxminus (n)_Z)$ is thus $b - b'$. The second component of each state is equal to the last guessed b to check that w does not contain two consecutive 1s. The set of final states is

$$F' := \{(q, x) : q \in F \text{ and } \exists q' \in F \text{ with } (q, x) \xrightarrow{0} (q', 0)\},$$

so that both statements are verified. The first condition $q \in F$ guarantees the first equality $[w \boxminus ((m)_Z \boxminus (n)_Z)]_Z = 0$. The second condition guarantees the second equality $[(w \boxminus ((m)_Z \boxminus (n)_Z))0] = 0$. □

Corollary 23. *There exists a deterministic automaton $\mathcal{D} = (Q, \bar{B}, \Gamma, q_0, \tau)$ with an output function $\tau : Q \to \bar{B}$ such that $\Gamma(q_0, 0) = q_0$ and $\tau(\Gamma(q_0, (m)_Z \boxminus (n)_Z)) = \delta(m - n, n)$ for all $m \geqslant n \geqslant 0$.*

Proof. It suffices to combine the three deterministic automata for $X_{\bar{1}}$, X_0 and X_1 obtained in Proposition 21. □

It should be noted that the set $\{(m)_Z \boxminus (n)_Z : m \geqslant n \geqslant 0\}$ is strictly contained in $\bar{B}^*$. Indeed, it only contains words with no consecutive occurrences of either the digit 1 or the digit $\bar{1}$. This gives us some freedom in the construction of an automaton $\mathcal{D}$ satisfying the statement of the previous corollary, and it explains why in $\mathcal{D}$, one cannot follow all possible words. We give in Figure 5 an automaton with 5 states computing the defect $\delta(m - n, n)$; the value of the output function τ appears inside the states.

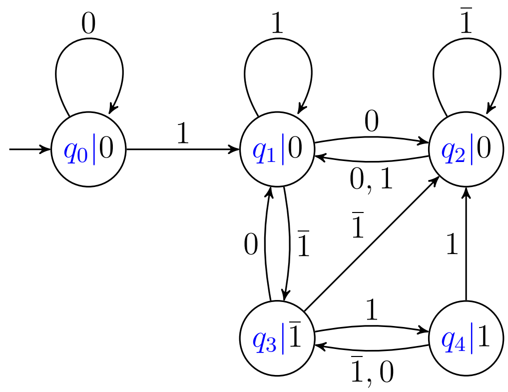


FIGURE 5. An automaton computing $\delta(m-n,n)$, given $(m)_Z \boxminus (n)_Z$.

3.4. **Z-Mahler equations and weighted automata.** The following definition of the operator Φ is exactly analogous to that for base-q defined in Section 1, but it is based on the function ϕ rather than the function $n \mapsto qn$. Let R be a commutative ring. The Z-Mahler operator $\Phi : R[\![x]\!] \to R[\![x]\!]$ is defined as follows.

$$\Phi\Big(\sum_{n\geqslant 0} f_n x^n\Big) := \sum_{n\geqslant 0} f_n x^{\phi(n)}. \tag{8}$$

Note that, unlike the q-Mahler case, we do not have $\Phi(fg) = \Phi(f)\Phi(g)$ in general, as the function ϕ is not linear. Nevertheless, as ϕ is linear over integers with disjoint support, we have that $\Phi(fg) = \Phi(f)\Phi(g)$ if for each pair of non-zero coefficients $f_m \neq 0$ and $g_n \neq 0$, m and n have disjoint support.

The equation

$$P(x,y) = \sum_{i=0}^{d} A_i(x)\Phi^i(y) = 0, \tag{9}$$

with A_i defined as in (5), is called a *Z-Mahler equation*, and if $f \in R[\![x]\!]$ satisfies $\sum_{i=0}^{d} A_i(x)\Phi^i(f) = 0$, then it is called *Z-Mahler*; we also say that f is a solution of the functional equation P. As in the case of standard Mahler equations, d is the *exponent* of the equation P, and the maximum degree h of the polynomials $A_0(x), \dots, A_d(x)$ the *height* of P. Also, in analogy to (6), a solution $y = \sum_{n\geqslant 0} f_n x^n$ to (9) satisfies

$$f_n = \sum_{\phi^i(k)+j=n} \alpha_{i,j} f_k\,. \tag{10}$$

and where here also we set $\alpha_{i,j} = 0$ for i, j outside the bounds given by the equation.

For example, the polynomial $f(x) = 1 + x$ is the solution of the Z-Mahler equation $(1+x^2)f(x) = (1+x)\Phi(f(x))$ because $\Phi(f(x)) = 1 + x^2$. The following is a slightly less trivial example, inspired by [Bec94, Proposition 1].

Example 24. We return to Example 17, whose automaton computes the number a_n of representations of n as a sum of distinct Fibonacci numbers. Consider the series $f(x) := \sum_{n\geqslant 0} a_n x^n$; we also have $f(x) := \prod_{n\geqslant 0}(1+x^{F_n})$. This series $f(x)$ is the solution of the equation $f(x) = (1+x)\Phi(f(x))$. It can be indeed verified that $\Phi(\prod_{n\geqslant 0}(1+z^{F_n})) = \prod_{n\geqslant 0}(1+z^{F_{n+1}}) = \prod_{n\geqslant 1}(1+z^{F_n})$ because the terms of the product have disjoint exponents.

It is not a coincidence that the solution in Example 24 can be computed using a weighted automaton, as we will see next.

3.4.1. *From isolating Z-Mahler equations to weighted automata.* As in the case of q-Mahler equations, a Z-Mahler equation $P(x,y)$ is *isolating* if $A_0 = 1$, i.e., $P(x,y) = y - \sum_{i=1}^{d} A_i(x)\Phi^i(y)$. The aim of this section is to show that any solution of an isolating Z-Mahler equation is Z-regular, i.e., if $f(x) = \sum_{n\geqslant 0} f_n x^n$ is the solution of P with initial condition f_0, then there exists a weighted automaton $\mathcal{A}$ such that $f_n = \mathrm{weight}_{\mathcal{A}}((n)_Z)$ for each integer $n \geqslant 0$.

The construction of the weighted automaton is similar to the one we gave in Section 2.4 for q-Mahler equations, but it is more involved due to the non-linearity of the function ϕ.

