Perfect state transfer on Cayley graphs over a group of order $8n$

Akash Kalita and Bikash Bhattacharjya
Department of Mathematics
Indian Institute of Technology Guwahati, India
akash.kalita@iitg.ac.in, b.bikash@iitg.ac.in

Abstract

The *transition matrix* of a graph Γ with adjacency matrix A is defined by $H(t) := \exp(-\mathbf{i}tA)$, where $t \in \mathbb{R}$ and $\mathbf{i} = \sqrt{-1}$. The graph Γ is said to admit *perfect state transfer* (PST) between a pair of vertices u and v if there exists $\tau(> 0) \in \mathbb{R}$ such that $|H(\tau)_{uv}| = 1$. Perfect state transfer has great importance due to its applications in quantum information processing, quantum communication networks and cryptography. In this paper, we study the existence of perfect state transfer on Cayley graphs over the group V_{8n}. we present some necessary and sufficient conditions for the existence of perfect state transfer on $\mathrm{Cay}(V_{8n}, S)$.

Keywords. Perfect state transfer; Cayley graph; Eigenvalues of a graph
Mathematics Subject Classifications: 05C25; 81P45; 81Q35

1 Introduction

The concept of quantum walks on graphs was first introduced by Farhi and Gutmann [14] in the year 1998. Quantum walks on finite graphs provide useful simple models for quantum transport phenomena which was first discovered by Bose [7] in 2003. Christandl et al. [10] proposed a class of qubit networks that admit perfect state transfer. Quantum walks are important tools in quantum computation and information theory and can be used to describe the fidelity of information transfer in a network of interacting qubits.

Let Γ be a finite simple connected graph with adjacency matrix A. Denoted by $V(\Gamma)$ the set of vertices of Γ. The *transition matrix* of Γ is defined by

$$H(t) = H_\Gamma(t) =: \exp(-\mathbf{i}tA) = \sum_{s=0}^{\infty} \frac{(-\mathbf{i}tA)^s}{s!} = (H_{u,v}(t))_{u,v\in V(\Gamma)},$$

where $t \in \mathbb{R}$ and $\mathbf{i} = \sqrt{-1}$.

A graph Γ is said to exhibit *perfect state transfer* (PST) from a vertex u to the vertex v if there exists $\tau(>0) \in \mathbb{R}$ such that the uv-th element of $H(\tau)$ has absolute value 1. We describe Γ as exhibiting *periodicity* at the vertex u at time τ if the uu-th element of $H(\tau)$ has absolute value 1. The graph Γ is considered to exhibit *periodicity* if it exhibits the property of periodicity across all its vertices simultaneously at the same time.

PST has been studied for several families of graphs. Angeles-Canul et al. [1] investigated PST in integral circulant graphs and the join of graphs. Coutinho and Godsil [11] explored PST in products and covers of graphs. Pal and Bhattacharjya [22] studied PST on NEPS of the path on three vertices. Godsil [17] offers a survey on perfect state transfer and related questions up to 2011. He [16] explains the close relationship between the existence of perfect state transfer on certain graphs and association schemes. Notably, in [18] Godsil presents a complete characterization of PST on simple connected graphs.

Cayley graphs are good candidates for exhibiting PST due to their nice algebraic structure. Among these results Basic et al. [4, 5, 3, 23] , Cheung and Godsil [9] and Bernasconi et al. [6] present some criterions on circulant graphs and cubelike graphs having PST. Tan et al. [25] presented a characterization on abelian Cayley graphs having PST. They showed that many of the previous results on periodicity and existence of PST in circulant graphs and cubelike graphs can be derived in unified and more simple ways. However, relatively little research has been carried out on Cayley graphs over non-abelian groups exhibiting PST. Cao and Feng [8] investigated PST on Cayley graphs over dihedral groups. Subsequently, Arezoomand et al. [2] and Luo et al. [21] explored PST on Cayley graphs over dicyclic groups and semi-dihedral groups, respectively. Recently, Khalilipour and Ghorbani [20] studied PST on Cayley graphs over the group $U_{6n} = \langle a, b \colon a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$.

In this paper, we consider the existence of PST on Cayley graphs over the group $V_{8n} = \langle a, b \colon a^{2n} = b^4 = 1,\ \ ba = a^{-1}b^{-1},\ b^{-1}a = a^{-1}b\rangle$, where n is a positive integer. Using the irreducible representations of V_{8n}, several necessary and sufficient conditions for a normal Cayley graph $\mathrm{Cay}(V_{8n}, S)$ exhibiting PST are carried out.

The rest of the current work is organized as follows. In Section 2, we give the description of the irreducible representations of V_{8n} and spectra of normal Cayley graphs $\mathrm{Cay}(V_{8n}, S)$. The existence of PST on normal Cayley graphs $\mathrm{Cay}(V_{8n}, S)$ is explored in Section 3.

2 Irreducible representations of the group V_{8n} and spectra of Cayley graphs Cay(V_{8n}, S)

A *representation* of a finite group G is a homomorphism $\theta \colon G \to GL(U)$, where $GL(U)$ is the group of all automorphisms of a finite-dimensional and non-zero complex vector space U. The dimension of U

is called the *degree* of θ. Two representations θ and ψ of G on U and W, respectively, are *equivalent*, denoted by $\theta \sim \psi$, if there is an isomorphism $T\colon U \to W$ such that $\theta(g) = T\psi(g)T^{-1}$ for all $g \in G$.

Let $\theta\colon G \to GL(U)$ be a representation. The *character* $\chi_\theta\colon G \to \mathbb{C}$ of θ is defined by setting $\chi_\theta(g) = \mathrm{Tr}(\theta(g))$ for all $g \in G$, where $\mathrm{Tr}(\theta(g))$ is the trace of the representation matrix of $\theta(g)$. A subspace W of U is said to be G-invariant if $\theta(g)w \in W$ for all $g \in G$ and $w \in W$. Obviously, $\{0\}$ and U are G-invariant subspaces, called trivial subspaces. If U has no non-trivial G-invariant subspaces, then θ is called an *irreducible representation* and χ_θ an *irreducible character* of G.

Let G be a finite group and S be a symmetric subset of G, that is, $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$ and $\mathbf{1} \notin S$. The *Cayley* graph of G with respect to S, denoted $\mathrm{Cay}(G, S)$, is a graph whose vertices are the elements of G and there exists an edge between distinct vertices $g, h \in G$ if $gh^{-1} \in S$. The set S is called the *connection* set. If $Sg = gS$ for all $g \in G$, then S is called a *normal Cayley subset* and $\mathrm{Cay}(G, S)$ a *normal* Cayley graph. Since S is symmetric, $\mathrm{Cay}(G, S)$ is a simple graph. We assume $G = \langle S \rangle$ to ensure that $\mathrm{Cay}(G, S)$ is a connected graph. The adjacency matrix of $\mathrm{Cay}(G, S)$ is defined by $A = (a_{g,h})_{g,h\in G}$, where

$$a_{g,h} = \begin{cases} 1 & \text{if } gh^{-1} \in S \\ 0 & \text{otherwise.} \end{cases}$$

For more properties about Cayley graphs, One can refer to [15].

Let n be a positive integer. The group V_{8n} is defined by

$$V_{8n} = \langle a, b\colon a^{2n} = b^4 = 1,\ \ ba = a^{-1}b^{-1},\ b^{-1}a = a^{-1}b\rangle.$$

Note that $V_{8n} = \{a^r, a^r b, a^r b^2, a^r b^3 \colon 0 \le r \le 2n-1\}$

For odd values of n, the group V_{8n} has $2n + 3$ conjugacy classes as follows:

$$\begin{aligned} &\{1\}, \{b^2\}, \{a^{2r+1}, a^{-2r-1}b^2\}, && r \in \{0, \ldots, n-1\}, \\ &\{a^{2s}, a^{-2s}\}, \{a^{2s}b^2, a^{-2s}b^2\}, && s \in \{1, \ldots, (n-1)/2\}, \\ &\{a^j b^k \colon j \text{ even},\ k = 1 \text{ or } 3\} \text{ and} && \\ &\{a^j b^k \colon j \text{ odd},\ k = 1 \text{ or } 3\}. && \end{aligned}$$

For even values of n, the group V_{8n} has $2n + 6$ conjugacy classes as follows:

$$\begin{aligned} &\{1\}, \{b^2\}, \{a^n\}, \{a^n b^2\}, && \\ &\{a^{2r+1}, a^{-(2r+1)}b^2\}, && r \in \{0, \ldots, n-1\} \\ &\{a^{2s}, a^{-2s}\}, \{a^{2s}b^2, a^{-2s}b^2\}, && s \in \{1, \ldots, n/2 - 1\} \\ &\{a^{2k}b^{(-1)^k} \colon 0 \le k \le n-1\}, && \{a^{2k}b^{(-1)^{k+1}} \colon 0 \le k \le n-1\}, \\ &\{a^{2k+1}b^{(-1)^k} \colon 0 \le k \le n-1\} \text{ and } \{a^{2k+1}b^{(-1)^{k+1}} \colon 0 \le k \le n-1\}. && \end{aligned}$$

Table 1: Irreducible representations of V_{8n}, for n odd

	a	b
χ_1	1	1
χ_2	1	-1
χ_3	-1	1
χ_4	-1	-1
ψ_j $(0 \leq j \leq n-1)$	$\begin{pmatrix} \omega^{2j} & 0 \\ 0 & -\omega^{-2j} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
ϕ_k $(1 \leq k \leq n-1)$	$\begin{pmatrix} \omega^{k} & 0 \\ 0 & \omega^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The following lemma presents the well-known irreducible representations and characters for the group V_{8n}.

Lemma 2.1. *[19, 13] Let n be a positive integer and $\omega = \exp(\frac{2\pi i}{2n})$ be a primitive $2n$-th root of unity.*
(1) The irreducible representations of V_{8n} are listed in Table 1 for n odd and in Table 2 for n even.
(2) The character table of V_{8n} is listed in Table 3 for n odd, in Table 4 for $n \equiv 0 \pmod 4$ and in Table 5 for $n \equiv 2 \pmod 4$.

The following lemma determines the eigenvalues and eigenvectors of the adjacency matrix of a normal Cayley graph.

