Mapping Cone and Morse Theory

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Abstract

On a smooth manifold, we associate to any closed differential form a mapping cone algebra. The cohomology of this mapping cone algebra can vary with the de Rham cohomology class of the closed form. We present a novel Morse theoretical description for the mapping cone cohomology. Specifically, we introduce a Morse complex for the mapping cone algebra which is generated by pairs of critical points with the differential defined by gradient flows and an integration of the closed form over spaces of gradient flow lines. We prove that the cohomology of our cone Morse complex is isomorphic to the mapping cone cohomology and hence independent of both the Riemannian metric and the Morse function used to define the complex. We also obtain sharp inequalities that bound the dimension of the mapping cone cohomology in terms of the number of Morse critical points and the properties of the specified closed form. Our results are widely applicable, especially for any manifold equipped with a geometric structure described by a closed differential form. We also obtain a bound on the difference between the number of Morse critical points and the Betti numbers.

Contents

1 Introduction

Manifolds with a geometric structure given by a closed differential form are widely studied. For example, a large class are the symplectic manifolds which by definition contain a non-degenerate, closed two-form. Another large class are special holonomy manifolds which carry a closed invariant form such as Kähler and G_2 manifolds. And even for complex manifolds that are non-Kähler, there often is a distinguished closed form. For instance, on a complex threefold that satisfies the balanced condition, the square of the hermitian form, is a distinguished closed four-form.

For a manifold (M, ψ), with $\psi \in \Omega^\ell(M)$ being a distinguished ℓ-form that is also d-closed, we seek invariants that are dependent on the geometric structure ψ. Certainly, without ψ being present, there is the well-known de Rham differential graded algebra $(\Omega^*, d, \wedge)$ that results in basic invariants, i.e. de Rham cohomology ring and Massey products, for any smooth manifold M. So we begin in this paper with a simple question: Is there a natural extension of the de Rham algebra that incorporates the additional geometric structure ψ? Indeed, there is the mapping cone differential graded algebra $\mathrm{Cone}(\psi)$ which provides invariants that in general depend on $[\psi] \in H^\ell(M)$.

1.1 Mapping cone algebra

This mapping cone algebra (or the "cone" algebra, for short) can be motivated as follows. For differential forms, the exterior (or wedge) product is a natural operation. So given $\psi \in H^\ell_{dR}(M)$, we can turn it into an operator on forms, $\psi\wedge : \Omega^k(M) \to \Omega^{k+\ell}(M)$. This operator can then be combined with the exterior differential d to give what is commonly called the "twisted" differential, $d + \psi$. Unfortunately, $d + \psi$ has two notable drawbacks if it were to be considered as the differential of a complex: (1) $d+\psi$ generally does not square to zero unless ℓ is odd; (2) $d + \psi$ does not preserves grading as it maps Ω^k to a mixture of $\Omega^k \oplus \Omega^{k+\ell}$. There is however a simple solution to alleviate these two issues. For instance, consider instead mapping both d and $\psi\wedge$ into Ω^{k+1}. This would require considering a pair of forms in $\Omega^k \oplus \Omega^{k-\ell+1}$ with the map

$$\begin{tikzcd} \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \\ & \Omega^{k-\ell+1}(M). \arrow[u, "\psi\wedge"'] \end{tikzcd}$$

This is suggestive of defining the space of cone forms, $\mathrm{Cone}^\bullet(\psi) = \Omega^\bullet(M) \oplus \Omega^{\bullet-\ell+1}(M)$, which consists of just pairs of differential forms. And with it, we can introduce the cone differential:

$$\begin{aligned} d_C : \mathrm{Cone}^k(\psi) &\longrightarrow \mathrm{Cone}^{k+1}(\psi) \\ \begin{pmatrix} \Omega^k \\ \Omega^{k-\ell+1} \end{pmatrix} &\longmapsto \begin{pmatrix} d & \psi\wedge \\ 0 & (-1)^{\ell-1}d \end{pmatrix} \begin{pmatrix} \Omega^{k+1} \\ \Omega^{k-\ell+1} \end{pmatrix} = \begin{pmatrix} d\,\Omega^k + \psi\wedge\Omega^{k-\ell+1} \\ (-1)^{\ell-1} d\,\Omega^{k-\ell+1} \end{pmatrix}. \end{aligned} \tag{1.1}$$

It is straightforward to check that $d_C\, d_C = 0$. This allows us to define the following cone cohomology with respect to ψ:

$$H^k(\mathrm{Cone}(\psi)) = \frac{\ker d_C \cap \mathrm{Cone}^k(\psi)}{\mathrm{im}\, d_C \cap \mathrm{Cone}^k(\psi)}\,.$$

This cone cohomology is not a topological invariant and in general is dependent on ψ. It contains the product structure information of the de Rham cohomology associated with ψ. To see this, let us express $\mathrm{Cone}^k(\psi) = \Omega^k(M) \oplus \Omega^{k-\ell+1}(M)$ in terms of an exact sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\iota_{dR}} \mathrm{Cone}^k(\psi) \xrightarrow{\pi_{dR}} \Omega^{k-\ell+1}(M) \longrightarrow 0\,,$$

where ι_{dR} is the inclusion map and π_{dR} is the projection onto the second component. This short exact sequence standardly leads to a long exact sequence of cohomologies

$$\ldots \longrightarrow H^{k-\ell}_{dR} \xrightarrow{[\psi]} H^k_{dR} \xrightarrow{[\iota_{dR}]} H^k(\mathrm{Cone}(\psi)) \xrightarrow{[\pi_{dR}]} H^{k-\ell+1}_{dR} \xrightarrow{[\psi]} H^{k+1}_{dR} \longrightarrow \ldots \tag{1.2}$$

which implies that

$$H^k(\mathrm{Cone}(\psi)) \cong \mathrm{coker}(\,[\psi]\colon H^{k-\ell}_{dR} \to H^k_{dR}\,) \,\oplus\, \ker(\,[\psi]\colon H^{k-\ell+1}_{dR} \to H^{k+1}_{dR}\,)\,. \tag{1.3}$$

Hence, we see that $H^k(\mathrm{Cone}(\psi))$ encodes the product structure of the de Rham cohomology under the linear map $[\psi] : H^\bullet_{dR} \to H^{\bullet+\ell}_{dR}$. And in general, the kernel and cokernel of such a map can vary as $[\psi]$ varies in de Rham cohomology.

To proceed with an algebra structure for $\mathrm{Cone}^\bullet(\psi)$, perhaps the easiest way to motivate its description is to introduce a formal object, θ, that acts like a differential $(\ell-1)$-form with two defining properties: (i) $d\theta = \psi$; (ii) $\theta \wedge \theta = 0$. Making use of θ, we can express $\mathrm{Cone}^k(\psi) = \Omega^k(M) \oplus \theta \wedge \Omega^{k-\ell+1}(M)$ which now has the same total degree grading on both components. Moreover, the cone differential d_C can be interpreted simply as an exterior derivative:

$$d(\Omega^k \oplus \theta \wedge \Omega^{k-\ell+1}) = (d\Omega^k + \psi \wedge \Omega^{k-\ell+1}) \oplus \theta \wedge (-d\Omega^{k-\ell+1}) = d_C \, \mathrm{Cone}(\psi)\,.$$

We can thus treat $\mathrm{Cone}^k(\psi)$ formally as a differential form space and define the product operation on cone forms by means of the standard wedge product:

$$\begin{aligned}\mathrm{Cone}^j(\psi) \times \mathrm{Cone}^k(\psi) :&= (\Omega^j \oplus \theta \wedge \Omega^{j+\ell-1}) \wedge (\Omega^k \oplus \theta \wedge \Omega^{k+\ell-1}) \\ &= (\Omega^j \wedge \Omega^k) \oplus \theta \wedge \left[\Omega^{j+\ell-1} \wedge \Omega^k + (-1)^{j(\ell-1)} \Omega^j \wedge \Omega^{k-\ell+1}\right]\end{aligned}$$

Altogether, $(\mathrm{Cone}^\bullet(\psi), d_C, \times)$ is a differential graded algebra that is dependent on ψ.