Now we describe the weighted automaton $\mathcal{A}$ computing the solution of an isolating Z-Mahler equation. Recall that $B = \{0,1\}$, $\bar{B} = \{\bar{1},0,1\}$ and $C = \{\bar{1},0,1,2\}$. Let $\mathcal{D} = (Q, \bar{B}, \Gamma, \{q_0\}, \tau)$ be a deterministic automaton given by Corollary 23, with $\tau : Q \to \bar{B}$ such that $\tau(\Gamma(q_0, (m)_Z \boxminus (n)_Z)) = \delta(m-n, n)$ for all integers $m \geqslant n \geqslant 0$; it can be assumed that $\Gamma(q_0, 0) = q_0$.

We now come to the definition of the weighted automaton associated to a Z-Mahler equation. Let $P(x,y) = \sum_{i=0}^{d} A_i(x)\Phi^i(y) = 0$ be an isolating Z-Mahler equation with $A_i(x) = \sum_{j=0}^{h} \alpha_{i,j} x^j$ for $1 \leqslant i \leqslant d$. Set

$$\tilde{h} := \left\lfloor \frac{h+3-\varphi}{\varphi-1} \right\rfloor \quad \text{and} \quad g := |(\tilde{h})_Z|.$$

We define the state set S as

$$S := \{s_{i,j,q,u} : 0 \leqslant i \leqslant d, 0 \leqslant j \leqslant \tilde{h}, q \in Q \text{ and } u \in B^g\}.$$

Define the function $\hat{\delta} : S \to \bar{B}$ by

$$\hat{\delta}(s_{i,j,q,u}) = \tau(\Gamma(q, u \boxminus (j)_Z)).$$

Note that if $q \neq q_0$, the automaton in Figure 5 can be fed with any word in $\bar{B}^*$, i.e., $\tau(\Gamma(q, u \boxminus (j)_Z))$ is well defined. If $q = q_0$, we will see in the second statement of Lemma 27 that then we will only be concerned with states $s_{i,j,q_0,u}$ where $[u]_Z \geqslant (j)_Z$ in which case $\hat{\delta}$

is also well defined. Define the transition set Δ as follows.

$$\Delta := \left\{ s_{i,j,q,au} \xrightarrow{b:1} s_{i+1,\ell,\Gamma(q,a),ub} : \begin{array}{c} 0 \leqslant i \leqslant d-1, \\ 0 \leqslant j \leqslant \tilde{h} \\ 0 \leqslant \ell = \phi(j) + \hat{\delta}(s_{i,j,q,au}) + b \leqslant \tilde{h} \\ a, b \in B, u \in B^{g-1} \end{array} \right\}$$
$$\cup \left\{ s_{i,j,q,au} \xrightarrow{b:\alpha_{i+1,\ell-k}} s_{0,k,\Gamma(q,a),ub} : \begin{array}{c} 0 \leqslant i \leqslant d-1 \\ 0 \leqslant j,k \leqslant \tilde{h} \\ 0 \leqslant \ell = \phi(j) + \hat{\delta}(s_{i,j,q,au}) + b \\ 0 \leqslant \ell - k \leqslant h \\ a, b \in B, u \in B^{g-1} \end{array} \right\}.$$

We set the initial and final weights I and F as

$$I(s_{i,j,q,u}) := \begin{cases} f_0 & \text{if } j = 0,\ q = q_0 \text{ and } u = 0^g, \\ 0 & \text{otherwise.} \end{cases}$$
$$F(s_{i,j,q,u}) := \begin{cases} 1 & \text{if } i = 0 \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We call the automaton $\mathcal{A} := \langle S, A, \Delta, I, F \rangle$ the *weighted automaton associated to* P *and* f_0, $\mathcal{A} = \mathcal{A}(P, f_0)$.

The first two components i and j of a state $s_{i,j,q,u}$ in $\mathcal{A}$ play the same role as in the base-q case. We will show how the last two components are needed to track the linearity defect. The following lemma describes the evolution of the third and fourth components of the states along a path.

Lemma 25. *Let* $s_{i',j',q',u'} \xrightarrow{w:r} s_{i,j,q,u}$ *be a path in the automaton* $\mathcal{A}(P, f_0)$*. Then* u *is the suffix of length* g *of* $u'w$ *and* $q = \Gamma(q', v)$ *where* $vu = u'w$*, that is,* v *is the prefix of length* $|w|$ *of* $u'w$*.*

Proof. The proof is a straightforward induction on the length of w. The base case $|w| = 0$ is trivial and the induction step follows directly from the definition of the transition set Δ. □

Rephrasing Lemma 25 when q' is the initial state q_0 of $\mathcal{D}$ and $u' = 0^g$ gives Corollary 26. In particular, it tells us that if $s_{i,j,q,u}$ is accessible from a state with non-zero initial weight, then the word u must be a suffix of the input word w. This implies that u cannot have consecutive occurrences of the digit 1. We have not taken this into account, in the definition of S, in order to simplify the definition of $\mathcal{A}$. However this fact is used to bound the number of states of $\mathcal{A}$ in Lemma 32.

Corollary 26. *Let* $s_{i',j',q_0,0^g} \xrightarrow{w:r} s_{i,j,q,u}$ *be a path in the automaton* $\mathcal{A}(P, f_0)$*. Then* u *is the suffix of length* g *of* $0^g w$ *and* $q = \Gamma(q_0, v)$ *where* $vu = 0^g w$*, that is,* v *is the prefix of length* $|w|$ *of* $0^g w$*.*

The following lemma states that the function $\hat{\delta}$ indeed tracks the linearity defect, starting from the initial state on input of any word w, as a function of the state where it arrives. Note that the proof of this lemma also justifies the definition of g.

Lemma 27. *Let $s_{i',0,q_0,0^g} \xrightarrow{w:r} s_{i,j,q,u}$ be a path in the automaton $\mathcal{A}(P, f_0)$.*

a) *There exists $k \geqslant 0$ such that $\phi^i(k) + j = [w]_Z$ and thus $j \leqslant [w]_Z$.*

b) *$\hat{\delta}(s_{i,j,q,u}) = \tau(\Gamma(q, u \boxminus (j)_Z))$ is well-defined and $\hat{\delta}(s_{i,j,q,u}) = \delta([w]_Z - j, j)$.*

Proof. The two statements are simultaneously proved by induction on the length of the word w. If the word w is empty, then $j = 0$ and the two statements trivially hold because $\phi^i(0) = 0$ for each $i \geqslant 0$.