Lemma 2.2. *[24] Let $G = \{g_1, \dots, g_n\}$ be a finite group and $\phi^{(1)}, \dots, \phi^{(t)}$ be a complete set of unitary representatives of the equivalence classes of irreducible representations of G. Let χ_k be the character of $\phi^{(k)}$ and d_k be the degree of $\phi^{(k)}$. Let S be a symmetric subset of G and assume further that S is conjugation-closed. Then the eigenvalues of the Cayley graph $Cay(G, S)$ are $\lambda_1, \dots, \lambda_t$, where*

$$\lambda_k = \frac{1}{d_k} \sum_{s \in S} \chi_k(s), \quad 1 \leq k \leq t,$$

and λ_k has multiplicity d_k^2. Moreover, the vectors

$$v_{ij}^{(k)} = \sqrt{\frac{d_k}{|G|}} \left(\phi_{ij}^{(k)}(g_1), \dots, \phi_{ij}^{(k)}(g_n) \right)^t, 1 \leq i, j \leq d_k$$

form an orthonormal basis for the eigenspace associated with the eigenvalue λ_k.

Table 2: Irreducible representations of V_{8n}, for n even

	a	b
χ_1	1	1
χ_2	$\mathbf{i}$	$-\mathbf{i}$
χ_3	-1	-1
χ_4	$-\mathbf{i}$	$\mathbf{i}$
χ_5	1	-1
χ_6	$\mathbf{i}$	$\mathbf{i}$
χ_7	-1	1
χ_8	$-\mathbf{i}$	$-\mathbf{i}$
$\psi_j \ (1 \leq j \leq n-1)$	$\begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}$
$\phi_k \ (1 \leq k \leq n-1)$	$\begin{pmatrix} \mathbf{i}\omega^k & 0 \\ 0 & \mathbf{i}\omega^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Table 3: Character table of V_{8n}, for n odd

	1	b^2	$a^{2r+1} \ (0 \leq r \leq n-1)$	a^{2s}	$a^{2s}b^2 \ (1 \leq s \leq (n-1)/2)$	b	ab
ξ_1	1	1	1	1	1	1	1
ξ_2	1	1	1	1	1	-1	-1
ξ_3	1	1	-1	1	1	1	-1
ξ_4	1	1	-1	1	1	-1	1
$\zeta_j \ (0 \leq j \leq n-1)$	2	-2	$\omega^{2j(2r+1)} - \omega^{-2j(2r+1)}$	$\omega^{4js} + \omega^{-4js}$	$-\omega^{4js} - \omega^{-4js}$	0	0
$\nu_k \ (1 \leq k \leq n-1)$	2	2	$\omega^{k(2r+1)} + \omega^{-k(2r+1)}$	$\omega^{2ks} + \omega^{-2ks}$	$\omega^{2ks} + \omega^{-2ks}$	0	0

Table 4: Character table of V_{8n}, for $n \equiv 0 \pmod 4$

	1	b^2	a^n	a^nb^2	a^{4m+1}	a^{4m+3}	a^{4s}	a^{4t+2}	$a^{4s}b^2$	$a^{4t+2}b^2$	b	b^{-1}	ab	ab^{-1}
ξ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_2	1	-1	-1	1	$\mathbf{i}$	-$\mathbf{i}$	1	-1	-1	1	-$\mathbf{i}$	$\mathbf{i}$	1	-1
ξ_3	1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
ξ_4	1	-1	-1	1	-$\mathbf{i}$	$\mathbf{i}$	1	-1	-1	1	$\mathbf{i}$	-$\mathbf{i}$	1	-1
ξ_5	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
ξ_6	1	-1	-1	1	$\mathbf{i}$	-$\mathbf{i}$	1	-1	-1	1	$\mathbf{i}$	-$\mathbf{i}$	-1	1
ξ_7	1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1
ξ_8	1	-1	-1	1	-$\mathbf{i}$	$\mathbf{i}$	1	-1	-1	1	-$\mathbf{i}$	$\mathbf{i}$	-1	1
ζ_j	2	2	$2(-1)^j$	$2(-1)^j$	$\alpha^{j(4m+1)}$	$\alpha^{j(4m+3)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0
ν_k	2	-2	$2(-1)^k$	-$2(-1)^k$	$\mathbf{i}\alpha^{j(4m+1)}$	-$\mathbf{i}\alpha^{j(4m+3)}$	$\alpha^{j(4s)}$	-$\alpha^{j(4t+2)}$	-$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0

$\alpha^{jr} = \omega^{jr} + \omega^{-jr} = 2\cos(\frac{\pi jr}{n})$, $\alpha^{kr} = \omega^{kr} + \omega^{-kr} = 2\cos(\frac{\pi kr}{n})$, $\omega = \exp(\frac{2\pi\mathbf{i}}{2n})$;
$m \in \{0, \ldots, n/2-1\}$, $s \in \{1, \ldots, n/4-1\}$, $t \in \{0, \ldots, n/4-1\}$, $j, k \in \{1, \ldots, n-1\}$.

Table 5: Character table of V_{8n}, for $n \equiv 2 \pmod 4$

	1	b^2	a^n	a^nb^2	a^{4m+1}	a^{4m+3}	a^{4s}	a^{4t+2}	$a^{4s}b^2$	$a^{4t+2}b^2$	b	b^{-1}	ab	ab^{-1}
ξ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_2	1	-1	1	-1	$\mathbf{i}$	-$\mathbf{i}$	1	-1	-1	1	-$\mathbf{i}$	$\mathbf{i}$	1	-1
ξ_3	1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
ξ_4	1	-1	1	-1	-$\mathbf{i}$	$\mathbf{i}$	1	-1	-1	1	$\mathbf{i}$	-$\mathbf{i}$	1	-1
ξ_5	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
ξ_6	1	-1	1	-1	$\mathbf{i}$	-$\mathbf{i}$	1	-1	-1	1	$\mathbf{i}$	-$\mathbf{i}$	-1	1
ξ_7	1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1
ξ_8	1	-1	1	-1	-$\mathbf{i}$	$\mathbf{i}$	1	-1	-1	1	-$\mathbf{i}$	$\mathbf{i}$	-1	1
ζ_j	2	2	$2(-1)^j$	$2(-1)^j$	$\alpha^{j(4m+1)}$	$\alpha^{j(4m+3)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0
ν_k	2	-2	-$2(-1)^k$	$2(-1)^k$	$\mathbf{i}\alpha^{j(4m+1)}$	-$\mathbf{i}\alpha^{j(4m+3)}$	$\alpha^{j(4s)}$	-$\alpha^{j(4t+2)}$	-$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0

$\alpha^{jr} = \omega^{jr} + \omega^{-jr} = 2\cos(\frac{\pi jr}{n})$, $\alpha^{kr} = \omega^{kr} + \omega^{-kr} = 2\cos(\frac{\pi kr}{n})$, $\omega = \exp(\frac{2\pi \mathbf{i}}{2n})$;

$m \in \{0, \dots, n/2-1\}$, $s \in \{1, \dots, n/4-1\}$, $t \in \{0, \dots, n/4-1\}$, $j, k \in \{1, \dots, n-1\}$.

2.1 Spectra of normal Cayley graphs Cay(V_{8n}, S), for odd n

We consider the fixed ordering $1, a, a^2, \ldots, a^{2n-1}, b, ab, \ldots, a^{2n-1}b, b^2, ab^2, \ldots, a^{2n-1}b^2, b^3, ab^3, \ldots, a^{2n-1}b^3$ for the elements of the group V_{8n}. The adjacency matrix of the normal Cayley graph Cay(V_{8n}, S) has the following eigenvectors corresponding to the one-dimensional irreducible representations χ_1, χ_2, χ_3 and χ_4.

$$
\begin{aligned}
u_1 =& \frac{1}{\sqrt{8n}}(1, \ldots, 1)^t,\\
u_2 =& \frac{1}{\sqrt{8n}}(1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1)^t,\\
u_3 =& \frac{1}{\sqrt{8n}}(1, -1, \ldots, 1, -1, 1, -1, \ldots, 1, -1, 1, -1, \ldots, 1, -1, 1, -1, \ldots, 1, -1)^t \text{ and}\\
u_4 =& \frac{1}{\sqrt{8n}}(1, -1, \ldots, 1, -1, -1, 1, \ldots, -1, 1, 1, -1, \ldots, 1, -1, -1, 1, \ldots, -1, 1)^t.
\end{aligned}
$$

Now we compute the eigenvectors of the normal Cayley graph Cay(V_{8n}, S) corresponding to the two-dimensional irreducible representations. The adjacency matrix of the normal Cayley graph Cay(V_{8n}, S) has the following eigenvectors corresponding to the two-dimensional representations ψ_j, for $0 \leq j \leq n-1$.

$$
\begin{aligned}
{u_j}^{(1)} &= \frac{1}{\sqrt{4n}}(\{\omega^{2rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{-\omega^{2rj}\}_{r=0}^{2n-1}, \mathbf{0})^t,\\
{u_j}^{(2)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{\omega^{2rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{-\omega^{2rj}\}_{r=0}^{2n-1})^t,\\
{u_j}^{(3)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{(-1)^{r+1}\omega^{-2rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{(-1)^{r}\omega^{-2rj}\}_{r=0}^{2n-1})^t \quad \text{and}\\
{u_j}^{(4)} &= \frac{1}{\sqrt{4n}}(\{(-1)^{r}\omega^{2rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{(-1)^{r+1}\omega^{2rj}\}_{r=0}^{2n-1}, \mathbf{0})^t.
\end{aligned}
$$

The adjacency matrix of the normal Cayley graph Cay(V_{8n}, S) has the following eigenvectors corresponding to the two-dimensional representations ϕ_k, for $1 \leq k \leq n-1$.

$$
\begin{aligned}
{v_k}^{(1)} &= \frac{1}{\sqrt{4n}}(\{\omega^{rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{\omega^{rk}\}_{r=0}^{2n-1}, \mathbf{0})^t,\\
{v_k}^{(2)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{\omega^{rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{\omega^{rk}\}_{r=0}^{2n-1})^t,\\
{v_k}^{(3)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{\omega^{-rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{\omega^{-rk}\}_{r=0}^{2n-1})^t \quad \text{and}\\
{v_k}^{(4)} &= \frac{1}{\sqrt{4n}}(\{\omega^{-rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{\omega^{-rk}\}_{r=0}^{2n-1}, \mathbf{0})^t.
\end{aligned}
$$

