

# CHORDAL MATROIDS ARISING FROM GENERALIZED PARALLEL CONNECTIONS II

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**ABSTRACT.** In 1961, Dirac showed that chordal graphs are exactly the graphs that can be constructed from complete graphs by a sequence of clique-sums. In an earlier paper, by analogy with Dirac's result, we introduced the class of  $GF(q)$ -chordal matroids as those matroids that can be constructed from projective geometries over  $GF(q)$  by a sequence of generalized parallel connections across projective geometries over  $GF(q)$ . Our main result showed that when  $q = 2$ , such matroids have no induced minor in  $\{M(C_4), M(K_4)\}$ . In this paper, we show that the class of  $GF(2)$ -chordal matroids coincides with the class of binary matroids that have none of  $M(K_4)$ ,  $M^*(K_{3,3})$ , or  $M(C_n)$  for  $n \geq 4$  as a flat. We also show that  $GF(q)$ -chordal matroids can be characterized by an analogous result to Rose's 1970 characterization of chordal graphs as those that have a perfect elimination ordering of vertices.

## 1. INTRODUCTION

The notation and terminology in this paper will follow [5] for graphs and [10] for matroids. Unless stated otherwise, all graphs and matroids considered here are simple. Thus every contraction of a set from a matroid is immediately followed by the simplification of the resulting matroid. We will also assume all matroids are binary unless otherwise specified. Following Cordovil, Forge, and Klein [4], we define a simple or non-simple matroid  $M$  to be *chordal* if, for each circuit  $D$  that has at least four elements, there are circuits  $D_1$  and  $D_2$  and an element  $e$  such that  $D_1 \cap D_2 = \{e\}$  and  $D = (D_1 \cup D_2) - e$ . Therefore, a simple binary matroid is chordal precisely when it has no member of  $\{M(C_n) : n \geq 4\}$  as a flat, where  $C_n$  is the  $n$ -edge cycle. In Section 2, we prove an assertion made in [7] that the class of such matroids coincides with the class of binary matroids with no  $M(C_4)$  as an induced minor, where an *induced minor* of a matroid  $M$  is any matroid that can be obtained from  $M$  by a sequence of contractions and restrictions to flats. This

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matroid notion is analogous to a graph notion, an induced minor of a graph  $G$  being a graph that can be obtained from  $G$  by a sequence of vertex deletions and edge contractions.

For a prime power  $q$ , we denote the projective geometry  $PG(r-1, q)$  by  $P_r$  when context makes the field clear. A matroid is  $GF(q)$ -chordal if it can be obtained by repeated generalized parallel connections across projective geometries over  $GF(q)$  starting with projective geometries over  $GF(q)$ . In Section 3, we prove the next theorem, which is the main result of the paper, and we give an analogous result for  $q > 2$ . The equivalence of (i) and (ii) was shown in [7].

**Theorem 1.1.** *The following are equivalent for a binary matroid  $M$ .*

- (i)  $M$  is  $GF(2)$ -chordal.
- (ii)  $M$  has no member of  $\{M(C_4), M(K_4)\}$  as an induced minor.
- (iii)  $M$  has no member of  $\{M(C_n) : n \geq 4\} \cup \{M(K_4), M^*(K_{3,3})\}$  as an induced restriction.

In Section 3, we also prove the following analog of Theorem 1.1 for all other primes.

**Theorem 1.2.** *For each prime  $p > 2$ , the following are equivalent for a  $GF(p)$ -representable matroid  $M$ .*

- (i)  $M$  is  $GF(p)$ -chordal.
- (ii)  $M$  has no member of  $\{U_{2,k} : 3 \leq k \leq p\}$  as an induced minor.
- (iii)  $M$  has no member of  $\{U_{n,n+1} : n \geq 2\} \cup \{U_{2+t,k+t} : 4 \leq k \leq p \text{ and } 0 \leq t \leq p+1-k\}$  as an induced restriction.

Chordal graphs have been characterized in several other ways apart from Dirac's [6] description. A *perfect elimination ordering* of a graph  $G$  is an ordering of  $V(G)$  such that, for every vertex  $v$ , the graph induced by  $v$  and all of its neighbors that occur after  $v$  in the ordering is a clique. In 1970, Rose [11] proved the following characterization.

**Theorem 1.3.** *A graph  $G$  chordal if and only if  $G$  has a perfect elimination ordering.*

A *perfect elimination ordering of cocircuits* of a matroid  $M$  is a collection  $C_1^*, C_2^*, \dots, C_r^*$  such that, for all  $i$  in  $[r]$ , the set  $C_i^*$  is a cocircuit of the matroid  $M_i$ , which is  $M \setminus (C_1^* \cup C_2^* \cup \dots \cup C_{i-1}^*)$ , and  $M|_{\text{cl}_{M_i}(C_i^*)}$  is a projective geometry. In Section 4, we prove the following analog of Theorem 1.3 for  $GF(q)$ -chordal matroids.

**Theorem 1.4.** *A matroid  $M$  is  $GF(q)$ -chordal if and only if  $M$  has a perfect elimination ordering of cocircuits.*

## 2. BINARY CHORDAL MATROIDS

In this section, we will show that the class of binary chordal matroids coincides with the class of matroids with no  $M(C_4)$  as an induced minor. We then give a constructive characterization of such matroids.