Now we suppose that $w = w'b$ where b is the last digit of w. The path $s_{i'',0,q_0,0^g} \xrightarrow{w:r} s_{i,j,q,u}$ can then be decomposed

$$s_{i'',0,q_0,0^g} \xrightarrow{w':r'} s_{i',j',q',u'} \xrightarrow{b:r''} s_{i,j,q,u}.$$

For the first statement, we distinguish two cases depending on the form of the last transition $s_{i',j',q',u'} \xrightarrow{b:r''} s_{i,j,q,u}$.

We first suppose that this last transition of the path is a transition of the form $s_{i',j',q',u'} \xrightarrow{b:1} s_{i,j,q,u}$ where $i = i' + 1$ and $j = \phi(j') + \hat{\delta}(s_{i',j',q',u'}) + b$. By the induction hypothesis, $\hat{\delta}(s_{i',j',q',u'}) = \delta([w']_Z - j', j')$ and there exists an integer $k \geqslant 0$ such that $\phi^{i'}(k) + j' = [w']$. Applying ϕ to the equality $\phi^{i'}(k) = [w']_Z - j'$ yields $\phi^i(k) = \phi([w']_Z - j') = \phi([w']_Z) - \phi(j') - \delta([w']_Z - j', j') = \phi([w']_Z) + b - j = [w]_Z - j$.

Next we suppose that this last transition of the path is a transition of the form $s_{i',j',q',u'} \xrightarrow{b:\alpha_{i'+1,\ell-j}} s_{0,j,q,u}$ where $\ell = \phi(j') + \hat{\delta}(s_{i',j',q',u'}) + b$. By definition, we have $j \leqslant \ell$ and $\ell = \phi([w']_Z) - \phi([w']_Z - j') + b$, and thus $j \leqslant \ell \leqslant \phi([w']_Z) + b = [w]_Z$. Since $i = 0$, $\phi^i(k) + j = [w]_Z$ where $k = [w]_Z - j$. This completes the proof of the first statement.

The only missing transition in the automaton pictured in Figure 5 is the transition with label $\bar{1}$ leaving state q_0. If the state q in $s_{i,j,q,u}$ is equal to q_0, then $[u]_Z = [w]_Z$. Therefore $j \leqslant [u]_Z$ and the most significant digit of $u \boxminus (j)_Z$ must be 1. This shows that $\hat{\delta}(s_{i,j,q,u}) = \tau(\Gamma(q, u \boxminus (j)_Z))$ is always well-defined.

Let m be the length of $w = w'b$. By Corollary 26, one has $q = \Gamma(q_0, v)$ where $vu = 0^g w$ and $|v| = m$. This can be rewritten $q = \Gamma(q_0, v \boxminus 0^m)$. The function $\hat{\delta}$ is defined by $\hat{\delta}(s_{i,j,q,u}) = \tau(\Gamma(q, u \boxminus (j)_Z))$. Since $|(j)_Z| \leqslant g$, $\Gamma(q, u \boxminus (j)_Z) = \Gamma(q_0, vu \boxminus 0^m 0^{g-|(j)_Z|}(j)_Z) = \Gamma(q_0, w \boxminus 0^{m-|(j)_Z|}(j)_Z)$. It follows that $\hat{\delta}(s_{i,j,q,u}) = \delta([w] - j, j)$. □

The following lemma is used in the proof of Proposition 29 below.

Lemma 28. *Let $s_{i'',0,q_0,0^g} \xrightarrow{w':r'} s_{i',j',q',au} \xrightarrow{b:r} s_{i,j,q,ub}$ be a path in the automaton $\mathcal{A}(P, f_0)$ where w' is a word and b is a digit. If $i \geqslant 1$ then the state $s_{i',j',q',au}$ is unique and does not depend on i''.*

Proof. By Corollary 26, au must be the suffix of length g of $0^g w'$ and the state q' is given by $q' = \Gamma(q_0, v)$ where $vau = 0^g w'$. Since $i \geqslant 1$, the last transition must be transition of the form $s_{i',j',q',au} \xrightarrow{b:1} s_{i,j,q,ub}$ where $i = i' + 1$ and $j = \phi(j') + \hat{\delta}(s_{i',j',q',u'}) + b$. This shows that $i' = i-1$ and that $r = 1$. It remains to show that j is also unique. By Lemma 27, $\hat{\delta}(s_{i',j',q',u'})$ is equal to $\delta([w']_Z - j', j')$ and the equality $j = \phi(j') + \hat{\delta}(s_{i',j',q',u'}) + b$ can be rewritten $\phi([w']_Z - j') = \phi([w']_Z) + b - j$. Since the function ϕ is one-to-one, there is at most one integer j' satisfying this equality. Note that this proof does not depend on i'', so the statement follows. ☐

We bring this together now to obtain the following version of Proposition 14.

Proposition 29. *Let R be a commutative ring, let $P(x,y) \in R[x,y]$ be an isolating Z-Mahler equation of exponent d and height h, and let $f = \sum_{n\geqslant 0} f_n x^n$ be a solution of $P = 0$. If $\mathcal{A}(P, f_0)$ is the weighted automaton associated to $P(x,y)$ and $f_0 \in R$, and if $w \in \{0,1\}^*$ has no consecutive occurrences of the digit 1, then*

$$\mathrm{weight}^*_{\mathcal{A},s_{i,j,q,u}}(w) = \begin{cases} f_k \ \textit{whenever} & \begin{cases} \phi^i(k) + j = [w]_Z, \\ q = \Gamma(q_0, w), \ \textit{and} \\ u \textit{ is the suffix of } 0^g w \end{cases} \\ 0 & \textit{otherwise.} \end{cases}$$

Note that i, j and w being given, there exists at most one integer k that satisfies the equation $\phi^i(k) + j = [w]_Z$, as the function ϕ is one-to-one. This means that the integer k implicitly given by $\phi^i(k) + j = [w]_Z$ is well-defined.

Proof. The proof is by induction on the length of the word w. If w is the empty word ε, then $\mathrm{weight}^*_{\mathcal{A},s_{i,j,q,u}}(\varepsilon) = I(s_{i,j,q,u})$; this equals f_0 if $j = 0$, $q = q_0$ and $u = 0^g$, and equals 0 otherwise, so the claim is proved for $w = \varepsilon$ since $[\varepsilon]_Z = 0$.