Let us make here two observations. First, the cone cohomology and algebra have appeared previously in the work of Tanaka-Tseng [TT18] within the context of a symplectic manifold (M^{2n}, ω) where ψ was specified to be $\psi = \omega^{p+1}$, for $p = 0, \ldots, n-1$. In this special setting, it was shown that $H(\mathrm{Cone}(\omega^{p+1}))$ is isomorphic to the symplectic cohomologies of differential forms called p-filtered cohomologies, $F^pH(M, \omega)$, introduced by Tsai-Tseng-Yau [TTY16] and the underlying algebras of these two cohomologies are quasi-isomorphic as A_∞ algebras. Of interest, the dimensions of $F^pH(M, \omega)$ have been shown to vary with the symplectic structure in various examples: a six-dimensional symplectic nilmanifold [TY12b], a three-torus product with a three-ball, $T^3 \times B^3$ [TW22], and for classes of open 4-manifolds that are homeomorphic but not diffeomorphic [GTV22]. By the isomorphism of cohomologies, we know that $H(\mathrm{Cone}(\omega^{p+1}))$ would also vary with ω in these examples.

Our second observation is especially worthy to emphasize. It is that the cone cohomology and algebra defined above only requires ψ to be d-closed and nothing more. Geometric structures such as a symplectic two-form or an associative three-form on G_2 manifolds often have additional properties like non-degeneracy besides d-closedness. Certainly, when

ψ has more properties besides being closed, the cone algebra may have added qualities as well. For instance, if ψ is an even-degree form and additionally an element of the integral cohomology $H^\ell(M, \mathbb{Z})$, then the cone algebra interestingly can be interpreted geometrically as the invariant de Rham algebra of $S^{\ell+1}$ sphere bundle with Euler class given by ψ and the formal θ representing the global angular form [TT18]. But the cone algebra is more widely applicable and it is even useful as we shall see to consider the case when ψ is d-exact. Though we may have been initially motivated to seek a cohomology/algebra with respect to a geometric structure, with the cone algebra at hand, it is useful to first study on its own right without imposing additional properties to ψ. And this we shall do below and proceed as our main focus in this paper to develop a Morse theory for the cone cohomology with respect to any d-closed ψ.

1.2 Morse complex for the mapping cone

On a Riemannian manifold, the de Rham cohomology can be described alternatively as the cohomology of a Morse complex (or also referred to as the Morse-Witten or Smale-Thom complex). Besides the Riemannian metric g, to define a Morse complex requires the introduction of a special function f on M, called a Morse function, which is defined by the property that the Hessian at each critical point is non-degenerate. The elements of the Morse complex $C^k(M, f)$ are then generated by the critical points of f, $q \in Crit(f)$, and grouped together by their index, $k = ind(q)$, the number of negative eigenvalues of the Hessian matrix at q. The differential of the complex, ∂, is defined by the gradient flow, $-\nabla f$, from one critical point to another. Explicitly, in local coordinates $\{x^i\}$, the gradient flow is $\dot{x}^i(t) = -g^{ij}\partial f/\partial x^j$ which involves the Riemannian metric g. (We shall assume throughout this paper that the metric g satisfies the usual Smale transversality condition, that is, the submanifolds that flow into or from the critical points are transverse.) The resulting cohomology of the Morse cohomology is famously known to be isomorphic to the standard cohomology, and therefore, the Morse cohomology generally does not depend on the choice of the Morse function f and metric g that are used to define it. As a corollary of this isomorphism, there are the well-known Morse inequalities which bound the Betti numbers of M in terms of the number of critical points of the Morse function.

Now, for a smooth manifold equipped with a geometric structure described by a closed ℓ-form (M, ψ), we have discussed the cone cohomology $H(\text{Cone}(\psi))$ which provides basic geometrical invariants that are dependent on $[\psi] \in H^\ell_{dR}(M)$. Given the connection between the mapping cone complex and the de Rham complex, it is natural to ask whether there

Complex	Cochains	Differential	Cohomology
de Rham	$\Omega^\bullet(M)$	d	$H^\bullet_{dR}(M)$
Morse	$C^\bullet(M,f)$	∂	$H^\bullet_{C(f)}(M)$
Cone	$\Omega^\bullet(M)\oplus\Omega^{\bullet-\ell+1}(M)$	$d_C=\begin{pmatrix} d & \psi \\ 0 & (-1)^{\ell-1}d\end{pmatrix}$	$H^\bullet(\mathrm{Cone}(\psi))(M)$
Cone Morse	$C^\bullet(M,f)\oplus C^{\bullet-\ell+1}(M,f)$	$\partial_C=\begin{pmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1}\partial\end{pmatrix}$	$H^\bullet(\mathrm{Cone}(c(\psi)))(M)$

Table 1: The relations between the de Rham and Morse cochain complexes and Cone and Cone-Morse complexes.

is also a Morse theory-type description for the mapping cone cohomology? Such a Morse description would necessarily require the involvement of ψ in some intrinsic way. And if a Morse theory for (M,ψ) exists, can we bound the dimensions of the cone cohomology by means of the critical points of a Morse function and their gradient flows?

In this paper, we answer both questions in the affirmative.

Motivated by the relationship between de Rham complex and the Morse cochain complex over $\mathbb{R}$ (see Table 1), we define in the following a *cone* Morse complex also over $\mathbb{R}$.

Definition 1.1. Let (M,g) be an oriented, Riemannian manifold and f a Morse function satisfying the Morse-Smale transversality condition. Let $C^k(M,f)$ be the $\mathbb{R}$-module with generators the critical points of f with index k. Given a d-closed form $\psi\in\Omega^\ell(M)$, we define the **cone Morse cochain complex** of ψ, $\mathrm{Cone}(c(\psi))=(C^\bullet(M,f)\oplus C^{\bullet-\ell+1}(M,f),\,\partial_C)$:

$$\dots\xrightarrow{\partial_C} C^k(M,f)\oplus C^{k-\ell+1}(M,f)\xrightarrow{\partial_C} C^{k+1}(M,f)\oplus C^{k-\ell+2}(M,f)\xrightarrow{\partial_C}\dots$$

with

$$\partial_C=\begin{pmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1}\partial\end{pmatrix}. \tag{1.4}$$

Here, ∂ is the standard Morse cochain differential defined by gradient flow, and $c(\psi)$:

	de Rham	Morse
Cochains	$\Omega^\bullet(M)$	$C^\bullet(M,f)$
Differential	d	∂ (gradient flow)
Cohomology	$H^k_{dR}(M) \cong H^k_{C(f)}(M)$	
Morse Inequalities	$b_k \le m_k$ $\sum_{k=0}^{j}(-1)^{j-k} b_k \le \sum_{k=0}^{j}(-1)^{j-k} m_k$	

Table 2: The relations between the de Rham and Morse cochain complexes and Morse inequalities, where $b_k = \dim H^k_{dR}(M)$ and $m_k = \dim C^k(M,f)$

	Cone	Cone Morse
Cochains	$\Omega^\bullet(M) \oplus \Omega^{\bullet-\ell+1}(M)$	$C^\bullet(M,f) \oplus C^{\bullet-\ell+1}(M,f)$
Differential	$d_C = \begin{pmatrix} d & \psi \\ 0 & (-1)^{\ell-1} d \end{pmatrix}$	$\partial_C = \begin{pmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1}\partial \end{pmatrix}$
Cohomology	$H^k(\mathrm{Cone}(\psi))(M) \cong H^k(\mathrm{Cone}(c(\psi)))(M)$	
Cone-Morse Inequalities	$b^\psi_k \le m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1}$ $\sum_{k=0}^{j}(-1)^{j-k} b^\psi_k \le \sum_{k=0}^{j}(-1)^{j-k}(m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1})$	

Table 3: The relations between the cone complex and the cone Morse complex and inequalities in the presence of a closed ℓ-form ψ, where $b^\psi_k = \dim H^k(\mathrm{Cone}(\psi))(M)$ and $v_k = \mathrm{rank}[c(\psi) : C^k(M,f) \to C^{k+\ell}(M,f)]$.

$C^k(M,f) \to C^{k+\ell}(M,f)$ acting on a critical point of index k is defined to be

$$c(\psi)\, q_k = \sum_{ind(r)=k+\ell} \left(\int_{\overline{\mathcal{M}(r_{k+\ell}, q_k)}} \psi \right) r_{k+\ell} \tag{1.5}$$

where $\mathcal{M}(r_{k+\ell}, q_k)$ is the ℓ-dimensional submanifold of M consisting of all flow lines from the index $k+\ell$ critical point, $r_{k+\ell}$, to q_k.

