GLOBAL REGULARITY FOR SOLUTIONS OF MAGNETOHYDRODYNAMIC EQUATIONS WITH LARGE INITIAL DATA

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Abstract. We study the existence of a strong solution to the initial value problem for the magnetohydrodynamic equations in $\mathbb{R}^N, N \geq 3$. We obtain a global in-time strong solution without any smallness assumptions on the initial data.

1. Introduction

Magneto-hydrodynamics (MHD) couples Maxwell's equations of electromagnetism with hydrodynamics to describe the macroscopic behavior of conducting fluids such as plasmas. It plays a very important role in solar physics, astrophysics, space plasma physics, and in laboratory plasma experiments. These equations read

$$\partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla p = \nu \Delta u \ \text{ in } \mathbb{R}^N \times (0,T) \equiv Q_T, \tag{1.1}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \eta \Delta b \ \text{ in } Q_T, \tag{1.2}$$

$$\nabla \cdot u = \nabla \cdot b = 0 \ \text{ in } Q_T, \tag{1.3}$$

where u denotes the velocity field, b the magnetic field, p the pressure, $\nu > 0$ the kinematic viscosity, and $\eta > 0$ the magnetic diffusivity. They are coupled with the initial conditions

$$u(x,0) = u^{(0)}(x), \quad b(x,0) = b^{(0)}(x) \ \text{ on } \mathbb{R}^N. \tag{1.4}$$

We refer the reader to [2] for a rather comprehensive survey on the subject of MHD. The MHD equations offer an interesting interplay between the Navier-Stokes equations of hydrodynamics and electromagnetism. Mathematical analysis of their rich solution properties has always attracted a lot of attention. However, we will not attempt to offer a review of exiting results. For that we refer the reader to [9]. One of the fundamental problems concerning the MHD equations is whether physically relevant regular solutions remain smooth for all time or they develop singularities in finite time. This problem is extremely difficult even in the 2D case [8]. It has remained elusive in spite of many partial results. See [4, 6, 12] and the references therein. The objective of this paper is to settle this open problem. More precisely, we have

Theorem 1.1. *Assume that*

$$|u^{(0)}| \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \ \text{ with } \ \nabla \cdot u^{(0)} = 0 \ \text{ and} \tag{1.5}$$

$$|b^{(0)}| \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \ \text{ with } \nabla \cdot b^{(0)} = 0. \tag{1.6}$$

Then there exists a global strong solution to (1.1)-(1.4). *Moreover, we have*

$$\|u\|_{\infty,Q_T} \leq c, \tag{1.7}$$

$$\|b\|_{\infty,Q_T} \leq c. \tag{1.8}$$

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Here and in what follows the constant c depends only on N, ν, η, and the initial data via the norms of the function spaces in (1.5) *and* (1.6)*.*

Obviously, under (1.7)-(1.8) higher regularity of the solution can be obtained via a bootstrap argument. In fact, a strong solution [7] is understood to be a weak solution with the additional properties (1.7)-(1.8) . A strong solution can be shown to satisfy system (1.1)-(1.2) in the a.e. sense [7]. We will not pursue the details here.

To describe our approach, let us first recall the classical regularity theorem for the initial value problem for linear parabolic equation of the form

$$\begin{aligned} \partial_t v - \Delta v &= \partial_{x_i} f_i(x,t) \ \text{ in } Q_T, \\ v(x,0) &= v^{(0)} \ \text{ on } \mathbb{R}^N. \end{aligned}$$

Here we have employed the notation convention of summing over repeated indices. It asserts that for each $q > N+2$ there is a positive number c such that

$$\|v\|_{\infty,Q_T} \le \|v^{(0)}\|_{\infty,\mathbb{R}^N} + c\sum_{i=1}^{N} \|f_i\|_{q,Q_T}. \tag{1.9}$$

In the incompressible Navier-Stokes equations f_i is roughly $v_i \sum_{j=1}^N v_j$. As a result, (1.9) becomes

$$\|v\|_{\infty,Q_T} \le \|v^{(0)}\|_{\infty,\mathbb{R}^N} + c\|v\|^2_{2q,Q_T}. \tag{1.10}$$

Thus, to be able to apply the classical theorem here, one must have $v \in L^{2q}(Q_T)$ for some $q > N+2$. Of course, this integrability requirement can be weakened due to the work of Serrin, Prodi, and Ladyzenskaja [7]. In spite of that, the gap between what is needed and what is available in terms of integrability is still substantial because we only have

$$v \in L^{\frac{2(N+2)}{N}}(Q_T).$$

for the Navier-Stokes. On the other hand, from the interpolation inequality

$$\|v\|_{2q,Q_T} \le \varepsilon\|v\|_{\infty,Q_T} + c(\varepsilon)\|v\|_{\frac{2(N+2)}{N},Q_T}, \quad \varepsilon > 0,$$

we see that it is the power 2 in the last term of (1.10) that causes the problem. If we somehow manage to lower the power to 1, then how large q is does not really matter. This is the key new observation in [10]. Also see [11]. How to reduce the power is based upon three ingredients: scaling of the modified equations by a suitable L^q-norm, interpolation inequalities for L^q-norms, and a carefully-designed iteration scheme of the De Giorgi type. The trick is to find the right combination of the parameters introduced. Our method seems to be very effective in dealing with the type of nonlinearity appearing in Navier-Stokes equations [10, 11].

Throughout this paper the following two inequalities will be used without acknowledgment:

$$(|a|+|b|)^\gamma \le \begin{cases} 2^{\gamma-1}(|a|^\gamma + |b|^\gamma) & \text{if } \gamma \ge 1, \\ |a|^\gamma + |b|^\gamma & \text{if } \gamma \le 1. \end{cases}$$

The remainder of this paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

As observed in [8], the standard approach to the global regularity problem for the incompressible MHD equations is to divide the process into two steps. The first step is to show the local well-posedness. In our case, this can be done via the contraction mapping principle and its variants such as successive approximations [7]. The second step is to extend the local solution of the first step into a global (in time) one by establishing suitable global a priori bounds on the solutions. That is, one needs to prove that the solution remains bounded for all $t > 0$. Once we have the global bounds, the standard Picard type extension theorem allows us to extend the local solution into a global one. Therefore, the global regularity problem on the MHD equations boils down to the global a priori bounds for smooth solutions. Thus, in our subsequent calculations we can assume that our solutions are sufficiently regular.

The free energy associated with (1.1)-(1.4) is well known. For completeness, we reproduce it in the following lemma.