**Lemma 2.1.** *Let  $n$  be the size of a largest circuit that is an induced minor of  $M$ . Then  $M$  has an  $n$ -element circuit as an induced restriction.*

*Proof.* We may assume that, for some independent set  $I$  of  $M$ , the matroid  $\text{si}(M/I)$  has an  $n$ -element circuit as a flat. If  $|I| = 0$ , then the result holds. Assume the result holds for  $|I| < k$ , and let  $|I| = k \geq 1$ . Take  $e \in I$ . Then  $\text{si}((M/e)/(I - e))$  has an  $n$ -element circuit as a flat. Certainly a largest circuit that occurs as an induced minor of  $\text{si}(M/e)$  has  $n$  elements. Thus, by the induction assumption,  $\text{si}(M/e)$  has, as an induced restriction, an  $n$ -element circuit  $C$  where  $C = \{e_1, e_2, \dots, e_n\}$ . Then  $C$  is a circuit of  $M/e \setminus Y$  for some set  $Y$ . Thus  $C$  or  $C \cup e$  is a circuit of  $M$ . We may assume that no  $n$ -element circuit is an induced restriction of  $M$ . Now view  $M$  as a restriction of the binary projective geometry  $P_r$  where  $r = r(M)$ . For each  $i$  in  $[n]$ , let  $f_i$  be the third point on the projective line  $\text{cl}_{P_r}(\{e, e_i\})$ . For each  $i$  in  $[n]$ , the set  $\{e_1, e_2, \dots, e_n\} - \{e_i\}$  is an independent set  $I_i$  of  $M/e$  and hence of  $M$ . Now  $\text{cl}_M(I_i)$  does not contain  $e$  otherwise  $I_i$  contains a circuit of  $M/e$ . Moreover,  $\text{cl}_M(I_i)$  does not contain  $e_i$  as  $M$  does not have an  $n$ -element circuit as an induced restriction. The projective flat  $\text{cl}_{P_r}(I_i)$  must meet  $\{e, f_i, e_i\}$  in  $P_r$ , so this intersection is  $f_i$ . If  $f_i$  is in  $E(M)$ , then  $M$  has  $I_i \cup f_i$  as an induced restriction that is an  $n$ -element circuit. Thus  $\{f_1, f_2, \dots, f_n\}$  avoids  $E(M)$ . We deduce that  $C \cup e$  is an  $(n+1)$ -element circuit of  $M$  that is an induced minor of  $M$ , a contradiction.  $\square$

In the next theorem, the equivalence of (i) and (iii) is an immediate consequence of the definition. In [7, Lemma 3.8], we had asserted the equivalence of (i) and (ii). Our proof of this relies on Lemma 2.1 and is more subtle than we originally thought, so we have included it.

**Theorem 2.2.** *The following are equivalent for a binary matroid  $M$ .*

- (i)  $M$  is chordal.
- (ii)  $M$  has no  $M(C_4)$  as an induced minor.
- (iii)  $M$  has no member of  $\{M(C_n) : n \geq 4\}$  as an induced restriction.

*Proof.* Clearly (iii) implies (ii). Now suppose that  $M$  has  $M(C_4)$  as an induced minor. Let  $n$  be the size of a largest circuit that is an induced minor of  $M$ . Since  $M$  has  $M(C_4)$  as an induced minor,  $n \geq 4$ . Then,

by Lemma 2.1,  $M$  has an  $n$ -element circuit as an induced restriction, that is,  $M$  has  $M(C_n)$  as an induced restriction for some  $n \geq 4$ .  $\square$

We now give a constructive characterization of binary chordal matroids. In a matroid  $M$ , denote a vertical  $k$ -separation  $(X, Y)$  by  $(X, G, Y)$  where  $G = \text{cl}_M(X) \cap \text{cl}_M(Y)$ . For a rank- $r$  binary matroid  $M$  that is viewed as a restriction of  $P_r$ , if  $X \subseteq E(P_r) - E(M)$ , we denote by  $M + X$  the matroid  $P_r|(E(M) \cup X)$ .

**Lemma 2.3.** *For some  $k \geq 2$ , let  $(X, G, Y)$  be a vertical  $k$ -separation of a binary matroid  $M$  where  $G = \emptyset$ . Then  $M$  has  $M(C_4)$  as an induced minor.*

*Proof.* Let  $C$  be a smallest circuit of  $\text{cl}_{P_r}(M)$  such that  $|C \cap (\text{cl}_{P_r}(X) \cap \text{cl}_{P_r}(Y))| = 1$  and  $C - (\text{cl}_{P_r}(X) \cap \text{cl}_{P_r}(Y)) \subseteq X$ . Let  $\{z\} = C \cap (\text{cl}_{P_r}(X) \cap \text{cl}_{P_r}(Y))$ , let  $a$  and  $b$  be distinct elements in  $C - z$ , and let  $M' = (M + z)/(C - \{a, b, z\})$  where we do not simplify here after doing this contraction. Then  $\{a, b, z\}$  is a triangle of  $M'$ , and, for some  $k' \leq k$ , the matroid  $M'$  has a vertical  $k'$ -separation  $(X', G', Y)$  where  $X' = X - (C - \{a, b, z\})$  and  $G' = \{z\}$ . To see this, observe that if  $|G'| > 1$ , then there is a circuit  $C' \cup z'$  in  $\text{cl}_{P_r}(M)$  where  $C'$  is contained in  $C - \{a, b, z\}$ , and  $\{z'\}$  is in  $\text{cl}_{P_r}(X) \cap \text{cl}_{P_r}(Y)$ . This contradicts the choice of  $C$  as a smallest circuit with these properties. Moreover, there is no element parallel to  $z$  in  $M'$ . Let  $D$  be a smallest circuit of  $M'$  such that  $D$  contains  $z$  and  $D - z$  is contained in  $Y$ . Note that  $D \cup \{a, b\}$  is a circuit of  $M'$ . Moreover,  $\text{cl}_{M'}(D \cup \{a, b, z\}) = D \cup \{a, b, z\}$ . Therefore,  $M' \setminus \{z\}$  has an  $n$ -element circuit as an induced restriction for some  $n \geq 4$ . By Theorem 2.2,  $M' \setminus \{z\}$  has  $M(C_4)$  as an induced minor. It follows that  $M$  has  $M(C_4)$  as an induced minor.  $\square$

**Lemma 2.4.** *For some  $k \geq 2$ , suppose  $(X, G, Y)$  is a vertical  $k$ -separation of a binary chordal matroid  $M$ . Then  $r_M(G) = k - 1$ .*

*Proof.* Suppose  $r(G) < k - 1$ . Then, for  $k' = k - r(G)$ , we see that  $k' \geq 2$  and  $M/G$  has a vertical  $k'$ -separation  $(X', G', Y')$  with  $G' = \emptyset$ . By Lemma 2.3,  $M$  has  $M(C_4)$  as an induced minor, a contradiction to Theorem 2.2. Therefore,  $r(G) = k - 1$ .  $\square$

In the next two proofs, we allow the matroids to be non-simple and we do not simplify after contracting. The next result seems unlikely to be new, but we include a proof for completeness.