Notice that the elements of the Morse cone complex, $\mathrm{Cone}^k(c(\omega)) = C^k(M,f) \oplus C^{k-l+1}(M,f)$, can be generated by *pairs* of critical points of index k and $k-l+1$. The differential ∂_C consists of the standard Morse differential ∂ from gradient flow couple with the $c(\psi)$ map which involves an integration of ψ over the space of gradient flow lines. This $c(\psi)$ map has appeared in Austin-Braam [AB95] and Viterbo [Vit95] to define a cup product on Morse cohomology. It satisfies the following Leibniz rule

$$\partial c(\psi) + (-1)^{\ell+1} c(\psi)\partial = -c(d\psi)\,. \tag{1.6}$$

A check of the $\pm$ signs of this equation together with a description of the orientation of $\overline{\mathcal{M}(r_{k+\ell+1}, q_k)}$ is given in Appendix A. With (1.6) and $\partial\partial = 0$, they together imply $\partial_C\, \partial_C = 0$.

We will prove in Section 2 that the cohomology of our cone Morse complex $\mathrm{Cone}(c(\psi))$ is isomorphic to the cohomology of the cone complex $\mathrm{Cone}(\psi)$ of differential forms.

Theorem 1.2. *Let M be a closed, oriented manifold and $\psi \in \Omega^l(M)$ a d-closed form. There exists a chain map $\mathcal{P}_C : (\mathrm{Cone}^\bullet(\psi), d_C) \to (\mathrm{Cone}^\bullet(c(\psi)), \partial_C)$ that is a quasi-isomorphism, and therefore, for any $k \in \mathbb{Z}$,*

$$H^k(\mathrm{Cone}(\psi)) \cong H^k(\mathrm{Cone}(c(\psi)))\,.$$

Theorem 1.2 importantly shows that the cohomology of the cone Morse complex is independent of the choice of both the Morse function f and the Riemannian metric g used to define $\mathrm{Cone}(c(\psi))$. It is also worthwhile to emphasize that the above theorem is a general one, applicable for any closed smooth manifold, odd or even dimensional, with respect to any closed differential form on the manifold.

Having obtained a cone Morse theory, we would like to write down the Morse-type inequalities that bounds the dimension of the cone cohomology which we will denote by $b_k^\psi = \dim H^k(\mathrm{Cone}(\psi))$. Specifically, we would like to bound the b_k^ψ's by the properties of the Morse functions. Recall that the standard Morse inequalities (for a reference, see e.g. [Mil63]) bounds the k-th Betti numbers $b_k = \dim H^k_{dR}(M)$ by m_k, the number of index k critical points of a Morse function. The usual Morse inequalities can be stated concisely as the existence of a polynomial $Q(t)$ with non-negative integer coefficients such that

$$\sum_{k=0} m_k\, t^k = \sum_{k=0} b_k\, t^k \;+\; (1+t)\, Q(t)\,. \tag{1.7}$$

This is equivalent to what is called the strong Morse inequalities

$$\sum_{i=0}^{k} (-1)^{k-i}\, b_i \;\leq\; \sum_{i=0}^{k} (-1)^{k-i}\, m_i\,, \qquad k = 0, \ldots, \dim M \tag{1.8}$$

which imply the weak Morse inequalities

$$b_k \leq m_k \,, \tag{1.9}$$

also for $k = 0, \ldots, \dim M$.

We can derive the analogous Morse-type inequalities results for the cone cohomology. We obtain the following:

Theorem 1.3. *Let (M, ψ, f, g) be a closed, oriented Riemannian manifold with Morse function f and Riemannian metric g and $\psi \in \Omega^\ell(M)$ a d-closed form. Then there exists a polynomial $Q(t)$ with non-negative integer coefficients such that*

$$(1 + t^{\ell-1}) \sum_{k=0} m_k \, t^k - (t^{\ell-1} + t^\ell) \sum_{k=0} v_k \, t^k = \sum_{k=0} b^\psi_k \, t^k \; + \; (1+t)\, Q(t)$$

where $b^\psi_k = \dim H^k(\mathrm{Cone}(\psi))$ and

$$v_k = \mathrm{rank}\, \Big(c(\psi) : C^k(M, f) \to C^{k+\ell}(M, f) \Big) \,. \tag{1.10}$$

Equivalently, we have the following inequalities:

(A) Weak cone Morse inequalities

$$b^\psi_k \; \leq \; m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1} \,, \qquad k = 0, \ldots, \dim M + \ell - 1; \tag{1.11}$$

(B) Strong cone Morse inequalities

$$\sum_{k=0}^{j} (-1)^{j-k} b^\psi_k \leq \sum_{k=0}^{j} (-1)^{j-k} (m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1}) \,, \tag{1.12}$$

for $j = 0, \ldots, \dim M + \ell - 1$.

In the special case where ψ is chosen to be a d-exact two-form, then the strong cone Morse inequalities of (1.12) imply an interesting bound for the difference between the number of critical points of any Morse function and the Betti numbers.

Corollary 1.4. *For ψ any d-exact two-form, we have the following bounds for $k =$*

$1, \ldots, \dim M$,

$$b_k \leq m_k - v_{k-1}\,. \tag{1.13}$$

This corollary gives an estimate for the difference between m_k and b_k in terms of v_{k-1} (1.10), which involves integration of ψ over the gradient flow lines. It represents an improvement to the classical Morse inequality (1.9) and can be used to quickly determine whether a Morse function is perfect or not. (Recall a perfect Morse function is one where $m_k = b_k$ for all k.) In considering the inequality (1.13), note that it applies for any exact two-form $\psi = d\alpha$ where $\alpha \in \Omega^1(M)$. Then v_{k-1} can only be non-zero if at least one moduli space of flow lines $\overline{\mathcal{M}(r_{k+1}, q_{k-1})}$, which is the domain of integration in (1.5), has a boundary. For if one $\partial\overline{\mathcal{M}} \neq 0$, then we can choose to work with a one-form α that takes value only along a small localized region along the boundary, such that the boundary integral of α is non-zero and thus generates $v_{k-1} > 0$.

It is worthwhile to point out that not all manifolds have perfect Morse functions. For instance, Morse functions on manifolds that has torsion in its homology class must satisfy the inequalities [Pit58]

$$b_k + \mu_k + \mu_{k-1} \leq m_k$$

where μ_k is the minimum number of generators of the torsion components of $H_k(M, \mathbb{Z})$. Hence our results implies that a manifold with torsion must have a moduli space of flow lines between two critical points that differ by index two that has non-trivial boundary.

The outline of the paper is the following. In Section 2, we define our cone Morse complex in detail and proof the isomorphism between the cone and cone-Morse cohomology of Theorem 1.2. In Section 3, we study the implication of the isomorphism and derive the cone Morse inequalities of Theorem 1.3. In Section 3, we demonstrate all the various properties of the cone Morse cohomology and inequalities in the simple, yet rich example of the two-sphere S^2.

Note added: Some of the results of this paper were announced in our arXiv preprint [CTTa]. We have since significantly developed that initial preprint resulting in two separate papers. The manuscript [CTTb] that has replaced the original preprint is focused on the Morse theory for symplectic manifolds and gives an analytic-based proof of Theorem 1.2 by means of the Witten deformation method for the special case where ψ is the symplectic structure. This paper incorporates the previous algebraic proof in [CTTa] and greatly expands on defining the general notion of a cone Morse theory with respect to any closed form on a manifold.

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2 Cone Morse complex: Cone($\mathbf{c}(\psi)$)

2.1 Preliminaries: Morse complex and $c(\psi)$

To begin, let f be a Morse function and g a Riemannian metric on M. We will assume throughout this paper that (f, g) satisfy the standard Morse-Smale transversality condition. The elements of the Morse cochain complex $C^\bullet(M, f)$ are $\mathbb{R}$-modules with generators critical points of f, graded by the index of the critical points, with boundary operator ∂ determined by the counting of gradient lines, i.e

$$\partial q_k = \sum_{ind(r)=k+1} n(r_{k+1}, q_k)\, r_{k+1},$$

where $n(r_{k+1}, q_k) = \#\widetilde{\mathcal{M}}(r_{k+1}, q_k)$ is a count of the moduli space of gradient flow lines with orientation modulo reparametrization.

Note that Morse theory is typically presented as a homology theory, and hence, flowing from index k to index $k-1$ critical points. To match up with the cochain complex of differential forms, we here work with the dual Morse cochain complex. Hence, our ∂ is the adjoint of the usual Morse boundary map under the inner product $\langle q_{k_i}, q_{k_j} \rangle = \delta_{ij}$.

