Free extensivity via distributivity

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Abstract

We consider the canonical pseudodistributive law between various free limit completion pseudomonads and the free coproduct completion pseudomonad. When the class of limits includes pullbacks, we show that this consideration leads to notions of extensive categories. More precisely, we show that extensive categories with pullbacks and infinitary lextensive categories are the pseudoalgebras for the pseudomonads resulting from the pseudodistributive laws. Moreover, we introduce the notion of doubly-infinitary lextensive category, and we establish that the freely generated such categories are cartesian closed. From this result, we further deduce that, in freely generated infinitary lextensive categories, the objects with a finite number of connected components are exponentiable. We conclude our work with remarks on descent theoretical aspects of this work, along with results concerning non-canonical isomorphisms.

Keywords: free (co)limit completion, free coproduct completion, exponentiable object, (co)lax idempotent pseudomonad, extensive category, pseudodistributive law, cartesian closed category

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Introduction

Two-dimensional monad theory [5, 23] is the categorical approach to two-dimensional universal algebra, which mainly deals with the problem of understanding *algebraic structures*, in a suitable sense, over objects in a 2-category.

Focusing on the case where the base 2-category is the 2-category of categories **CAT**, this leads to the systematic study of categories with additional (algebraic) structures (or properties) [5, 21]. The 2-categories of interest usually arise as 2-categories of pseudoalgebras or lax algebras of a given pseudomonad – we refer, for instance, to [24, 26, 27] for the definitions of these concepts.

There are many well-known examples of such 2-categories of interest; namely:

- the 2-category of monoidal categories, monoidal functors and monoidal natural transformations is the 2-category of pseudoalgebras for the free monoid 2-monad (also known as the *list* 2-monad) on **CAT** [5, 16];
- the 2-category of monads is given by the 2-category of lax algebras w.r.t. the identity 2-monad on CAT;
- 2-categories of pseudofunctors and pseudonatural transformations between two suitable cocomplete 2-categories [5, 26, 28];
- the 2-category of categories with Φ -(co)limits and Φ -(co)limit preserving functors is the 2-category of pseudoalgebras and pseudomorphisms w.r.t. a suitable pseudomonad on **CAT** [21, 35, 40].

The framework of two-dimensional monad theory is well-suited for studying the age-old problem of distributivity between limits and colimits of a given category. Specifically, our focus lies on the canonical pseudodistributive law [36, 37] between various sorts of free limit completion pseudomonads and the free coproduct completion pseudomonad. Previous considerations of such distributivity properties include (infinitary) distributive categories [8], completely distributive categories [38], and doubly-infinitary distributive categories [34]. In this paper, we show that a similar analysis gives rise to well-known and novel notions of extensive categories.

Recall that, if C has (in)finite coproducts, C is said to be an (infinitary) extensive category [8] if the functor

$$\prod_{i \in I} \mathcal{C} \downarrow X_i \xrightarrow{\sum} \mathcal{C} \downarrow \sum_{i \in I} X_i$$

is an equivalence of categories for every (in)finite family $(X_i)_{i\in I}$ of objects in \mathcal{C} . It has been observed in [9] and [43, Section 7] that "(infinitary) extensivity" can be viewed as a distributivity condition of pullbacks over (infinitary) coproducts.

The present work, which is a sequel to [34], aims to study categories with a given class of limits, small coproducts, and a (pseudo)distributive law between them. More precisely, given a class Φ of diagrams, there is a canonical pseudodistributive law between the free Φ -limit completion pseudomonad and the free coproduct completion, denoted by **Fam** [48, 22, 35]. The pseudoalgebras for the composite pseudomonad can be given as categories with Φ -limits and coproducts such that the coproduct functor

$$\sum : \mathbf{Fam}(\mathcal{C}) \to \mathcal{C} \tag{1}$$

preserves Φ -limits.

One of our key contributions is the observation that various flavors of infinitary extensive categories are pseudoalgebras for such composites of pseudomonads; namely, let \mathcal{C} be a category with coproducts:

- if \mathcal{C} has pullbacks, and (1) preserves them, then \mathcal{C} is infinitary extensive with pullbacks;
- if \mathcal{C} has finite limits, and (1) preserves them, then \mathcal{C} is infinitary lextensive;
- if \mathcal{C} has small limits, and (1) preserves them, we say that the category \mathcal{C} is doubly-infinitary lextensive. We observe that \mathcal{C} satisfies such properties if and only if \mathcal{C} is simultaneously a doubly-infinitary distributive category [34] as well as a lextensive category.

The observations presented above, coupled with the findings of [33], contribute to the understanding of extensive categories and distributive categories through the prism of 2-dimensional universal algebra, adding to the comparison of these notions started originally in [8].

Motivated by [33, 32], we further study the exponentiable objects of the free pseudoalgebras for the pseudomonads we considered; namely, we find that:

- in a freely generated infinitary lextensive category, objects with a finite number of connected components are exponentiable;
- freely generated doubly-infinitary are cartesian closed.

Outline: We revisit the notion of free Φ -colimit completions for a class Φ of diagrams (small categories) in Section 1. Several authors have worked on free (co)limit completions; namely, we have [15, 45, 1] for ordinary categories, and [20, 3] in the context of enriched category theory. We also have the accounts [22, 48, 35] which study free Φ -(co)limit completions from the perspective of 2-dimensional monad theory [5, 26, 27], which is the approach we employ, so some familiarity with these methods is assumed. We focus specifically on four classes of free (co)limit completions:

- the free *coproduct* completion, denoted **Fam**,
- the free small limit completion, denoted \mathcal{L} ,
- the free *finite limit* completion, denoted \mathcal{L}_{fin} ,
- the free pullback completion, denoted \mathcal{L}_{pb} .

In Section 2, we study the distributivity of Φ -limits over coproducts. Similar work has been carried out in [2, 38, 46] and in the prequel [34]. After recalling the necessary concepts pertaining to pseudodistributive laws [36, 37, 47], we confirm that there is a pseudodistributive law between any free Φ -limit completion pseudomonad and the free coproduct completion pseudomonad **Fam** (Lemma 2.3). Instantiating this result with each of the aforementioned free limit completions, we obtain the composite pseudomonads **Fam** $\circ \mathcal{L}_{\text{fin}}$, **Fam** $\circ \mathcal{L}_{\text{pb}}$, and **Fam** $\circ \mathcal{L}$.

The study of these pseudomonads and their pseudoalgebras have given us novel characterizations of (infinitary) extensivity. More specifically, we prove that:

- (Fam $\circ \mathcal{L}_{fin}$)-pseudoalgebras are precisely the lextensive categories (Theorem 2.5),
- (Fam $\circ \mathcal{L}_{pb}$)-pseudoalgebras are precisely the extensive categories with pullbacks (Theorem 2.7),

- (Fam $\circ \mathcal{L}$)-pseudoalgebras are precisely the doubly-infinitary lextensive categories; that is, lextensive categories that are also doubly-infinitary distributive [34] (Theorem 2.8).

Our study of exponentiable objects in freely generated categorical structures is the content of Section 3. This includes our main results, which respectively state that

- freely generated doubly-infinitary lextensive categories are cartesian closed (Theorem 3.4).
- in freely generated infinitary lextensive categories, finite coproducts of connected objects are exponentiable (Theorem 3.7),

We also supplement this section with examples and non-examples of cartesian closed and doubly-infinitary lextensive categories.

In Section 5, we show that analogous results also hold for the *free finite coproduct completion* pseudomonad, leading to similar novel characterisations of (finitely) extensive categories. Further, we discuss possible avenues for future work, descent theoretical considerations of our findings, and we note a result on non-canonical isomorphisms, as a direct consequence of the work of [30].