For $|w| > 0$, we consider first the easier case of a state $s_{i,j,q,ub}$ with $i \geqslant 1$. We are only concerned with states, and transitions into these states that are part of a path starting at an initial state. By Lemma 28, there is a unique such transition $s_{i_0,0,q_0,0^g} \xrightarrow{w:r} s_{i-1,j',q',au} \xrightarrow{b:1} s_{i,j,q,ub}$. Here $j = \phi(j') + \hat{\delta}(s_{i-1,j',q',au}) + b = \phi(j') + b + \delta([w]_Z - j', j')$, by Lemma 27. Then

$$\mathrm{weight}^*_{\mathcal{A},s_{i,j,q,ub}}(wb) = \mathrm{weight}^*_{\mathcal{A},s_{i-1,j',q',au}}(w) = f_k,$$

where $\phi^{i-1}(k) + j' = [w]_Z$. Applying ϕ to this last equality, we obtain

$$\begin{aligned} \phi^i(k) &= \phi([w]_Z - j') \\ &= \phi([w]_Z) - \phi(j') - \delta([w]_Z - j', j') \\ &= [wb]_Z - b - \phi(j') - \delta([w]_Z - j', j') \\ &= [wb]_Z - j, \end{aligned}$$

from which the statement of the proposition follows for the case $i > 0$.

Now we consider the case case $i = 0$, so that our state is $s_{0,j,q,ub}$. As the transitions that concern us are of the form $s_{i',j',q',au} \xrightarrow{b:\alpha_{i'+1,\ell'-j}} s_{0,j,q,ub}$, henceforth we will implicitly only sum over states $s_{i',j',q',au}$ which satisfy this. Also, as we are only concerned with paths that commence at an initial state, we have by Lemma 27 that $\ell' = \phi(j') + \delta([w]_Z - j', j') + b$.

Thus

$$\begin{aligned}\mathrm{weight}^*_{\mathcal{A},s_{0,j,q,ub}}(wb) &= \sum_{i',j',q',a} \alpha_{i'+1,\ell'-j}\,\mathrm{weight}^*_{\mathcal{A},s_{i',j',q',au}}(w)\\ &= \sum_{\phi^{i'}(k)+j'=[w]_Z} \alpha_{i'+1,\ell'-j} f_k\end{aligned}$$

where to obtain the last equality we have applied the inductive hypothesis. Since ϕ is one-to-one, the equality $\phi^{i'}(k) = [w]_Z - j'$ is equivalent to the equality

$$\begin{aligned}\phi^{i'+1}(k) = \phi([w]_Z - j') &= \phi([w]_Z) - \phi(j') - \delta([w]_Z - j', j')\\ &= \phi([w]_Z) - \ell' + b.\end{aligned}$$

Setting $\ell = \ell' - j$, we obtain $\phi^{i'+1}(k) + \ell = [wb] - j$ and thus

$$\begin{aligned}\mathrm{weight}^*_{\mathcal{A},s_{0,j,q,ub}}(wb) &= \sum_{\phi^{i'+1}(k)+\ell=[wb]-j} \alpha_{i'+1,\ell} f_k\\ &= \sum_{\phi^{i}(k)+\ell=[wb]-j} \alpha_{i,\ell} f_k\\ &= f_{[wb]_Z-j}\end{aligned}$$

where in the penultimate line we set $i = i' + 1$, and where we used Equation 10 to get to the last line. □

The following theorem states that the automaton $\mathcal{A}(P, f_0)$ defined above indeed computes the unique solution of the equation $P(x, f(x))$ satisfying $f(0) = f_0$.

Theorem 30. *Let R be a commutative ring, and let $P(x, y) \in R[x, y]$ be an isolating Z-Mahler equation of exponent d and height h. Then the automaton $\mathcal{A} = \mathcal{A}(P, f_0)$ associated to P and f_0 satisfies*

$$\mathrm{weight}_{\mathcal{A}}(w) = f_{[w]_Z}$$

where $f = \sum_{n\geqslant 0} f_n x^n$ is a solution of $P(x, f(x)) = 0$. Consequently if R is finite, then there exists a deterministic automaton, and a constant C, depending only on φ, with at most $|R|^{Cdh^2}$ states that generates $f(x)$.

In the classical q-numeration, if P is an isolating Ore polynomial over $\mathbb{F}_q$ of degree q^d and height h, then a minimal deterministic automaton generating a solution of P will have at most $|\mathbb{F}_q|^{dh}$ states. This should be compared to the bound above, where the exponent $Cdh^2 = 320dh^2$ has an extra factor of $320h$, with 320 being a function of φ (see Lemma 32). This extra factor arises because we need to carry extra information, in the form of a word of length g, which is used to compute the linearity defect.

Proof. The proof of the theorem follows directly from Proposition 29. The states with a non-zero final weight are the states of the form $s_{0,0,q,u}$ whose final weight is given by $F(s_{0,0,q,u}) = 1$. For each input word w, that is, with no consecutive occurrences of the digit 1, there exists exactly one state $s_{0,0,q,u}$ where $\mathrm{weight}^*_{\mathcal{A},s_{0,0,q,u}}(w)$ is non-zero. This unique state is the state $s_{0,0,q,u}$ where u is the suffix of length g of $0^g w$ and q is given by $q = \Gamma(q_0, w)$. Therefore $\mathrm{weight}_{\mathcal{A}}(w)$ is equal to $\mathrm{weight}^*_{\mathcal{A},s_{0,0,q,u}}(w)$ where u and q satisfy the required properties. Therefore $\mathrm{weight}_{\mathcal{A}}(w) = f_{[w]_Z}$. The bound on the number of states follows from Lemmas 31 and 32 below. □

The following lemma is the analog of Lemma 10. It justifies the choice $\tilde{h} = \frac{h+3-\varphi}{\varphi-1}$, as this is an upper bound on indices for states that are the range of paths of positive weight.