Following Austin-Braam [AB95] and Viterbo [Vit95], we define

$$c(\psi) q_k = \sum_{ind(r)=k+\ell} \left(\int_{\overline{\mathcal{M}(r_{k+\ell}, q_k)}} \psi \right) r_{k+\ell}$$

where $\psi \in \Omega^\ell(M)$ is an ℓ-form and $\mathcal{M}(r_{k+\ell}, q_k)$ is the submanifold of all points that flow from $r_{k+\ell}$ to q_k, oriented as in [AB95]. From Appendix A, we have the Leibniz-type product relation

$$\partial c(\psi) + (-1)^{\deg(\psi)+1} c(\psi)\partial = -c(d\psi)$$

specifying a sign convention that is ambiguous in Austin-Braam [AB95] and Viterbo [Vit95]. Thus, for instance, for $\psi = \omega$, the symplectic structure, we have the relation

$$\partial c(\omega) - c(\omega)\partial = -c(d\omega) = 0$$

2.2 Chain map between $\mathrm{Cone}(\psi)$ and $\mathrm{Cone}(c(\psi))$

As explained by Bismut, Zhang and Laudenbach [BZ92, Zha01], there is a chain map $\mathcal{P} : \Omega^k(M) \to C^k(M,f)$ between differential forms and the Morse cochain complex given by

$$\mathcal{P}\phi = \sum_{q_k \in Crit(f)} \left(\int_{\overline{U_{q_k}}} \phi \right) q_k$$

where $\phi \in \Omega^k(M)$ and U_q is the set of all points on a gradient flow away from q. Being a chain map,

$$\partial\, \mathcal{P} = \mathcal{P}\, d\,. \tag{2.1}$$

Bismut, Zhang and Laudenbach, in [BZ92, Theorem 2.9] (see also [Zha01, Theorem 6.4]), also proved the following:

$$\mathcal{P} : H^k_{dR}(M) \to H^k_{C(f)}(M) \text{ is an isomorphism.} \tag{2.2}$$

Furthermore, Austin-Braam [AB95, Section 3.5] showed that $\mathcal{P}(\psi \wedge \gamma)$ and $c(\psi)\mathcal{P}\gamma$ are cohomologous:

$$[\mathcal{P}][\psi] = [c(\psi)][\mathcal{P}]\,. \tag{2.3}$$

We wish to find an analogous chain map relating $\mathrm{Cone}(\psi) = (\Omega^\bullet(M) \oplus \Omega^{\bullet-\ell+1}(M),\, d_C)$ with $\mathrm{Cone}(\psi) = (C^\bullet(M,f) \oplus C^{\bullet-\ell+1}(M,f),\, \partial_C)$, where as given in (1.1) and Definition 1.1,

$$\begin{aligned} &d_C : \Omega^k(M) \oplus \Omega^{k-l+1}(M) \to \Omega^{k+1}(M) \oplus \Omega^{k-l+2}(M) \\ &\partial_C : C^k(M,f) \oplus C^{k-l+1}(M,f) \to C^{k+1}(M,f) \oplus C^{k-l+2}(M,f) \end{aligned}$$

with

$$d_C = \begin{pmatrix} d & \psi \\ 0 & (-1)^{\ell-1} d \end{pmatrix}, \qquad \partial_C = \begin{pmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1}\partial \end{pmatrix}.$$

The chain map, which we will label by $\mathcal{P}_C$, that links the two cone complexes will need to satisfy $\partial_C \, \mathcal{P}_C = \mathcal{P}_C \, d_C$. In fact, such a map exists and can be expressed in an upper-triangular matrix form.

Definition 2.1. Let $\mathcal{P}_C : \mathrm{Cone}^\bullet(\psi) \to \mathrm{Cone}^\bullet(c(\psi))$ be the upper-triangular matrix map

$$\mathcal{P}_C = \begin{pmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{pmatrix}$$

where $K : \Omega^{k-\ell+1}(M) \to C^k(M,f)$ acting on $\xi \in \Omega^{k-\ell+1}(M)$ is defined by

$$K\xi = (-1)^\ell \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right) d^* G\xi \, + \, \partial^{-1}_{k,\perp} \left(\left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right) \mathcal{H}\xi\,\right), \tag{2.4}$$

in terms of the Hodge decomposition with respect to the de Rham Laplacian $\Delta = dd^* + d^*d$:

$$\xi = (\mathcal{H} + \Delta G)\xi = \mathcal{H}\xi + dd^*G\xi + d^*dG\xi\,,$$

where $\mathcal{H}\xi$ is the harmonic component and G is the Green's operator.

We explain the notation $\partial^{-1}_{k,\perp}$ in the second term for the definition of K in ((2.4)).

Let γ be a closed $(k-\ell+1)$-form. Then from (2.3), we know that $\mathcal{P}(\psi\wedge\gamma)$ and $c(\psi)\mathcal{P}\gamma$ are cohomologous, and therefore, $\mathcal{P}(\psi\wedge\gamma) - c(\psi)\mathcal{P}\gamma = \partial b$ for some $b \in C^k(M,f)$. Note that $C^k(M,f)$ is an inner product space under $\langle q_{k_i}, q_{k_j}\rangle = \delta_{ij}$, so we have an orthogonal splitting, $C^k(M,f) = \ker\partial_k \oplus (\ker\partial_k)^\perp$, and that ∂_k gives an isomorphism between $\left(C^k(M,f)/\ker\partial_k\right) \cong (\ker\partial_k)^\perp$ and $\mathrm{im}\,\partial_k \subset C^{k+1}(M,f)$. Thus, it follows from the finite-dimensional assumption on $C^k(M,f)$ and $C^{k+1}(M,f)$ that we can define a right inverse $\partial^{-1}_{k,\perp} : \mathrm{im}\,\partial_k \to (\ker\partial_k)^\perp \subset C^k(M,f)$, and $\partial^{-1}_{k,\perp}(\mathcal{P}(\psi\wedge\gamma) - c(\psi)\mathcal{P}\gamma) \in C^k(M,f)$. For the second term of K in (2.4), $\gamma = \mathcal{H}\xi$ is the closed form that is the harmonic component of ξ.

With $\mathcal{P}_C$ defined, we now show that it is a chain map.

Theorem 2.2. $\mathcal{P}_C : \mathrm{Cone}^\bullet(\psi) \to \mathrm{Cone}^\bullet(c(\psi))$ *is a chain map. In particular,*

$$\partial_C \, \mathcal{P}_C = \mathcal{P}_C \, d_C\,. \tag{2.5}$$

Proof. The right and the left hand side of (2.5) acting on $\eta + \theta\xi \in Cone^k(\psi)$ give

$$\mathcal{P}_C\, d_C = \begin{pmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{pmatrix} \begin{pmatrix} d & \psi \\ 0 & (-1)^{\ell-1} d \end{pmatrix} = \begin{pmatrix} \mathcal{P}d & \mathcal{P}\psi + (-1)^{\ell-1} K d \\ 0 & (-1)^{\ell-1}\mathcal{P}d \end{pmatrix},$$

$$\partial_C\, \mathcal{P}_C = \begin{pmatrix} \partial & c(\psi) \\ 0 & (-1)^{\ell-1}\partial \end{pmatrix} \begin{pmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{pmatrix} = \begin{pmatrix} \partial\mathcal{P} & c(\psi)\mathcal{P} + \partial K \\ 0 & (-1)^{\ell-1}\partial\mathcal{P} \end{pmatrix}.$$

Since $\mathcal{P}$ is a chain map (2.1), i.e. $d\,\mathcal{P} = \mathcal{P}\,\partial$, the only entry we need to check comes from the off-diagonal one,

$$\mathcal{P}\psi + (-1)^{\ell-1} K d = c(\psi)\mathcal{P} + \partial K\,,$$

or equivalently, we need to show that

$$\mathcal{P}\psi - c(\psi)\mathcal{P} = \partial K + (-1)^{\ell} K d\,, \tag{2.6}$$

or K a graded chain homotopy. To compute $Kd\xi$, note first that $\mathcal{H}d\xi = 0\,,\ \forall \xi \in \Omega^{k-l+1}(M)$. Therefore, we find that

$$K d\xi = (-1)^{\ell} \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right) d^* G d\xi = (-1)^{\ell} \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right) d^* d G\xi\,,$$

having used (2.4) and the fact that $Gd = dG$. Now, for the $\partial K\xi$ term, we have

$$\begin{aligned} \partial K\xi &= (-1)^{\ell}\partial\left[\left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right) d^* G\xi\right] + \partial\left[\partial^{-1}_{k,\perp}\left(\left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right)\mathcal{H}\xi\,\right)\right] \\ &= (-1)^{\ell}\left[(-1)^{\ell}\left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right) d d^* G\xi\right] + \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right)\mathcal{H}\xi \\ &= \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right)\left(d d^* G\xi + \mathcal{H}\xi\right) \end{aligned}$$

where in the second line, we have applied the graded commutative properties: $\partial\mathcal{P} = \mathcal{P}d$ and $\partial c(\psi) = (-1)^{\ell} c(\psi)\partial$ for ψ a d-closed ℓ-form. Altogether, we find for the right-hand side of (2.6)

$$\begin{aligned} \partial K\xi + (-1)^{\ell} K d\xi &= \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right)\left(d d^* G\xi + d^* d G\xi + \mathcal{H}\xi\right) \\ &= \left(\mathcal{P}\psi - c(\psi)\mathcal{P}\right)\xi. \end{aligned}$$