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1 Free colimit completions

Let **CAT** be the 2-category of locally small (**Set**-enriched) categories. Any other category considered in this work is assumed to be an object of **CAT**.

Let Φ be a class of small categories. We say a category \mathcal{C} has Φ -colimits if any functor $D \colon \mathcal{J} \to \mathcal{C}$ with $\mathcal{J} \in \Phi$ has a colimit in \mathcal{C} . Moreover, if $F \colon \mathcal{C} \to \mathcal{D}$ is a functor between categories with Φ -colimits, we have a morphism

$$\operatorname{colim} FD \to F(\operatorname{colim} D) \tag{2}$$

which is natural in $D: \mathcal{J} \to \mathcal{C}$ for $\mathcal{J} \in \Phi$. We say that F preserves Φ -colimits if (2) is a natural isomorphism.

We let Φ -Colim be the 2-category of categories with Φ -colimits, Φ -colimit preserving functors and natural transformations. We have a forgetful 2-functor

$$\Phi\text{-Colim} \longrightarrow \mathbf{CAT} \tag{3}$$

which is pseudomonadic – we let \mathcal{P}_{Φ} be the left biadjoint to (3), as well as the induced pseudomonad by the biadjunction – the *free* Φ -colimit completion pseudomonad. We can justify this abuse of notation, by noting that a category \mathcal{C} has Φ -colimits if and only if the (fully faithful) unit of \mathcal{P}_{Φ} at \mathcal{C} , denoted by $\mathfrak{y}: \mathcal{C} \to \mathcal{P}_{\Phi}(\mathcal{C})$, has a left adjoint [3]. Thus, being a \mathcal{P}_{Φ} -pseudoalgebra becomes a property of the category \mathcal{C} , as opposed to structure [21]. In other words, \mathcal{P}_{Φ} is a lax idempotent pseudomonad [22] (also known as Kock- $Z\"{o}berlein pseudomonad$).

Dually, we say a category \mathcal{C} has Φ -limits whenever $\mathcal{C}^{\mathsf{op}}$ has Φ -colimits, and we say a functor $F: \mathcal{C} \to \mathcal{D}$ between categories with Φ -limits preserves Φ -limits if $F^{\mathsf{op}}: \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ preserves Φ -colimits. We denote by Φ -**Lim** the 2-category of categories with Φ -limits, Φ -limit preserving functors and natural transformations. We also have a pseudomonadic 2-functor

$$\Phi\text{-Lim} \longrightarrow \mathbf{CAT} \tag{4}$$

whose left biadjoint and induced pseudomonad are denoted by \mathcal{L}_{Φ} , so that we have a biequivalence \mathcal{L}_{Φ} -PsAlg $\simeq \Phi$ -Lim. In fact, we note that $\mathcal{L}_{\Phi}(\mathcal{C}) = \mathcal{P}_{\Phi}(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$. We likewise denote the (fully faithful) unit at a category \mathcal{C} by $\mathfrak{y} \colon \mathcal{C} \to \mathcal{L}_{\Phi}(\mathcal{C})$. This unit has a right adjoint if and only if \mathcal{C} has Φ -limits.

Remark 1.1. In [20, 3], the notions of Φ -colimits and Φ -colimit completions were worked out in the more general setting of enriched category theory, where Φ is taken to be a class of small weights instead (that is, functors $\mathcal{J}^{\mathsf{op}} \to \mathcal{V}$ with \mathcal{J} small), where \mathcal{V} is the base monoidal category.

In our setting, the notions we provided correspond to the classes Φ of weights that are constant functors to the terminal object. We leave the consideration of our results in an enriched setting for future work.

As argued in [20, 3], the free Φ -colimit completion $\mathcal{P}_{\Phi}(\mathcal{C})$ of a category \mathcal{C} is most succinctly described as the smallest full subcategory of $\mathbf{CAT}(\mathcal{C}^{\mathsf{op}}, \mathbf{Set})$ that has Φ -colimits. Dually, $\mathcal{L}_{\Phi}(\mathcal{C})$ is the smallest full subcategory of $\mathbf{CAT}(\mathcal{C}, \mathbf{Set})^{\mathsf{op}}$ that has Φ -limits. With this, we can obtain an expression for the hom-sets of Φ -(co)limit completions:

Lemma 1.2. Let Φ be a class of small categories, let C be an object of C, and let $E \colon \mathcal{K} \to \mathcal{P}_{\Phi}(C)$ be a diagram with $\mathcal{K} \in \Phi$. We have a natural isomorphism

$$(\mathcal{P}_{\Phi}\mathcal{C})(\mathfrak{y}(C), \underset{k \in \mathcal{K}}{\mathsf{colim}} Ek) \cong \underset{k \in \mathcal{K}}{\mathsf{colim}}(\mathcal{P}_{\Phi}\mathcal{C})(\mathfrak{y}(C), Ek), \tag{5}$$

and dually, for a diagram $F: \mathcal{K} \to \mathcal{L}_{\Phi}(\mathcal{C})$,

$$(\mathcal{L}_{\Phi}\mathcal{C})(\lim_{k \in \mathcal{K}} Fk, \mathfrak{y}(C)) \cong \underset{k \in \mathcal{K}}{\mathsf{colim}}(\mathcal{L}_{\Phi}\mathcal{C})(Fk, \mathfrak{y}(C)). \tag{6}$$

Proof. We have

$$\begin{split} &(\mathcal{P}_{\Phi}\mathcal{C})(\mathfrak{y}(C), \operatorname*{colim}_{k \in \mathcal{K}} Ek) \\ &\cong \mathbf{CAT}(\mathcal{C}^{\mathsf{op}}, \mathbf{Set})(\mathcal{C}(-, C), \operatorname*{colim}_{k \in \mathcal{K}} Ek) \\ &\cong (\operatorname*{colim}_{k \in \mathcal{K}} Ek)C \qquad \qquad \text{Yoneda lemma,} \\ &\cong \operatorname*{colim}_{k \in \mathcal{K}} ((Ek)C) \qquad \qquad \text{componentwise colimits,} \\ &\cong \operatorname*{colim}_{k \in \mathcal{K}} \mathbf{CAT}(\mathcal{C}^{\mathsf{op}}, \mathbf{Set})(\mathcal{C}(-, C), Ek) \qquad \qquad \text{Yoneda lemma,} \\ &\cong \operatorname*{colim}_{k \in \mathcal{K}} (\mathcal{P}_{\Phi}\mathcal{C})(\mathfrak{y}(C), Ek). \end{split}$$

This leads to the following formulas for the homsets, based on the observation that representable functors preserve limits.

Corollary 1.3. Let Φ be a class of small categories. If $\mathcal{J}, \mathcal{K} \in \Phi$, and $F: \mathcal{J} \to \mathcal{C}, G: \mathcal{K} \to \mathcal{P}_{\Phi}(\mathcal{C})$, then

$$\mathcal{P}_{\Phi}(\mathcal{C})(\operatorname*{colim}_{j\in\mathcal{J}}Fj,\operatorname*{colim}_{k\in\mathcal{K}}Gk)\cong \lim_{j\in\mathcal{J}}\operatorname*{colim}_{k\in\mathcal{K}}\mathcal{P}_{\Phi}(\mathcal{C})(Fj,Gk),\tag{7}$$

and dually, if $H: \mathcal{K} \to \mathcal{L}_{\Phi}(\mathcal{C})$, then

$$\mathcal{L}_{\Phi}(\mathcal{C})(\lim_{k \in \mathcal{K}} Hk, \lim_{j \in \mathcal{J}} Fj) \cong \lim_{j \in \mathcal{J}} \operatorname{colim}_{k \in \mathcal{K}} \mathcal{L}_{\Phi}(\mathcal{C})(Hk, Fj), \tag{8}$$

where we identify an object of C with its image in $\mathcal{P}_{\Phi}(C)$ and $\mathcal{L}_{\Phi}(C)$.