2.2 Spectra of normal Cayley graphs Cay(V_{8n}, S), for even n

The adjacency matrix of the normal Cayley graph Cay(V_{8n}, S) has the following eigenvectors corresponding to the one-dimensional irreducible representations $\chi_1, \ldots, \chi_8$.

$$\begin{aligned}
u_1 &= \frac{1}{\sqrt{8n}}(1, \ldots, 1)^t, \\
u_2 &= \frac{1}{\sqrt{4n}}(\{\mathbf{i}^r\}_{r=0}^{2n-1}, \{\mathbf{i}^r(-\mathbf{i})\}_{r=0}^{2n-1}, \{\mathbf{i}^r(-1)\}_{r=0}^{2n-1}, \{\mathbf{i}^r(\mathbf{i})\}_{r=0}^{2n-1})^t, \\
u_3 &= \frac{1}{\sqrt{8n}}(1, -1, \ldots, 1, -1, -1, 1, \ldots, -1, 1, 1, -1, \ldots, 1, -1, -1, 1, \ldots, -1, 1)^t, \\
u_4 &= \frac{1}{\sqrt{4n}}(\{(-\mathbf{i})^r\}_{r=0}^{2n-1}, \{(-\mathbf{i})^r(\mathbf{i})\}_{r=0}^{2n-1}, \{(-\mathbf{i})^r(-1)\}_{r=0}^{2n-1}, \{(-\mathbf{i})^r(-\mathbf{i})\}_{r=0}^{2n-1})^t, \\
u_5 &= \frac{1}{\sqrt{8n}}(1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1)^t, \\
u_6 &= \frac{1}{\sqrt{4n}}(\{\mathbf{i}^r\}_{r=0}^{2n-1}, \{\mathbf{i}^r(\mathbf{i})\}_{r=0}^{2n-1}, \{\mathbf{i}^r(-1)\}_{r=0}^{2n-1}, \{\mathbf{i}^r(-\mathbf{i})\}_{r=0}^{2n-1})^t, \\
u_7 &= \frac{1}{\sqrt{8n}}(1, -1, \ldots, 1, -1, 1, -1, \ldots, 1, -1, 1, -1, \ldots, 1, -1, 1, -1, \ldots, 1, -1)^t \text{ and} \\
u_8 &= \frac{1}{\sqrt{4n}}(\{(-\mathbf{i})^r\}_{r=0}^{2n-1}, \{(-\mathbf{i})^r(-\mathbf{i})\}_{r=0}^{2n-1}, \{(-\mathbf{i})^r(-1)\}_{r=0}^{2n-1}, \{(-\mathbf{i})^r(\mathbf{i})\}_{r=0}^{2n-1})^t.
\end{aligned}$$

Now we compute the eigenvectors of the normal Cayley graph Cay(V_{8n}, S) corresponding to the two-dimensional irreducible representations. The adjacency matrix of the normal Cayley graph Cay(V_{8n}, S) has the following eigenvectors corresponding to the two-dimensional representations ψ_j, for $1 \leq j \leq n-1$.

$$\begin{aligned}
{u_j}^{(1)} &= \frac{1}{\sqrt{4n}}(\{\omega^{rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{\omega^{rj}\}_{r=0}^{2n-1}, \mathbf{0})^t, \\
{u_j}^{(2)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{\mathbf{i}\omega^{rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{\mathbf{i}\omega^{rj}\}_{r=0}^{2n-1})^t, \\
{u_j}^{(3)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{-\mathbf{i}\omega^{-rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{-\mathbf{i}\omega^{-rj}\}_{r=0}^{2n-1})^t \quad \text{and} \\
{u_j}^{(4)} &= \frac{1}{\sqrt{4n}}(\{\omega^{-rj}\}_{r=0}^{2n-1}, \mathbf{0}, \{\omega^{-rj}\}_{r=0}^{2n-1}, \mathbf{0})^t.
\end{aligned}$$

The adjacency matrix of the normal Cayley graph Cay(V_{8n}, S) has the following eigenvectors corresponding to the two-dimensional representations ϕ_k, for $1 \leq k \leq n-1$.

$$\begin{aligned}
{v_k}^{(1)} &= \frac{1}{\sqrt{4n}}(\{\mathbf{i}^r\omega^{rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{-\mathbf{i}^r\omega^{rk}\}_{r=0}^{2n-1}, \mathbf{0})^t, \\
{v_k}^{(2)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{\mathbf{i}^r\omega^{rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{-\mathbf{i}^r\omega^{rk}\}_{r=0}^{2n-1})^t, \\
{v_k}^{(3)} &= \frac{1}{\sqrt{4n}}(\mathbf{0}, \{-\mathbf{i}^r\omega^{-rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{\mathbf{i}^r\omega^{-rk}\}_{r=0}^{2n-1})^t \quad \text{and} \\
{v_k}^{(4)} &= \frac{1}{\sqrt{4n}}(\{\mathbf{i}^r\omega^{-rk}\}_{r=0}^{2n-1}, \mathbf{0}, \{-\mathbf{i}^r\omega^{-rk}\}_{r=0}^{2n-1}, \mathbf{0})^t.
\end{aligned}$$

3 PST on Cayley graphs

Let Γ be a simple graph with n vertices and $\mathrm{Spec}(\Gamma)$ denotes the set of all the eigenvalues of Γ. Let A be the adjacency matrix of Γ and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Let $P = (v_1, \dots, v_n)$ be an unitary matrix, where v_i is an eigenvector corresponding to the eigenvalue λ_i $(1 \leq i \leq n)$. Then the spectral decomposition of A is given by

$$A = \lambda_1 E_1 + \cdots + \lambda_n E_n,$$

where $E_i = v_i v_i^*$ $(1 \leq i \leq n)$ satisfies

$$E_i E_j = \begin{cases} E_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the spectral decomposition of the transition matrix $H(t)$ is given by

$$H(t) = \exp(-\mathbf{i}\lambda_1 t)E_1 + \cdots + \exp(-\mathbf{i}\lambda_n t)E_n.$$

The *2-adic exponential valuation* of rational numbers is denoted by Υ_2 and is a mapping defined by

$$\Upsilon_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, \text{ such that } \Upsilon_2(0) = \infty, \text{ and } \Upsilon_2(2^l \frac{a}{b}) = l, \text{ where } a, b, l \in \mathbb{Z} \text{ and } 2 \nmid ab.$$

We assume that $\infty + \infty = \infty + l = \infty$ and $\infty > l$ for any $l \in \mathbb{Z}$. Then Υ_2 has the following properties. For $\beta, \beta' \in \mathbb{Q}$,

1. $\Upsilon_2(\beta\beta') = \Upsilon_2(\beta) + \Upsilon_2(\beta')$ and
2. $\Upsilon_2(\beta + \beta') \geq \min(\Upsilon_2(\beta), \Upsilon_2(\beta'))$ and the equality holds if $\Upsilon_2(\beta) \neq \Upsilon_2(\beta')$.

We write the vertex set of $\mathrm{Cay}(V_{8n}, S)$ as $V_1 \cup V_2 \cup V_3 \cup V_4$, where

$$\begin{aligned} V_1 &= \{0, 1, \dots 2n-1\}, \\ V_2 &= \{2n, 2n+1, \dots, 4n-1\}, \\ V_3 &= \{4n, 4n+1, \dots, 6n-1\} \text{ and} \\ V_4 &= \{6n, 6n+1, \dots, 8n-1\}. \end{aligned}$$

3.1 PST on normal Cayley graphs over the group V_{8n}, for odd n

We want to state the main result of this section.

Theorem 3.1. *Let S be a non-empty subset of V_{8n} such that $\mathbf{1} \notin S$ and $Sg = gS$ for all $g \in V_{8n}$. Let $\Gamma = Cay(V_{8n}, S)$ be a connected Cayley graph with connection set S, where n is odd. Then Γ has four distinct eigenvalues which corresponds to the one-dimensional representations χ_1, χ_2, χ_3 and χ_4, respectively, with one is $\alpha_1 = |S|$ and the other three eigenvalues are denoted by α_2, α_3 and α_4, and some* *multiple eigenvalues corresponding to the two-dimensional representations ψ_j and ϕ_k, denoted by β_j and γ_k, respectively, for $0 \le j \le n-1$ and $1 \le k \le n-1$.*

1. If $u \in V_1, v \in V_2$ or $u \in V_1, v \in V_4$ or $u \in V_2, v \in V_1$ or $u \in V_2, v \in V_3$ or $u \in V_3, v \in V_2$ or $u \in V_3, v \in V_4$ or $u \in V_4, v \in V_1$ or $u \in V_4, v \in V_3$, then Γ cannot have PST between two distinct vertices u and v.

2. If $u, v \in V_1$ or $u, v \in V_2$ or $u, v \in V_3$ or $u, v \in V_4$, then Γ cannot have PST between two distinct vertices u and v.

3. If $u \in V_1, v \in V_3$ or $u \in V_2, v \in V_4$ or $u \in V_3, v \in V_1$ or $u \in V_4, v \in V_2$, then Γ has PST between the vertices u and v if and only if the following three conditions hold.

(i) All the eigenvalues of Γ are integers, namely, Γ is integral,

(ii) $u = v + 4n$ and

(iii) $\Upsilon_2(\alpha_1 - \beta_j)$ is a constant, say μ, and $\Upsilon_2(\alpha_1 - \alpha_2)$, $\Upsilon_2(\alpha_1 - \alpha_3)$, $\Upsilon_2(\alpha_1 - \alpha_4)$ and $\Upsilon_2(\alpha_1 - \gamma_k)$ are all bigger than μ, for $0 \le j \le n-1$ and $1 \le k \le n-1$.

Furthermore, when the conditions (i), (ii) and (iii) hold, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where $M = \gcd(\alpha - \alpha_1 \colon \alpha \in \mathrm{Spec}(\Gamma) \setminus \{\alpha_1\})$.