Thus, K is a graded chain homotopy of $\mathcal{P}\psi$ and $c(\psi)\mathcal{P}$, and therefore, $\mathcal{P}_C\, d_C = \partial_C\, \mathcal{P}_C$. $\square$

2.3 Isomorphism of cohomologies via Five Lemma

A mapping cone cochain complex can be described by a short exact sequence of chain maps. For the differential forms case, we have

$$0 \longrightarrow (\Omega^k(M), d) \xrightarrow{\iota_{dR}} (\mathrm{Cone}^k(\psi), d_C) \xrightarrow{\pi_{dR}} (\Omega^{k-\ell+1}(M), (-1)^{\ell-1}d) \longrightarrow 0 \tag{2.7}$$

where ι_{dR} is the inclusion into the first component $\iota_{dR}(\eta) = \begin{pmatrix} \eta \\ 0 \end{pmatrix}$ and π_{dR} is the projection of the second component $\pi_{dR}\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \xi$. It is easy to check that these maps are chain maps:

$$\iota_{dR} d\eta = \begin{pmatrix} d\eta \\ 0 \end{pmatrix} = d_C \iota_{dR} \eta$$

and

$$\pi_{dR}\, d_C \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \pi_{dR} \begin{pmatrix} d\eta + \psi \wedge \xi \\ (-1)^{\ell-1} d\xi \end{pmatrix} = (-1)^{\ell-1} d\xi = (-1)^{\ell-1} d \left\{ \pi_{dR} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \right\} .$$

The short exact sequence (2.7) implies the following long exact sequence for the cohomology of $\mathrm{Cone}(\psi)$

$$\ldots \longrightarrow H^{k-\ell}_{dR}(M) \xrightarrow{[\psi]} H^k_{dR}(M) \xrightarrow{[\iota_{dR}]} H^k(\mathrm{Cone}(\psi)) \xrightarrow{[\pi_{dR}]} H^{k-\ell+1}_{dR}(M) \longrightarrow \ldots \tag{2.8}$$

Analogously, for $\mathrm{Cone}(c(\psi))$, we also have the short exact sequence of chain maps

$$0 \longrightarrow (C^k(M,f), \partial) \xrightarrow{\iota_{C(f)}} (\mathrm{Cone}^k(c(\psi)), \partial_C) \xrightarrow{\pi_{C(f)}} (C^{k-\ell+1}(M,f), (-1)^{\ell-1}\partial) \longrightarrow 0 \tag{2.9}$$

and the long exact sequence of cohomology

$$\ldots \longrightarrow H^{k-\ell}_{C(f)}(M) \xrightarrow{[c(\psi)]} H^k_{C(f)}(M) \xrightarrow{[\iota_{C(f)}]} H^k(\mathrm{Cone}(c(\psi))) \xrightarrow{[\pi_{C(f)}]} H^{k-\ell+1}_{C(f)}(M) \longrightarrow \ldots \tag{2.10}$$

The two short exact sequences, (2.7) and (2.9), fit into a commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & (\Omega^k(M), d) \arrow[r, "\iota_{dR}"] \arrow[d, "\mathcal{P}"'] & (\mathrm{Cone}^k(\psi)), d_C) \arrow[r, "\pi_{dR}"] \arrow[d, "\mathcal{P}_C"'] & (\Omega^{k-\ell+1}(M), (-1)^{\ell-1} d) \arrow[r] \arrow[d, "\mathcal{P}"'] & 0 \\
0 \arrow[r] & (C^k(M, f), \partial) \arrow[r, "\iota_{C(f)}"] & (\mathrm{Cone}^k(c(\psi)), \partial_C) \arrow[r, "\pi_{C(f)}"] & (C^{k-\ell+1}(M, f), (-1)^{\ell-1}\partial) \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

The commutativity of the above diagram can be checked as follows:

$$\iota_{C(f)}(\mathcal{P}(\eta)) = \begin{pmatrix} \mathcal{P}\eta \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{P} & K \\ 0 & \mathcal{P} \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix} = \mathcal{P}_C(\iota_{dR}(\eta))\,,$$

$$\pi_{C(f)} \left(\mathcal{P}_C \begin{pmatrix} \eta \\ \xi \end{pmatrix} \right) = \pi_{C(f)} \begin{pmatrix} \mathcal{P}\eta + K\xi \\ \mathcal{P}\xi \end{pmatrix} = \mathcal{P}\xi = \mathcal{P} \left(\pi_{dR} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \right).$$

The short exact commutative diagram (2.11) gives a long commutative diagram of cohomologies:

$$\begin{tikzcd}
H^{k-\ell}_{dR}(M) \arrow[r, "{[\psi]}"] \arrow[d, "{[\mathcal{P}]}"'] & H^k_{dR}(M) \arrow[r, "{[\iota_{dR}]}"] \arrow[d, "{[\mathcal{P}]}"'] & H^k(\mathrm{Cone}(\psi)) \arrow[r, "{[\pi_{dR}]}"] \arrow[d, "{[\mathcal{P}_C]}"'] & H^{k-\ell+1}_{dR}(M) \arrow[r, "{[\psi]}"] \arrow[d, "{[\mathcal{P}]}"'] & H^{k+1}_{dR}(M) \arrow[d, "{[\mathcal{P}]}"'] \\
H^{k-\ell}_{C(f)}(M) \arrow[r, "{[c(\psi)]}"] & H^k_{C(f)}(M) \arrow[r, "{[\iota_{C(f)}]}"] & H^k(\mathrm{Cone}(c(\psi))) \arrow[r, "{[\pi_{C(f)}]}"] & H^{k-\ell+1}_{C(f)}(M) \arrow[r, "{[c(\psi)]}"] & H^{k+1}_{C(f)}(M)
\end{tikzcd} \tag{2.12}$$

We can check that each square commutes. The outer squares commute since $\mathcal{P}(\psi \wedge \xi)$ and $c(\psi)\mathcal{P}\xi$ are cohomologous when both ξ and ψ are d-closed, by ((2.3)), as was shown by Austin-Braam in [AB95, Section 3.5]. The middle two squares commute follows from the commutativity of the chain maps in (2.11). Furthermore, the vertical map $[\mathcal{P}]$ is an isomorphism (2.2) as shown by Bismut-Zhang and Laudenbach [BZ92, Theorem 2.9] (see also [Zha01, Theorem 6.4]).

We can now apply the Five Lemma to (2.12) which implies that the middle vertical map $[\mathcal{P}_C]$ is also an isomorphism on cohomology, and thus we prove Theorem 1.2.

Theorem 2.3. $\mathcal{P}_C : (\mathrm{Cone}^\bullet(\psi), d_C) \to (\mathrm{Cone}^\bullet(c(\psi)), \partial_C)$ *is a* $\mathbb{Z}$ *graded quasi-isomorphism.*

3 Cone Morse inequalities

Having established the quasi-isomorphism between the complexes, $\mathrm{Cone}(\psi)$ and $\mathrm{Cone}(c(\psi))$, we will proceed now to prove Theorem 1.3, which gives the Morse-type inequalities for the $\mathrm{Cone}(\psi)$ complex analogous to those in (1.7)-(1.9) for the de Rham complex.