Alternatively, one may constuct $\mathcal{P}_{\Phi}(\mathcal{C})$, and, dually, $\mathcal{L}_{\Phi}(\mathcal{C})$, via transfinite induction [20, 3], by iteratively adjoining (co)limits of diagrams with domain in Φ , and taking unions at limit ordinals. In certain important cases, such as those of small (or finite) (co)limit or (co)product completions (see below), the induction stabilises after only one step.

Therefore, if Φ is a class of diagrams such that the transfinite construction converges in one step, every object in $\mathcal{P}_{\Phi}(\mathcal{C})$ is obtained as the Φ -colimit of a diagram in \mathcal{C} , from which we obtain the following characterisation of the Φ -(co)limit completion of \mathcal{C} ; $\mathcal{P}_{\Phi}(\mathcal{C})$ consists of

- diagrams $F: \mathcal{J} \to \mathcal{C}$ with $\mathcal{J} \in \Phi$ as objects.
- hom-sets given by the formula¹

$$\mathcal{P}_{\Phi}(\mathcal{C})(F,G) = \lim_{j \in \mathcal{J}} \underset{k \in \mathcal{K}}{\operatorname{colim}} \mathcal{C}(Fj,Gk)$$
(9)

for diagrams $F: \mathcal{J} \to \mathcal{C}, G: \mathcal{K} \to \mathcal{C}$ with $\mathcal{J}, \mathcal{K} \in \Phi$.

Dually, in case every object in $\mathcal{L}_{\Phi}(\mathcal{C})$ is obtained as the Φ -limit of a diagram in \mathcal{C} , the free limit completion of a category \mathcal{C} is given by $\mathcal{L}_{\Phi}(\mathcal{C}) = \mathcal{P}_{\Phi}(\mathcal{C}^{op})^{op}$. Explicitly, it consists of

- diagrams $F: \mathcal{J} \to \mathcal{C}$ with $\mathcal{J} \in \Phi$ as objects,

¹See [45, Section 1], and compare with (7).

- hom-sets given by the formula

$$\mathcal{L}_{\Phi}(\mathcal{C})(F,G) = \lim_{k \in \mathcal{K}} \underset{j \in \mathcal{J}}{\text{colim}} \, \mathcal{C}(Fj,Gk) \tag{10}$$

for diagrams $F: \mathcal{J} \to \mathcal{C}, G: \mathcal{K} \to \mathcal{C}$ with $\mathcal{J}, \mathcal{K} \in \Phi$.

Such a characterisation is appropriate, for example, when Φ consists of the class of all small (resp. finite) discrete categories, yielding small (resp. finite) coproduct and product completions, or if Φ consists of the class of all small (resp. finite) categories, yielding small (resp. finite) colimit and limit completions.

Free coproduct completion: When Φ is the class of discrete small categories (sets), Φ -Colim is the 2-category of categories with coproducts, and coproduct-preserving functors between them. In this case, we write $\mathbf{Fam} = \mathcal{P}_{\Phi}$.

We can explicitly describe the objects of $\mathbf{Fam}(\mathcal{C})$ – these are given by set-indexed families of objects $(X_i)_{i\in I}$, with $X_i \in \mathcal{C}$. Using the representation coming out of Corollary 1.3, we can also describe the hom-sets of morphisms from $(X_i)_{i\in I}$ to $(Y_j)_{j\in J}$ as

$$\prod_{i\in I}\sum_{j\in J}\mathcal{C}(X_i,Y_j).$$

There is a wealth of literature studying free coproduct completions and their properties. For instance, we refer the reader to [8, 1], [6, Chapter 6], [43, Section 7], and [32].

Free (co)limit completion: When Φ consists of all small categories, Φ -Colim is the 2-category of categories with small colimits and small-colimit preserving functors.

Given a category \mathcal{C} , its free colimit completion $\mathcal{P}(\mathcal{C})$ is the full subcategory of $\mathbf{CAT}(\mathcal{C}^{\mathsf{op}}, \mathbf{Set})$ consisting of the essentially small or accessible functors [20]. When \mathcal{C} is itself essentially small, we have $\mathcal{P}(\mathcal{C}) \simeq \mathbf{CAT}(\mathcal{C}^{\mathsf{op}}, \mathbf{Set})$.

Alternatively, as noted above, we can characterise $\mathcal{P}(\mathcal{C})$ as the category with small diagrams $F: \mathcal{J} \to \mathcal{C}$ as objects and homsets $\mathcal{P}(\mathcal{C})(F,G) = \lim_{j \in \mathcal{J}} \operatorname{colim}_{k \in \mathcal{K}} \mathcal{C}(Fj,Gk)$.

Free finite limit completion: We consider the class $\Phi = \text{fin}$ of all finite categories, in which case Φ -Lim is the 2-category of categories with finite limits and the functors that preserve them. We denote the free finite limit completion pseudomonad by \mathcal{L}_{fin} .

For any given category \mathcal{C} , the category $\mathcal{L}_{fin}(\mathcal{C})$ also admits a description as a category of diagrams, similar to $\mathcal{L}(\mathcal{C})$.

Free pullback completion: We consider the class $\Phi = \mathsf{pb}$ with consisting of a single element, the cospan diagram: $\cdot \to \cdot \leftarrow \cdot$

The 2-category Φ -**Lim** is the 2-category of categories with pullbacks and pullback preserving functors between them, and we denote the free pullback completion pseudomonad by \mathcal{L}_{pb} .

Unlike previous examples, not every object in $\mathcal{L}_{pb}(\mathcal{C})$ can be obtained by taking the pullback of a diagram in $\mathcal{L}_{pb}(\mathcal{C})$ of objects in the essential image of $\mathfrak{y} \colon \mathcal{C} \to \mathcal{L}_{pb}(\mathcal{C})$, so we cannot recover any formulae analogous to (10); we refer the interested reader to [3, Section 7] for further details.

2 Three pseudomonads

Let \mathcal{T} be a pseudomonad on **CAT**. We consider the following instance of the main result from [47]:

Lemma 2.1. The following are equivalent:

- (i) Fam lifts to a (lax idempotent) pseudomonad $Fam_{\mathcal{T}}$ on \mathcal{T} -PsAlg.
- (ii) There exists a pseudodistributive law $\delta \colon \mathcal{T} \circ \mathbf{Fam} \to \mathbf{Fam} \circ \mathcal{T}$.

Proof. Since **Fam** is a lax idempotent pseudomonad [22], we may instantiate [47, Theorem 35] with $\mathcal{P} = \mathbf{Fam}$.

In the presence of a pseudodistributive law $\delta \colon \mathcal{T} \circ \mathbf{Fam} \to \mathbf{Fam} \circ \mathcal{T}$, the composite $\mathbf{Fam} \circ \mathcal{T}$ also has the structure of a pseudomonad on \mathbf{CAT} [36, Section 5]. We also recall the following result from [37, Section 6]:

Lemma 2.2. We have a biequivalence $\operatorname{Fam}_{\mathcal{T}}\operatorname{-PsAlg} \simeq (\operatorname{Fam} \circ \mathcal{T})\operatorname{-PsAlg}$.