Proof. The adjacency matrix A of the normal Cayley graph $\mathrm{Cay}(V_{8n}, S)$ has the eigenvectors $u_i, {u_j}^{(i)}$ and ${v_k}^{(i)}$ $(1 \le i \le 4,\ 0 \le j \le n-1$ and $1 \le k \le n-1)$ which are introduced in section 2.1. Hence we have the following unitary matrix

$$P = (u_1, u_2, u_3, u_4, {u_0}^{(1)}, {u_0}^{(2)}, {u_0}^{(3)}, {u_0}^{(4)}, \ldots, {u_{n-1}}^{(1)}, {u_{n-1}}^{(2)}, {u_{n-1}}^{(3)}, {u_{n-1}}^{(4)},$$
$$ {v_1}^{(1)}, {v_1}^{(2)}, {v_1}^{(3)}, {v_1}^{(4)}, \ldots, {v_{n-1}}^{(1)}, {v_{n-1}}^{(2)}, v_{n-1}(3), {v_{n-1}}^{(4)}).$$

Let J_m be the all-one matrix of order m. Then the projective matrices corresponding to the eigenvectors u_1, u_2, u_3 and u_4 are given by

$$\begin{aligned} E_1 &= u_1 u_1^* = \frac{1}{8n} J_{8n}, \\ E_2 &= u_2 u_2^* = \frac{1}{8n} \begin{pmatrix} J_{2n} & -J_{2n} & J_{2n} & -J_{2n} \\ -J_{2n} & J_{2n} & -J_{2n} & J_{2n} \\ J_{2n} & -J_{2n} & J_{2n} & -J_{2n} \\ -J_{2n} & J_{2n} & -J_{2n} & J_{2n} \end{pmatrix}, \\ E_3 &= u_3 u_3^* = \frac{1}{8n}((-1)^{u+v}), u, v \in \{0, 1 \ldots, 8n-1\} \text{ and} \\ E_4 &= u_4 u_4^* = \frac{1}{8n}(e_4(u, v)), \end{aligned}$$

where

(i) $e_4(u, v) = (-1)^{u+v}$, when $u, v \in V_1 \cup V_3$ or $u, v \in V_2 \cup V_4$

(ii) $e_4(u,v) = (-1)^{u+v+1}$, when $u \in V_1 \cup V_3, v \in V_2 \cup V_4$ or $u \in V_2 \cup V_4, v \in V_1 \cup V_3$,

The projective matrices corresponding to the eigenvectors $u_j^{(1)}, u_j^{(2)}, u_j^{(3)}$ and $u_j^{(4)}$, where $0 \leq j \leq n-1$ are given by

$$E_j{}^{(1)} = u_j^{(1)} u_j^{(1)*} = \frac{1}{4n} \begin{pmatrix} X_1 & 0 & -X_1 & 0 \\ 0 & 0 & 0 & 0 \\ -X_1 & 0 & X_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_j{}^{(2)} = u_j^{(2)} u_j^{(2)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & -X_1 \\ 0 & 0 & 0 & 0 \\ 0 & -X_1 & 0 & X_1 \end{pmatrix},$$

$$E_j{}^{(3)} = u_j^{(3)} u_j^{(3)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_2 & 0 & -X_2 \\ 0 & 0 & 0 & 0 \\ 0 & -X_2 & 0 & X_2 \end{pmatrix} \text{ and } E_j{}^{(4)} = u_j^{(4)} u_j^{(4)*} = \frac{1}{4n} \begin{pmatrix} X_2 & 0 & -X_2 & 0 \\ 0 & 0 & 0 & 0 \\ -X_2 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The projective matrices corresponding to the eigenvectors $v_k^{(1)}, v_k^{(2)}, v_k^{(3)}$ and $v_k^{(4)}$, where $1 \leq k \leq n-1$ are given by

$$F_k{}^{(1)} = v_k^{(1)} v_k^{(1)*} = \frac{1}{4n} \begin{pmatrix} Y_1 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 0 \\ Y_1 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_k{}^{(2)} = v_k^{(2)} v_k^{(2)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_1 & 0 & Y_1 \\ 0 & 0 & 0 & 0 \\ 0 & Y_1 & 0 & Y_1 \end{pmatrix},$$

$$F_k{}^{(3)} = v_k^{(3)} v_k^{(3)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_2 & 0 & Y_2 \\ 0 & 0 & 0 & 0 \\ 0 & Y_2 & 0 & Y_2 \end{pmatrix} \text{ and } F_k{}^{(4)} = v_k^{(4)} v_k^{(4)*} = \frac{1}{4n} \begin{pmatrix} Y_2 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & 0 \\ Y_2 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} 1 & \omega^{-2j} & \cdots & \omega^{-2(2n-1)j} \\ \omega^{2j} & 1 & \cdots & \omega^{-2(2n-2)j} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{2(2n-1)j} & \omega^{2(2n-2)j} & \cdots & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & -\omega^{2j} & \cdots & -\omega^{2(2n-1)j} \\ -\omega^{-2j} & 1 & \cdots & \omega^{2(2n-2)j} \\ \vdots & \vdots & \vdots & \vdots \\ -\omega^{-2(2n-1)j} & \omega^{-2(2n-2)j} & \cdots & 1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 1 & \omega^{-k} & \cdots & \omega^{-(2n-1)k} \\ \omega^{k} & 1 & \cdots & \omega^{-(2n-2)k} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{(2n-1)k} & \omega^{(2n-2)k} & \cdots & 1 \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} 1 & \omega^{k} & \cdots & \omega^{(2n-1)k} \\ \omega^{-k} & 1 & \cdots & \omega^{(2n-2)k} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{-(2n-1)k} & \omega^{-(2n-2)k} & \cdots & 1 \end{pmatrix}.$$

Therefore, the transition matrix $H(t)$ of the normal Cayley graph $\mathrm{Cay}(V_{8n}, S)$ is given by

$$
\begin{aligned}
H(t) &= \exp(-\mathbf{i}t\alpha_1)E_1 + \exp(-\mathbf{i}t\alpha_2)E_2 + \exp(-\mathbf{i}t\alpha_3)E_3 + \exp(-\mathbf{i}t\alpha_4)E_4 \\
&\quad + \sum_{j=0}^{n-1} \exp(-\mathbf{i}t\beta_j)(E_j^{(1)} + E_j^{(2)} + E_j^{(3)} + E_j^{(4)}) + \sum_{k=1}^{n-1} \exp(-\mathbf{i}t\gamma_k)(F_k^{(1)} + F_k^{(2)} + F_k^{(3)} + F_k^{(4)}).
\end{aligned}
$$

Now we compute the (u, v)-th element of the transition matrix. We have the following three cases.

Case 1. If $u \in V_1, v \in V_2$ or $u \in V_1, v \in V_4$ or $u \in V_2, v \in V_1$ or $u \in V_2, v \in V_3$ or $u \in V_3, v \in V_2$ or $u \in V_3, v \in V_4$ or $u \in V_4, v \in V_1$ or $u \in V_4, v \in V_3$, then

$$
\begin{aligned}
H(t)_{uv} &= \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) - \exp(-\mathbf{i}t\alpha_2) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) + (-1)^{u+v+1}\exp(-\mathbf{i}t\alpha_4)) \\
&\quad + \frac{1}{4n}\sum_{j=0}^{n-1}\exp(-\mathbf{i}t\beta_j)(0+0+0+0) + \frac{1}{4n}\sum_{k=1}^{n-1}\exp(-\mathbf{i}t\gamma_k)(0+0+0+0) \\
&= \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) - \exp(-\mathbf{i}t\alpha_2) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_4) + (-1)^{u+v+1}\exp(-\mathbf{i}t\alpha_4)).
\end{aligned}
$$

$$\text{This implies that } \mid H(t)_{uv} \mid \leq \frac{1}{8n} \times 4 = \frac{1}{2n} < 1.$$

Therefore, PST cannot occur in this case.

Case 2. If $u, v \in V_1$ or $u, v \in V_2$ or $u, v \in V_3$ or $u, v \in V_4$, then

$$
\begin{aligned}
H(t)_{uv} &= \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) + \exp(-\mathbf{i}t\alpha_2) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_4)) \\
&\quad + \frac{1}{4n}\sum_{j=0}^{n-1}\exp(-\mathbf{i}t\beta_j)(\omega^{2(u-v)j} + (-1)^{u+v}\omega^{2(v-u)j}) + \frac{1}{4n}\sum_{k=1}^{n-1}\exp(-\mathbf{i}t\gamma_k)(\omega^{(u-v)k} + \omega^{(v-u)k}).
\end{aligned}
$$

$$\text{This implies that } \mid H(t)_{uv} \mid \leq \frac{1}{8n} \times 4 + \frac{1}{4n} \times 2n + \frac{1}{4n} \times 2(n-1) = \frac{2}{4n} + \frac{2n+2n-2}{4n} = \frac{4n}{4n} = 1$$

Therefore, $\mid H(t)_{uv} \mid \leq 1$. Thus $\mid H(t)_{uv} \mid = 1$ if and only if for $0 \leq j \leq n-1$ and $1 \leq k \leq n-1$, it holds that,

$$
\begin{aligned}
&\exp(-\mathbf{i}t\alpha_1) = \exp(-\mathbf{i}t\alpha_2) \\
&\exp(-\mathbf{i}t\alpha_1) = (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) \\
&\exp(-\mathbf{i}t\alpha_1) = (-1)^{u+v}\exp(-\mathbf{i}t\alpha_4) \\
&\exp(-\mathbf{i}t\alpha_1) = \omega^{2(u-v)j}\exp(-\mathbf{i}t\beta_j) \\
&\exp(-\mathbf{i}t\alpha_1) = (-1)^{u+v}\omega^{2(v-u)j}\exp(-\mathbf{i}t\beta_j) \\
&\exp(-\mathbf{i}t\alpha_1) = \omega^{(u-v)k}\exp(-\mathbf{i}t\gamma_k) \\
&\exp(-\mathbf{i}t\alpha_1) = \omega^{(v-u)k}\exp(-\mathbf{i}t\gamma_k)
\end{aligned}
$$

From the last two equations, since ω is the $2n$-th root of unity, we get that n divides $u-v$. Without loss of generality, we assume that $u \geq v$. This implies that either $u = v$ or $u - v = n$. Let $t = 2\pi T$. Then we have

$$(\alpha_1 - \alpha_2)T \in \mathbb{Z} \tag{1}$$

$$(\alpha_1 - \alpha_3)T - \frac{u+v}{2} \in \mathbb{Z} \tag{2}$$

$$(\alpha_1 - \alpha_4)T - \frac{u+v}{2} \in \mathbb{Z} \tag{3}$$

$$(\alpha_1 - \beta_j)T \in \mathbb{Z} \tag{4}$$

$$(\alpha_1 - \beta_j)T - \frac{u+v}{2} \in \mathbb{Z} \tag{5}$$

$$(\alpha_1 - \gamma_k)T - \frac{k}{2} \in \mathbb{Z} \tag{6}$$

$$(\alpha_1 - \gamma_k)T + \frac{k}{2} \in \mathbb{Z} \tag{7}$$

Since $0 = \mathrm{tr}(A) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 4\sum_{j=0}^{n-1}\beta_j + 4\sum_{k=1}^{n-1}\gamma_k$, we have that $8n\alpha_1 T \in \mathbb{Z}$, and since $\alpha_1 = |S|$ is a positive integer, we have $T \in \mathbb{Q}$. This implies that all the eigenvalues of the graph are rational numbers. It is well known that any rational eigenvalue of a graph is an integer. Therefore in this case the graph is integral.