For a closed, oriented manifold M and a d-closed form $\psi \in \Omega^\ell(M)$, let us denote by $b_k^\psi = \dim H^k(\mathrm{Cone}(\psi))$. From (1.3), we know that

$$H^k(\mathrm{Cone}(\psi)) \cong \mathrm{coker}\left([\psi]\colon H_{dR}^{k-\ell} \to H_{dR}^k\right) \oplus \ \ker\left([\psi]\colon H_{dR}^{k-\ell+1} \to H_{dR}^{k+1}\right) \tag{3.1}$$

which implies

$$\begin{aligned} b_k^\psi &= \dim\left[\mathrm{coker}\left([\psi] : H_{dR}^{k-\ell} \to H_{dR}^k\right)\right] + \dim\left[\ker\left([\psi] : H_{dR}^{k-\ell+1} \to H_{dR}^{k+1}\right)\right] \\ &= b_k - r_{k-\ell} + b_{k-\ell+1} - r_{k-\ell+1} \end{aligned} \tag{3.2}$$

where $b_k = \dim H_{dR}^k(M)$ and

$$r_k = \mathrm{rank}\left([\psi] : H_{dR}^k(M) \to H_{dR}^{k+\ell}(M)\right) . \tag{3.3}$$

We would like to bound b_k^ψ by means of the Morse function and properties of the cone Morse complex $\mathrm{Cone}(c(\psi))$. That $H(\mathrm{Cone}(\psi))$ as expressed above is related to the cokernel and kernel of the ψ map is suggestive that we should look for an analogous relationship between $H(\mathrm{Cone}(c(\psi)))$ with the cokernel and kernel of the $c(\psi)$ map. Indeed, such a relationship exists for any cone complex. (See [Wei94] or Appendix B for a review.) For the Morse complex $(C^\bullet(M,f),\partial)$, we will make use of two subcomplexes, the kernel and cokernel complex, associated to the map $c(\psi)$:

- The kernel complex of $c(\psi)$, $(\ker c(\psi), \partial)$, is the complex consisting of $\ker^j c(\psi) = \{b \in C^j(M,f) \,|\, c(\psi)b = 0\}$, with differential ∂.
- The cokernel complex of $c(\psi)$, $(\mathrm{coker}\, c(\psi), \partial^\pi)$, is the complex $\mathrm{coker}^j c(\psi) = \{[a] \in C^j / \operatorname{im} c(\psi)\}$ with differential $\partial^\pi[a] = [\partial a] \in C/\operatorname{im} c(\psi)$.

The cohomologies of these two subcomplexes together with $H(\mathrm{Cone}(c(\psi)))$ forms a long exact sequence (B.6)

$$\ldots \longrightarrow H^{k-\ell+1}(\ker c(\psi)) \xrightarrow{h_{ker}^{k-\ell+1}} H^k(\mathrm{Cone}(c(\psi))) \xrightarrow{h_{Cone}^k} H^k(\mathrm{coker}\, c(\psi)) \xrightarrow{h_{coker}^k} \ldots \tag{3.4}$$

The precise definitions of the maps in the long exact sequence will not be needed in our discussion here. From (3.4), we can immediately obtain the following weak cone-Morse inequality.

Theorem 3.1 (Weak Cone Morse Inequalities)**.** *On a closed manifold M with $\psi \in H^\ell_{dR}(M)$, let $b^\psi_k = \dim H^k(\mathrm{Cone}(\psi))$ and m_k the number of index k critical points of a Morse function on M. Then, we have for $k = 0, 1, \ldots, (\dim M + \ell - 1)$,*

$$b^\psi_k \le m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1}\,, \tag{3.5}$$

where

$$v_k = \mathrm{rank}\Big(c(\psi) : C^k(M,f) \to C^{k+\ell}(M,f)\Big)\,. \tag{3.6}$$

Proof. From (3.4), we have

$$\begin{aligned} b^\psi_k &\le \dim H^k(\mathrm{coker}\, c(\psi)) + \dim H^{k-\ell+1}(\ker c(\psi)) \\ &\le \dim(\mathrm{coker}^k c(\psi)) + \dim(\ker^{k-\ell+1} c(\psi)) \\ &= m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1}\,. \end{aligned}$$

□

In general, the number $v_k = \mathrm{rank}\, c(\psi)|_{C^k(M,f)}$ is not equal to $r_k = \mathrm{rank}\, [\psi]|_{H^k_{dR}(M)}$ as defined in (3.3). However, we have the following relations.

Lemma 3.2. *Let $r_k = \mathrm{rank}\, [\psi]|_{H^k_{dR}(M)}$ and $v_k = \mathrm{rank}\, c(\psi)|_{C^k(M,f)}$ as defined in* (3.3) *and* (3.6)*, respectively. Then for $k = 0, 1, \ldots, (\dim M + \ell - 1)$,*

(a) $r_k = \mathrm{rank}\Big([c(\psi)] : H^k_{C(f)}(M) \to H^{k+\ell}_{C(f)}(M)\Big)$*;*

(b) $r_k \le v_k$*;*

(c) $b_k - v_{k-\ell} + b_{k-\ell+1} - v_{k-\ell+1} \le b^\psi_k \le m_k - r_{k-\ell} + m_{k-\ell+1} - r_{k-\ell+1}$*;*

(d) $(v_{k-\ell} - r_{k-\ell}) + (v_{k-\ell+1} - r_{k-\ell+1}) \le (m_k - b_k) + (m_{k-\ell+1} - b_{k-\ell+1})\,.$

Proof. Property (a) follows from (2.2)-(2.3) which implies that rank $[\psi] =$ rank $[c(\psi)]$. For (b), if $\{[c(\psi)a_1], ...[c(\psi)a_{r_k}]\}$ gives a basis for $\mathrm{im}\,[c(\psi)] \subset H^{k+\ell}_{C(f)}(M)$, then $\{c(\psi)a_1, ..., c(\psi)a_{r_k}\}$

must constitute a linearly independent set of elements in $\operatorname{im} c(\psi) \subset C^{k+\ell}(M,f)$, and therefore,

$$r_k \leq \dim\Big(\operatorname{im} c(\psi) \cap C^{k+\ell}(M,f)\Big) = v_k\,.$$

Applying property (b) to (3.2) and (3.5) results in property (c). Lastly, property (d) follows from combining (3.2) and (3.5). $\square$

From the standard Morse inequality, $b_k \leq m_k$, and Lemma 3.2(b), $r_k \leq v_k$, we see that the relation in Lemma 3.2(d)

$$(v_{k-\ell} - r_{k-\ell}) + (v_{k-\ell+1} - r_{k-\ell+1}) \leq (m_k - b_k) + (m_{k-\ell+1} - b_{k-\ell+1}) \tag{3.7}$$

consist of sums of two non-negative terms on both sides. In particular, if the Morse function f is perfect, i.e. $m_k = b_k$, then (3.7) implies the following result.

Corollary 3.3. *If f is a perfect Morse function, then $v_k = r_k$ and*

$$b_k^\psi = (m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1}) \tag{3.8}$$

for $k = 0, 1, \ldots, \dim M + \ell - 1$.

Hence, for a perfect Morse function, the weak cone-Morse inequality becomes an equality. And this is as expected since a perfect Morse function implies for the Morse complex that $\dim H^k_{C(f)}(M) = \dim C^k(M,f)$, and therefore, $[c(\psi)]|_{H_{C(f)}}$ and $c(\psi)|_{C^k(M,f)}$ are the same map. Clearly, equation (3.7) constrains the deviations of the v_k's from the r_k's by the deviations of the m_k's from the b_k's.

We now proceed to prove the strong cone Morse inequalities.

Theorem 3.4 (Strong Cone Morse Inequalities)**.** *On a closed manifold M with $\psi \in H^\ell_{dR}(M)$, let $b_k^\psi = \dim H^k(Cone(\psi))$, m_k be the number of index k critical points of a Morse function on M, and $v_k = \operatorname{rank} c(\psi)|_{C^k(M,f)}$. Then, we have for $j = 0, 1, \ldots, (\dim M + \ell - 1)$,*

$$\sum_{k=0}^{j} (-1)^{j-k} b_k^\psi \leq \sum_{k=0}^{j} (-1)^{j-k} (m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1}). \tag{3.9}$$

Proof. The $j = 0$ case is just the weak inequality of (3.5). So we can assume $j \geq 1$. We note first that (3.4) implies

$$0 \to \operatorname{im} h_{ker}^{k-\ell+1} \to H^k(\operatorname{Cone}(c(\psi))) \to \operatorname{im} h_{Cone}^k \to 0 \tag{3.10}$$
$$0 \to \operatorname{im} h_{Cone}^k \to H^k(\operatorname{coker} c(\psi)) \to \operatorname{im} h_{coker}^k \to 0 \tag{3.11}$$
$$0 \to \operatorname{im} h_{coker}^{k-1} \to H^{k-\ell+1}(\ker c(\psi)) \to \operatorname{im} h_{ker}^{k-\ell+1} \to 0 \tag{3.12}$$