Lemma 31. *If $j \geqslant \frac{h+3-\varphi}{\varphi-1} = h\varphi + 3\varphi - \varphi^2$ then there are no paths with positive weight from $s_{i,j,q,u}$ to $s_{0,0,q'u'}$.*

Proof. Let $s_{i,j,q,u} \xrightarrow{b:\alpha} s_{i',j',q,u}$ be a transition in the weighted automaton $\mathcal{A}$ associated to $(\alpha_{i,j})_{i\geqslant 1, j\geqslant 0}$ where $\alpha_{i,j} = 0$ if $j \geqslant h$. We claim that if $\alpha \neq 0$ and $j \geqslant \frac{h+3-\varphi}{\varphi-1}$ then $j' \geqslant \frac{h+3-\varphi}{\varphi-1}$. If $s_{i,j,q,u} \xrightarrow{b:\alpha} s_{i',j',q',u'}$ is a transition of the form $s_{i,j,q,u} \xrightarrow{b:1} s_{i+1,\ell,q',u'}$ then $\ell = \phi(j) + \hat{\delta}(s_{i,j,q,u}) + b$ and the claim is clear. Indeed, if $j = 0$, then $\hat{\delta}(s_{i,j,q,u}) = 0$ and if $j > 0$, then $\phi(j) \geqslant j+1$. If it is a transition of the form $s_{i,j} \xrightarrow{b:\alpha_{i+1,\ell-j'}} s_{0,j'}$ where $\ell = \phi(j) + \hat{\delta}(s_{i,j,q,u}) + b$, then, since $\alpha_{i+1,\ell-j'}$ is assumed non-zero, we have $\phi(j) + \hat{\delta}(s_{i,j,q,u}) + b - j' \leqslant h$, and so

$$j' \geqslant \phi(j) - 1 - h \geqslant j\varphi + \varphi - 3 - h \geqslant \frac{h+3-\varphi}{\varphi - 1}$$

The statement of the lemma follows. □

The following lemma provides an upper bound of the number of states of $\mathcal{A}$.

Lemma 32. *Let $\mathcal{A}$ be the weighted automaton associated to a Z-Mahler equation of height h and exponent d. Then the number of states $|S|$ of $\mathcal{A}$ is bounded by $320dh^2$ and asymptotically, $|S| \leqslant 5\varphi^4 dh^2(1 + o(1))$ as h tends to ∞.*

Proof. Recall that one can take the automaton $\mathcal{D}$, that computes $\delta(m-n, n)$, to have at most 5 states. Now $\tilde{h} = \lfloor \frac{h+3-\varphi}{\varphi-1} \rfloor \leqslant 4h$ and asymptotically, as $h \to \infty$, we have $\tilde{h} = \varphi h(1 + o(1))$. The number of words of length g with no consecutive occurrences of 1 is equal to the Fibonacci number F_{g+1}. Since $F_{n+1} \leqslant 2F_n$ for each $n \geqslant 0$ and $F_{n+1} = \varphi F_n(1 + o(1))$ and because $F_{g-1} \leqslant \tilde{h}$ by definition of g, then $F_{g+1} \leqslant 4\tilde{h}$ and asymptotically, F_{g+1} is bounded by $\varphi^2 \tilde{h}(1 + o(1))$. The number of states is thus bounded by $5 \cdot 4\tilde{h} \cdot d\tilde{h} = 20d\tilde{h}^2 \leqslant 320dh^2$ and asymptotically, is bounded by $5\varphi^4 dh^2(1 + o(1))$. □

3.4.2. *From weighted automata to Z-Mahler equations.* This section is almost standard, but we include it for completeness. We first redo Example 9, considering again the automaton of Figure 1, except that here we use it to generate the term $a_n := \mathrm{weight}_{\mathcal{A}}((n)_Z)$. The weights of this automaton are in the field $\mathbb{F}_2$ but the computation of the equation satisfied by $t(x) = \sum_{n\geqslant 0} a_n x^n$ can be carried out in any ring R. Let $t(x) = \sum_{n\geqslant 0} a_n x^n$, and $s = s(x)$ in $R[\![x]\!]$ be defined similarly as in Example 9. Instead of (3), we have

$$t = \Phi(t) + x\Phi^2(t) + x\Phi^2(s) \quad \text{and} \quad s = \Phi(s) + x\Phi^2(s). \tag{11}$$

Here, we need to shift twice to add 1 without carry. Applying Φ to the equations (11), we have

$$\begin{aligned}\Phi(t) &= \Phi^2(t) + \Phi(x\Phi^2(t)) + \Phi(x\Phi^2(s))\\ &= \Phi^2(t) + x^2\Phi^3(t) + x^2\Phi^3(s),\end{aligned}$$

as x and $\Phi^2(t)$ or $\Phi^2(s)$ have disjoint support, and similarly,

$$\begin{aligned}\Phi(s) &= \Phi^2(s) + x^2\Phi^3(s)\\ \Phi^2(t) &= \Phi^3(t) + x^3\Phi^4(t) + x^3\Phi^4(s), \text{ and}\\ \Phi^2(s) &= \Phi^3(s) + x^3\Phi^4(s).\end{aligned}$$

Re-arranging, we obtain

$$\begin{aligned}t &= (1+x+x^2)\Phi^3(t) + (x^3+x^4)\Phi^4(t) + (x+x^2)\Phi^3(s) + (x^3+2x^4)\Phi^4(s)\\ \Phi(t) &= (1+x^2)\Phi^3(t) + x^3\Phi^4(t) + x^2\Phi^3(s) + x^3\Phi^4(s)\end{aligned}$$

from which we obtain

$$xt - (1+x)\Phi(t) + (1-2x^2)\Phi^2(t) + 2x^2\Phi^3(t) + x^5\Phi^4(t) = 0$$

i.e., the Z-Thue-Morse power series is a solution of the isolating Mahler Z-equation $P(x,y) = xy - (1+x)\Phi(y) + (1-2x^2)\Phi^2(y) + 2x^2\Phi^3(y) + x^5\Phi^4(y)$.