In [37] we also find a description of the $\mathbf{Fam}_{\mathcal{T}}$ -pseudoalgebras; they are the categories \mathcal{C} together with

- a \mathcal{T} -pseudoalgebra structure $\Lambda \colon \mathcal{TC} \to \mathcal{C}$ on \mathcal{C} ,
- a **Fam**-pseudoalgebra structure Σ : **Fam**(\mathcal{C}) $\to \mathcal{C}$ on \mathcal{C} in other words, \mathcal{C} is a category with coproducts,
- The coproduct functor $\Sigma \colon \mathbf{Fam}(\mathcal{C}) \to \mathcal{C}$ lifts to a \mathcal{T} -pseudomorphism.

Moreover, a $\mathbf{Fam}_{\mathcal{T}}$ -pseudomorphism $F \colon \mathcal{C} \to \mathcal{D}$ is a functor F that preserves coproducts and is a \mathcal{T} -pseudomorphism in a compatible way (up to natural isomorphism).

Our work focuses on pseudomonads \mathcal{T} that are free Φ -limit completions for a class Φ of small categories. In this setting, we have the following result.

Lemma 2.3. For a class Φ of small categories, Fam lifts to a pseudomonad Fam $_{\mathcal{L}_{\Phi}}$ on \mathcal{L}_{Φ} -PsAlg.

Proof. Since Fam(\mathcal{C}) has whichever Φ-limits that \mathcal{C} has and Fam(F) is Φ-limit preserving whenever F is [14, Section 4], we conclude that Fam lifts to an endo-2-functor on \mathcal{L}_{Φ} -PsAlg, and $\mathfrak{y} \colon \mathcal{C} \to$ Fam(\mathcal{C}) preserves Φ-limits. Moreover, since we have a fully faithful adjoint string

$$\operatorname{\mathbf{Fam}} \cdot \mathfrak{n} \dashv \mathfrak{m} \dashv \mathfrak{n} \cdot \operatorname{\mathbf{Fam}},$$

we note that, in particular, \mathfrak{m} is a right adjoint, and therefore preserves Φ -limits.

As a particular instance, we have the work of [34], which studies the pseudodistributive law of free product completion pseudomonad $\mathcal{L}_{\mathbf{Set}} = \mathbf{Fam}((-)^{\mathsf{op}})^{\mathsf{op}}$ over \mathbf{Fam} , taking \mathbf{Set} to be the class of small, discrete categories. In this case, the composite pseudomonad $\mathbf{Fam} \circ \mathcal{L}_{\mathbf{Set}} = \mathbf{Dist}$ is the pseudomonad whose pseudoalgebras are the doubly-infinitary distributive categories.

2.1 Infinitary lextensive categories

We recall that a category with small coproducts C is *infinitary extensive* if it has pullbacks along coproduct inclusions, and if the coproducts are *disjoint* and *pullback-stable*. This can be expressed in three conditions:

(a) for every pair of objects $A, B \in \mathcal{C}$, we have a pullback diagram:

$$\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & A + B
\end{array}$$

(b) for each morphism $f\colon Y\to \sum_{i\in I}X_i$, if we take pullbacks along the coproduct inclusions $X_i\stackrel{\iota_i}{\longrightarrow} \sum_{i\in I}X_i$,

$$Y_{i} \xrightarrow{\iota_{i}} Y$$

$$\downarrow f$$

$$X_{i} \xrightarrow{\iota_{i}} \sum_{i \in I} X_{i}$$

we have that $Y_i \xrightarrow{\iota_i} Y$ form a coproduct diagram as well, and

(c) for every family $(f_i: Y_i \to X_i)_{i \in I}$ of morphisms, the following commutative square

$$\begin{array}{ccc} Y_i & \xrightarrow{\iota_i} & \sum_{i \in I} Y_i \\ f_i & & & \downarrow \sum_{i \in I} f_i \\ X_i & \xrightarrow{\iota_i} & \sum_{i \in I} X_i \end{array}$$

is a pullback diagram.

We also make use of the following notation: if \mathcal{C} is a category with coproducts and a terminal object, let $-\cdot 1$: **Set** $\to \mathcal{C}$ be the functor left adjoint to $\mathcal{C}(1,-)$: $\mathcal{C} \to \mathbf{Set}$. We highlight that if \mathcal{C} has a terminal object, then so does $\mathbf{Fam}(\mathcal{C})$, so we have a functor $-\cdot 1$: $\mathbf{Set} \to \mathbf{Fam}(\mathcal{C})$.

The following result, appearing in [9] and [43], is an important step in the characterization of the $\mathbf{Fam}_{\mathcal{L}_{fin}}$ -pseudoalgebras:

Lemma 2.4. Let C be a category with finite limits and coproducts. Then the following are equivalent:

- (i) C is infinitary lextensive.
- (ii) Σ : Fam(\mathcal{C}) $\to \mathcal{C}$ preserves finite limits.

Proof. For an infinitary lextensive \mathcal{C} , [43, Lemma 7.1] guarantees that we have an equivalence $\mathbf{Fam}(\mathcal{C}) \simeq (\mathcal{C} \downarrow (-\cdot 1))$, and that the projection $(\mathcal{C} \downarrow (-\cdot 1)) \to \mathcal{C}$ preserves finite limits. Moreover, we also establish that the composite

$$\mathbf{Fam}(\mathcal{C}) \xrightarrow{\simeq} (\mathcal{C} \downarrow (-\cdot 1)) \longrightarrow \mathcal{C}$$

corresponds to the coproduct functor $\mathbf{Fam}(\mathcal{C}) \to \mathcal{C}$. This shows that (i) \implies (ii).

Now, if we assume (ii), it follows in particular that \sum preserves pullbacks. So, we consider the following pullback diagrams in $\mathbf{Fam}(\mathcal{C})$

for objects $A_0, A_1 \in \mathcal{C}$, a morphism $f: Y \to \sum_{i \in I} X_i$ in \mathcal{C} , and a family of morphisms $(g_j: V_j \to W_j)_{j \in J}$ in \mathcal{C} .

Since the coproduct functor preserves pullbacks, it can be composed with each diagram (11) to respectively obtain the pullback diagrams in (a), (b) and (c). Hence, we witness the infinitary extensivity of C, thereby confirming that (ii) \Longrightarrow (i).

Now, by Lemma 2.2 and the description for $\mathbf{Fam}_{\mathcal{L}_{\mathsf{fin}}}$ -pseudoalgebras, we conclude, as a corollary, that:

Theorem 2.5. The 2-category (Fam $\circ \mathcal{L}_{fin}$)-PsAlg consists of infinitary lextensive categories, and functors preserving coproducts and finite limits.

2.2 Infinitary extensive categories with pullbacks

As one might expect, we can still obtain results analogous to Lemma 2.4 even in the absence of terminal objects.

Lemma 2.6. Let C be a category with coproducts and pullbacks. The following are equivalent:

- (i) The coproduct functor \sum : Fam(\mathcal{C}) $\to \mathcal{C}$ preserves pullbacks.
- (ii) C is infinitary extensive.

Proof. If C is infinitary extensive and has pullbacks, then $C \downarrow X$ is infinitary lextensive for all objects X. Thus, we may apply Lemma 2.4 to conclude that

$$\mathbf{Fam}(\mathcal{C})\downarrow X\simeq\mathbf{Fam}(\mathcal{C}\downarrow X)\stackrel{\sum}{-\!-\!-\!-\!-}\mathcal{C}\downarrow X$$

preserves finite limits. Since $Fam(\mathcal{C})$ is infinitary extensive, we have

$$\mathbf{Fam}(\mathcal{C}) \downarrow (X_i)_{i \in I} \simeq \prod_{i \in I} \mathbf{Fam}(\mathcal{C}) \downarrow X_i,$$

and a product of finite limit preserving functors preserves finite limits as well. Thus, we deduce that $\sum : \mathbf{Fam}(\mathcal{C}) \to \mathcal{C}$ preserves pullbacks, confirming that (ii) \Longrightarrow (i).