By Theorem 1.2, $b_k^\psi = \dim H^k(\operatorname{Cone}(c(\psi)))$. Thus, we can use (3.10) to write

$$\begin{aligned}
\sum_{k=0}^{j}(-1)^{j-k} b_k^\psi &= \sum_{k=0}^{j}(-1)^{j-k}\left[\dim(\operatorname{im} h_{Cone}^k) + \dim(\operatorname{im} h_{ker}^{k-\ell+1})\right] \\
&= \dim(\operatorname{im} h_{Cone}^j) - \sum_{k=0}^{j}(-1)^{j-k}\left[\dim(\operatorname{im} h_{Cone}^{k-1}) - \dim(\operatorname{im} h_{ker}^{k-\ell+1})\right] \\
&= \dim(\operatorname{im} h_{Cone}^j) - \sum_{k=0}^{j}(-1)^{j-k}\left[\dim H^{k-1}(\operatorname{coker} c(\psi)) - \dim H^{k-\ell+1}(\ker c(\psi))\right] \\
&\leq \sum_{k=0}^{j}(-1)^{j-k}\left[\dim H^k(\operatorname{coker} c(\psi)) + \dim H^{k-\ell+1}(\ker c(\psi))\right] \qquad (3.13)
\end{aligned}$$

where in the third line, we used (3.11)-(3.12), and in the fourth line (3.11) again. Now, because M is assumed to be a closed manifold, both the $\ker c(\psi)$ and the $\operatorname{coker} c(\psi)$ complex are finitely generated. In general, for any finitely-generated cochain complex $0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} \dots$, the dimensions of the associated cohomologies and that of the cochains satisfy the following inequality:

$$\sum_{k=0}^{j}(-1)^{j-k}\dim H^k(C) \leq \sum_{k=0}^{j}(-1)^{j-k}\dim C^k\,.$$

Applying this relation to (3.13) results in

$$\begin{aligned}
\sum_{k=0}^{j}(-1)^{j-k} b_k^\psi &\leq \sum_{k=0}^{j}(-1)^{j-k}\left[\dim(\operatorname{coker}^k c(\psi)) + \dim(\ker^{k-\ell+1} c(\psi))\right] \\
&= \sum_{k=0}^{j}(-1)^{j-k}(m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1})
\end{aligned}$$

thus, we obtain the strong cone-Morse inequality for the cone complex. □

In the case where f is a perfect Morse function, Corollary 3.3 immediately implies the following.

Corollary 3.5. *If f is a perfect Morse function, then the strong cone Morse inequalities become equalities:*

$$\sum_{k=0}^{j}(-1)^{j-k}b_k^{\psi} = \sum_{k=0}^{j}(-1)^{j-k}(m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1})\,. \tag{3.14}$$

More generally, when the Morse function is not perfect, Theorem 3.4 implies an analogous strong-version of the inequalities in Lemma 3.2(d).

Corollary 3.6. *For $j = 0, \ldots, \dim M + \ell - 1$,*

$$\begin{aligned}\sum_{k=0}^{j}(-1)^{j-k}((v_{k-\ell} - r_{k-\ell}) + (v_{k-\ell+1} - r_{k-\ell+1}))&\\ \leq \sum_{k=0}^{j}(-1)^{j-k}((m_k - b_k) + (m_{k-\ell+1} - b_{k-\ell+1})).&\end{aligned}$$

Proof. By (3.2) and Theorem 3.4, we have

$$\begin{aligned}\sum_{k=0}^{j}(-1)^{j-k}(b_k - r_{k-\ell} + b_{k-\ell+1} - r_{k-\ell+1}) &= \sum_{k=0}^{j}(-1)^{j-k}b_k^{\psi}\\ &\leq \sum_{k=0}^{j}(-1)^{j-k}(m_k - v_{k-\ell} + m_{k-\ell+1} - v_{k-\ell+1})\,.\end{aligned}$$

The relation of the Corollary is then obtained by moving the b_k's to the right-hand-side and the v_k's to the left-hand-side. $\square$

In the special case when ψ is a closed two-form, i.e. $\ell = 2$, Theorem 3.5 results in an interesting relation.

Corollary 3.7. *For a closed two-form ψ, we have the bounds for $k = 0, \ldots, \dim M - 1$,*

$$0 \leq v_k - r_k \leq m_{k+1} - b_{k+1}\,.$$

Proof. When $\ell = 2$, equation (3.2) implies for $j \geq 1$

$$\sum_{k=0}^{j}(-1)^{j-k}b_k^{\psi} = \sum_{k=0}^{j}(-1)^{j-k}(b_k + r_{k-2} - b_{k-1} - r_{k-1}) = b_j - r_{j-1}\,, \tag{3.15}$$

and similarly, Theorem 3.4 implies for $j \geq 1$

$$\sum_{k=0}^{j}(-1)^{j-k}b_k^{\psi} \leq \sum_{k=0}^{j}(-1)^{j-k}(m_k - v_{k-2} + m_{k-1} - v_{k-1}) = m_j - v_{j-1}\,. \tag{3.16}$$

Combining (3.15)-(3.16) gives the relation $v_{j-1} - r_{j-1} \leq m_j - b_j$. □

Corollary 3.7 interestingly shows that when $\ell = 2$, the rank of the $c(\psi)$ map on $C^k(M,f)$ is constrained, not just by m_k, the number of critical points of index k, as would be expected, but also by m_{k+1}, relative to b_{k+1}. Corollary 3.7 also gives a bound for the difference between m_k and b_k. The bound becomes especially simple in the case when ψ is an exact form. For an exact ψ implies $r_k = 0$ and we thus obtain the relation in Corollary 1.4

$$b_k \leq m_k - v_{k-1}\,, \qquad k = 1, \ldots, \dim M\,. \tag{3.17}$$

4 Examples of the two-sphere

We will analyze the two-sphere in details. We will give an explicit examples where the cone cohomology $H^k(\mathrm{Cone}(\psi))$ varies with ψ, and how the cone Morse bounds can vary with ψ, the metric, and Morse function.

Consider the two-sphere $M = S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$, described as the unit sphere in $\mathbb{R}^3$. We will let $\psi \in \Omega^2(S^2)$. We note in two dimensions, all two forms are trivially closed.

We consider first $b_k^{\psi} = \dim H^k(Cone(\psi))$. From (3.2), we find

$$b_k^{\psi} = \begin{cases} 1 & k = 0,3 \\ 1 - r_0 & k = 1,2 \end{cases} \tag{4.1}$$

where $r_0 = \mathrm{rank}[\psi]|_{H^0_{dR}}$. If $\psi = \omega_0 := d\phi \wedge \sin\phi\, d\theta$, the standard symplectic structure of the round two-sphere, then $r_0 = 1$.

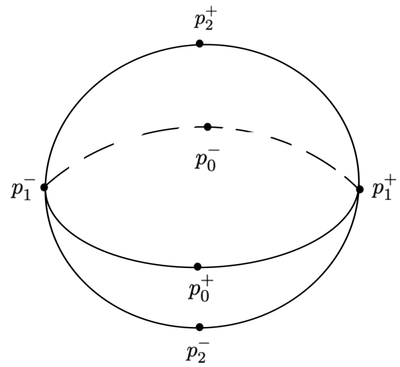

Figure 1: Two sphere in $\mathbb{R}^3$ with six critical points of the Morse function $f = x^2+2y^2+3z^2$.

In fact, $r_0 = 1$ whenever $[\psi] \in H^2_{dR}(S^2)$ is a non-trivial class.

Example 4.1. [Change in $b_k^\psi = \dim H^k(\mathrm{Cone}(\psi))$ as $[\psi]$ varies.] Consider the one-parameter family

$$\psi_s = (z+s)\,\omega_0\,, \qquad\qquad s \in [-1,1]\,. \tag{4.2}$$

Note that ψ_s is not symplectic as it vanishes along $z = -s$. In fact, the cohomology class $[\psi_s] = s[\omega_0]$, and therefore, $r_0 = 1$ for $s \neq 0$ and $r_0 = 0$ for $s = 0$. It follows from (4.1) that at the special value of $s = 0$, $b_k^{\psi_s}$ increases by one for $k = 1, 2$.