From (4) and (5) it follows that $u+v$ is even. Since $u+v$ and $u-v$ have the same parity, so $u-v$ cannot be equal to n. Hence, $u = v$. Therefore, Γ cannot have PST between distinct vertices u and v.

Case 3. If $u \in V_1, v \in V_3$ or $u \in V_2, v \in V_4$ or $u \in V_3, v \in V_1$ or $u \in V_4, v \in V_2$, then

$$\begin{aligned} H(t)_{uv} = {} & \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) + \exp(-\mathbf{i}t\alpha_2) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) + \exp(-\mathbf{i}t\alpha_4)) \\ & + \frac{1}{4n}\sum_{j=0}^{n-1}\exp(-\mathbf{i}\ t\beta_j)(-\omega^{2(u-v)j} + (-1)^{u+v+1}\omega^{2(v-u)j}) + \frac{1}{4n}\sum_{k=1}^{n-1}\exp(-\mathbf{i}t\gamma_k)(\omega^{(u-v)k} + \omega^{(v-u)k}). \end{aligned}$$

This implies that $\mid H(t)_{uv} \mid \leq \frac{1}{8n} \times 4 + \frac{1}{4n} \times 2n + \frac{1}{4n} \times 2(n-1) = \frac{2}{4n} + \frac{2n+2n-2}{4n} = \frac{4n}{4n} = 1.$

Therefore, $\mid H(t)_{uv} \mid \leq 1$. Thus, $\mid H(t)_{uv} \mid = 1$ if and only if for $0 \leq j \leq n-1$ and $1 \leq k \leq n-1$, it holds that,

$$\begin{aligned} \exp(-\mathbf{i}t\alpha_1) &= \exp(-\mathbf{i}t\alpha_2) \\ \exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) \\ \exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v}\exp(-\mathbf{i}t\alpha_4) \\ \exp(-\mathbf{i}t\alpha_1) &= -\omega^{2(u-v)j}\exp(-\mathbf{i}t\beta_j) \\ \exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v+1}\omega^{2(v-u)j}\exp(-\mathbf{i}t\beta_j) \\ \exp(-\mathbf{i}t\alpha_1) &= \omega^{(u-v)k}\exp(-\mathbf{i}t\gamma_k) \\ \exp(-\mathbf{i}t\alpha_1) &= \omega^{(v-u)k}\exp(-\mathbf{i}t\gamma_k) \end{aligned}$$

From the last two equations, since ω is the $2n$-th root of unity, we get that n divides $u-v$. Without loss of generality, we assume that $u > v$. This implies that either $u-v=3n$ or $u-v=4n$ or $u-v=5n$. Let $t=2\pi T$,. Then we have

$$(\alpha_1-\alpha_2)T \in \mathbb{Z} \tag{8}$$
$$(\alpha_1-\alpha_3)T - \frac{u+v}{2} \in \mathbb{Z} \tag{9}$$
$$(\alpha_1-\alpha_4)T - \frac{u+v}{2} \in \mathbb{Z} \tag{10}$$
$$(\alpha_1-\beta_j)T + \frac{1}{2} \in \mathbb{Z} \tag{11}$$
$$(\alpha_1-\beta_j)T + \frac{u+v+1}{2} \in \mathbb{Z} \tag{12}$$
$$(\alpha_1-\gamma_k)T + \frac{(u-v)k}{2n} \in \mathbb{Z} \tag{13}$$
$$(\alpha_1-\gamma_k)T + \frac{(v-u)k}{2n} \in \mathbb{Z} \tag{14}$$

Since $0=\operatorname{tr}(A)=\alpha_1+\alpha_2+\alpha_3+\alpha_4+4\sum_{j=0}^{n-1}\beta_j+4\sum_{k=1}^{n-1}\gamma_k$, we have that $8n\alpha_1 T\in\mathbb{Z}$, and since $\alpha_1=|S|$ is a positive integer, we have $T\in\mathbb{Q}$. This implies that all the eigenvalues of the graph are rational numbers. It is well known that any rational eigenvalue of a graph is an integer. Therefore in this case the graph is integral.

From (11) and (12) it follows that $u+v$ is even. Since $u+v$ and $u-v$ have the same parity, so $u-v$ is neither $3n$ nor $5n$. Therefore $u-v=4n$.
Conditions (8) to (14) can be written as follows

$$(\alpha_1-\alpha_2)T \in \mathbb{Z} \tag{15}$$
$$(\alpha_1-\alpha_3)T \in \mathbb{Z} \tag{16}$$
$$(\alpha_1-\alpha_4)T \in \mathbb{Z} \tag{17}$$
$$(\alpha_1-\beta_j)T - \frac{1}{2} \in \mathbb{Z} \tag{18}$$
$$(\alpha_1-\gamma_k)T \in \mathbb{Z} \tag{19}$$

Suppose that $(\alpha_1-\beta_r)T$, $(\alpha_1-\beta_s)T\in\frac{1}{2}+\mathbb{Z}$, for $r,s\in\{0,\dots,n-1\}$, then $\Upsilon_2((\alpha_1-\beta_r)T)=\Upsilon_2((\alpha_1-\beta_s)T)=-1$. This implies that $\Upsilon_2(\alpha_1-\beta_r)=\Upsilon_2(\alpha_1-\beta_s)=-1-\Upsilon_2(T)$. Thus for all $j\in\{0,\dots,n-1\}$, $\Upsilon_2(\alpha_1-\beta_j)$ is a constant, $\mu=-1-\Upsilon_2(T)$. From (15) it follows that $\Upsilon_2((\alpha_1-\alpha_2)T)\geq 0$. Therefore, $\Upsilon_2(\alpha_1-\alpha_2)\geq\mu+1$. Simillarly it can be shown that $\Upsilon_2(\alpha_1-\alpha_3)$, $\Upsilon_2(\alpha_1-\alpha_4)$ and $\Upsilon_2(\alpha_1-\gamma_k)$ are also bigger than μ, for all $k\in\{1,\dots,n-1\}$.

Suppose that (i), (ii) and (iii) hold. Let $M_1=\gcd(\alpha_1-\alpha_2,\alpha_1-\alpha_3,\alpha_1-\alpha_4,\alpha_1-\gamma_k\colon 1\leq k\leq n-1)$ and $M_2=\gcd(\alpha_1-\beta_j\colon 0\leq j\leq n-1)$. It can be easily seen that $\Upsilon_2(M_2)=\mu$.

Then the conditions $(15),(16),(17)$ and (19) implies that $T\in\frac{1}{M_1}\mathbb{Z}$ and condition (18) imply that

$T \in \frac{1}{M_2}(\frac{1}{2}+\mathbb{Z})$. let $\tilde{T} = \{t > 0 : \Gamma$ has PST between u and v at time $t\}$. Then we get

$$\begin{aligned}\tilde{T} &= \left(\frac{2\pi}{M_1}\mathbb{Z}\right) \cap \left(\frac{2\pi}{M_2}(\frac{1}{2}+\mathbb{Z})\right) \cap \mathbb{R}_{>0} &(20)\\ &= \frac{\pi}{M_1M_2}(2M_2\mathbb{Z} \cap M_1(1+2\mathbb{Z})) \cap \mathbb{R}_{>0} &(21)\end{aligned}$$

For every $z \in \mathbb{Z}$, it is easy to check that

$$\begin{aligned}&z \in 2M_2\mathbb{Z} \cap M_1(1+2\mathbb{Z})\\ &\Leftrightarrow z = 2M_2x_0 = M_1(1+2y_0), \text{ for some } x_0, y_0 \in \mathbb{Z}\\ &\Leftrightarrow 2M_2x - 2M_1y = M_1 \text{ has a solution}\\ &\Leftrightarrow \gcd(2M_1, 2M_2) \mid M_1\\ &\Leftrightarrow \Upsilon_2(M_1) \geq \mu + 1 \text{ (since } \Upsilon_2(M_2) = \mu).\end{aligned}$$

Let $M = \gcd(M_1, M_2)$. Write $M_1 = m_1M$, $M_2 = m_2M$. Then $\gcd(m_1, m_2) = 1$. From $\Upsilon_2(M_1) \geq \mu+1$, $\Upsilon_2(M_2) = \mu$, we get that $\Upsilon_2(M) = \mu$ and m_1 is even, m_2 is odd. Then $2M_2x - 2M_1y = M_1 \Leftrightarrow m_2x - m_1y = \frac{m_1}{2}$. Since $\gcd(m_1, m_2) = 1$, so the solutions of the Diophantine equation $m_2x - m_1y = \frac{m_1}{2}$ are given by

$$\begin{aligned}x &= \frac{m_1}{2} + m_1l\\ y &= \frac{m_2-1}{2} + m_2l, \text{ where } l \in \mathbb{Z}.\end{aligned}$$

Thus

$$z = 2M_2x = 2M_2\frac{m_1}{2}(1+2l) = \frac{M_1M_2}{M}(1+2l),$$

and

$$2M_2\mathbb{Z} \cap M_1(1+2\mathbb{Z}) = \frac{M_1M_2}{M}(1+2Z).$$

By (21), we get

$$\begin{aligned}\tilde{T} &= (\frac{\pi}{M} + \frac{2\pi}{M}\mathbb{Z}) \cap \mathbb{R}_{>0}\\ &= \{\frac{\pi}{M} + \frac{2\pi}{M}l \colon l \in \mathbb{N} \cup \{0\}\}.\end{aligned}$$

Therefore, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$.

This completes the proof. $\square$

3.2 PST on normal Cayley graphs over the group V_{8n}, n even

The main result of this section is as follows.