Conversely, if \sum : $\mathbf{Fam}(\mathcal{C}) \to \mathcal{C}$ preserves pullbacks, we follow the same argument used for Lemma 2.4: we compose the coproduct functor with each of the diagrams (11) to respectively obtain (a), (b) and (c), exhibiting infinitary extensiveness. This proves that (i) \Longrightarrow (ii).

As a consequence, by Lemma 2.2 and the description of $\mathbf{Fam}_{\mathcal{L}_{pb}}$ -pseudoalgebras, we conclude that:

Theorem 2.7. The 2-category (Fam $\circ \mathcal{L}_{pb}$)-PsAlg consists of infinitary extensive categories with pullbacks, and functors which preserve coproducts and pullbacks.

2.3 Doubly infinitary lextensive categories

Inspired by the terminology of [34], we call the ($\mathbf{Fam} \circ \mathcal{L}$)-pseudoalgebras doubly-infinitary lextensive categories.

Theorem 2.8. Let C be a category with coproducts and limits. The following are equivalent:

- (i) The coproduct functor Σ : $\mathbf{Fam}(\mathcal{C}) \to \mathcal{C}$ preserves limits;
- (ii) C is doubly infinitary extensive;
- (iii) C is lextensive and doubly infinitary distributive.

Proof. We have the equivalence (i) \iff (ii) by definition.

The equivalence (iii) \iff (ii) follows by Lemma 2.6 and [34, Lemma 3.1]. We use the basic facts that any limit can be obtained via pullbacks and arbitrary products, and that infinitary extensive categories with products are, in particular, infinitary distributive (see [8, Proposition 4.5]).

3 Exponentiability in freely generated structures

The purpose of this section is to study the exponentiable objects of the free completions $\mathbf{Fam}(\mathcal{L}_{fin}(\mathcal{C}))$ and $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$, which constitute the main results of this work. Aiming for a self-contained account of exponentiability, we begin by recalling the definition of *exponentiable object*, as well as some elementary properties.

In order to fix notation, we recall that an object E in a category \mathcal{C} with finite products is exponentiable at X if there exists an object $E \Rightarrow X$ and a natural isomorphism

$$\mathcal{C}(-\times E, X) \cong \mathcal{C}(-, E \Rightarrow X). \tag{12}$$

We say E is exponentiable if (12) holds naturally for every object X in \mathcal{C} .

We revisit the the following elementary observation about exponentiable objects:

Lemma 3.1. Let C be a category with finite products and \mathcal{J} -limits for a small category \mathcal{J} . If $F \colon \mathcal{J} \to \mathcal{C}$ is a diagram, and E is an object such that E is exponentiable at Fj for each j in \mathcal{J} , then E is exponentiable at $\lim_{j \in \mathcal{J}} F$ and

$$E \Rightarrow \lim_{j \in \mathcal{J}} Fj \cong \lim_{j \in \mathcal{J}} (E \Rightarrow Fj)$$

Proof. For each object A, we have a natural isomorphism

$$\mathcal{C}(A\times E, \lim_{j\in\mathcal{J}}Fj)\cong \lim_{j\in\mathcal{J}}\mathcal{C}(A\times E, F_j)\cong \lim_{j\in\mathcal{J}}\mathcal{C}(A, E\Rightarrow F_j)\cong \mathcal{C}(A, \lim_{j\in\mathcal{J}}(E\Rightarrow F_j))$$

as desired. \Box

We recall from [6, Definition 6.1.3] that an object x of a category \mathcal{C} is connected if the homfunctor $\mathcal{C}(x,-)$ preserves coproducts. It is an immediate consequence of Lemma 1.2 that the objects in the essential image of $\mathfrak{y} \colon \mathcal{C} \to \mathbf{Fam}(\mathcal{C})$ are precisely the connected objects in $\mathbf{Fam}(\mathcal{C})$. We confirm an analogous characterization is available for the internal hom-functor:

Lemma 3.2. If C is a category with finite products, and C is an exponentiable object in Fam(C), then the following are equivalent:

- (i) C is connected.
- (ii) $C \Rightarrow -$ preserves coproducts.

Proof. Let $(A_i)_{i\in I}$ be a family of objects in \mathcal{C} , and let $(X_j)_{j\in J}$ be a family of objects in $\mathbf{Fam}(\mathcal{C})$.

If $\mathcal C$ is connected, then we have natural isomorphisms

$$\mathbf{Fam}(\mathcal{C})\Big((A_{i})_{i\in I}\times C, \sum_{j\in J}X_{j}\Big)$$

$$\cong \mathbf{Fam}(\mathcal{C})\Big((A_{i}\times C)_{i\in I}, \sum_{j\in J}X_{j}\Big) \qquad \text{products in } \mathbf{Fam}(\mathcal{C}),$$

$$\cong \prod_{i\in I}\sum_{j\in J}\mathbf{Fam}(\mathcal{C})(A_{i}\times C, X_{j}) \qquad (7),$$

$$\cong \prod_{i\in I}\sum_{j\in J}\mathbf{Fam}(\mathcal{C})(A_{i}, C\Rightarrow X_{j}) \qquad C \text{ exponentiable,}$$

$$\cong \mathbf{Fam}(\mathcal{C})\Big((A_{i})_{i\in I}, \sum_{j\in J}(C\Rightarrow X_{j})\Big) \qquad (7).$$

Hence, we conclude that

$$\sum_{j \in J} (C \Rightarrow X_j) \cong C \Rightarrow \sum_{j \in J} X_j,$$

which confirms that (i) \implies (ii).

Conversely, if $C \Rightarrow -$ preserves coproducts, then for a family $(X_j)_{j \in J}$ of objects in $\mathbf{Fam}(\mathcal{C})$, we have

$$\begin{aligned} \mathbf{Fam}(\mathcal{C})\Big(C, \sum_{j \in J} X_j\Big) &\cong \mathbf{Fam}(\mathcal{C})\Big(1, C \Rightarrow \sum_{j \in J} X_j\Big) & C \text{ exponentiable,} \\ &\cong \mathbf{Fam}(\mathcal{C})\Big(1, \sum_{j \in J} C \Rightarrow X_j\Big) & \text{by hypothesis (ii),} \\ &\cong \sum_{j \in J} \mathbf{Fam}(\mathcal{C})(1, C \Rightarrow X_j) & \text{terminal connected,} \\ &\cong \sum_{j \in J} \mathbf{Fam}(\mathcal{C})(C, X_j), \end{aligned}$$

hence, we conclude that (ii) \implies (i).

Let Φ be a class of small categories that includes all finite, discrete categories, so that every \mathcal{L}_{Φ} -pseudoalgebra has finite products. For the sake of succinctness, we say an object of $\mathbf{Fam}(\mathcal{L}_{\Phi}(\mathcal{C}))$ is a *generator* if it is in the essential image of the inclusion $\mathcal{C} \to \mathbf{Fam}(\mathcal{L}_{\Phi}(\mathcal{C}))$.