**Lemma 2.5.** *If  $G$  is a flat of a matroid  $N$  and  $G - g$  is a modular flat of  $N/g$  for some  $g$  in  $G$ , then  $G$  is modular flat of  $N$ .*

*Proof.* The result is immediate if  $g$  is a loop of  $N$ , so assume  $r(\{g\}) = 1$ . For all flats  $F$  of  $N/g$ ,

$$r_{N/g}(F) + r_{N/g}(G - g) = r_{N/g}(F \cap (G - g)) + r_{N/g}(F \cup (G - g)). \quad (2.1)$$

We shall show that, if  $H$  is a flat of  $N$ , then

$$r_N(H) + r_N(G) - r_N(H \cap G) - r_N(H \cup G) = 0. \quad (2.2)$$

Suppose  $g \in H$ . Then  $r_{N/g}(H - g) = r_N(H) - 1$  and  $r_{N/g}((H \cup G) - g) = r_N(H \cup G) - 1$ . Therefore, since (2.1) holds, so does (2.2).

Now suppose  $g \notin H$ . Then  $r_{N/g}(H) = r_N(H \cup g) - 1$  and  $r_{N/g}(H \cap G) = r_N((H \cap G) \cup g) - 1$ . Since  $H$  and  $G$  are flats of  $N$ , it follows that  $H \cap G$  is a flat of  $N$ . Therefore,  $r_N(H \cup g) = r_N(H) + 1$  and  $r_N((H \cap G) \cup g) = r_N(H \cap G) + 1$ . Thus (2.2) holds and the lemma is proved.  $\square$

**Lemma 2.6.** *For some  $k \geq 2$ , let  $(X, G, Y)$  be a vertical  $k$ -separation of a binary chordal matroid  $M$ . Then  $G$  is a modular flat of  $M|cl(X)$  or of  $M|cl(Y)$ .*

*Proof.* If  $k = 2$ , then, by Lemma 2.4,  $r(G) = 1$ . Moreover,  $M$  is a parallel connection of  $M|cl(X)$  and  $M|cl(Y)$ . By Theorem 2.2, both of these matroids are chordal. As  $G$  is a single point, it is a modular flat in both  $M|cl(X)$  and  $M|cl(Y)$  and the result holds. Suppose the result holds for all  $k < n$  and let  $(X, G, Y)$  be a vertical  $n$ -separation of  $M$ . Then, for any  $g$  in  $G$ , the matroid  $M/g$  has a vertical  $(n - 1)$ -separation  $(X - g, G - g, Y - g)$ . By the induction assumption,  $G - g$  is a modular flat in either  $(M/g)|cl(X - g)$  or  $(M/g)|cl(Y - g)$ . Therefore, by Lemma 2.5, it follows that  $G$  is a modular flat of  $M|cl(X)$  or of  $M|cl(Y)$ .  $\square$

**Theorem 2.7.** *All binary chordal matroids can be obtained by starting with round binary chordal matroids and repeatedly taking generalized parallel connections of two previously constructed matroids across a set that is a modular flat of one of them.*

*Proof.* Let  $M$  be a binary chordal matroid. If  $M$  has no vertical  $k$ -separations for any  $k$ , then  $M$  is round and the result holds. If  $M$  has a vertical  $k$ -separation  $(X, G, Y)$  for some  $k > 1$ , then, by Lemma 2.6,  $M$  is a generalized parallel connection of  $M|cl(X)$  and  $M|cl(Y)$  across  $M|cl(G)$ , and  $G$  is a modular flat of  $M|cl(X)$  or of  $M|cl(Y)$ .  $\square$

### 3. GF(q)-CHORDAL MATROIDS

The goal of this section is to prove Theorem 1.1. Recall that a matroid is  $GF(2)$ -chordal if it can be built from binary projective geometries by repeated generalized parallel connections across projective

geometries. In [7, Theorem 1.3], we showed that the class of  $GF(2)$ -chordal matroids is closed under taking induced minors and therefore, the class is also closed under taking induced restrictions. For a binary matroid  $M$ , we may uniquely specify  $M$  by describing its complement in the projective geometry  $P_{r(M)}$ . In general, Brylawski and Lucas [3] (see also [10, Proposition 10.1.7]) showed that the complement of any uniquely  $GF(q)$ -representable matroid  $M$  is well defined in any projective geometry of rank at least  $r(M)$ . We will first examine the matroids that have  $M(K_4)$  as an induced minor.