Lemma 2.1. *There holds*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N}\left(|u|^2+|b|^2\right)dx+\int_{Q_T}\left(|\nabla u|^2+|\nabla b|^2\right)dxdt\le c(\nu,\eta)\int_{\mathbb{R}^N}\left(|u^{(0)}|^2+|b^{(0)}|^2\right)dx. \tag{2.1}$$

Proof. We easily verify that

$$\partial_t u\cdot u = \frac{1}{2}\partial_t|u|^2, \tag{2.2}$$

$$u\cdot\nabla u\cdot u = u_j\partial_{x_j}u_iu_i=\frac{1}{2}u\cdot\nabla|u|^2, \tag{2.3}$$

$$\nu\Delta u\cdot u = \frac{\nu}{2}\Delta|u|^2-\nu|\nabla u|^2. \tag{2.4}$$

When taking account of (1.3), we further have that

$$b\cdot\nabla b\cdot u = b_j\partial_{x_j}b_iu_i=\nabla\cdot(b\otimes bu)-b\cdot\nabla u\cdot b, \tag{2.5}$$

$$\nabla p\cdot u = \nabla\cdot(pu). \tag{2.6}$$

With these in mind, we use u as a test function in (1.1) to drive

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|u|^2dx+\nu\int_{\mathbb{R}^N}|\nabla u|^2dx=-\int_{\mathbb{R}^N}b\cdot\nabla u\cdot b\ dx. \tag{2.7}$$

Similarly, by using b as a test function in (1.2), we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|b|^2dx+\eta\int_{\mathbb{R}^N}|\nabla b|^2dx=\int_{\mathbb{R}^N}b\cdot\nabla u\cdot b\ dx.$$

Add this to (2.7) to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\left(|u|^2+|b|^2\right)dx+\nu\int_{\mathbb{R}^N}|\nabla u|^2dx+\eta\int_{\mathbb{R}^N}|\nabla b|^2dx=0.$$

An integration with respect to t yields the lemma. □

Recall that the Sobolev inequality in the whole space asserts

$$\|f\|_{\frac{2N}{N-2},\mathbb{R}^N}\le c(N)\|\nabla f\|_{2,\mathbb{R}^N}\quad\text{for each } f\in H^1(\mathbb{R}^N). \tag{2.8}$$

This together with (2.1) implies

$$\begin{aligned}\int_{Q_T}|u|^{\frac{2(N+2)}{N}}dxdt &\le \int_0^T\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\ &\le \sup_{0\le t\le T}\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{2}{N}}\int_0^T\left(\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\ &\le c(N)\sup_{0\le t\le T}\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{2}{N}}\int_{Q_T}|\nabla u|^2dxdt\le c. \end{aligned}\tag{2.9}$$

By the same token, we also have

$$\int_{Q_T}|b|^{\frac{2(N+2)}{N}}dxdt\le c. \tag{2.10}$$

The integrability of u, b is not very satisfactory in that the number $\frac{2(N+2)}{N}$ is too small to apply the classical parabolic regularity theory [10]. To address this issue, we define

$$w = |u|^2 + |b|^2, \quad A_r = \|w\|_{r,Q_T} \quad \text{for } r > 1. \tag{2.11}$$

Then consider the functions

$$\varphi = \frac{|u|^2}{A_r}, \quad \psi = \frac{|b|^2}{A_r}. \tag{2.12}$$

We proceed to derive the differential inequalities satisfied by these two functions. To this end, we take the dot product of (1.1) with u to obtain

$$\partial_t u \cdot u + u \cdot \nabla u \cdot u - b \cdot \nabla b \cdot u + \nabla p \cdot u = \nu \Delta u \cdot u.$$

Substitute (2.2)-(2.6) into the above equation to get

$$\begin{aligned} &\partial_t |u|^2 + u \cdot \nabla |u|^2 - \nu \Delta |u|^2 + 2\nu |\nabla u|^2 \\ &= \quad 2\nabla \cdot (b \otimes bu) - 2b \cdot \nabla u \cdot b - 2\nabla \cdot (pu) \\ &\leq \quad 2\nu |\nabla u|^2 + \frac{1}{2\nu} |b|^4 + 2\nabla \cdot (b \otimes bu) - 2\nabla \cdot (pu), \end{aligned}$$

from whence follows

$$\partial_t |u|^2 + u \cdot \nabla |u|^2 - \nu \Delta |u|^2 \leq \frac{1}{2\nu} |b|^4 + 2\nabla \cdot (b \otimes bu) - 2\nabla \cdot (pu). \tag{2.13}$$

Divide through (2.13) by A_r to derive

$$\partial_t \varphi + u \cdot \nabla \varphi - \nu \Delta \varphi \leq \frac{A_r^{-1}}{2\nu} |b|^4 + 2A_r^{-1} \nabla \cdot (b \otimes bu) - 2A_r^{-1} \nabla \cdot (pu). \tag{2.14}$$

Similarly, take the dot product of (1.2) with b to get

$$\partial_t b \cdot b + u \cdot \nabla b \cdot b - b \cdot \nabla u \cdot b = \eta \Delta b \cdot b.$$

We can easily infer from our earlier calculations that

$$\begin{aligned} \partial_t |b|^2 + u \cdot \nabla |b|^2 - \eta \Delta |b|^2 + 2\eta |\nabla b|^2 &= 2b \cdot \nabla u \cdot b \\ &= 2\nabla \cdot (b \otimes bu) - 2b \cdot \nabla b \cdot u \\ &\leq 2\eta |\nabla b|^2 + \frac{1}{2\eta} |b|^2 |u|^2 + 2\nabla \cdot (b \otimes bu). \end{aligned}$$

Consequently,

$$\partial_t |b|^2 + u \cdot \nabla |b|^2 - \eta \Delta |b|^2 \leq \frac{1}{2\eta} |b|^2 |u|^2 + 2\nabla \cdot (b \otimes bu).$$

That is, ψ satisfies

$$\partial_t \psi + u \cdot \nabla \psi - \eta \Delta \psi \leq \frac{A_r^{-1}}{2\eta} |b|^2 |u|^2 + 2A_r^{-1} \nabla \cdot (b \otimes bu). \tag{2.15}$$

Next, we show that we can apply a De Giorgi iteration scheme to (2.14). The following lemma is the foundation of a De Giorgi iteration scheme, whose proof can be found in ([3], p.12).

Lemma 2.2. *Let $\{y_n\}, n = 0, 1, 2, \cdots$, be a sequence of positive numbers satisfying the recursive inequalities*

$$y_{n+1} \leq c b^n y_n^{1+\alpha} \quad \textit{for some } b > 1, c, \alpha \in (0, \infty).$$

If

$$y_0 \leq c^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then $\lim_{n \to \infty} y_n = 0$.

Proof of Theorem 1.1. We design a De Giorgi iteration scheme. To this end, select

$$k \geq 2\|\varphi(\cdot,0)+\psi(\cdot,0)\|_{\infty,\mathbb{R}^N} \tag{2.16}$$

as below. Define

$$k_n = k - \frac{k}{2^{n+1}} \quad \text{for } n = 0, 1, \cdots.$$

Obviously,

$$\{k_n\} \text{ is increasing and } \tfrac{k}{2} \leq k_n \leq k \text{ for each } n. \tag{2.17}$$

We can easily verify that

$$(\ln\varphi - \ln k_n)^+$$

is a legitimate test function for (2.14). Upon using it, we derive

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}\int_{k_n}^{\varphi}(\ln\mu - \ln k_n)^+\, d\mu dx + \nu\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi}|\nabla\varphi|^2dx \\
&\leq -\int_{\mathbb{R}^N} u\cdot\nabla\varphi\,(\ln\varphi - \ln k_n)^+\,dx + \frac{A_r^{-1}}{2\nu}\int_{\mathbb{R}^N}|b|^4\,(\ln\varphi-\ln k_n)^+\,dx \\
&\quad -2A_r^{-1}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{b\otimes bu\cdot\nabla\varphi}{\varphi}dx + 2A_r^{-1}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{p}{\varphi}u\cdot\nabla\varphi dx.
\end{aligned}\tag{2.18}$$