We will give an inductive perspective on exponentials in $\mathbf{Fam}(\mathcal{L}_{\Phi}(\mathcal{C}))$, and the following result is our cornerstone (see [34, Remark 1]):

Lemma 3.3. If X is a generator and D is connected in $\mathbf{Fam}(\mathcal{L}_{\Phi}(\mathcal{C}))$, then D is exponentiable at X and we have

$$D \Rightarrow X \cong X + \hat{\mathcal{C}}(D, X) \cdot 1$$

where $\hat{\mathcal{C}} = \mathbf{Fam}(\mathcal{L}_{\Phi}(\mathcal{C})).$

Proof. Let $(E_i)_{i\in I}$ be a family of objects in $\mathcal{L}_{\Phi}(\mathcal{C})$. We have natural isomorphisms

$$\hat{\mathcal{C}}((E_{i})_{i \in I} \times D, X)
\cong \hat{\mathcal{C}}((E_{i} \times D)_{i \in I}, X) \qquad \text{products in } \hat{\mathcal{C}}
\cong \prod_{i \in I} \hat{\mathcal{C}}(E_{i} \times D, X) \qquad \hat{\mathcal{C}}(-, X) \text{ preserves products,}
\cong \prod_{i \in I} (\mathcal{L}_{fin}\mathcal{C})(E_{i} \times D, X) \qquad \text{full faithfulness,}
\cong \prod_{i \in I} (\mathcal{L}_{fin}\mathcal{C})(E_{i}, X) + (\mathcal{L}_{fin}\mathcal{C})(D, X) \qquad (8),
\cong \prod_{i \in I} \hat{\mathcal{C}}(E_{i}, X) + \hat{\mathcal{C}}(D, X) \qquad \text{full faithfulness,}
\cong \hat{\mathcal{C}}((E_{i})_{i \in I}, X + \hat{\mathcal{C}}(D, X) \cdot 1) \qquad (7)$$

3.1 Exponentials for free doubly infinitary lextensive categories

Having reviewed the elementary properties of exponentiable objects, we proceed to prove our main result on exponentiability of the objects of freely generated doubly-infinitary lextensive categories:

Theorem 3.4. The category $Fam(\mathcal{L}(\mathcal{C}))$ is cartesian closed.

Proof. First, we note that connected objects are exponentiable:

- By Lemma 3.3, we have that any connected object in $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$ is exponentiable at the generators.
- Any connected object is a limit of generators, so by Lemma 3.1 we conclude that connected objects are exponentiable at any connected object in $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$.
- Since any object in $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$ is a coproduct of connected objects, we simply apply Lemma 3.2 to deduce our claim.

Now, let $(E_i)_{i\in I}$ and $(D_j)_{j\in J}$ be families of objects in $\mathcal{L}(\mathcal{C})$, and X any object in $\hat{\mathcal{C}}$. We have natural isomorphisms

$$\hat{\mathcal{C}}((E_i)_{i \in I} \times (D_j)_{j \in J}, X)
\cong \hat{\mathcal{C}}((E_i \times D_j)_{(i,j) \in I \times J}, X)$$
binary products in $\hat{\mathcal{C}}$,
$$\cong \prod_{i \in I} \prod_{j \in J} \hat{\mathcal{C}}(E_i \times D_j, X)$$

$$\cong \prod_{i \in I} \prod_{j \in J} \hat{\mathcal{C}}(E_i, D_j \Rightarrow X)$$

$$\cong \prod_{i \in I} \hat{\mathcal{C}}(E_i, D_j \Rightarrow X)$$

$$\cong \prod_{i \in I} \hat{\mathcal{C}}(E_i, \prod_{j \in J} (D_j \Rightarrow X))$$

$$\cong \hat{\mathcal{C}}((E_i)_{i \in I}, \prod_{j \in J} (D_j \Rightarrow X))$$

$$\cong \hat{\mathcal{C}}((E_i)_{i \in I}, \prod_{j \in J} (D_j \Rightarrow X))$$

Thus, we obtain

$$(D_j)_{j\in J} \Rightarrow X \cong \prod_{j\in J} (D_j \Rightarrow X),$$

confirming that coproducts of connected objects are exponentiable. But every object in $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$ is a coproduct of connected objects, hence the result follows.

3.2 Explicit descriptions of the exponentials

Let $(D_j)_{j\in J}$ and $(E_k\colon \mathcal{A}_k\to \mathcal{C})_{k\in K}$ be families of objects in \mathcal{L} , where \mathcal{A}_k is a small category for each $k\in K$.

The results of the previous subsection can be used to calculate an explicit expression for the exponential $(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K}$ in $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$: via Lemmas 3.1–3.3, one of Theorems 3.7 or 3.4, and the key ideas of the proof of [34, Theorem 2.3], we obtain

$$(D_j)_{j \in J} \Rightarrow (E_k)_{k \in K} \cong \left(\prod_{j \in J} \lim_{l \in \mathcal{A}_{f_j^K}} \Delta_{f,j,l} \right)_{f \in \Omega}$$
(13)

where

$$\begin{split} \Omega &= \prod_{j \in J} \sum_{k \in K} \lim_{l \in \mathcal{A}_k} (1 + \mathcal{L}(\mathcal{C})(D_j, E_{k,l})), \\ \Delta_{f,j,l} &= \begin{cases} E_{f_j^K,l} & \text{if } f_j(l) \in 1 \\ 1 & \text{if } f_j(l) \in \mathcal{L}(\mathcal{C})(D_j, E_{f_j^K,l}) \end{cases} \end{split}$$

and f_i^K is the projection of f_j onto K for each $f \in \Omega$, $j \in J$.

Remark 3.5. As long as \mathcal{C} has an initial object, the exponentials may be given explicitly by

$$(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K} \cong \left(\prod_{j\in J} \mathfrak{d}^c(\pi_2(f(j)))\right)_{f\in \Omega}$$

where

$$\Omega = \prod_{j \in J} \sum_{k \in K} (\mathcal{L}_{fin}(\mathcal{C}))(D_j \times \mathbb{O}, E_k),$$

and $\mathfrak{d}^c(g)$ is defined via the following pushout in $\mathcal{L}_{\mathsf{fin}}(\mathcal{C})$, by co-extensivity:

$$D_i \times \mathbb{O} \xrightarrow{g} E_{\pi_1(f(i))}$$

$$\downarrow^{\pi_2} \qquad \qquad \downarrow^{\sigma}$$

$$\mathbb{O} \xrightarrow{\mathfrak{O}^c(g)}.$$

3.3 Exponentials for free infinitary lextensive categories

As we remarked in Section 1, we have a fully faithful, finite limit preserving functor

$$\mathfrak{u}\colon \mathcal{L}_{\mathsf{fin}}(\mathcal{C}) \to \mathcal{L}(\mathcal{C})$$

for every category C. By studying the fully faithful functor

$$\overline{\mathfrak{u}} = \mathbf{Fam}(\mathfrak{u}) \colon \mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathcal{C})) \to \mathbf{Fam}(\mathcal{L}(\mathcal{C})),$$

we can deduce results about exponentiability of objects in $\mathbf{Fam}(\mathcal{L}_{fin}(\mathcal{C}))$. More precisely, we have Lemma 3.6. The functor $\overline{\mathfrak{u}}$ reflects exponentials of finite coproducts of connected objects.