**Lemma 3.1.** *If  $M$  is a simple binary matroid with  $r(M) = 4$  and  $|E(M)| > 9$ , then either  $M$  has  $M(C_4)$  or  $M(K_4)$  as an induced restriction, or  $M$  does not contain  $M(K_4)$  as an induced minor.*

*Proof.* Let  $P_4$  be the rank-4 binary projective geometry. Suppose  $T$  is a largest subset of  $E(P_4)$  such that  $P_4|T$  contains  $M(K_4)$  as an induced minor, but  $T$  does not contain one of  $M(C_4)$  or  $M(K_4)$  as an induced restriction. Certainly,  $P_4$  does not contain  $M(K_4)$  as an induced minor and so  $|T| < 15$ . If  $|T| \in \{13, 14\}$ , then  $P_4|T$  certainly has  $M(K_4)$  as an induced restriction. If  $|T| = 12$ , then  $P_4 \setminus T$  is either  $P_2$  or  $U_{3,3}$ . If  $P_4 \setminus T$  is  $P_2$ , then  $P_4|T$  has  $M(C_4)$  as an induced restriction. If  $P_4 \setminus T$  is  $U_{3,3}$ , then  $P_4|T$  will again contain  $M(K_4)$  as an induced restriction. If  $|T| = 11$ , then  $P_4 \setminus T$  is  $M(C_4)$ ,  $U_{4,4}$ , or  $P_2 \oplus U_{1,1}$ . When  $P_4 \setminus T$  is  $M(C_4)$ , we see that  $P_4|T \cong P_{P_2}(P_3, P_3)$ , so it has no element whose contraction gives  $M(K_4)$ . If  $P_4 \setminus T$  is  $U_{4,4}$ , then  $P_4|T$  has a plane with exactly six points, so  $P_4|T$  has  $M(K_4)$  as an induced restriction. If  $P_4 \setminus T$  is  $P_2 \oplus U_{1,1}$ , then  $P_4|T$  has  $M(C_4)$  as an induced restriction. If  $|T| = 10$ , then  $P_4 \setminus T$  is  $M(K_4 - e)$ ,  $M(C_4) \oplus U_{1,1}$ , or  $P_2 \oplus U_{2,2}$ . In each case,  $P_4|T$  has  $M(K_4)$  as an induced restriction.  $\square$

**Lemma 3.2.** *If  $M/f$  has  $M(K_4)$  as an induced restriction for some  $f$  in  $E(M)$ , then  $M$  has  $M(C_4)$ ,  $M(K_4)$ , or  $M^*(K_{3,3})$  as an induced restriction.*

*Proof.* It suffices to show the result holds for  $r(M) = 4$  since we may restrict to a rank-4 flat  $F$  such that  $(M|F)/f \cong M(K_4)$ . Assume  $M$  does not have any of  $M(C_4)$ ,  $M(K_4)$ , or  $M^*(K_{3,3})$  as an induced restriction. Since  $M/f$  is isomorphic to  $M(K_4)$ , it follows that  $|E(M)| \geq 7$ , and, by Lemma 3.1, we have  $|E(M)| \leq 9$ . Certainly  $M$  is connected. If  $M$  is not 3-connected, then  $M$  is isomorphic to the 2-sum or the parallel connection of  $U_{2,3}$  and  $M(K_4)$ . In the former case,  $M$  has  $M(C_4)$  as an induced restriction, and, in the latter case,  $M$  has  $M(K_4)$  as an induced restriction. Thus we may assume  $M$  is 3-connected. Suppose  $M$  is graphic. Then  $M$  is obtained from  $M(K_5)$  by deleting at most

two edges. If two incident edges have been deleted from  $K_5$ , then  $M$  is not 3-connected. Therefore  $M$  is isomorphic to  $M(K_5 \setminus e)$  for some edge  $e$ , or  $M$  is isomorphic to  $M(W_4)$ . In the first case,  $M$  has  $M(K_4)$  as an induced restriction; in the second case,  $M$  has  $M(C_4)$  as an induced restriction.

We may now assume that  $M$  is not graphic. Then  $M$  has one of  $M^*(K_5)$ ,  $M^*(K_{3,3})$ ,  $F_7$ , or  $F_7^*$  as a minor by a theorem of Tutte [13] (see also [10, Theorem 6.6.7]). Since  $|E(M)| \leq 9$ , we have  $M \not\cong M^*(K_5)$ . If  $M \cong M^*(K_{3,3})$ , the result holds. If  $r^*(M) = 3$ , then  $M \cong F_7^*$ , and thus  $M$  has  $M(C_4)$  as an induced restriction, a contradiction. If  $r^*(M) = 4$ , then, by a result of Seymour [12] (see also [10, Lemma 12.2.4]),  $M$  is isomorphic to either  $AG(3, 2)$  or  $S_8$ , and, in each case,  $M$  has  $M(C_4)$  as an induced restriction, a contradiction. By the Splitter Theorem, if  $|E(M)| = 9$ , then  $M$  is an extension of  $AG(3, 2)$  or  $S_8$ . Since the rank-4 complements of  $AG(3, 2)$  and  $S_8$  are  $F_7$  and  $M(K_4) \oplus U_{1,1}$ , respectively, there are exactly two possible 9-element matroids as extensions of  $AG(3, 2)$  and  $S_8$ , that is,  $M$  is the rank-4 complement of  $M(K_4)$  or the rank-4 complement of  $M(K_4 \setminus e) \oplus U_{1,1}$ . In each case,  $M$  has  $M(C_4)$  as an induced restriction, a contradiction.  $\square$

To complete the proof of Theorem 1.1, we only need to consider the matroids with  $M^*(K_{3,3})$  as an induced minor.