We proceed to analyze each term in the above inequality. First, note from (1.3) that

$$-\int_{\mathbb{R}^N} u\cdot\nabla\varphi\,(\ln\varphi-\ln k_n)^+\,dx = -\int_{\mathbb{R}^N} u\cdot\nabla\int_{k_n}^{\varphi}(\ln\mu-\ln k_n)^+\,d\mu = 0.$$

The Sobolev inequality (2.8) asserts that

$$\begin{aligned}
&\frac{A_r^{-1}}{2\nu}\int_{\mathbb{R}^N}|b|^4\,(\ln\varphi-\ln k_n)^+\,dx \\
&\leq \frac{A_r^{-1}}{2\nu}\left\|(\ln\varphi-\ln k_n)^+\right\|_{\frac{2N}{N-2},\mathbb{R}^N}\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^{\frac{8N}{N+2}}dx\right)^{\frac{N+2}{2N}} \\
&\leq cA_r^{-1}\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi^2}|\nabla\varphi|^2dx\right)^{\frac{1}{2}}\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^{\frac{8N}{N+2}}dx\right)^{\frac{N+2}{2N}} \\
&\leq cA_r^{-1}k_n^{-\frac{1}{2}}\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi}|\nabla\varphi|^2dx\right)^{\frac{1}{2}}\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^{\frac{8N}{N+2}}dx\right)^{\frac{N+2}{2N}} \\
&\leq \frac{\nu}{2}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi}|\nabla\varphi|^2dx + cA_r^{-2}k^{-1}\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^{\frac{8N}{N+2}}dx\right)^{\frac{N+2}{N}}.
\end{aligned}$$

The last step is due to the Cauchy's inequality and (2.17). We can infer from (2.12) that

$$\begin{aligned}
-2A_r^{-1}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{b\otimes bu\cdot\nabla\varphi}{\varphi}dx &\leq 2A_r^{-\frac{1}{2}}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{|b|^2|\nabla\varphi|}{\varphi^{\frac{1}{2}}}dx \\
&\leq \frac{\nu}{4}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi}|\nabla\varphi|^2dx + cA_r^{-1}\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^4dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
2A_r^{-1}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{p}{\varphi}u\cdot\nabla\varphi dx &\leq 2A_r^{-\frac{1}{2}}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi^{\frac{1}{2}}}|p||\nabla\varphi|dx\\
&\leq \frac{\nu}{8}\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi}|\nabla\varphi|^2dx+cA_r^{-1}\int_{\{\varphi(\cdot,t)\geq k_n\}}|p|^2dx.
\end{aligned}$$

Collect the preceding results in (2.18) and integrate the resulting inequality with respect to t to derive

$$\begin{aligned}
&\sup_{0\leq t\leq T}\int_{\mathbb{R}^N}\int_{k_n}^{\varphi}(\ln\mu-\ln k_n)^+\,d\mu dx+\int_{\{\varphi\geq k_n\}}\frac{1}{\varphi}\left|\nabla\varphi\right|^2dxdt\\
&\leq cA_r^{-1}\int_{\{\varphi\geq k_n\}}p^2dxdt+cA_r^{-1}\int_{\{\varphi\geq k_n\}}|b|^4dxdt+cA_r^{-2}k^{-1}\int_0^T\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^{\frac{8N}{N+2}}dx\right)^{\frac{N+2}{N}}dt. \qquad (2.19)
\end{aligned}$$

Here we have used the fact that

$$\left.\int_{k_n}^{\varphi}(\ln\mu-\ln k_n)^+\,d\mu\right|_{t=0}=0,$$

which can easily be inferred from (2.17) and (2.16). Furthermore, it is also elementary to verify that

$$\begin{aligned}
\int_{\{\varphi(\cdot,t)\geq k_n\}}\frac{1}{\varphi}\left|\nabla\varphi\right|^2dx &= 4\int_{\mathbb{R}^N}\left|\nabla\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right|^2dx,\\
\int_{k_n}^{\varphi}(\ln\mu-\ln k_n)^+\,d\mu &\geq 2\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^2.
\end{aligned}$$

Combine them with (2.19) and keep in mind (2.11) to deduce

$$\begin{aligned}
&\sup_{0\leq t\leq T}\int_{\mathbb{R}^N}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^2dx+\int_{Q_T}\left|\nabla\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right|^2dxdt\\
&\leq cA_r^{-1}\int_{\{\varphi\geq k_n\}}p^2dxdt+cA_r^{-1}\int_{\{\varphi\geq k_n\}}|b|^4dxdt+cA_r^{-2}k^{-1}\int_0^T\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}|b|^{\frac{8N}{N+2}}dx\right)^{\frac{N+2}{N}}dt\\
&\leq cA_r^{-1}\left(\int_{\{\varphi\geq k_n\}}p^2dxdt+\int_{\{\varphi\geq k_n\}}w^2dxdt+A_r^{-1}k^{-1}\int_0^T\left(\int_{\{\varphi(\cdot,t)\geq k_n\}}w^{\frac{4N}{N+2}}dx\right)^{\frac{N+2}{N}}dt\right)\\
&\equiv cA_r^{-1}I.
\end{aligned}$$

As in (2.9), we calculate that

$$\begin{aligned}
&\int_{Q_T}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^{\frac{4}{N}+2}dxdt\\
&\leq \int_0^T\left(\int_{\mathbb{R}^N}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^2dx\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt\\
&\leq c\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^N}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^2dx\right)^{\frac{2}{N}}\int_{Q_T}\left|\nabla\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right|^2dxdt\\
&\leq cI^{\frac{N+2}{N}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{Q_T}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^{\frac{4}{N}+2}dxdt &\geq \int_{\{\varphi\geq k_{n+1}\}}\left[\left(\sqrt{\varphi}-\sqrt{k_n}\right)^+\right]^{\frac{4}{N}+2}dxdt\\ &\geq \left(\sqrt{k_{n+1}}-\sqrt{k_n}\right)^{\frac{4}{N}+2}|\{\varphi\geq k_{n+1}\}|\\ &\geq c\left(\frac{\sqrt{k}}{2^{n+2}}\right)^{\frac{4}{N}+2}|\{\varphi\geq k_{n+1}\}|\\ &\geq \frac{c|\{\varphi\geq k_{n+1}\}|k^{\frac{(N+2)}{N}}}{2^{(\frac{4}{N}+2)n}}.\end{aligned}$$

Putting them together yields

$$\begin{aligned}|\{\varphi\geq k_{n+1}\}| &= |\{\varphi\geq k_{n+1}\}|^{\frac{N}{N+2}+\frac{2}{N+2}}\\ &\leq \frac{c4^nI}{kA_r}|\{\varphi\geq k_{n+1}\}|^{\frac{2}{N+2}}.\end{aligned}\tag{2.20}$$

A similar estimate can be obtained for ψ. Indeed, we use $(\ln\psi-\ln k_n)^+$ as a test function in (2.15) to deduce

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}\int_0^\psi(\ln s-\ln k_n)^+\,dsdx+\eta\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{1}{\psi}|\nabla\psi|^2dx\\ &\leq -\int_{\mathbb{R}^N}u\cdot\nabla\psi\,(\ln\psi-\ln k_n)^+\,dx+\frac{A_r^{-1}}{2\eta}\int_{\mathbb{R}^N}|b|^2|u|^2\,(\ln\psi-\ln k_n)^+\,dx\\ &\quad -2A_r^{-1}\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{1}{\psi}b\otimes bu\cdot\nabla\psi dx.\end{aligned}\tag{2.21}$$