Proof. Let $(D_j)_{j\in J}$ be a *finite* family of objects in $\mathcal{L}_{fin}(\mathcal{C})$, and let $(E_k: \mathcal{A}_k \to \mathcal{C})_{k\in K}$ be a family of objects in $\mathcal{L}_{fin}(\mathcal{C})$, where \mathcal{A}_k is a finite category for each $k \in K$. Given any object X in $Fam(\mathcal{L}_{fin}(\mathcal{C}))$, we have

$$\begin{aligned} \mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathcal{C}))(X \times (D_j)_{j \in J}, (E_k)_{k \in K}) \\ & \cong \hat{\mathcal{C}}\Big(\overline{\mathfrak{u}}(X \times (D_j)_{j \in J}), \overline{\mathfrak{u}}((E_k)_{k \in K})\Big) & \overline{\mathfrak{u}} \text{ fully faithful,} \\ & \cong \hat{\mathcal{C}}\Big(\overline{\mathfrak{u}}(X) \times \overline{\mathfrak{u}}((D_j)_{j \in J}), \overline{\mathfrak{u}}((E_k)_{k \in K})\Big) & \overline{\mathfrak{u}} \text{ preserves binary products} \\ & \cong \hat{\mathcal{C}}\Big(\overline{\mathfrak{u}}(X), \overline{\mathfrak{u}}((D_j)_{j \in J}) \Rightarrow \overline{\mathfrak{u}}((E_k)_{k \in K})\Big) & \mathbf{Fam}(\mathcal{L}(\mathcal{C})) \text{ is cartesian closed,} \end{aligned}$$

where $\hat{\mathcal{C}} = \mathbf{Fam}(\mathcal{L}(\mathcal{C}))$. Moreover, we have

$$\overline{\mathfrak{u}}((D_i)_{i\in J})\Rightarrow \overline{\mathfrak{u}}((E_k)_{k\in K})\cong (\mathfrak{u}(D_i))_{i\in J}\Rightarrow (\mathfrak{u}(E_k))_{k\in K},$$

and calculating the exponential as in (13), we obtain

$$(\mathfrak{u}(D_j))_{j\in J}\Rightarrow (\mathfrak{u}(E_k))_{k\in K}\cong \Big(\prod_{j\in J}\lim_{l\in\mathcal{L}_{f_j^K}}\mathfrak{u}(\Gamma_{f,j,l})\Big)_{f\in\Xi}$$

where

$$\Xi = \prod_{j \in J} \sum_{k \in K} \lim_{l \in \mathcal{L}_k} \Big(1 + \mathcal{L}_{\mathsf{fin}}(\mathcal{C})(D_j, E_{k,l}) \Big),$$

$$\Gamma_{f,j,l} = \begin{cases} E_{f_j^K,l} & \text{if } f_j(l) \in 1\\ 1 & \text{if } f_j(l) \in \mathcal{L}_{\text{fin}}(\mathcal{C})(D_j, E_{f_j^K,l}), \end{cases}$$

and f_i^K is the projection of f_j onto K for each $f \in \Xi$, $j \in J$.

Now, since \mathfrak{u} is fully faithful and preserves finite limits, it must reflect them as well. Since we are given that J is finite, as well as A_k for all $k \in K$, we have

$$\prod_{j\in J} \lim_{l\in \mathcal{A}_{f_j^K}} \mathfrak{u}(\Gamma_{f,j,l}) \cong \mathfrak{u}\Big(\prod_{j\in J} \lim_{l\in \mathcal{A}_{f_j^K}} \Gamma_{f,j,l}\Big).$$

and thus

$$\overline{\mathfrak{u}}((D_j)_{j\in J})\Rightarrow\overline{\mathfrak{u}}((E_k)_{k\in K})\cong\overline{\mathfrak{u}}\Big(\prod_{j\in J}\lim_{l\in\mathcal{L}_{f_j^K}}\Gamma_{f,j,l}\Big)_{f\in\Xi}$$

so, since $\overline{\mathfrak{u}}$ is fully faithful, we conclude that the exponential $(D_j)_{j\in J}\Rightarrow (E_k)_{k\in K}$ in $\mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathcal{C}))$ exists and

$$(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K} \cong \Big(\prod_{j\in J} \lim_{l\in \mathcal{L}_{f_j^K}} \Gamma_{f,j,l}\Big)_{f\in\Xi},$$

as desired.

As an immediate corollary, we obtain our second main result:

Theorem 3.7. Finite coproducts of connected objects in $Fam(\mathcal{L}_{fin}(\mathcal{C}))$ are exponentiable.

4 Examples

We end with a discussion of examples of categories with certain classes of limits and coproducts that distribute over each other, as well as a discussion of the exponentiability of their objects. Our goal is to relate well-known results with the work developed herein.

By Theorem 2.8, doubly-infinitary lextensive categories are precisely the doubly-infinitary distributive categories which are also lextensive. With this in mind, we will consider some examples of doubly-infinitary distributive categories from [34], and verify whether they are extensive.

Fundamental examples: The category $\mathbf{Set} \simeq \mathbf{Fam}(1) \simeq \mathbf{Fam}(\mathcal{L}(0))$ of sets is the initial object in the 2-category of $(\mathbf{Fam} \circ \mathcal{L})$ -pseudoalgebras. Similarly, any presheaf category $[\mathcal{A}^{\mathsf{op}}, \mathbf{Set}]$ is lextensive and doubly-infinitary distributive for any category \mathcal{A} .

The category $\mathbf{Fam}(\mathbf{Set}^{\mathsf{op}}) \simeq \mathbf{Fam}(\mathcal{L}(1)) \simeq \mathbf{Dist}(1)$ of polynomials is a free doubly-infinitary lextensive category and was known to be cartesian closed [4].

The two previous examples also happen to be free doubly-infinitary distributive categories. In fact, they are particular cases of a more general phenomenon when the generating category is discrete. To be precise, if \mathcal{C} is a small discrete category, we have

$$\mathcal{L}(\mathcal{C}) \simeq \mathbf{CAT}(\mathcal{C}, \mathbf{Set})^{\mathsf{op}} \simeq \mathbf{CAT}(\mathcal{C}, \mathbf{Set}^{\mathsf{op}}) \simeq \mathbf{Fam}(\mathcal{C})^{\mathsf{op}} \simeq \mathbf{Fam}(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$$

so, for small discrete categories, the free limit completion coincides with the free product completion. Hence, for any such discrete category \mathcal{C} , we have that $\mathbf{Dist}(\mathcal{C}) = \mathbf{Fam}(\mathcal{L}(\mathcal{C}))$ is a (free) doubly-infinitary lextensive category.

Doubly-infinitary lextensive categories via free coproduct completions: As we have shown in Lemma 2.3, Fam lifts to a pseudomonad Fam_{\mathcal{L}_{Φ}} on \mathcal{L}_{Φ} -PsAlg. Thus, if a category \mathcal{C} has Φ-limits, then Fam(\mathcal{C}) has Φ-limits as well, which are preserved by the coproduct \mathfrak{m} : Fam(Fam(\mathcal{C})) \rightarrow Fam(\mathcal{C}). In particular,

- if \mathcal{C} has pullbacks, then $\mathbf{Fam}(\mathcal{C})$ is infinitary extensive with pullbacks,
- if \mathcal{C} has finite limits, then $\mathbf{Fam}(\mathcal{C})$ is infinitary lextensive,
- if \mathcal{C} has small limits, then $\mathbf{Fam}(\mathcal{C})$ is doubly-infinitary lextensive,
- if \mathcal{C} has products, then $\mathbf{Fam}(\mathcal{C})$ is doubly-infinitary distributive by [34, Example 1].

So, even if a category \mathcal{C} with products does not have small limits, we can still establish that the category $\mathbf{Fam}(\mathcal{C})$ is doubly-infinitary distributive, and it is extensive [8] by virtue of being a free coproduct completion. Hence, if $\mathbf{Fam}(\mathcal{C})$ has small limits, we conclude that it is doubly-infinitary lextensive, by Theorem 2.8.

Before discussing our examples, we let $\mathbf{Conn}(\mathcal{C})$ be the full subcategory of a category \mathcal{C} with coproducts consisting of the *connected objects* [6, Definition 6.1.3].