**Lemma 3.3.** *If  $M/e$  has  $M^*(K_{3,3})$  as an induced minor, then  $M$  contains one of  $M(C_4)$ ,  $M(K_4)$ , or  $M^*(K_{3,3})$  as an induced restriction.*

*Proof.* It suffices to show the result holds when  $r(M) = 5$  since we may restrict to the relevant rank-5 flat. Assume  $M$  does not have any member of  $\{M(C_4), M(K_4), M^*(K_{3,3})\}$  as an induced restriction. Let  $e_1, e_2, e_3, e_4$  be the standard basis for  $P_4$ . Then we may assume that the ground set  $Z$  of the  $M^*(K_{3,3})$ -restriction of  $M/e$  is  $\{e_1, e_2, e_3, e_4, e_1 + e_2, e_2 + e_3, e_3 + e_4, e_1 + e_4, e_1 + e_2 + e_3 + e_4\}$ . Then the ground set of  $M$  may only contain  $e$ , elements of  $Z$ , and elements of the form  $e + f$  where  $f$  is an element of  $Z$ . Let  $T$  a rank-4 flat of  $M$  such that  $(M/e)|_{\text{cl}_{M/e}(T)} \cong M^*(K_{3,3})$  and  $|T|$  is maximal with this property. Certainly  $|T| < 9$ . Observe that, for every rank-4 flat  $F$  of  $M$  that avoids  $e$ , we have  $(M/e)|_{\text{cl}_{M/e}(F)} \cong M^*(K_{3,3})$ .

**3.3.1.**  $|T| < 8$ .

If  $|T| = 8$ , then  $M|_T$  is isomorphic to a 4-wheel, which has  $M(C_4)$  as an induced restriction, a contradiction. Thus 3.3.1 holds.

**3.3.2.**  $|T| < 7$ .

If  $|T| = 7$ , then, to avoid having  $M(C_4)$  as an induced restriction, we may assume  $T = \{e_1, e_3, e_4, e_1 + e_4, e_2 + e_3, e_3 + e_4, e_1 + e_2 + e_3 + e_4\}$ . Therefore,  $E(M)$  must contain  $\{e, e + e_2, e + e_1 + e_2\}$ . Look at the flat containing  $\{e_3, e_2 + e_3, e + e_2, e\}$ , noting that  $e_2$  is absent. Since this flat is not isomorphic to  $M(C_4)$ , it must contain either  $e + e_3$  or  $e + e_2 + e_3$ . Suppose that  $e + e_2 + e_3$  is present. Then  $M$  contains  $\{e_1, e_1 + e_4, e_4, e_3 + e_4, e_3, e + e_2 + e_3, e + e_2, e + e_1 + e_2\}$  as eight elements of a rank-4 flat  $F$  that avoids  $e$ . By 3.3.1,  $|F| \leq 7$ , a contradiction. We deduce that  $e + e_2 + e_3$  is absent, and hence  $e + e_3$  must be present. In this case,  $M$  has  $\{e_1, e_4, e_1 + e_4, e_1 + e_2 + e_3 + e_4, e_2 + e_3, e + e_3, e + e_2, e + e_1 + e_2\}$  as eight elements of a rank-4 flat that avoids  $e$ , and again we contradict 3.3.1. We conclude that, 3.3.2 holds.

**3.3.3.** *The set  $Z - T$  does not contain a triangle.*

Without loss of generality, suppose  $Z - T$  contains the triangle  $\{e_1, e_2, e_1 + e_2\}$ . Then  $M$  must contain the 4-element circuit  $\{e, e + e_1, e + e_2, e + e_1 + e_2\}$  as a flat, a contradiction. Therefore, 3.3.3 holds.

**3.3.4.**  $|T| < 6$ .

If  $|T| = 6$ , then, since  $M$  does not have  $M(C_4)$  as an induced restriction and  $Z - T$  does not contain a triangle, we may assume  $T$  is missing  $\{e_2, e_1 + e_2, e_2 + e_3\}$ . Then  $M$  contains  $\{e + e_2, e + e_1 + e_2, e + e_2 + e_3\}$ . Thus  $M$  contains  $\{e_1, e + e_1 + e_2, e + e_2, e + e_2 + e_3, e_3, e_3 + e_4, e_4, e_1 + e_4\}$  as eight elements of a rank-4 flat of  $M$  that avoids  $e$ , a contradiction to 3.3.1. Therefore, 3.3.4 holds.

**3.3.5.**  $|T| < 5$ .

If  $|T| = 5$ , then, since  $M$  does not have  $M(C_4)$  as an induced restriction and  $Z - T$  does not contain a triangle, we may assume  $T$  is missing one of the following four sets of points.

- (a)  $\{e_2, e_3, e_3 + e_4, e_1 + e_2\}$
- (b)  $\{e_2, e_2 + e_3, e_3 + e_4, e_1 + e_2\}$
- (c)  $\{e_1, e_2, e_3 + e_4, e_1 + e_2 + e_3 + e_4\}$

In case (a),  $M$  has  $\{e_1, e + e_1 + e_2, e + e_2, e + e_3, e_2 + e_3, e_1 + e_2 + e_3 + e_4, e_1 + e_4, e_4\}$  as eight elements of a rank-4 flat of  $M$  that avoids  $e$ , which contradicts 3.3.1. In case (b),  $\{e_1, e + e_1 + e_2, e + e_2, e + e_2 + e_3, e_3, e_4, e_1 + e_4\}$  spans a rank-4 flat  $F$  of  $M$  that avoids  $e$  and has at least seven elements, which contradicts 3.3.2. In case (c),  $M$  has  $\{e + e_2, e + e_2 + e_3, e_3, e_4, e + e_1, e_1 + e_2\}$  contained in a rank-4 flat of  $M$  avoiding  $e$ , which contradicts 3.3.4. Therefore, 3.3.5 holds.

If  $|T| = 4$ , then, as  $T$  must be independent and  $Z - T$  contains no triangles, we may assume that  $T = \{e_1, e_1 + e_2, e_3, e_1 + e_2 + e_3 + e_4\}$ .



This implies that  $M$  has  $\{e_1, e + e_2, e + e_2 + e_3, e_3, e + e_3 + e_4\}$  spanning a rank-4 flat of  $M$  that avoids  $e$ , a contradiction to 3.3.5.  $\square$

The next result establishes a connection between the excluded induced minors of a class of matroids and the excluded induced restrictions of that class of matroids.