The three terms on the right-hand side can be handled almost the same way as before. That is, first,

$$\int_{\mathbb{R}^N}u\cdot\nabla\psi\,(\ln\psi-\ln k_n)^+\,dx=\int_{\mathbb{R}^N}u\cdot\nabla\int_0^\psi(\ln s-\ln k_n)^+\,dsdx=0.$$

Second,

$$\begin{aligned}&\frac{A_r^{-1}}{2\eta}\int_{\mathbb{R}^N}|b|^2|u|^2\,(\ln\psi-\ln k_n)^+\,dx\\ &\leq \frac{A_r^{-1}}{2\eta}\|\,(\ln\psi-\ln k_n)^+\,\|_{\frac{2N}{N-2},\mathbb{R}^N}\left(\int_{\{\psi(\cdot,t)\geq k_n\}}(|b||u|)^{\frac{4N}{N+2}}\,dx\right)^{\frac{N+2}{2N}}\\ &\leq cA_r^{-1}k_n^{-\frac{1}{2}}\left(\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{1}{\psi}|\nabla\psi|^2dx\right)^{\frac{1}{2}}\left(\int_{\{\psi(\cdot,t)\geq k_n\}}(|b||u|)^{\frac{4N}{N+2}}\,dx\right)^{\frac{N+2}{2N}}\\ &\leq \frac{\eta}{2}\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{1}{\psi}|\nabla\psi|^2dx+cA_r^{-2}k^{-1}\left(\int_{\{\psi(\cdot,t)\geq k_n\}}w^{\frac{4N}{N+2}}dx\right)^{\frac{N+2}{N}}.\end{aligned}$$

The last term in (2.21) is estimated as follows:

$$\begin{aligned}-2A_r^{-1}\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{1}{\psi}b\otimes bu\cdot\nabla\psi dx &\leq 2A_r^{-\frac{1}{2}}\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{|b||u|}{\psi^{\frac{1}{2}}}|\nabla\psi|dx\\ &\leq \frac{\eta}{4}\int_{\{\psi(\cdot,t)\geq k_n\}}\frac{1}{\psi}|\nabla\psi|^2dx+cA_r^{-1}\int_{\{\psi(\cdot,t)\geq k_n\}}w^2dx.\end{aligned}$$

Plug the preceding three estimates into (2.21), then follow our earlier calculations, and thereby derive

$$\begin{aligned}
&|\{\psi \geq k_{n+1}\}| \\
&\quad = \quad |\{\psi \geq k_{n+1}\}|^{\frac{N}{N+2}+\frac{2}{N+2}} \\
&\quad \leq \quad \frac{c4^n}{kA_r}\left(\int_{\{\psi \geq k_n\}} w^2 dxdt + A_r^{-1}k^{-1}\int_0^T \left(\int_{\{\psi(\cdot,t)\geq k_n\}} w^{\frac{4N}{N+2}}dx\right)^{\frac{N+2}{N}} dt\right)|\{\psi \geq k_{n+1}\}|^{\frac{2}{N+2}} \\
&\quad \leq \quad \frac{c4^n}{kA_r} I |\{\psi \geq k_{n+1}\}|^{\frac{2}{N+2}}.
\end{aligned}$$

Add this to (2.20) to get

$$y_{n+1} \equiv |\{\varphi \geq k_{n+1}\}| + |\{\psi \geq k_{n+1}\}| \leq \frac{c4^n I}{kA_r} y_n^{\frac{2}{N+2}}. \tag{2.22}$$

Now we turn our attention to p. Take the divergence of both sides of (1.1) to obtain

$$-\Delta p = \nabla \cdot (u \cdot \nabla u - b \cdot \nabla b).$$

In view of the classical representation theorem ([5], p. 17), we can write p as

$$p(x,t) = \int_{\mathbb{R}^N} \Gamma(y-x)\left[u \cdot \nabla u - b \cdot \nabla b\right] dy.$$

We observe from (1.3) that

$$\int_{\mathbb{R}^N} \Gamma(y-x)\nabla \cdot (u \cdot \nabla u - b \cdot \nabla b) dy = \int_{\mathbb{R}^N} \Gamma_{y_i y_j}(y-x)\left(u_i u_j - b_i b_j\right) dy.$$

It is a well known fact that $\partial^2_{y_i y_j}\Gamma(y)$ is a Calderón-Zygmund kernel. A result of [1] asserts that for each $\ell \in (1,\infty)$ there is a positive number c_ℓ determined by N and ℓ such that

$$\left\|\int_{\mathbb{R}^N} \Gamma_{y_i y_j}(y-x)\left(u_i u_j - b_i b_j\right) dy\right\|_{\ell,\mathbb{R}^N} \leq c_\ell \|u_i u_j - b_i b_j\|_{\ell,\mathbb{R}^N} \leq c\|w\|_{\ell,\mathbb{R}^N}.$$

Subsequently,

$$\|p\|_{\ell,\mathbb{R}^N} \leq c\|w\|_{\ell,\mathbb{R}^N} \quad \text{for each } \ell > 1. \tag{2.23}$$

We pick

$$q > N+2. \tag{2.24}$$

Subsequently, by (2.23), we have

$$\int_{\{\varphi \geq k_n\}} p^2 dxdt \leq \|p\|^2_{q,Q_T}|\{\varphi \geq k_n\}|^{1-\frac{2}{q}} \leq c\|w\|^2_{q,Q_T}|\{\varphi \geq k_n\}|^{1-\frac{2}{q}}.$$

Similarly,

$$
\begin{aligned}
\int_{\{\varphi\geq k_n\}} w^2 dxdt &\leq \|w\|_{q,Q_T}^2 \{\varphi \geq k_n\}|^{1-\frac{2}{q}},\\
\int_0^T \left(\int_{\{\varphi(\cdot,t)\geq k_n\}} w^{\frac{4N}{N+2}} dx\right)^{\frac{N+2}{N}} dt &\leq \int_0^T \left(\int_{\mathbb{R}^N} w^{\frac{2Nq}{N+q}} dx\right)^{\frac{2(q+N)}{qN}} |\{\varphi(\cdot,t)\geq k_n\}|^{\frac{q-2}{q}} dt\\
&\leq \left(\int_0^T \left(\int_{\mathbb{R}^N} w^{\frac{2Nq}{N+q}} dx\right)^{\frac{q+N}{N}} dt\right)^{\frac{2}{q}} |\{\varphi \geq k_n\}|^{1-\frac{2}{q}}\\
&\equiv \|w\|^4_{\frac{2Nq}{N+q},2q,Q_T} y_n^{1-\frac{2}{q}}.
\end{aligned}
$$

Substitute the preceding estimates into (2.22) to get

$$
y_{n+1} \leq \frac{c4^n}{A_r k}\left(\|w\|_{q,Q_T}^2 + A_r^{-1}k^{-1}\|w\|^4_{\frac{2Nq}{N+q},2q,Q_T}\right) y_n^{1+\alpha}, \tag{2.25}
$$

where

$$
\alpha = -\frac{2}{q} + \frac{2}{N+2} > 0 \quad \text{due to (2.24).} \tag{2.26}
$$