We begin by noting that the category $\mathbf{Cat} \simeq \mathbf{Fam}(\mathbf{Conn}(\mathbf{Cat}))$ of small categories is doubly-infinitary lextensive, as it is both doubly-infinitary distributive and extensive, and \mathbf{Cat} has small limits. Likewise, we can prove that the category ω - $\mathbf{CPO} \simeq \mathbf{Fam}(\mathbf{Conn}(\omega$ - $\mathbf{CPO}))$ of ω -complete partial orders is also a doubly-infinitary lextensive category.

Again similarly, the category **LocConTop** of locally connected topological spaces and continuous functions is doubly-infinitary lextensive. Indeed, from [34, Example 8], we learn that $\mathbf{LocConTop} \simeq \mathbf{Fam}(\mathbf{Conn}(\mathbf{LocConTop}))$ is both doubly-infinitary distributive and extensive, as the free coproduct completion of a category with products. Moreover, $\mathbf{LocConTop}$ is a coreflective subcategory of \mathbf{Top} [13], therefore, $\mathbf{LocConTop}$ has small limits, letting us conclude that $\mathbf{LocConTop}$ is doubly-infinitary lextensive.

Doubly-infinitary distributive categories that are not extensive. As observed in [34], a distributive lattice \mathcal{D} (seen as a distributive, thin category) is extensive if and only if $\mathcal{D} \simeq 1$, so any non-trivial example of a completely distributive lattice \mathcal{D} will be doubly-infinitary distributive, but not extensive.

Another example is the full subcategory \mathbf{Set}^2_{\bullet} of $\mathbf{Set} \times \mathbf{Set}$ consisting of those pairs of sets that are either both empty, or both non-empty. Since coproducts and products are calculated componentwise in \mathbf{Set}^2_{\bullet} , this category is doubly-infinitary distributive as well, but it is not extensive.

Cartesian closedness vs. doubly-infinitary lextensivity The category Fam(Top) is an example of a doubly-infinitary lextensive category that is *not* cartesian closed. We note that the category Top of topological spaces is infinitary distributive, but not cartesian closed. So, by [34, Theorem 4.2], we conclude that Fam(Top) is not cartesian closed as well. However, Fam(Top) is doubly-infinitary lextensive, since Top has small limits.

An example of a cartesian closed category with all coproducts and limits, but not doubly-infinitary lextensive, is given in [34, Counter-example 2], the category of Quasi-Borel spaces.

5 Epilogue

Motivated particularly by the insights from [32, 34], the present work explores the distributive properties of limits over coproducts through the lens of two-dimensional monad theory [5, 26].

We have demonstrated that the canonical (pseudo)distributivity of pullbacks over coproducts leads to a pseudomonad whose pseudoalgebras are precisely the infinitary extensive categories equipped with pullbacks. Similarly, the distributivity of finite limits over coproducts leads to the notion of a pseudomonad whose 2-category of pseudoalgebras is precisely the 2-category of infinitary lextensive categories. Finally, we showed that the distributivity of limits over coproducts leads to the concept of doubly-infinitary lextensivity, characterized as infinitary extensive categories that are also doubly-infinitary distributive as introduced in [34].

We also studied the exponential objects of the free completions $\mathbf{Fam}(\mathcal{L}_{fin}(\mathcal{C}))$ and $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$, confirming that the latter is a cartesian closed category for any category \mathcal{C} . These free completions enjoy various other known properties since they end up being the free coproduct completion of a well-behaved category – we refer the reader to [8, 1, 32, 33, 34] for further results.

Free finite coproduct completion

By replacing the free coproduct pseudomonad **Fam** with its *finite* counterpart **FinFam**, we recover nearly all of our results, provided we make some adaptations to be finitary setting. Namely, we obtain a pseudodistributive law

$$\mathcal{L}_{\Phi} \circ \mathbf{FinFam} \to \mathbf{FinFam} \circ \mathcal{L}_{\phi},$$

for any class Φ of *finite* diagrams, by reworking the proof of Lemma 2.3. We then obtain two more characterizations:

- the (**FinFam** $\circ \mathcal{L}_{pb}$)-pseudoalgebras are precisely the extensive categories with pullbacks,
- the (FinFam $\circ \mathcal{L}_{fin}$)-pseudoalgebras are precisely the lextensive categories.

Most consequentially, an adaptation of our exponentiability results will confirm that $\mathbf{FinFam}(\mathcal{L}_{fin}(\mathcal{C}))$ is a cartesian closed category whenever \mathcal{C} is *locally finite* as opposed to locally small.

Descent theory

Effective descent morphisms [12, 18] (see also [31, Sections 3 and 4]) are the backbone of Grothendieck's descent theory [17, 28], which has impactful consequences ranging various fields [39, 44, 7]. Besides their wide range of applications, effective descent morphisms hold intrinsic interest, as their purpose is the reconstruction of data over the codomain from given data over the domain, plus some additional algebraic structure.

Of particular relevance to the present work are effective descent morphisms of freely generated categorical structures. For instance, [41, Section 4], studied categories of descent data for families of morphisms $\phi \colon (X_i)_{i \in I} \to Y$, as well as conditions under which ϕ is an effective descent morphism in $\mathbf{Fam}(\mathcal{C})$, provided that \mathcal{C} has finite limits. Namely, it was shown that all such descent data is a coproduct of connected descent data, which provided simpler conditions for a morphism $\phi \colon (X_i)_{i \in I} \to Y$ to be of effective descent – this gives evidence that $\mathbf{Fam}(\mathcal{C})$ is a good proxy for the study effective descent morphisms of \mathcal{C} . This perspective was consequential in the study of effective descent functors between enriched categories, establishing precise connections between the work of [25], [28, Theorem 9.11], [42], and the work of [44, 10, 11].

Since the free completions $\mathbf{Dist}(\mathcal{C})$ and $\mathbf{Fam}(\mathcal{L}(\mathcal{C}))$ are even better behaved categories, enjoying properties such as cartesian closedness, an inquiry on whether studying effective descent morphisms in such free completions seems to be a reasonable avenue for future work.

Non-canonical isomorphisms

In analogy with [34, Subsection 5.2], we may use the results of [30] to prove that a category C is a (Fam $\circ \mathcal{L}_{\Phi}$)-pseudoalgebra if it has coproducts, Φ -limits, and there exists a(ny) invertible natural isomorphism

$$\sum_{x \in \lim_{i \in \mathcal{I}} UF} \lim_{j \in \mathcal{J}} F_{j,x_j} \stackrel{\cong}{\longrightarrow} \lim_{j \in \mathcal{J}} \sum_{x \in UFj} F_{j,x}$$

for any functor $F: \mathcal{J} \to \mathbf{Fam}(\mathcal{C})$ and $\mathcal{J} \in \Phi$, where we let $U: \mathbf{Fam}(\mathcal{C}) \to \mathbf{Set}$ be the functor that outputs the underlying indexing set.

More generally, if we have a pseudomonad \mathcal{T} on **CAT** and a pseudodistributive law $\delta \colon \mathcal{T} \circ \mathbf{Fam} \to \mathbf{Fam} \circ \mathcal{T}$, then for any category \mathcal{C} with coproducts and the structure of a \mathcal{T} -pseudoalgebra, the coproduct functor

$$\sum\colon \mathbf{Fam}(\mathcal{C})\to \mathcal{C}$$

is an oplax \mathcal{T} -morphism by doctrinal adjunction [19, 29]. The (codual version of the) techniques of non-canonical isomorphisms from [30] can be applied just as well to this setting.

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