Lemma 33. *Let $F = \{s_1, \dots, s_m\}$ be a family of formal power series such that there exist two families of coefficients $(\alpha_{i,j})_{1\leqslant i,j\leqslant m}$ and $(\beta_{i,j})_{1\leqslant i,j\leqslant m}$ satisfying for each $1 \leqslant i \leqslant m$,*

$$s_i = \sum_{j=1}^{m} \alpha_{i,j}\Phi(s_j) + x\sum_{j=1}^{m} \beta_{i,j}\Phi^2(s_j).$$

Then for each pair of integers k, n with $0 \leqslant k \leqslant n$, there exist two families of polynomials $(p_{i,j}^{k,n})_{1\leqslant i,j\leqslant m}$ and $(q_{i,j}^{k,n})_{1\leqslant i,j\leqslant m}$ such that for each $1 \leqslant i \leqslant m$,

$$\Phi^k(s_i) = \sum_{j=1}^{m} p_{i,j}^{k,n}\Phi^n(s_j) + \sum_{j=1}^{m} q_{i,j}^{k,n}\Phi^{n+1}(s_j)$$

Proof. Note that the hypothesis is the case $k = 0$ and $n = 1$ of the statement. The proof is by induction on the difference $n - k$. The case $n = k$ is trivial and gives

$$p_{i,j}^{k,k} := \begin{cases} 1 & \text{if } i = j\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad q_{i,j}^{k,k} := 0$$

For $n = k + 1$, we start from the hypothesis

$$s_i = \sum_{j=1}^{m} \alpha_{i,j}\Phi(s_j) + x\sum_{j=1}^{m} \beta_{i,j}\Phi^2(s_j).$$

to get the following equality by applying Φ^k to both members.

$$\Phi^k(s_i) = \sum_{j=1}^{m} \alpha_{i,j}\Phi^{k+1}(s_j) + \Phi^k(x)\sum_{j=1}^{m} \beta_{i,j}\Phi^{k+2}(s_j),$$

where we note that the polynomial x and the series $\Phi^2(s_j)$ are indeed disjoint and that $\Phi^k(x\Phi^2(s_j)) = x^{\phi^k(1)}\Phi^{k+2}(s_j) = x^{F_k}\Phi^{k+2}(s_j)$. It follows that

$$p_{i,j}^{k,k+1} := \alpha_{i,j} \qquad \text{and} \qquad q_{i,j}^{k,k+1} := \beta_{i,j}x^{F_k}.$$

Now we suppose $k+2 \leqslant n$, and assume that the induction hypothesis holds for $n-(k+1)$ and $n-(k+2)$; we show the required statement holds for $n-k$. From the induction hypothesis applied to $\Phi^{k+1}(s_\ell)$ and $\Phi^{k+2}(s_\ell)$, we have the following equalities:

$$\Phi^{k+1}(s_\ell) = \sum_{j=1}^{m} p_{\ell,j}^{k+1,n}\Phi^n(s_j) + \sum_{j=1}^{m} q_{\ell,j}^{k+1,n}\Phi^{n+1}(s_j)$$
$$\Phi^{k+2}(s_\ell) = \sum_{j=1}^{m} p_{\ell,j}^{k+2,n}\Phi^n(s_j) + \sum_{j=1}^{m} q_{\ell,j}^{k+2,n}\Phi^{n+1}(s_j)$$

Combining these two equalities with the equality

$$\Phi^k(s_i) = \sum_{\ell=1}^{m} \alpha_{i,\ell}\Phi^{k+1}(s_\ell) + x^{F_k}\sum_{\ell=1}^{m} \beta_{i,\ell}\Phi^{k+2}(s_\ell)$$

we get the following equalities, defining the required polynomials $p_{i,j}^{k,n}$ and $q_{i,j}^{k,n}$ by

$$p_{i,j}^{k,n} := \sum_{\ell=1}^{m} \alpha_{i,\ell}p_{\ell,j}^{k+1,n} + x^{F_k}\sum_{\ell=1}^{m} \beta_{i,\ell}p_{\ell,j}^{k+2,n}$$
$$q_{i,j}^{k,n} := \sum_{\ell=1}^{m} \alpha_{i,\ell}q_{\ell,j}^{k+1,n} + x^{F_k}\sum_{\ell=1}^{m} \beta_{i,\ell}q_{\ell,j}^{k+2,n}.$$

□

Corollary 34. *A Z-regular series is the solution of a Z-Mahler equation.*

Proof. Suppose that the series $f = \sum_{n\geqslant 0} f_n x^n$ is computed by the weighted automaton $\mathcal{A}$, that is $f_n = \mathrm{weight}_{\mathcal{A}}((n)_Z)$ for each $n \geqslant 0$. Let $\langle I, \mu, F\rangle$ be a matrix representation of dimension m of the weighted automaton $\mathcal{A}$ as given in Section 2.2.1. For $1 \leqslant i \leqslant m$, let s_i be the series computed by the weighted automaton whose matrix representation is $\langle I, \mu, G_i\rangle$ where G_i is the vector having 1 in its i-th coordinate and 0 in all other coordinates. Note that f is equal to $f = \sum_{i=1}^m F_i s_i$ where F_i is the i-th entry of the column vector F. Let $s_0 = 1$ be the constant series which is the solution of the equation $s_0 = \Phi(s_0)$. Let the two families of coefficients $(\alpha_{i,j})_{1\leqslant i,j\leqslant m}$ and $(\beta_{i,j})_{1\leqslant i,j\leqslant m}$ be defined by $\alpha_{i,j} := \mu(0)_{j,i}$

and $\beta_{i,j} := \mu(01)_{j,i} = (\mu(0)\mu(1))_{j,i}$. Note the inversion of the indices i and j. For each $1 \leqslant i \leqslant m$ the series s_i satisfies

$$s_i = \sum_{j=1}^{m} \alpha_{i,j}\Phi(s_j) + x\sum_{j=1}^{m} \beta_{i,j}\Phi^2(s_j) + (I_i - \sum_{j \neq i} \alpha_{i,j} I_j)\Phi(s_0)$$

These equations come from the fact that the Zeckendorf representation of a positive integer ends either with 0 or 01. The last term of the right hand side deals with the values $s_i(0)$. Said differently, each non-negative integer is either of the form $\phi(k)$ or $\phi^2(k) + 1$ for some $k \geqslant 0$. Let n be the integer $2m + 1$. By Lemma 33, the series $\Phi^k(s_i)$ for $0 \leqslant k \leqslant n + 1 = 2m + 2$ and $1 \leqslant i \leqslant m$ are linear combinations of the $2m + 2$ series $\Phi^n(s_i)$ and $\Phi^{n+1}(s_i)$ for $0 \leqslant i \leqslant m$. It follows that the $2m + 3$ series $\Phi^k(f)$ for $0 \leqslant k \leqslant 2m + 2$ are also linear combinations of the $2m + 2$ series $\Phi^n(s_i)$ and $\Phi^{n+1}(s_i)$ for $0 \leqslant i \leqslant m$ and that the series $\Phi^k(f)$ for $0 \leqslant k \leqslant 2m + 2$ are not linearly independent.