**Lemma 3.4.** *Suppose  $\mathcal{N}$  is a class of matroids such that if a matroid  $M$  has an element  $e$  such that  $M/e$  is isomorphic to a member of  $\mathcal{N}$ , then  $M$  has a member of  $\mathcal{N}$  as an induced restriction. Let  $\mathcal{N}'$  be the class of induced-minor-minimal members of  $\mathcal{N}$ . Then the class of matroids with no member of  $\mathcal{N}'$  as an induced minor coincides with the class of matroids with no member of  $\mathcal{N}$  as an induced restriction.*

*Proof.* Clearly if  $M$  has a member of  $\mathcal{N}$  as an induced restriction, then  $M$  has a member of  $\mathcal{N}'$  as an induced minor. Conversely, suppose  $M$  has a member of  $\mathcal{N}'$  as an induced minor. We may assume that  $M$  has no member of  $\mathcal{N}'$  as an induced restriction. Then  $M$  has a flat  $F$  and a nonempty independent set  $T$  such that  $(M|F)/T$  is isomorphic to a member of  $\mathcal{N}'$  and hence a member of  $\mathcal{N}$ . We argue by induction on  $|T|$  that  $M$  has a member of  $\mathcal{N}$  as an induced restriction.

When  $|T| = 1$ , we see that the assertion holds by the definition of  $\mathcal{N}$ . Now assume the result holds for  $|T| < n$ , and let  $|T| = n \geq 2$ . Let  $M_1 = M|F$ , and take  $e$  in  $T$ . Then  $(M_1/e)/(T - e)$  is isomorphic to a member of  $\mathcal{N}$ . Thus, by the induction assumption,  $M_1/e$  has a member of  $\mathcal{N}$  as an induced restriction. Then, by the induction assumption again,  $M_1$  has a member of  $\mathcal{N}$  as an induced restriction, and hence  $M$  has a member of  $\mathcal{N}$  as an induced restriction.  $\square$

*Proof of Theorem 1.1.* Let  $\mathcal{N} = \{M(C_n) : n \geq 4\} \cup \{M(K_4), M^*(K_{3,3})\}$ . Then, by Lemmas 2.1, 3.2, and 3.3, the set  $\mathcal{N}$  has the property that if  $M/e$  is isomorphic to a member of  $\mathcal{N}$ , then  $M$  has a member of  $\mathcal{N}$  as an induced minor. Since  $\{M(C_4), M(K_4)\}$  is the set of induced-minor-minimal members of  $\mathcal{N}$ , the result holds by Lemma 3.4.  $\square$

Next we prove the analog of Theorem 1.1 when  $q > 2$ . Although the result is rather unattractive, we include it for completeness. The equivalence of (i) and (ii) was shown in [7]. Because  $U_{3,q+2}$  is  $GF(q)$ -representable if and only if  $q$  is even, when  $q$  is odd, we can omit  $U_{3,q+2}$  from (ii). This corrects a small error in Theorem 1.4 of [7]. Note that some matroids in (iii) may be excluded as they fail to be  $GF(q)$ -representable. But the problem of determining precisely which uniform matroids are  $GF(q)$ -representable is open. For results on exactly which uniform matroids are known to be  $GF(q)$ -representable, we direct the

reader to [1, 2, 8, 9] (see also [10, Conjecture 6.5.20]). Recall that the class of  $GF(q)$ -chordal matroids is the class of matroids that can be obtained by a sequence of generalized parallel connections across projective geometries over  $GF(q)$  starting with projective geometries over  $GF(q)$ .

**Theorem 3.5.** *When  $q > 2$ , the following are equivalent for a  $GF(q)$ -representable matroid  $M$ .*

- (i)  $M$  is  $GF(q)$ -chordal.
- (ii)  $M$  has no member of  $\{U_{2,k} : 2 < k \leq q\} \cup \{U_{3,q+2}\}$  as an induced minor.
- (iii)  $M$  has no member of  $\{U_{n,n+1} : n \geq 2\} \cup \{U_{2+t,k+t} : 4 \leq k \leq q \text{ and } 0 \leq t \leq q-3\} \cup \{U_{3,q+2}\}$  as an induced restriction.

**Lemma 3.6.** *Suppose  $M$  is  $GF(q)$ -representable. If  $M/e$  has  $U_{2,n}$  as an induced minor for some  $n$  with  $3 \leq n \leq q$ , then  $M$  has a member of  $\{U_{2,k} : 3 \leq k \leq q\} \cup \{U_{3,n+1}\}$  as an induced restriction.*

*Proof.* It suffices to show that the result holds for  $r(M) = 3$  since we may restrict to the relevant rank-3 flat. Let  $E(M/e) = \{x_1, x_2, \dots, x_k\}$ . If  $M$  has a  $(q+1)$ -point line, then  $e$  must be on that line, otherwise,  $M/e$  would be  $U_{2,q+1}$ . Without loss of generality, suppose  $\text{cl}(\{e, x_k\})$  is a  $(q+1)$ -point line,  $L_1$ . Then the line  $L_2 = \text{cl}(\{x_1, x_2\})$  must intersect  $L_1$  at a point different from  $e$ . Therefore,  $L_2$  is either isomorphic to  $U_{2,k}$  for some  $k$  with  $3 \leq k \leq q$ , in which case the result holds, or  $L_2$  is a  $(q+1)$ -point line, and  $M/e$  is isomorphic to  $U_{2,q+1}$ , a contradiction. Therefore, we may assume that  $M$  has no  $(q+1)$ -point lines. Let  $X$  be a 3-element subset of  $E(M)$ . Since  $M$  has no member of  $\{U_{2,k} : 3 \leq k \leq q\}$  as an induced restriction, and  $M$  has no  $(q+1)$ -point lines, it follows that  $X$  is independent. Therefore,  $M$  is isomorphic to  $U_{3,n+1}$ .  $\square$

To complete the proof Theorem 3.5, we consider the following general result for uniform matroids.