By the interpolation inequality for L^q norms in ([5], p.146),

$$
\|w\|_{\frac{2Nq}{N+q},\mathbb{R}^N} \leq \|w\|^{1-\lambda}_{\frac{(2N-1)q}{N},\mathbb{R}^N}\|w\|^{\lambda}_{1,\mathbb{R}^N}, \quad \lambda = \frac{\frac{N+q}{2Nq} - \frac{N}{(2N-1)q}}{1-\frac{N}{(2N-1)q}} = \frac{1}{2N}.
$$

Consequently,

$$
\begin{aligned}
\|w\|^4_{\frac{2Nq}{N+q},2q,Q_T} &\leq \left(\int_0^T \|w\|^{2q(1-\lambda)}_{\frac{(2N-1)q}{N},\mathbb{R}^N}\|w\|^{2q\lambda}_{1,\mathbb{R}^N} dt\right)^{\frac{2}{q}}\\
&\leq \sup_{0\leq t\leq T}\|w\|^{4\lambda}_{1,\mathbb{R}^N}\left(\int_{Q_T} w^{\frac{(2N-1)q}{N}} dxdt\right)^{\frac{2}{q}} \leq c\|w\|^{\frac{2(2N-1)}{N}}_{\frac{(2N-1)q}{N},Q_T}.
\end{aligned}
$$

The last step is due to (2.1). Substitute this into (2.25) to obtain

$$
y_{n+1} \leq \frac{c4^n}{A_r k}\left(\|w\|_{q,Q_T}^2 + A_r^{-1}k^{-1}\|w\|^{\frac{2(2N-1)}{N}}_{\frac{(2N-1)q}{N},Q_T}\right) y_n^{1+\alpha}, \tag{2.27}
$$

Obviously, we wish to decrease the power of $\|w\|_{\frac{(2N-1)q}{N},Q_T}$ in the above inequality. It turns out that we can do this through k. Observe from (2.1) that

$$
\|w\|_{\frac{(2N-1)q}{N},Q_T} \leq \|w\|^{1-\frac{N+2}{(2N-1)q}}_{\infty,Q_T}\|w\|^{\frac{N+2}{(2N-1)q}}_{\frac{N+2}{N},Q_T} \leq c\|w\|^{1-\frac{N+2}{(2N-1)q}}_{\infty,Q_T}.
$$

That is,

$$
\|w\|^{\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T} \leq c\|w\|_{\infty,Q_T}. \tag{2.28}
$$

Thus, if we choose k so that

$$
L_0 A_r^{-1}\|w\|^{\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T} \leq k, \tag{2.29}
$$

we can turn (2.27) into

$$
y_{n+1} \leq \frac{c4^n}{A_r k}\left(\|w\|_{q,Q_T}^2 + L_0^{-1}\|w\|^{\frac{2(2N-1)}{N}-\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}\right) y_n^{1+\alpha}. \tag{2.30}
$$

Here L_0 is a positive number to be determined. Later we shall see that L_0 must be suitably small. Unfortunately, the number $\frac{(2N-1)q}{(2N-1)q-N-2}$ is not as big as we want it to be. To compensate for this, we must find a way to duplicate the feat. To this end, invoke the L^q interpolation inequality again to get

$$\|w\|_{q,Q_T}\leq \|w\|^{1-\lambda_1}_{\frac{(2N-1)q}{N},Q_T}\|w\|^{\lambda_1}_{\frac{N+2}{N},Q_T},\quad \lambda_1=\frac{\frac{1}{q}-\frac{N}{(2N-1)q}}{\frac{N}{N+2}-\frac{N}{(2N-1)q}}=\frac{(N+2)(N-1)}{N[(2N-1)q-N-2]}.$$

Plug this into (2.30) to deduce

$$\begin{aligned}
y_{n+1} &\leq \frac{c4^n}{A_rk}\left(\|w\|^{2(1-\lambda_1)}_{\frac{(2N-1)q}{N},Q_T}\|w\|^{2\lambda_1}_{\frac{N+2}{N},Q_T}+L_0^{-1}\|w\|^{\frac{2(2N-1)}{N}-\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}\right)y_n^{1+\alpha}\\
&\leq \frac{c4^n}{A_rk}\left(c(q,N)\|w\|^{c(q,N)}_{\frac{N+2}{N},Q_T}+\left(c(q,N)+L_0^{-1}\right)\|w\|^{\frac{2(2N-1)}{N}-\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}\right)y_n^{1+\alpha}.
\end{aligned}\tag{2.31}$$

Here we have applied Young's inequality. This can be done because we obviously have

$$\frac{2(2N-1)}{N}-\frac{(2N-1)q}{(2N-1)q-N-2}>2>2(1-\lambda_1)\quad\text{for } q>N+2.$$

Without any loss of generality, we may assume that the second term in the last pair of parentheses in (2.31) is larger than or equal the first term there, i.e.,

$$c(q,N)\|w\|^{c(q,N)}_{\frac{N+2}{N},Q_T}\leq\left(c(q,N)+L_0^{-1}\right)\|w\|^{\frac{2(2N-1)}{N}-\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}.\tag{2.32}$$

Subsequently, (2.31) becomes

$$\begin{aligned}
y_{n+1} &\leq \frac{c\left(c(q,N)+L_0^{-1}\right)4^n}{A_rk}\|w\|^{\frac{2(2N-1)}{N}-\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}y_n^{1+\alpha}\\
&\leq cL_0^{-1}\left(c(q,N)+L_0^{-1}\right)4^n\|w\|^{\frac{2(2N-1)}{N}-\frac{2(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}y_n^{1+\alpha}.
\end{aligned}\tag{2.33}$$

The last step here is due to (2.29).

Next, we use the previous power reduction idea for a different L^q norm. To do this, recall (2.11) and (2.12) to derive

$$\|\varphi+\psi\|_{r,Q_T}=A_r^{-1}\|w\|_{r,Q_T}=1.\tag{2.34}$$

Subsequently, for each

$$\ell>r,\tag{2.35}$$

there holds

$$\|\varphi+\psi\|^{\frac{\ell}{\ell-r}}_{\ell,Q_T}\leq\left(\|\varphi+\psi\|^{1-\frac{r}{\ell}}_{\infty,Q_T}\|\varphi+\psi\|^{\frac{r}{\ell}}_{r,Q_T}\right)^{\frac{\ell}{\ell-r}}=\|\varphi+\psi\|_{\infty,Q_T}.\tag{2.36}$$

As in (2.29), we choose k so large that

$$L_1\|\varphi+\psi\|^{\frac{\ell}{\ell-r}}_{\ell,Q_T}\leq k,\tag{2.37}$$

where L_1 is a positive number to be determined. Under our assumptions (2.37) and (2.29), for each

$$j>0$$

there holds

$$L_1^{j\alpha}\|\varphi+\psi\|^{\frac{j\alpha\ell}{\ell-r}}_{\ell,Q_T}\leq k^{j\alpha},\quad L_0^{j\alpha}\|w\|^{-j\alpha}_{r,Q_T}\|w\|^{\frac{j\alpha(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}\leq k^{j\alpha}.$$

Use this in (2.33) to deduce

$$\begin{aligned} y_{n+1} &\leq cL_0^{-1}\left(c(q,N)+L_0^{-1}\right)4^n\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{2(2N-1)}{N}-\frac{2(2N-1)q}{(2N-1)q-N-2}}y_n^{1+\alpha} \\ &\leq \frac{cL_0^{-1}\left(c(q,N)+L_0^{-1}\right)4^n\|w\|_{r,Q_T}^{j\alpha}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{d}k^{2j\alpha}}{(L_1L_0)^{j\alpha}\|\varphi+\psi\|_{\ell,Q_T}^{\frac{j\alpha\ell}{\ell-r}}}y_n^{1+\alpha}, \end{aligned}$$

where

$$d=\frac{2(2N-1)}{N}-\frac{(j\alpha+2)(2N-1)q}{(2N-1)q-N-2}. \tag{2.38}$$

The introduction of j here is very crucial. Obviously, by choosing j suitably large, we can even make d negative. The trade-off is that the power of $\|w\|_{r,Q_T}$ becomes large. The trick is to manipulate the various parameters in these exponents so that they fit our needs.