□

3.4.3. *Dumas' result.* As we have noted, Becker and Dumas each showed that a solution of an isolating q-Mahler equation is q-regular. In fact Dumas obtained a more general version, which extends to the Z-numeration as follows.

Theorem 35. *Let $f(x)$ be the solution of an isolating equation*

$$f(x) = \sum_{i=0}^{d} A_i(x)\Phi^i(f(x)) + g(x) \tag{12}$$

where $g(x)$ is Z-regular. Then f is also Z-regular.

A weighted automaton is called *normalized* if it has a unique state with a non-zero final weight and there is no transition with non-zero weight going out of this state. The following result is very classical, see eg [Sak09, Proposition 2.14]. Recall that two weighted automata are *equivalent* if they assign the same weight to each word.

Lemma 36. *For each weighted automaton with n states, there is an equivalent normalized weighted automaton with $n + 1$ states.*

Lemma 37. *$f(x)$ is Z-regular if and only if $xf(x)$ is Z-regular.*

Proof. By Theorem 8, the class of Z-regular functions is closed under taking products. Therefore if $f(x)$ is Z-regular, then $xf(x)$ is also Z-regular since each polynomial is obviously Z-regular.

Conversely, suppose that the series $xf(x)$ is computed by the weighted automaton $\mathcal{A}$. By a variant of Theorem 20, there exists an automaton $\mathcal{B}$ over the alphabet $\bar{B} = \{-1, 0, 1\}$ accepting

$$\{w \boxminus w' : w, w' \in \{0,1\}^* \text{ and } (w)_Z = (w')_Z + 1\}$$

By combining the automaton $\mathcal{B}$ and the weighted automaton $\mathcal{A}$, it is possible to construct a weighted automaton $\mathcal{C}$ such that $\mathcal{C}((n)_Z) = \mathcal{A}((n-1)_Z)$ for each $n \geqslant 1$ and $\mathcal{C}(0) = 0$. This completes the proof of the converse. □

Now we come to the proof of the theorem.

Proof of Theorem 35. Let us suppose that the series g is computed by the weighted automaton $\mathcal{B}$. By Lemma 37, there is, for each integer $j \geqslant 0$, a weighted automaton $\mathcal{B}_j$ such that $\mathcal{B}_j((n)_Z) = g_{n-j}$ for each $n \geqslant j$, and $\mathcal{B}_j((n)_Z) = 0$ otherwise. By Lemma 36, it can be assumed that each weighted automaton $\mathcal{B}_j$ is normalized. The weighted automaton to compute the solution $f(x)$ of Equation (12) is obtained by combining the automaton $\mathcal{A}(P, f_0)$ with the automata $\mathcal{B}_j$ for $0 \leqslant j \leqslant \tilde{h}$. The automaton is the disjoint union of these automata except that for each integer $0 \leqslant j \leqslant \tilde{h}$, the unique final state of $\mathcal{B}_j$ is removed and that all transitions ending in that state now end in the states $s_{0,j,q,u}$ for all possible choices of q and u. □

3.4.4. *A Z-Mahler series which is not Z-regular.* As in the case for q-Mahler series [Bec94, Proposition 1], a Z-Mahler series is not necessarily Z-regular, as the following example shows.

Proposition 38. *The solution $f(x) = \sum_n f_n x^n$ with $f_0 = 1$ of the Z-Mahler equation*

$$(1 - x)f(x) = \Phi(f(x)). \tag{13}$$

is not Z-regular.

In [Bec94, Prop. 1], Becker shows that the solution of the analogous q-Mahler equation $(1-x)f(x) = f(z^q)$ is also not q-regular. The proof that we provide below is different from the one given in [Bec94]: it is based on the growth of coefficients.

To prove Proposition 38, we define the function $\lambda : \mathbb{N} \to \mathbb{N}$, which is the analogue in Zeckendorf numeration of the function $n \mapsto \lfloor n/q \rfloor$ is base q. If $(n)_Z = b_k \cdots b_0$, set

$$\lambda(n) = \lambda\left(\sum_{i=0}^{k} b_i F_i\right) := \sum_{i=1}^{k} b_i F_{i-1}.$$

The function λ is almost an inverse of the function ϕ as $\lambda(\phi(n)) = n$ for each integer $n \geqslant 0$, and

$$\phi(\lambda(n)) = \begin{cases} n & \text{if } n \equiv 0 \bmod Z \\ n - 1 & \text{otherwise.} \end{cases}$$

The following lemma follows from these relations between the functions λ and ϕ, and Lemma 18.

Lemma 39. *There is a positive constant c such that $\lambda(n) \geqslant n/\varphi - c$ for each integer $n \geqslant 0$.*

Proof of Proposition 38. It follows from (13) that each coefficient f_i for $i > 0$ satisfies the following equality.

$$f_i = \begin{cases} f_{i-1} + f_{\lambda(i)} & \text{if } i \in \varphi(\mathbb{N}) \\ f_{i-1} & \text{otherwise} \end{cases}$$

Summing up these equations for $i = 0, \cdots, n$ yields

$$\sum_{i=0}^{n} f_i = \sum_{i=0}^{n-1} f_i + \sum_{i=0}^{\lambda(n)} f_i$$

and thus

$$f_n = \sum_{i=0}^{\lambda(n)} f_i$$

We now prove by induction on k that for each integer k, there exists a positive constant C_k such that $f_n \geqslant C_k n^k$ holds for each integer $n \geqslant 0$. It follows easily from the relation above that each coefficient f_n satisfies $f_n \geqslant 0$ and thus $f_n \geqslant f_{n-1}$ for $n \geqslant 1$. Since $f_0 = 1$, each coefficient satisfies $f_n \geqslant 1$ and the result is proved for $k = 0$ with $C_0 = 1$. Suppose that the statement is true for some $k \geqslant 0$. From the previous equation, we get

$$f_n \geqslant C_k \sum_{i=0}^{\lambda(n)} i^k$$

Using the relation stated in Lemma 39, there is a positive constant C_{k+1} such that $f_n \geqslant C_{k+1} n^{k+1}$ holds for each $n \geqslant 0$. This proves the statement for $k + 1$. By Lemma 5, f is not Z-regular. □