**Lemma 3.7.** *Let  $M$  be  $GF(q)$ -representable for some  $q > 2$  and suppose  $M/e \cong U_{r,n}$  for some  $r$  and  $n$  with  $2 < r < n$ . Then  $M$  has a member of  $\{U_{2+t,k+t} : 3 \leq k \leq q \text{ and } 0 \leq t \leq q-3\} \cup \{U_{r,n}, U_{r+1,n+1}\}$  as an induced restriction.*

*Proof.* Assume the result fails. Observe that, since  $U_{2,q+2}$  is not  $GF(q)$ -representable, we must have  $n \leq q + r - 1$ . Next note that if  $M$  has a hyperplane  $H_0$  that avoids  $e$ , then  $M|H_0$  must be isomorphic to a uniform matroid for if  $H_0$  contains a non-spanning circuit  $C$ , then  $M/e$  must also contain  $C$  as a non-spanning circuit. Moreover,  $H_0$  can contain at most  $n$  elements as  $|E(M/e)| = n$ . Therefore,  $M|H_0$

is isomorphic to  $U_{r,s}$  for some  $s$  with  $r \leq s \leq n$ . Suppose that  $H_0$  can be chosen so that  $s > r$ . If  $s = n$ , then  $M$  has  $U_{r,n}$  as an induced restriction, a contradiction. Thus we may suppose that  $s < n$ . Observe that if  $r > q-1$ , then, since  $U_{q,q+2}$  is not  $GF(q)$ -representable, we must have  $n = r+1$ , so  $s = n$ , a contradiction. Thus we may assume that  $r \leq q-1$ . Then we obtain a contradiction by taking  $t = r-2$  and  $k = s-r+2$  for then  $t \leq q-3$  and  $k < n-r+2 \leq (q+r-1)-r+2 = q+1$ . Hence we may assume that every hyperplane that avoids  $e$  is an independent set of size  $r$ .

Let  $E(M/e) = \{x_1, x_2, \dots, x_n\}$ . If  $M$  has a  $(q+1)$ -point line, this line must contain  $e$  as  $M/e$  does not have a  $(q+1)$ -point line. Without loss of generality, suppose  $\text{cl}_M(\{e, x_n\})$  is a  $(q+1)$ -point line,  $L$ . Then  $\text{cl}_M(\{x_1, x_2, \dots, x_r\})$  is a hyperplane  $H$  of  $M$  that meets  $L$  at a point different from  $e$ . Therefore  $H$  is a hyperplane that avoids  $e$  and contains a circuit of  $M$ , a contradiction. We conclude that  $M$  has no  $(q+1)$ -point lines. Therefore, either  $e$  is contained in a  $U_{2,k}$  induced restriction for some  $k$  with  $3 \leq k \leq q$ , and the result holds, or every circuit containing  $e$  has size  $r+2$ . Let  $X$  be a subset of  $E(M) - e$ . If  $|X| < r+1$ , then  $X$  is certainly independent as  $X$  must be independent in  $M/e$ . Suppose  $|X| = r+1$  and that  $X$  is dependent. Then  $\text{cl}_M(X)$  is a hyperplane that avoids  $e$  and contains a circuit of  $M$ , a contradiction. Therefore,  $X$  is an independent set of size  $r+1$ , and  $M$  is isomorphic to  $U_{r+1,n+1}$ .  $\square$

*Proof of Theorem 3.5.* It suffices to show the equivalence of (ii) and (iii). Let  $\mathcal{N} = \{U_{n,n+1} : n \geq 2\} \cup \{U_{2+t,k+t} : 4 \leq k \leq q \text{ and } 0 \leq t \leq q-3\} \cup \{U_{3,q+2}\}$ . Then, since  $U_{4,q+3}$  is not  $GF(q)$ -representable for any  $q$ , it follows by Lemmas 3.6 and 3.7 that  $\mathcal{N}$  has the property that if  $M/e$  is isomorphic to a member of  $\mathcal{N}$ , then  $M$  has a member of  $\mathcal{N}$  as an induced minor. Since  $\{U_{2,k} : 3 \leq k \leq q\} \cup \{U_{3,q+2}\}$  is the set of induced-minor-minimal members of  $\mathcal{N}$ , the result holds by Lemma 3.4.

Note if  $M$  has  $U_{2+t,k+t}$  as an induced restriction for some  $t$  and  $k$  with  $t \geq q-2$  and  $k \geq 4$ , then  $M$  has  $U_{q,q+2}$  as a minor, so  $M$  is not  $GF(q)$ -representable, a contradiction.  $\square$

Theorem 1.2 follows from Theorem 3.5 and a result of Ball [1] (see also [10, Theorem 6.5.21]), which establishes precisely when a uniform matroid is  $GF(p)$ -representable for  $p$  a prime.

When  $p = 3$ , Theorem 1.2 gives the following.

**Corollary 3.8.** *The following are equivalent for a ternary matroid  $M$ .*

- (i)  $M$  is  $GF(3)$ -chordal.
- (ii)  $M$  has no  $M(C_3)$  induced minor.

- (iii)  $M$  has no member of  $\{M(C_n) : n \geq 3\}$  as an induced restriction.

Similarly, Theorem 3.5 gives the following when  $q = 4$ .

**Corollary 3.9.** *The following are equivalent for a matroid  $M$  that is representable over  $GF(4)$ .*

- (i)  $M$  is  $GF(4)$ -chordal.
- (ii)  $M$  has no member of  $\{U_{2,3}, U_{2,4}, U_{3,6}\}$  as an induced minor.
- (iii)  $M$  has no member of  $\{U_{n,n+1} : n \geq 2\} \cup \{U_{2,4}, U_{3,5}, U_{3,6}\}$  as an induced restriction.