To see that Lemma 2.2 is applicable here, we first recall (2.22) and (2.34) to deduce

$$\begin{aligned} y_0 &= |Q_0| = \left|\left\{\varphi\geq\frac{k}{2}\right\}\right|+\left|\left\{\psi\geq\frac{k}{2}\right\}\right| \\ &\leq \int_{Q_T}\left(\frac{2\varphi}{k}\right)^r dxdt+\int_{Q_T}\left(\frac{2\psi}{k}\right)^r dxdt \\ &= =\frac{2^r}{k^r}\int_{Q_T}\left(\varphi^r+\psi^r\right)dxdt\leq\frac{2^{r+1}}{k^r}\int_{Q_T}(\varphi+\psi)^r\,dxdt=\frac{2^{r+1}}{k^r}. \end{aligned}$$

Assume that

$$r>2j. \tag{2.39}$$

Subsequently, we can pick k so large that

$$\frac{2^r}{k^{r-2j}}\leq\frac{(L_1L_0)^j\|\varphi+\psi\|_{\ell,Q_T}^{\frac{j\ell}{\ell-r}}}{\left[cL_0^{-1}\left(c(q,N)+L_0^{-1}\right)\right]^{\frac{1}{\alpha}}4^{\frac{1}{\alpha^2}}\|w\|_{r,Q_T}^{j}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{d}{\alpha}}}. \tag{2.40}$$

Lemma 2.2 asserts

$$\lim_{n\to\infty}y_n=|\{\varphi\geq k\}|+|\{\psi\geq k\}|=0.$$

That is,

$$\sup_{Q_T}(\varphi+\psi)\leq 2k. \tag{2.41}$$

According to (2.16), (2.37), (2.40), and (2.29), it is enough for us to take

$$\begin{aligned} k &= 2\|\varphi(\cdot,0)+\psi(\cdot,0)\|_{\infty,\mathbb{R}^N}+L_1\|\varphi+\psi\|_{\ell,Q_T}^{\frac{\ell}{\ell-r}}+L_0\|w\|_{r,Q_T}^{-1}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{(2N-1)q}{(2N-1)q-N-2}} \\ &\quad+\left[cL_0^{-1}\left(c+L_0^{-1}\right)\right]^{\frac{1}{\alpha(r-2j)}}4^{\frac{1}{\alpha^2(r-2j)}}(L_1L_0)^{-\frac{j}{r-2j}}\|\varphi+\psi\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{\frac{j}{r-2j}}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{d}{\alpha(r-2j)}}. \end{aligned}$$

Plug this into (2.41), take $L_1=\frac{1}{4}$ in the resulting inequality, and make use of (2.36) to yield

$$\begin{aligned} \|\varphi+\psi\|_{\infty,Q_T} &\leq 4\|\varphi(\cdot,0)+\psi(\cdot,0)\|_{\infty,\mathbb{R}^N}+4L_0\|w\|_{r,Q_T}^{-1}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{(2N-1)q}{(2N-1)q-N-2}} \\ &\quad+c\left[L_0^{-1}\left(c+L_0^{-1}\right)\right]^{\frac{1}{\alpha(r-2j)}}L_0^{-\frac{j}{r-2j}}\|\varphi+\psi\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{\frac{j}{r-2j}}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{d}{\alpha(r-2j)}}. \end{aligned}$$

Recall (2.11) and (2.12) to deduce

$$\|w\|_{\infty,Q_T} \leq 4\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + 4L_0\|w\|^{\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T}$$
$$+c\left[L_0^{-1}\left(c+L_0^{-1}\right)\right]^{\frac{1}{\alpha(r-2j)}} L_0^{-\frac{j}{r-2j}}\|w\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{1+\frac{j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|^{\frac{d}{\alpha(r-2j)}}_{\frac{(2N-1)q}{N},Q_T}. \tag{2.42}$$

In view of (2.28), we may choose L_0 suitably small so that

$$4L_0\|w\|^{\frac{(2N-1)q}{(2N-1)q-N-2}}_{\frac{(2N-1)q}{N},Q_T} \leq \frac{1}{2}\|w\|_{\infty,Q_T}.$$

Utilize the preceding estimate in (2.42) to deduce

$$\|w\|_{\infty,Q_T} \leq 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{1+\frac{j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|^{\frac{d}{\alpha(r-2j)}}_{\frac{(2N-1)q}{N},Q_T}. \tag{2.43}$$

We proceed to show that we can extract enough information from this inequality by making suitable choice of the parameters

We first pick

$$r > \frac{(2N-1)q}{N}. \tag{2.44}$$

Then we can invoke the interpolation inequality for L^q norms again to deduce that

$$\|w\|_{r,Q_T} \leq \|w\|_{\ell,Q_T}^{1-\lambda_2}\|w\|^{\lambda_2}_{\frac{(2N-1)q}{N},Q_T}, \quad \lambda_2 = \frac{(2N-1)q(\ell-r)}{r[N\ell-(2N-1)q]}. \tag{2.45}$$

Use this in (2.43) to derive

$$\|w\|_{\infty,Q_T} \leq 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}$$
$$+c\|w\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}+(1-\lambda_2)\left(\frac{r-j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}\right)}\|w\|^{\frac{d}{\alpha(r-2j)}+\lambda_2\left(\frac{r-j}{r-2j}+\frac{j\ell}{(r-2j)(\ell-r)}\right)}_{\frac{(2N-1)q}{N},Q_T}. \tag{2.46}$$

We further require that the last exponent in the above inequality be 0, i.e.,

$$\frac{d}{\alpha(r-2j)} + \lambda_2\left(\frac{r-j}{r-2j} + \frac{j\ell}{(r-2j)(\ell-r)}\right) = 0. \tag{2.47}$$

Substitute λ_2 in (2.45) into this equation and then simplify to deduce

$$[Nd + (2N-1)\alpha q]\,\ell = (2N-1)\alpha qr - (2N-1)\alpha qj + (2N-1)qd. \tag{2.48}$$

Later we will show how to pick our parameters so that this new condition can be made consistent with our old assumptions. Assume that this is the case for the moment. Substitute (2.47) and (2.48) into (2.46) to deduce

$$\begin{aligned}
\|w\|_{\infty,Q_T} &\leq 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{\frac{r-j}{r-2j}+\frac{d}{\alpha(r-2j)}}\\
&= 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1+\frac{\alpha j+d}{\alpha(r-2j)}}\\
&= 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1+\frac{(2N-1)q(\alpha j+d)}{[Nd+(2N-1)\alpha q]\ell-(2N-1)q(d+\alpha j)}}\\
&\equiv 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1+\frac{M}{\ell-M}}.
\end{aligned} \tag{2.49}$$