The same technique can be applied to show that solutions of other Z-Mahler equations are also not Z-regular. For instance, the non-zero solution of the equation $(1 - x)f(x) = \frac{1}{2}\left(\Phi(f(x)) + \Phi^2(f(x))\right)$, as well as the solution of the equation is $(1 - x^2)f(x) = \Phi(f(x))$ are also not Z-regular. It also follows from the proof of Proposition 38] that the solution of the Z-Mahler equation $(1 - \alpha x)f(x) = \Phi(f(x))$ for $\alpha \geqslant 1$ is also not Z-regular because the coefficients obviously grow faster than the ones of the solution of $(1 - x)f(x) = \Phi(f(x))$. The same approach does not seem to work for the similar Z-Mahler equation $(1+x^2)f(x) = \Phi(f(x))$ although it's reasonable to assume that the solution of this equation is also not Z-regular. Note that the solution of $(1 + x)f(x) = \Phi(f(x))$ turns out to be Z-regular because it is the polynomial $1 - x$.

Conclusion

In conclusion, we mention a few open problems. Allouche and Shallit prove in [AS92, Thm 2.11] that a geometric series $f = \sum_{n \geqslant 0} \alpha^n x^n$ is q-regular if and only if α is either zero or a root of unity. We do not have a similar result for Z-regular series. Using this, in [BCCD19], the authors characterise q-regular series in terms of the q-Mahler equations they satisfy. We do not know if there is a similar characterisation for Z-Mahler equations.

In [AB17, Prop 7.8], Adamczewski and Bell give a series which is q-regular but which is not the solution of an isolating q-Mahler equation (q-Becker in their terminology). We do not have such an example, of a series which is Z-regular but which is not a solution of an isolating Z-Mahler equation.

Other questions include whether one can extend existing Cobham type results. For example, in [AB17] and [SS19], it is shown that a series which is both k- and l-Mahler over a field of characteristic zero, with k and l multiplicatively independent, must be rational. Which series are both k- and Z-Mahler?

References

[AB17] Boris Adamczewski and Jason P. Bell, *A problem about Mahler functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17** (2017), no. 4, 1301–1355. MR 3752528

[Ada19] Boris Adamczewski, *Mahler's method*, Doc. Math. (2019), no. Extra vol.: Mahler Selecta, 95–122. MR 4604998

[All92] Jean-Paul Allouche, *q-regular sequences and other generalizations of q-automatic sequences*, LATIN '92 (São Paulo, 1992), Lecture Notes in Comput. Sci., vol. 583, Springer, Berlin, 1992, pp. 15–23. MR 1253343

[AS92] Jean-Paul Allouche and Jeffrey Shallit, *The ring of k-regular sequences*, Theoret. Comput. Sci. **98** (1992), no. 2, 163–197. MR 1166363

[AS03] ———, *Automatic sequences*, Cambridge University Press, Cambridge, 2003, Theory, applications, generalizations. MR 1997038

[BCCD19] Jason P. Bell, Frédéric Chyzak, Michael Coons, and Philippe Dumas, *Becker's conjecture on Mahler functions*, Trans. Amer. Math. Soc. **372** (2019), no. 5, 3405–3423. MR 3988615

[Bec94] Paul-Georg Becker, *k-regular power series and Mahler-type functional equations*, J. Number Theory **49** (1994), no. 3, 269–286. MR 1307967

[BR11] Jean Berstel and Christophe Reutenauer, *Noncommutative rational series with applications*, Encyclopedia of Mathematics and its Applications, vol. 137, Cambridge University Press, Cambridge, 2011. MR 2760561

[Chr79] Gilles Christol, *Ensembles presque periodiques k-reconnaissables*, Theoret. Comput. Sci. **9** (1979), no. 1, 141–145. MR 535129

[CKMFR80] Gilles Christol, Teturo Kamae, Michel Mendès France, and Gérard Rauzy, *Suites algébriques, automates et substitutions*, Bull. Soc. Math. France **108** (1980), no. 4, 401–419. MR 614317

[CS18] Michael Coons and Lukas Spiegelhofer, *Number theoretic aspects of regular sequences*, Sequences, groups, and number theory, Trends Math., Birkhäuser/Springer, Cham, 2018, pp. 37–87. MR 3799924

[Dum93] Philippe Dumas, *Récurrences mahlériennes, suites automatiques, études asymptotiques*, Institut National de Recherche en Informatique et en Automatique (INRIA), Rocquencourt, 1993, Thèse, Université de Bordeaux I, Talence, 1993. MR 1346304

[Fro92] Christiane Frougny, *Representations of numbers and finite automata*, Math. Systems Theory **25** (1992), no. 1, 37–60. MR 1139094

[FS92] Christiane Frougny and Boris Solomyak, *Finite beta-expansions*, Ergodic Theory Dynam. Systems **12** (1992), no. 4, 713–723. MR 1200339

[Fur67] Harry Furstenberg, *Algebraic functions over finite fields*, J. Algebra **7** (1967), 271–277. MR 215820

[Lot02] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002. MR 1905123

[LS12] Jeong-Yup Lee and Boris Solomyak, *Pisot family self-affine tilings, discrete spectrum, and the Meyer property*, Discrete Contin. Dyn. Syst. **32** (2012), no. 3, 935–959. MR 2851885

[Mah82] Kurt Mahler, *Fifty years as a mathematician*, J. Number Theory **14** (1982), no. 2, 121–155. MR 655723

[Mey72] Yves Meyer, *Algebraic numbers and harmonic analysis*, North-Holland Mathematical Library, Vol. 2, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1972. MR 485769

[Nis96] Kumiko Nishioka, *Mahler functions and transcendence*, Lecture Notes in Mathematics, vol. 1631, Springer-Verlag, Berlin, 1996. MR 1439966

[Rig00] Michel Rigo, *Generalization of automatic sequences for numeration systems on a regular language*, Theoret. Comput. Sci. **244** (2000), no. 1-2, 271–281. MR 1774400

[Sak09] Jacques Sakarovitch, *Elements of automata theory*, Cambridge University Press, 2009.

[Sha88] Jeffrey Shallit, *A generalization of automatic sequences*, Theoret. Comput. Sci. **61** (1988), no. 1, 1–16. MR 974766

[SS19] Reinhard Schäfke and Michael Singer, *Consistent systems of linear differential and difference equations*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 9, 2751–2792. MR 3985611

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