Theorem 3.2. *Let S be a non-empty subset of V_{8n} such that $\mathbf{1} \notin S$ and $Sg = gS$ for all $g \in V_{8n}$. Let $\Gamma = Cay(V_{8n}, S)$ be a connected Cayley graph with connection set S, where n is even. Then Γ has eight distinct eigenvalues which corresponds to the one-dimensional representations $\chi_1, \ldots, \chi_8$, respectively, with one is $\alpha_1 = |S|$ and the other three eigenvalues are denoted by $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ and α_8, and some multiple eigenvalues corresponding to the two-dimensional representations ψ_j and ϕ_k, denoted by β_j and γ_k, respectively, for $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$.*

1. If $u \in V_1, v \in V_2$ or $u \in V_1, v \in V_4$ or $u \in V_2, v \in V_1$ or $u \in V_2, v \in V_3$ or $u \in V_3, v \in V_2$ or $u \in V_3, v \in V_4$ or $u \in V_4, v \in V_1$ or $u \in V_4, v \in V_3$, then Γ cannot have PST between distinct vertices u and v.

2. If $u, v \in V_1$ or $u, v \in V_2$ or $u, v \in V_3$ or $u, v \in V_4$, then Γ has PST between distinct vertices u and v if and only if the following three conditions hold.

(i) All the eigenvalues of Γ are integers.

(ii) $u = v + n$.

(iii) (a) If $n \equiv 0 \pmod 4$, then $\Upsilon_2(\alpha_1 - \beta_{2j'-1})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'-1})$ are the same, say μ_1, and $\Upsilon_2(\alpha_1-\alpha_2)$, $\Upsilon_2(\alpha_1-\alpha_3)$, $\Upsilon_2(\alpha_1-\alpha_4)$, $\Upsilon_2(\alpha_1-\alpha_5)$, $\Upsilon_2(\alpha_1-\alpha_6)$, $\Upsilon_2(\alpha_1-\alpha_7)$, $\Upsilon_2(\alpha_1-\alpha_8)$, $\Upsilon_2(\alpha_1-\beta_{2j'})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'})$ are all strictly greater than μ_1, for $1 \leq j' \leq \frac{n-1}{2}$ and $1 \leq k' \leq \frac{n-1}{2}$.

(b) If $n \equiv 2 \pmod 4$, then $\Upsilon_2(\alpha_1 - \alpha_2)$, $\Upsilon_2(\alpha_1 - \alpha_4)$, $\Upsilon_2(\alpha_1 - \alpha_6)$, $\Upsilon_2(\alpha_1 - \alpha_8)$, $\Upsilon_2(\alpha_1 - \beta_{2j'-1})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'})$ are the same, say μ_2, and $\Upsilon_2(\alpha_1 - \alpha_3)$, $\Upsilon_2(\alpha_1 - \alpha_5)$, $\Upsilon_2(\alpha_1 - \alpha_7)$, $\Upsilon_2(\alpha_1 - \beta_{2j'})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'-1})$ are all strictly greater than μ_2, for $1 \leq j' \leq \frac{n-1}{2}$ and $1 \leq k' \leq \frac{n-1}{2}$.

3. If $u \in V_1, v \in V_3$ or $u \in V_2, v \in V_4$ or $u \in V_3, v \in V_1$ or $u \in V_4, v \in V_2$, then Γ has PST between distinct vertices u and v if and only if the following three conditions hold.

(i) All the eigenvalues of Γ are integers.

(ii) $u = v + 4n$.

(iii) $\Upsilon_2(\alpha_1 - \alpha_2)$, $\Upsilon_2(\alpha_1 - \alpha_4)$, $\Upsilon_2(\alpha_1 - \alpha_6)$, $\Upsilon_2(\alpha_1 - \alpha_8)$ and $\Upsilon_2(\alpha_1 - \gamma_k)$ are the same, say μ_3, and $\Upsilon_2(\alpha_1 - \alpha_3)$, $\Upsilon_2(\alpha_1 - \alpha_5)$, $\Upsilon_2(\alpha_1 - \alpha_7)$ and $\Upsilon_2(\alpha_1 - \beta_j)$ are all strictly greater than μ_3, for $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$.

Furthermore, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where $M = \gcd(\alpha - \alpha_1 \colon \alpha \in \mathrm{Spec}(\Gamma) \setminus \{\alpha_1\})$.

Proof. The adjacency matrix A of the normal Cayley graph $\mathrm{Cay}(V_{8n}, S)$ has the eigenvectors $u_i, {u_j}^{(i)}$ and ${v_k}^{(i)}$ $(1 \leq i \leq 8,\ 1 \leq j \leq n-1$ and $1 \leq k \leq n-1)$ which are introduced in subsection 2.2. Hence we have the following unitary matrix

$$P = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, {u_1}^{(1)}, {u_1}^{(2)}, {u_1}^{(3)}, {u_1}^{(4)}, \ldots, {u_{n-1}}^{(1)}, {u_{n-1}}^{(2)}, {u_{n-1}}^{(3)}, {u_{n-1}}^{(4)},$$
$${v_1}^{(1)}, {v_1}^{(2)}, {v_1}^{(3)}, {v_1}^{(4)}, \ldots, {v_{n-1}}^{(1)}, {v_{n-1}}^{(2)}, v_{n-1}(3), {v_{n-1}}^{(4)}).$$

Let J_m be the all-one matrix of order m. Then the projective matrices corresponding to the eigenvectors u_1, u_3, u_5, u_7 are given by

$$E_1 = u_1 u_1^* = \frac{1}{8n} J_{8n},$$
$$E_3 = u_4 u_4^* = \frac{1}{8n}(e_3(u, v)),$$
$$E_5 = u_2 u_2^* = \frac{1}{8n}\begin{pmatrix} J_{2n} & -J_{2n} & J_{2n} & -J_{2n} \\ -J_{2n} & J_{2n} & -J_{2n} & J_{2n} \\ J_{2n} & -J_{2n} & J_{2n} & -J_{2n} \\ -J_{2n} & J_{2n} & -J_{2n} & J_{2n} \end{pmatrix},$$
$$E_7 = u_3 u_3^* = \frac{1}{8n}((-1)^{u+v}), u, v \in \{0, 1 \ldots, 8n-1\}$$

where

(i) $e_4(u, v) = (-1)^{u+v}$, when $u, v \in V_1 \cup V_3$ or $u, v \in V_2 \cup V_4$

(ii) $e_4(u, v) = (-1)^{u+v+1}$, when $u \in V_1 \cup V_3, v \in V_2 \cup V_4$ or $u \in V_2 \cup V_4, v \in V_1 \cup V_3$,

and the projective matrices corresponding to the eigenvectors u_2, u_4, u_6 and u_8 are given by

$$E_2 = u_2{u_2}^* = \frac{1}{8n}\begin{pmatrix} X & \mathbf{i}X & -X & -\mathbf{i}X \\ -\mathbf{i}X & X & \mathbf{i}X & -X \\ -X & -\mathbf{i}X & X & \mathbf{i}X \\ \mathbf{i}X & -X & -\mathbf{i}X & X \end{pmatrix}, E_4 = u_4{u_4}^* = \frac{1}{8n}\begin{pmatrix} Y & -\mathbf{i}Y & -Y & \mathbf{i}Y \\ \mathbf{i}Y & Y & -\mathbf{i}Y & -Y \\ -Y & \mathbf{i}Y & Y & -\mathbf{i}Y \\ -\mathbf{i}Y & -Y & \mathbf{i}Y & Y \end{pmatrix},$$
$$E_6 = u_6{u_6}^* = \frac{1}{8n}\begin{pmatrix} X & -\mathbf{i}X & -X & \mathbf{i}X \\ \mathbf{i}X & X & -\mathbf{i}X & -X \\ -X & \mathbf{i}X & X & -\mathbf{i}X \\ -\mathbf{i}X & -X & \mathbf{i}X & X \end{pmatrix} \text{ and } E_8 = u_8{u_8}^* = \frac{1}{8n}\begin{pmatrix} Y & \mathbf{i}Y & -Y & -\mathbf{i}Y \\ -\mathbf{i}Y & Y & \mathbf{i}Y & -Y \\ -Y & -\mathbf{i}Y & Y & \mathbf{i}Y \\ \mathbf{i}Y & -Y & -\mathbf{i}Y & Y \end{pmatrix},$$

where $X_{uv} = \mathbf{i}^{u-v}$ and $Y_{uv} = (-\mathbf{i})^{u-v}$.

The projective matrices corresponding to the eigenvectors $u_j^{(1)}, u_j^{(2)}, u_j^{(3)}$ and $u_j^{(4)}$, where $1 \le j \le n-1$ are given by

$$E_j{}^{(1)} = u_j^{(1)} u_j^{(1)*} = \frac{1}{4n} \begin{pmatrix} X_1 & 0 & X_1 & 0 \\ 0 & 0 & 0 & 0 \\ X_1 & 0 & X_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_j{}^{(2)} = u_j^{(2)} u_j^{(2)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & X_1 \\ 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & X_1 \end{pmatrix},$$

$$E_j{}^{(3)} = u_j^{(3)} u_j^{(3)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_2 & 0 & X_2 \\ 0 & 0 & 0 & 0 \\ 0 & X_2 & 0 & X_2 \end{pmatrix} \text{ and } \quad E_j{}^{(4)} = u_j^{(4)} u_j^{(4)*} = \frac{1}{4n} \begin{pmatrix} X_2 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 \\ X_2 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The projective matrices corresponding to the eigenvectors $v_k^{(1)}, v_k^{(2)}, v_k^{(3)}$ and $v_k^{(4)}$, where $1 \leq k \leq n-1$ are given by

$$F_k{}^{(1)} = v_k^{(1)} v_k^{(1)*} = \frac{1}{4n} \begin{pmatrix} Y_1 & 0 & -Y_1 & 0 \\ 0 & 0 & 0 & 0 \\ -Y_1 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_k{}^{(2)} = v_k^{(2)} v_k^{(2)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_1 & 0 & -Y_1 \\ 0 & 0 & 0 & 0 \\ 0 & -Y_1 & 0 & Y_1 \end{pmatrix},$$