That is, we have set

$$M = \frac{(2N-1)q(\alpha j+d)}{Nd+(2N-1)\alpha q}. \tag{2.50}$$

Plug d in (2.38) into M to obtain

$$\begin{aligned} M &= \frac{(2N-1)q\left[\frac{2(2N-1)[2(N-1)q-N-2]}{N[(2N-1)q-N-2]}-\frac{(N+2)\alpha j}{(2N-1)q-N-2}\right]}{(2N-1)\alpha q+\frac{2(2N-1)[(N-1)q-N-2]}{2(N-1)q-N-2}-\frac{\alpha jN(2N-1)q}{(2N-1)q-N-2}} \\ &= \frac{(N+2)\left[\frac{2(2N-1)[2(N-1)q-N-2]}{N(N+2)\alpha}-j\right]}{N\left[\frac{\alpha q[2(N-1)q-N-2]+2[(N-1)q-N-2]}{\alpha Nq}-j\right]}. \end{aligned} \tag{2.51}$$

It is easy to verify that

$$\frac{2(2N-1)[2(N-1)q-N-2]}{N(N+2)\alpha} > \frac{\alpha q[2(N-1)q-N-2]+2[(N-1)q-N-2]}{\alpha Nq} \tag{2.52}$$

at least for q large. We shall choose

$$j\in\left(0,\ \frac{\alpha q[2(N-1)q-N-2]+2[(N-1)q-N-2]}{\alpha Nq}\right). \tag{2.53}$$

Clearly, M is an increasing function of j. However, j can not be too close to 0 because (2.47) asserts that

$$d<0. \tag{2.54}$$

No matter what j is, we always have

$$M>\frac{N+2}{N}.$$

Since (2.9) and (2.10) only imply that

$$w\in L^{\frac{N+2}{N}}(Q_T),$$

(2.49) is not enough for the boundedness of w. To achieve this via (2.49), we must have

$$w\in L^{M+\varepsilon}(Q_T)\ \text{ for some } \varepsilon>0.$$

Indeed, (2.49) asserts

$$\begin{aligned} \|w\|_{\infty,Q_T} &\le 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\left(\|w\|_{\infty,Q_T}^{1-\frac{M+\varepsilon}{\ell}}\|w\|_{M+\varepsilon,Q_T}^{\frac{M+\varepsilon}{\ell}}\right)^{1+\frac{M}{\ell-M}} \\ &= 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+c\|w\|_{\infty,Q_T}^{\frac{\ell-M-\varepsilon}{\ell-M}}\|w\|_{M+\varepsilon,Q_T}^{\frac{M+\varepsilon}{\ell-M}} \\ &\le 8\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}+\delta\|w\|_{\infty,Q_T}+c(\delta)\|w\|_{M+\varepsilon,Q_T}^{\frac{M+\varepsilon}{\varepsilon}},\quad \delta>0. \end{aligned} \tag{2.55}$$

The last step is due to Young's inequality. In spite of this gap, we shall see that (2.49) still plays a crucial role in the eventual proof of Theorem 1.1. This is mainly due to the fact that we can choose j so that

$$\lim_{q\to\infty}M<\infty.$$

Let us explore this. First, under (2.53), both the numerator and denominator of the fraction in (2.51) are positive. That is,

$$\alpha j+d>0,\quad Nd+(2N-1)\alpha q>0.$$

We easily see from (2.48) that

$$(Nd+(2N-1)\alpha q)(\ell-r)=-Ndr-(2N-1)\alpha qj+(2N-1)dq>0.$$

This together with (2.54) asserts that

$$r>\frac{(2N-1)q(\alpha j-d)}{-Nd}. \tag{2.56}$$

For simplicity, we take

$$d = -1.$$

Subsequently, by (2.38)

$$j = \frac{(3N-2)(2N-1)q - (5N-2)(N+2)}{\alpha N(2N-1)q}. \tag{2.57}$$

This number satisfies (2.53) if q is large enough. The right-hand side of (2.56) becomes

$$\frac{(2N-1)\alpha qj - (2N-1)dq}{-Nd} = \frac{2(2N-1)^2 q - (5N - 2(N+2))}{N^2}.$$

Evaluate M from (2.50) to get

$$M_0 = M|_{d=-1} = \frac{2(N-1)(2N-1)q - (5N-2)(N+2)}{N[(2N-1)\alpha q - N]} \rightarrow \frac{(N-1)(N+2)}{N} \quad \text{as } q \rightarrow \infty. \tag{2.58}$$

In summary, we have shown:

Claim 2.3. *Let q be given as in (2.24). Define j and M_0 as in (2.57) and (2.58), respectively. Then there is a $q_0 > N+2$ such that for each $q > q_0$ and each*

$$\ell > \max\left\{\frac{(2N-1)\alpha qj + (2N-1)q}{N}, \frac{(2N-1)\alpha qj - (2N-1)q}{-N + (2N-1)\alpha q}\right\} \tag{2.59}$$

there is a constant c such that

$$\|w\|_{\infty,Q_T} \le c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1+\frac{M_0}{\ell - M_0}}. \tag{2.60}$$

Proof. We pick $q_0 > N+2$ so that (2.52), (2.53), and

$$-N + (2N-1)\alpha q > 0$$

all hold for $q > q_0$. Under the conditions of the claim, $d = -1$. Define r via (2.48), i.e.,

$$r = \frac{[-N + (2N-1)\alpha q]\,\ell + (2N-1)\alpha qj + (2N-1)q}{(2N-1)\alpha q} < \ell. \tag{2.61}$$

The last step is due to (2.59). It is easy to see from the proof of (2.56) that (2.44) holds. It remains to be seen that $r > 2j$. To this end, we derive from (2.61) and (2.59) that

$$(2N-1)\alpha q(r-2j) \quad = \quad [-N + (2N-1)\alpha q]\,\ell - (2N-1)\alpha qj + (2N-1)q > 0.$$

That is, (2.39) holds. Thus, all the previous arguments are valid. The claim follows. □

To complete the proof of (1.7) and (1.8), we need to make use of the fact that j can be large. That is, we must choose our parameters differently than what we did earlier. Let q be given as in Claim 2.3. Fix

$$\varepsilon_0 > 0.$$

Then we pick r, j so that

$$\alpha_1 \equiv \varepsilon_0 + \frac{j}{r-2j} \quad = \quad -\frac{d}{\alpha(r-2j)}, \tag{2.62}$$

where α is defined as in (2.26). Substitute (2.38) into this equation to yield

$$\alpha j \quad = \quad \frac{\varepsilon_0 N[(2N-1)q - N - 2]\alpha r + 2(2N-1)[(N-1)q - (N+2)]}{2\varepsilon_0 N[(2N-1)q - N - 2] + N(N+2)}. \tag{2.63}$$

Subsequently,

$$\alpha(r-2j) \quad = \quad \frac{N(N+2)\alpha r - 4(2N-1)[(N-1)q - N - 2]}{2\varepsilon_0 N[(2N-1)q - N - 2] + N(N+2)}.$$

Hence, we need to assume

$$\begin{aligned} r &> \frac{4(2N-1)[(N-1)q-N-2]}{\alpha N(N+2)} \\ &= \frac{2(2N-1)[(N-1)q-N-2]q}{(q-N-2)N} \\ &> \frac{(2N-1)q}{N}. \end{aligned} \tag{2.64}$$

As a result, (2.44) is guaranteed. We choose ℓ so that both (2.59) and (2.35) are satisfied.