#### 4. PERFECT ELIMINATION ORDERING

In this section, we will prove Theorem 1.4. Recall that a perfect elimination ordering of cocircuits of a matroid  $M$  is a collection of sets  $C_1^*, C_2^*, \dots, C_r^*$  such that, for all  $i$  in  $[r]$ , the set  $C_i^*$  is a cocircuit of the matroid  $M_i = M \setminus (C_1^* \cup C_2^* \cup \dots \cup C_{i-1}^*)$  and  $M|_{\text{cl}_{M_i}(C_i^*)}$  is a projective geometry. Observe that  $M|_{\text{cl}_{M_i}(C_i^*)} = M_i|_{\text{cl}_{M_i}(C_i^*)}$  and that  $E(M_i)$  is a flat of  $M$  of rank  $r(M) - i + 1$ . We omit the straightforward proof of the next result.

**Lemma 4.1.** *Projective geometries have a perfect elimination ordering of cocircuits.*

We now prove the second main result of the paper.

*Proof of Theorem 1.4.* Suppose  $M$  is a  $GF(q)$ -chordal matroid that does not have a perfect elimination ordering of cocircuits and has  $|E(M)|$  a minimum among such matroids. Since  $M$  is  $GF(q)$ -chordal,  $M$  may be written as a  $P_N(M_1, M_2)$  where  $N$  is a projective geometry and  $M_2$  is a projective geometry. Therefore,  $M_2$  has a perfect elimination ordering of cocircuits. Choose  $C_1^*$  to be a cocircuit of  $M_2$  that avoids  $E(N)$ . By the minimality of  $M$ , the matroid  $M \setminus C_1^*$  has a perfect elimination ordering of cocircuits  $C_2^*, C_3^*, \dots, C_r^*$ . Therefore,  $M$  has a perfect elimination ordering.

Suppose  $M$  has a perfect elimination ordering of cocircuits labeled  $C_1^*, C_2^*, \dots, C_r^*$ . Let  $M_1 = M$ , and let  $M_i = M \setminus (C_1^* \cup C_2^* \cup \dots \cup C_{i-1}^*)$  for each  $i$  with  $2 \leq i \leq r$ . Certainly,  $M_r$  is a projective geometry, and therefore is  $GF(q)$ -chordal. Let  $k$  be the smallest integer such that  $M_k$  is not  $GF(q)$ -chordal. Then  $M_{k+1}$  is  $GF(q)$ -chordal,  $M_k|_{\text{cl}_{M_k}(C_k^*)}$  is a projective geometry, and  $M_{k+1}$  is a hyperplane of  $M_k$ . Thus  $E(M_{k+1}) \cap E(M_k|_{\text{cl}_{M_k}(C_k^*)})$  is a flat,  $N$ , of a projective geometry and so  $M|_N$  is a projective geometry. Hence  $M_k$  is a generalized parallel connection of  $GF(q)$ -chordal matroids across a projective geometry,

namely  $P_{M|N}(M_{k+1}, M|_{\text{cl}_{M_k}}(C_k^*))$ . Thus,  $M_k$  is  $GF(q)$ -chordal, a contradiction.  $\square$

On combining Theorems 1.1 and 1.4, we obtain the following.

**Corollary 4.2.** *The following are equivalent for a binary matroid  $M$ .*

- (i)  *$M$  is  $GF(2)$ -chordal.*
- (ii)  *$M$  has no member of  $\{M(C_4), M(K_4)\}$  as an induced minor.*
- (iii)  *$M$  has no member of  $\{M(C_n) : n \geq 4\} \cup \{M(K_4), M^*(K_{3,3})\}$  as an induced restriction.*
- (iv)  *$M$  has a perfect elimination ordering of cocircuits.*

## REFERENCES

- [1] Ball, S., On sets of vectors of a finite vector space in which every subset of basis size is a basis, *J. Eur. Math. Soc.* **14** (2012), 733–748
- [2] Ball, S., and De Beule, J., On sets of vectors of a finite vector space in which every subset of basis size is a basis II, *Des. Codes Cryptogr.* **65** (2012), 5–14
- [3] Brylawski, T.H., and Lucas, D., Uniquely representable combinatorial geometries, *Teorie Combinatorie* (Proc. 1973 Internat. Colloq.), pp. 83–104, Accademia Nazionale dei Lincei, Rome, 1976
- [4] Cordovil, R., Forge, D., and Klein, S., How is a chordal graph like a supersolvable binary matroid?, *Discrete Math.* **288** (2004), 167–172.
- [5] Diestel, R., *Graph Theory*, Third Edition, Springer, Berlin, 2005.
- [6] Dirac, G.A., On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25** (1961), 71–76.
- [7] Douthitt, J.D., and Oxley, J., Chordal matroids arising from generalized parallel connections, *Adv. in Appl. Math.* **153** (2024) 102631, 10 pp.
- [8] Hirschfeld, J.W.P., and Storme, L., The packing problem in statistics, coding theory and finite projective spaces: update 2001, *Finite Geometries*, 201–246, *Dev. Math.* **3**, Kluwer Acad. Publ., Dordrecht, 2001.
- [9] Hirschfeld, J.W.P., and Thas, J.A., *General Galois Geometries*, Oxford University Press, New York, 1991.
- [10] Oxley, J., *Matroid Theory*, Second Edition, Oxford University Press, New York, 2011.
- [11] Rose, D.J., Triangulated graphs and the elimination process, *J. Math. Anal. Appl.* **32** (1970), 597–609.
- [12] Seymour, P.D., Minors of 3-connected matroids. *European J. Combin.* **6** (1985), 375–382
- [13] Tutte, W.T., Matroids and graphs, *Trans. Amer. Math. Soc.* **90** (1959), 527–552.

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