$$F_k{}^{(3)} = v_k^{(3)} v_k^{(3)*} = \frac{1}{4n} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_2 & 0 & -Y_2 \\ 0 & 0 & 0 & 0 \\ 0 & -Y_2 & 0 & Y_2 \end{pmatrix} \quad \text{and} \quad F_k{}^{(4)} = v_k^{(4)} v_k^{(4)*} = \frac{1}{4n} \begin{pmatrix} Y_2 & 0 & -Y_2 & 0 \\ 0 & 0 & 0 & 0 \\ -Y_2 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} 1 & \omega^{-j} & \cdots & \omega^{-(2n-1)j} \\ \omega^{j} & 1 & \cdots & \omega^{-(2n-2)j} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{(2n-1)j} & \omega^{(2n-2)j} & \cdots & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & \omega^{j} & \cdots & \omega^{(2n-1)j} \\ \omega^{-j} & 1 & \cdots & \omega^{(2n-2)j} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{-(2n-1)j} & \omega^{-(2n-2)j} & \cdots & 1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 1 & \mathbf{i}^{-1}\omega^{-k} & \cdots & \mathbf{i}^{-(2n-1)}\omega^{-(2n-1)k} \\ \mathbf{i}\omega^{k} & 1 & \cdots & \mathbf{i}^{-(2n-2)}\omega^{-(2n-2)k} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{i}^{(2n-1)}\omega^{(2n-1)k} & \mathbf{i}^{(2n-2)}\omega^{(2n-2)k} & \cdots & 1 \end{pmatrix} \quad \text{and}$$

$$Y_2 = \begin{pmatrix} 1 & \mathbf{i}^{-1}\omega^{k} & \cdots & \mathbf{i}^{-(2n-1)}\omega^{(2n-1)k} \\ \mathbf{i}\omega^{-k} & 1 & \cdots & \mathbf{i}^{-(2n-2)}\omega^{(2n-2)k} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{i}^{(2n-1)}\omega^{-(2n-1)k} & \mathbf{i}^{(2n-2)}\omega^{-(2n-2)k} & \cdots & 1 \end{pmatrix}.$$

Therefore, the transition matrix $H(t)$ of the normal Cayley graph $\mathrm{Cay}(V_{8n}, S)$ is given by

$$\begin{aligned} H(t) &= \exp(-\mathbf{i}t\alpha_1)E_1 + \exp(-\mathbf{i}t\alpha_2)E_2 + \exp(-\mathbf{i}t\alpha_3)E_3 + \exp(-\mathbf{i}t\alpha_4)E_4 \\ &\quad + \exp(-\mathbf{i}t\alpha_5)E_5 + \exp(-\mathbf{i}t\alpha_6)E_6 + \exp(-\mathbf{i}t\alpha_7)E_7 + \exp(-\mathbf{i}t\alpha_8)E_8 \\ &\quad + \sum_{j=1}^{n-1} \exp(-\mathbf{i}t\beta_j)(E_j^{(1)} + E_j^{(2)} + E_j^{(3)} + E_j^{(4)}) + \sum_{k=1}^{n-1} \exp(-\mathbf{i}t\gamma_k)(F_k^{(1)} + F_k^{(2)} + F_k^{(3)} + F_k^{(4)}). \end{aligned}$$

Now we compute the (u, v)-th element of the transition matrix. The following three cases arise.

Case 1. If $u \in V_1, v \in V_2$, then

$$\begin{aligned} H(t)_{uv} &= \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) + \exp(-\mathbf{i}t\alpha_2)\mathbf{i}^{u-v+1} + (-1)^{u+v+1}\exp(-\mathbf{i}t\alpha_3) + (-\mathbf{i})^{u-v+1}\exp(-\mathbf{i}t\alpha_4) \\ &\quad - \exp(-\mathbf{i}t\alpha_5) + \mathbf{i}^{u-v+3}\exp(-\mathbf{i}t\alpha_6) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_7) + (-\mathbf{i})^{u-v+3}\exp(-\mathbf{i}t\alpha_8)). \end{aligned}$$

This implies that $\mid H(t)_{uv} \mid \leq \frac{1}{8n} \times 8 = \frac{1}{n} < 1$.

Simillarly, it can be shown that for $u \in V_1, v \in V_4$ or $u \in V_2, v \in V_1$ or $u \in V_2, v \in V_3$ or $u \in V_3, v \in V_2$ or $u \in V_3, v \in V_4$ or $u \in V_4, v \in V_1$ or $u \in V_4, v \in V_3$, $\mid H(t)_{uv} \mid < 1$.

Therefore, in this case PST cannot occur between the vertices u and v.

Case 2. If $u, v \in V_1$ or $u, v \in V_2$ or $u, v \in V_3$ or $u, v \in V_4$, then

$$\begin{aligned} H(t)_{uv} &= \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) + \mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_2) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) + (-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_4) \\ &\quad + \exp(-\mathbf{i}t\alpha_5) + \mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_6) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_7) + (-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_8)) \\ &\quad + \frac{1}{4n}\sum_{j=1}^{n-1}\exp(-\mathbf{i}t\beta_j)(\omega^{(u-v)j} + \omega^{-(u-v)j}) + \frac{1}{4n}\sum_{k=1}^{n-1}\exp(-\mathbf{i}t\gamma_k)(\mathbf{i}^{u-v}\omega^{(u-v)k} + \mathbf{i}^{u-v}\omega^{-(u-v)k}). \end{aligned}$$

This implies that $\mid H(t)_{uv} \mid \leq \frac{1}{8n} \times 8 + \frac{1}{4n} \times (2n-2) + \frac{1}{4n} \times (2n-2) = 1$.

Therefore, $\mid H(t)_{uv} \mid \leq 1$. Thus $\mid H(t)_{uv} \mid = 1$ if and only if for $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$, it holds that,

$$
\begin{aligned}
\exp(-\mathbf{i}t\alpha_1) &= \mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_2)\\
\exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3)\\
\exp(-\mathbf{i}t\alpha_1) &= (-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_4)\\
\exp(-\mathbf{i}t\alpha_1) &= \exp(-\mathbf{i}t\alpha_5)\\
\exp(-\mathbf{i}t\alpha_1) &= \mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_6)\\
\exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v}\exp(-\mathbf{i}t\alpha_7)\\
\exp(-\mathbf{i}t\alpha_1) &= (-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_8)\\
\exp(-\mathbf{i}t\alpha_1) &= \omega^{(u-v)j}\exp(-\mathbf{i}t\beta_j)\\
\exp(-\mathbf{i}t\alpha_1) &= \omega^{-(u-v)j}\exp(-\mathbf{i}t\beta_j)\\
\exp(-\mathbf{i}t\alpha_1) &= \mathbf{i}^{u-v}\omega^{(u-v)k}\exp(-\mathbf{i}t\gamma_k)\\
\exp(-\mathbf{i}t\alpha_1) &= \mathbf{i}^{u-v}\omega^{-(u-v)k}\exp(-\mathbf{i}t\gamma_k)
\end{aligned}
$$

From the last two equations, since ω is the $2n$-th root of unity, we get that n divides $u-v$. Without loss of generality, we assume that $u > v$. This implies that $u - v = n$. Let $t = 2\pi T$. Then the preceding equations implies that

$$(\alpha_1 - \alpha_2)T - \frac{n}{4} \in \mathbb{Z} \tag{22}$$

$$(\alpha_1 - \alpha_3)T \in \mathbb{Z} \tag{23}$$

$$(\alpha_1 - \alpha_4)T - \frac{n}{4} \in \mathbb{Z} \tag{24}$$

$$(\alpha_1 - \alpha_5)T \in \mathbb{Z} \tag{25}$$

$$(\alpha_1 - \alpha_6)T - \frac{n}{4} \in \mathbb{Z} \tag{26}$$

$$(\alpha_1 - \alpha_7)T \in \mathbb{Z} \tag{27}$$

$$(\alpha_1 - \alpha_8)T - \frac{n}{4} \in \mathbb{Z} \tag{28}$$

$$(\alpha_1 - \beta_j)T - \frac{j}{2} \in \mathbb{Z} \tag{29}$$

$$(\alpha_1 - \gamma_k)T - \frac{n}{4} - \frac{k}{2} \in \mathbb{Z} \tag{30}$$

Since $0 = \mathrm{tr}(A) = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+4\sum_{j=1}^{n-1}\beta_j+4\sum_{k=1}^{n-1}\gamma_k$, we have that $8n\alpha_1 T \in \mathbb{Z}$, and since $\alpha_1 = |S|$ is a positive integer, we have $T \in \mathbb{Q}$. This implies that all the eigenvalues of the graph are rational numbers. It is well known that any rational eigenvalue of a graph is an integer. Therefore, in this case the graph is integral.

Our next discussion is distinguished into two parts according as $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$.

(i) $n \equiv 0 \pmod 4$. In this case, conditions (22) to (30) turns into

$$(\alpha_1 - \alpha_2)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_3)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_4)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_5)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_6)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_7)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_8)T \in \mathbb{Z}$$
$$(\alpha_1 - \beta_j)T - \frac{j}{2} \in \mathbb{Z}$$
$$(\alpha_1 - \gamma_k)T - \frac{k}{2} \in \mathbb{Z}$$

The preceding conditions implies that, $\Upsilon_2(\alpha_1 - \beta_{2j'-1})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'-1})$ are the same, say μ_1, and $\Upsilon_2(\alpha_1 - \alpha_2)$, $\Upsilon_2(\alpha_1 - \alpha_3)$, $\Upsilon_2(\alpha_1 - \alpha_4)$, $\Upsilon_2(\alpha_1 - \alpha_5)$, $\Upsilon_2(\alpha_1 - \alpha_6)$, $\Upsilon_2(\alpha_1 - \alpha_7)$, $\Upsilon_2(\alpha_1 - \alpha_8)$, $\Upsilon_2(\alpha_1 - \beta_{2j'})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'})$ are all strictly greater than μ_1, for $1 \leq j' \leq \frac{n-1}{2}$ and $1 \leq k' \leq \frac{n-1}{2}$.