Under our new choice of parameters, (2.43) reduces to

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{-\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{r,Q_T}^{1-\varepsilon_0+\alpha_1+\frac{j\ell}{(r-2j)(\ell-r)}}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{-\alpha_1}. \tag{2.65}$$

Observe that the total sum of the last three exponents is $1-\varepsilon_0$. However, we must make the last three different norms into a single one. The first step in this direction is still (2.45). Upon using it in (2.65), we derive

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0+\beta_2}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{-\beta_2}, \tag{2.66}$$

where

$$\beta_2 = -\lambda_2\left(1-\varepsilon_0+\frac{j\ell}{(r-2j)(\ell-r)}\right)+(1-\lambda_2)\alpha_1.$$

We claim

$$\beta_2 > 0. \tag{2.67}$$

To see this, we calculate from (2.45) and (2.62) that

$$\begin{aligned} &\beta_2 \\ &= -\lambda_2(1-\varepsilon_0) - \frac{\lambda_2 j\ell}{(r-2j)(\ell-r)} + (1-\lambda_2)\left(\varepsilon_0+\frac{j}{r-2j}\right) \\ &= \varepsilon_0 - \frac{(2N-1)q(\ell-r)}{r[N\ell-(2N-1)q]} - \frac{(2N-1)qj\ell}{r[N\ell-(2N-1)q](r-2j)} + \frac{\ell[Nr-(2N-1)q]j}{r[N\ell-(2N-1)q](r-2j)} \\ &= \frac{N\ell}{N\ell-(2N-1)q}\left(\frac{[N\ell-(2N-1)q]\varepsilon_0}{N\ell}+\frac{[Nr-2(2N-1)q]\ell j-(2N-1)q(\ell-r)(r-2j)}{Nr\ell(r-2j)}\right) \\ &\ \frac{N\ell}{N\ell-(2N-1)q}\left(\frac{[N\ell-(2N-1)q]\varepsilon_0}{N\ell}+\frac{N\ell\left(j-\frac{(2N-1)q}{N}\right)+(2N-1)q(r-2j)}{N\ell(r-2j)}\right). \end{aligned} \tag{2.68}$$

It is easy to see that under (2.64) αj given in (2.63) is an increasing function of ε_0. That is, we have

$$j > \frac{2(2N-1)[(N-1)q-N-2]}{N(N+2)\alpha} = \frac{(2N-1)[(N-1)q-N-2]q}{N(q-N-2)} > \frac{(2N-1)q}{N}.$$

This gives (2.67), from which follows that

$$\begin{aligned} \|w\|_{\ell,Q_T}^{\frac{N\ell\beta_2}{(2N-1)q}} &\leq \left(\|w\|_{\infty,Q_T}^{1-\frac{(2N-1)q}{N\ell}}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\frac{(2N-1)q}{N\ell}}\right)^{\frac{N\ell\beta_2}{(2N-1)q}} \\ &\leq \|w\|_{\infty,Q_T}^{\frac{[N\ell-(2N-1)q]\beta_2}{(2N-1)q}}\|w\|_{\frac{(2N-1)q}{N},Q_T}^{\beta_2}. \end{aligned}$$

Utilizing this in (2.66) yields

$$\begin{aligned}
\|w\|_{\infty,Q_T} &\leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0+\beta_2-\frac{N\ell\beta_2}{(2N-1)q}}\|w\|_{\infty,Q_T}^{\frac{[N\ell-(2N-1)q]\beta_2}{(2N-1)q}} \\
&\leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0}\left(\frac{\|w\|_{\infty,Q_T}}{\|w\|_{\ell,Q_T}}\right)^{\frac{[N\ell-(2N-1)q]\beta_2}{(2N-1)q}}. \qquad (2.69)
\end{aligned}$$

We would be tempted to take the limit $\ell \to \infty$ here. Unfortunately, the last exponent in the above inequality also goes to infinity with ℓ. To address this issue, we appeal to (2.60). Since ℓ, q satisfy the conditions of Claim 2.3, (2.60) is indeed available to us. Divide through the inequality by $\|u\|_{\ell,Q_T}$ to obtain

$$\frac{\|w\|_{\infty,Q_T}}{\|w\|_{\ell,Q_T}} \leq \frac{c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N}}{\|w\|_{\ell,Q_T}} + c\|w\|_{\ell,Q_T}^{\frac{M_0}{\ell-M_0}}. \qquad (2.70)$$

Without any loss of generality, we may assume

the second term in (2.70) $\leq$ the third term there for all ℓ satisfying (2.59).

Otherwise, we would have nothing more to prove. Equipped with this, we deduce

$$\left(\frac{\|w\|_{\infty,Q_T}}{\|w\|_{\ell,Q_T}}\right)^{\frac{[N\ell-(2N-1)q]\beta_2}{(2N-1)q}} \leq c\|w\|_{\ell,Q_T}^{\frac{M_0[N\ell-(2N-1)q]\beta_2}{(2N-1)q(\ell-M_0)}},$$

Substitute this into (2.69) to get

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T}^{1-\varepsilon_0+\frac{M_0[N\ell-(2N-1)q]\beta_2}{(2N-1)q(\ell-M_0)}}.$$

We must show that we can choose our parameters so that

$$\alpha_2 \equiv 1-\varepsilon_0+\frac{M_0[N\ell-(2N-1)q]\beta_2}{(2N-1)q(\ell-M_0)} = 1. \qquad (2.71)$$

To this end, we evaluate from (2.68) that

$$\begin{aligned}
&\frac{M_0[N\ell-(2N-1)q]\beta_2}{(2N-1)q(\ell-M_0)} \\
&= \frac{M_0N\ell}{(2N-1)q(\ell-M_0)}\left(\frac{[N\ell-(2N-1)q]\varepsilon_0}{N\ell} + \frac{N\ell\left(j-\frac{(2N-1)q}{N}\right)+(2N-1)q(r-2j)}{N\ell(r-2j)}\right) \\
&= \frac{M_0[N\ell-(2N-1)q]\varepsilon_0}{(2N-1)q(\ell-M_0)} + \frac{M_0\left[N\ell\left(j-\frac{(2N-1)q}{N}\right)+(2N-1)q(r-2j)\right]}{(2N-1)q(\ell-M_0)(r-2j)}.
\end{aligned}$$

In view of (2.58), we may choose q large enough so that

$$\frac{M_0[N\ell-(2N-1)q]}{(2N-1)q(\ell-M_0)} < 1.$$

As a result, we can find a positive ε_0 to satisfy (2.71). That is, we have

$$\begin{aligned}
\|w\|_{\infty,Q_T} &\leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\ell,Q_T} \\
&\leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\infty,Q_T}^{1-\frac{(N+2)}{N\ell}}\|w\|_{\frac{N+2}{N},Q_T}^{\frac{(N+2)}{N\ell}}.
\end{aligned}$$

An application of Young's inequality ([5], p. 145) produces

$$\|w\|_{\infty,Q_T} \leq c\|w(\cdot,0)\|_{\infty,\mathbb{R}^N} + c\|w\|_{\frac{N+2}{N},Q_T} \leq c.$$

If (2.32) is not true, we can use the result in (2.31) to obtain the boundedness of u. The proof is rather standard. We shall omit here. This completes the proof of Theorem 1.1. □

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