(ii) $n \equiv 2 \pmod 4$. In this case, conditions (22) to (30) turns into

$$(\alpha_1 - \alpha_2)T - \frac{1}{2} \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_3)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_4)T - \frac{1}{2} \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_5)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_6)T - \frac{1}{2} \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_7)T \in \mathbb{Z}$$
$$(\alpha_1 - \alpha_8)T - \frac{1}{1} \in \mathbb{Z}$$
$$(\alpha_1 - \beta_j)T - \frac{j}{2} \in \mathbb{Z}$$
$$(\alpha_1 - \gamma_k)T - \frac{1}{2} - \frac{k}{2} \in \mathbb{Z}$$

The preceding conditions implies that, $\Upsilon_2(\alpha_1 - \alpha_2)$, $\Upsilon_2(\alpha_1 - \alpha_4)$, $\Upsilon_2(\alpha_1 - \alpha_6)$, $\Upsilon_2(\alpha_1 - \alpha_8)$, $\Upsilon_2(\alpha_1 - \beta_{2j'-1})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'})$ are the same, say μ_2, and $\Upsilon_2(\alpha_1 - \alpha_3)$, $\Upsilon_2(\alpha_1 - \alpha_5)$, $\Upsilon_2(\alpha_1 - \alpha_7)$, $\Upsilon_2(\alpha_1 - \beta_{2j'})$ and $\Upsilon_2(\alpha_1 - \gamma_{2k'-1})$ are all strictly greater than μ_2, for $1 \leq j' \leq \frac{n-1}{2}$ and $1 \leq k' \leq \frac{n-1}{2}$.

Case 3. If $u \in V_1, v \in V_3$ or $u \in V_2, v \in V_4$ or $u \in V_3, v \in V_1$ or $u \in V_4, v \in V_2$, then

$$\begin{aligned}H(t)_{uv} &= \frac{1}{8n}(\exp(-\mathbf{i}t\alpha_1) - \mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_2) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3) - (-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_4) \\ &+ \exp(-\mathbf{i}t\alpha_5) - \mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_6) + (-1)^{u+v}\exp(-\mathbf{i}t\alpha_7) - (-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_8)) \\ &+ \frac{1}{4n}\sum_{j=1}^{n-1}\exp(-\mathbf{i}t\beta_j)(\omega^{(u-v)j} + \omega^{-(u-v)j}) + \frac{1}{4n}\sum_{k=1}^{n-1}\exp(-\mathbf{i}t\gamma_k)(-\mathbf{i}^{u-v}\omega^{(u-v)k} - \mathbf{i}^{u-v}\omega^{-(u-v)k}).\end{aligned}$$

This implies that $\mid H(t)_{uv} \mid \leq \frac{1}{8n} \times 8 + \frac{1}{4n} \times (2n-2) + \frac{1}{4n} \times (2n-2) = 1$.
Therefore, $\mid H(t)_{uv} \mid \leq 1$. Thus $\mid H(t)_{uv} \mid = 1$ if and only if for $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$, it holds that,

$$\begin{aligned}\exp(-\mathbf{i}t\alpha_1) &= -\mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_2)\\ \exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v}\exp(-\mathbf{i}t\alpha_3)\\ \exp(-\mathbf{i}t\alpha_1) &= -(-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_4)\\ \exp(-\mathbf{i}t\alpha_1) &= \exp(-\mathbf{i}t\alpha_5)\\ \exp(-\mathbf{i}t\alpha_1) &= -\mathbf{i}^{u-v}\exp(-\mathbf{i}t\alpha_6)\\ \exp(-\mathbf{i}t\alpha_1) &= (-1)^{u+v}\exp(-\mathbf{i}t\alpha_7)\\ \exp(-\mathbf{i}t\alpha_1) &= -(-\mathbf{i})^{u-v}\exp(-\mathbf{i}t\alpha_8)\\ \exp(-\mathbf{i}t\alpha_1) &= \omega^{(u-v)j}\exp(-\mathbf{i}t\beta_j)\\ \exp(-\mathbf{i}t\alpha_1) &= \omega^{-(u-v)j}\exp(-\mathbf{i}t\beta_j)\\ \exp(-\mathbf{i}t\alpha_1) &= -\mathbf{i}^{u-v}\omega^{(u-v)k}\exp(-\mathbf{i}t\gamma_k)\\ \exp(-\mathbf{i}t\alpha_1) &= -\mathbf{i}^{u-v}\omega^{-(u-v)k}\exp(-\mathbf{i}t\gamma_k)\end{aligned}$$

From the last two equations, we get that $u - v = 4n$. Let $t = 2\pi T$. Then the preceding equations implies that

$$(\alpha_1 - \alpha_2)T - \frac{1}{2} \in \mathbb{Z} \tag{31}$$

$$(\alpha_1 - \alpha_3)T \in \mathbb{Z} \tag{32}$$

$$(\alpha_1 - \alpha_4)T - \frac{1}{2} \in \mathbb{Z} \tag{33}$$

$$(\alpha_1 - \alpha_5)T \in \mathbb{Z} \tag{34}$$

$$(\alpha_1 - \alpha_6)T - \frac{1}{2} \in \mathbb{Z} \tag{35}$$

$$(\alpha_1 - \alpha_7)T \in \mathbb{Z} \tag{36}$$

$$(\alpha_1 - \alpha_8)T - \frac{1}{2} \in \mathbb{Z} \tag{37}$$

$$(\alpha_1 - \beta_j)T \in \mathbb{Z} \tag{38}$$

$$(\alpha_1 - \gamma_k)T - \frac{1}{2} \in \mathbb{Z} \tag{39}$$

Since $0 = \mathrm{tr}(A) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + 4\sum_{j=1}^{n-1}\beta_j + 4\sum_{k=1}^{n-1}\gamma_k$, we have that $8n\alpha_1 T \in \mathbb{Z}$, and since $\alpha_1 = |S|$ is a positive integer, we have $T \in \mathbb{Q}$. This implies that all the eigenvalues of the graph are rational numbers. It is well known that any rational eigenvalue of a graph is an integer. Therefore, in this case the graph is integral.

The preceding conditions implies that, $\Upsilon_2(\alpha_1 - \alpha_2)$, $\Upsilon_2(\alpha_1 - \alpha_4)$, $\Upsilon_2(\alpha_1 - \alpha_6)$, $\Upsilon_2(\alpha_1 - \alpha_8)$ and $\Upsilon_2(\alpha_1 - \gamma_k)$ are the same, say μ_3, and $\Upsilon_2(\alpha_1 - \alpha_3)$, $\Upsilon_2(\alpha_1 - \alpha_5)$, $\Upsilon_2(\alpha_1 - \alpha_7)$ and $\Upsilon_2(\alpha_1 - \beta_j)$ are all strictly greater than μ_3, for $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$.

Using the same technique in Section 3.1, we can show that the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where $M = \gcd(\alpha - \alpha_1 \colon \alpha \in \mathrm{Spec}(\Gamma) \setminus \{\alpha_1\})$.

This completes the proof. $\square$

References

[1] R.J. Angeles-Canul, R. Norton, M. Opperman, C. Paribello, M. Russell and C. Tamon. Perfect state transfer, integral circulants and join of graphs. *Quantum. Inf. Comput.* 10 (2010), 325-342.

[2] M. Arezoomand, F. Shafiei and M. Ghorbani. Perfect state transfer on Cayley graphs over the dicyclic group. *Linear Algebra Appl.* 639 (2022), 116-134.

[3] M. Basic and M.D. Petkovic. Perfect state transfer in integral circulant graphs of non-square-free order. *Linear Algebra Appl.* 433(1) (2010), 149-163.

[4] M. Basic, M.D. Petkovic and D. Stevanovic. Perfect state transfer in integral circulant graphs. *Appl. Math. Lett.* 22(7) (2009), 1117-1121.

[5] M. Basic and M.D. Petkovic. Some classes of integral circulant graphs either allowing or not allowing perfect state transfer. *Appl. Math. Lett.* 22(10) (2009), 1609-1615.

[6] A. Bernasconi, C. Godsil and S. Severini. Quantum networks on cubelike graphs. *Phys. Rev. A.* (3) 78(5) (2008), 052320.

[7] S. Bose. Quantum communication through an unmodulated spin chain. *Phys. Rev. lett.* 91(20) (2003), 207901.

[8] X. Cao and K. Feng. Perfect state transfer on Cayley graphs over dihedral groups. *Linear Multilinear Algebra.* 69(2) (2021), 343-360.

[9] W. Cheung and C. Godsil. Perfect state transfer in cubelike graphs. *Linear Algebra Appl.* 435(10) (2011), 2468-2474.

[10] M. Christandl, N. Datta, T. Dorlas, A. Ekert, A. Kay and A. Landahl. Perfect transfer of arbitrary states in quantum spin networks. *Phys. Rev. A.* 71 (2005), 032312.

[11] G. Coutinho and C. Godsil. Perfect state transfer in products and covers of graphs. *Linear Multilinear Algebra.* 64(2) (2016), 235-246.

[12] G. Coutinho. Quantum state transfer in graphs [PhD dissertation]. University of Waterloo, (2014).

[13] M. R. Darafsheh and N. S. Poursalavati. On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups. *SUT J. Math.* 37(1) (2001), 1-17.

[14] E. Farhi and S. Gutmann. Quantum computation and decision trees. *Phys. Rev. A.* (3) 58 (1998), 915-928.

[15] C. Godsil and G. Royle. Algebraic Graph Theory. Graduate Texts in Mathematics. Vol. 207. Springer-Verlag, New York, (2001).

[16] C. Godsil. Periodic graphs. *Electron. J. Comb.* 18(1) (2011), #P23.

[17] C. Godsil. State transfer on graphs. *Discrete Math.* 312(1) (2012), 129-147.

[18] C. Godsil. When can perfect state transfer occur? *Electron. J. Linear Algebra.* 23 (2012), 877-890.

[19] G. James and M. Liebeck. Representations and Characters of Groups. Second Edition. Cambridge University Press, New York, (2001).

[20] A. Khalilipour and M. Ghorbani. Perfect state transfer on Cayley graphs over groups. *J. Disc. Math. Appl.* 8(4) (2023), 213-226.

[21] G. Luo, X. Cao, D. Wang and X. Wu. Perfect quantum state transfer on Cayley graphs over semi-dihedral groups. *Linear Multilinear Algebra.* 70(21) (2022), 6358-6374.

[22] H. Pal and B. Bhattacharjya. Perfect state transfer on NEPS of the path on three vertices. *Discrete Math.* 339 (2016), 831-838.

[23] M.D. Petkovic and M. Basic. Further results on the perfect state transfer in integral circulant graphs. *Comput. Math. Appl.* 61 (2011), 300-312.

[24] B. Steinberg. Representation Theory of Finite Groups. Universitext, Springer, New York, 2012.

[25] T. Tan, K. Feng and X. Cao. Perfect state transfer on abelian Cayley graphs. *Linear Algebra Appl.* 563 (2019), 331-352.