We now consider the cone Morse inequalities. In order to do so, we introduce a Morse function on S^2. The generic Morse function is not perfect. An example of a non-perfect Morse function which we will use for the remainder of this section is

$$f = x^2 + 2y^2 + 3z^2 \tag{4.3}$$

restricted to S^2. This Morse function has six critical points which can be easily seen by expressing f in terms of only two variables applying the unit circle condition (see Figure 1):

$$\begin{aligned}
&f = 1 + y^2 + 2z^2 \qquad (y^2+z^2 \leq 1) \qquad\qquad \text{index 0 critical points: } p_0^\pm = (\pm 1, 0, 0)\,;\\
&f = 2 - x^2 + z^2 \qquad (x^2+z^2 \leq 1) \qquad\qquad \text{index 1 critical points: } p_1^\pm = (0, \pm 1, 0)\,;\\
&f = 3 - 2x^2 - y^2 \qquad (x^2+y^2 \leq 1) \qquad\qquad \text{index 2 critical points: } p_2^\pm = (0, 0, \pm 1)\,.
\end{aligned}$$

Clearly, $m_k \neq b_k(S^2)$.

As for the Riemannian metric, let $g_0 = d\phi^2 + \sin^2\phi \, d\theta^2$, the standard round metric induced from the standard Euclidean metric on $\mathbb{R}^3$. The pair (f, g_0) determines the moduli space of gradient flow lines which goes into the calculation of $c(\psi) : C^k(S^2, f) \to C^{k+2}(S^2, f)$. Only for $k = 0$ is the $c(\psi)$ map non-trivial and this corresponds to integrating ψ over the moduli space $\overline{\mathcal{M}(p_2^\pm, p_0^\pm)}$ which are the four quarter spheres determined by ($x \leq 0$ or $x \geq 0$) and ($z \leq 0$ or $z \geq 0$).

The weak cone-Morse inequality in (3.5) gives for (S^2, f, g_0)

$$b_k^\psi \leq \begin{cases} 2 & k = 0, 3 \\ 4 - v_0 & k = 1, 2 \end{cases} \tag{4.4}$$

where $v_0 = \operatorname{rank} c(\psi)|_{C^0(S^2,f)}$.

Example 4.2. [Change in the cone-Morse bound for b_k^ψ as $[\psi]$ varies.] We will take ψ to be again the one-parameter family ψ_s introduced in (4.2) of the previous example. To obtain the weak cone-Morse bound for $b_k^{\psi_s}$, we calculate $v_0 = \operatorname{rank} c(\psi_s)|_{C^0(S^2,f)}$. The operator $c(\psi_s)$ acting on the two index zero critical points (p_0^+, p_0^-) can be found by integrating ψ_s over the quarter spheres, and has the following matrix form:

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = c(\psi_s) \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix} = \begin{pmatrix} 1/2 + s\,\pi & -1/2 + s\,\pi \\ -1/2 + s\,\pi & 1/2 + s\,\pi \end{pmatrix} \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix} .$$

We see that the rank $v_0 = 2$ for $s \neq 0$, and $v_0 = 1$ for $s = 0$. Hence, by (4.4), the weak cone-Morse bound for $b_k^{\psi_s}$ for $k = 1, 2$ increases by one at $s = 0$. This coincides exactly with the increase in $b_k^{\psi_s}$ at $s = 0$ as calculated in Example 4.1.

Example 4.3. [Change in the cone-Morse bound of b_k^ψ as ψ varies within a fixed de Rham class.] The weak cone-Morse bound can also vary within the same de Rham cohomology class $[\psi]$. Consider the following one-parameter family of ψ:

$$\psi_t = (1 + tx + ty)\,\omega_0\,, \qquad -\frac{1}{2} < t < \frac{1}{2}\,. \tag{4.5}$$

Note that $[\psi_t] = [\omega_0] \in H^2_{dR}(S^2)$ for all $t \in (-.5, .5)$, and so $b_k^{\psi_t}$ does not vary. However,

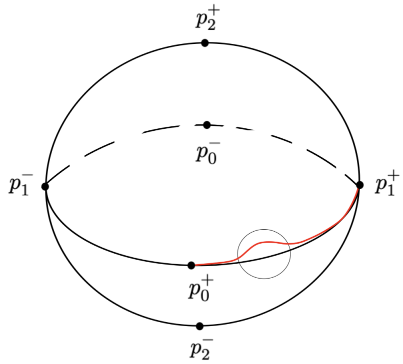

Figure 2: Two sphere with modified metric deforming the flow line from p_1^+ to p_0^+.

for the weak cone-Morse bound, the $c(\psi_t)$ map takes the form

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = c(\psi_t) \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix} = 4\pi \begin{pmatrix} 1+\frac{2t}{3} & 1 \\ 1 & 1-\frac{2t}{3} \end{pmatrix} \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix} .$$

and has rank $v_0 = 2$ for $t \neq 0$, and $v_0 = 1$ for $t = 0$. This gives the bound for $k = 1, 2$

$$b_k^{\psi_t} \leq \begin{cases} 3 & t = 0 \\ 4 & t \neq 0 \end{cases}$$

even though $b_k^{\psi_t}$ remains constant.

Remark 4.4. Notice that the one-parameter family of closed two-forms ψ_t in (4.5) are all non-degenerate, and hence, symplectic. Being in the same cohomology class, Moser's theorem implies the existence of a one-parameter family of symplectomorphism $\varphi_t : S^2 \to S^2$ such that $\varphi_t^* \omega_t = \omega_0$. We can use this symplectomorphism to pull back (S^2, ω_t, f, g_0) to $(S^2, \omega_0, \varphi_t^* f, \varphi_t^* g_0)$. As symplectomorphisms leave unchanged m_k's and v_k's, we can reinterpret the above example as varying the Morse-Smale pair $(\varphi_t^* f, \varphi_t^* g_0)$ while keeping fixed the closed-form $\psi = \omega_0$ on S^2. It thus also represents an example where the cone-Morse bound b_k^ψ changes when the Morse-Smale pair (f, g) is varied.

Example 4.5. [Change in the cone-Morse bound of b_k^ψ as the Riemannian metric g varies.] We demonstrate here that the bound for b_k^ψ can jump just by varying the metric. Let $\psi = \omega_0$ the standard area form on S^2. For the standard round metric, g_0, the moduli space $\mathcal{M}(p_2^{\pm}, p_0^{\pm})$ of flow lines are just the four quarters of the sphere. Suppose we modify this round metric within a small neighborhood of a point that is on the flow line between p_1^+ and p_0^+ (the small circle on S^2 in Figure 2). In doing so, we can change the gradient flow lines, so the new boundary is the red line above. As such that we remove an area of ϵ that is between the red flow line and the original black line. Thus we subtract ϵ from $\mathcal{M}(p_2^+, p_0^+)$ and add it to $\mathcal{M}(p_2^-, p_0^+)$

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = c(\omega_0) \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix} = \pi \begin{pmatrix} 1-\epsilon & 1 \\ 1+\epsilon & 1 \end{pmatrix} \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix}$$

In this case, the rank of $c(\omega_0)|_{C^0(S^2,f)}$ jumps to $v_0 = 2$ when $\epsilon \neq 0$. By (4.4), this correspondingly decreases the bound on $b_k^{\omega_0}$ for $k = 1, 2$, by one, and hence, gives an explicit example where the bound varies with the metric.

Remark 4.6. The cone cohomology dimension b_k^{ψ} depends only on the cohomology class $[\psi]$. We have seen above how the cone Morse inequalities, can explicitly depend on the Morse function f, the metric g and even the representative form ψ in $[\psi]$. We can improve the bound by varying g and ψ within the class $[\psi]$ to maximize $v_k = \operatorname{rank} c(\psi)|_{C^k(M,f)}$. Changing the Morse function f would change m_k. In the above examples, the variations considered improved the bounds but did not reach the actual value of b_k^{ψ} as given in (4.1). If we have chosen to work with a perfect Morse function on S^2, then by Corollary 3.3, we would have obtained the expected b_k^{ψ} exactly.

Finally, we show using the same S^2 example with the non-perfect Morse function $f = x^2+2y^2+3z^2$ how we can bound the Betti number by considering different exact differential forms as in (3.17).

Example 4.7. [Change in the Morse bound for an exact 1-form.] Let $\psi = d\alpha$ be an exact two-form on the sphere. We evaluate

$$c(\psi)p = \sum \left(\int_{\overline{\mathcal{M}(p,q)}} d\alpha \right) q = \sum \left(\int_{\partial \overline{\mathcal{M}(p,q)}} \alpha \right) q \, .$$

We will use the notation where a gradient flow curve from p_i^+ to p_j^- is labelled by γ_{ij}^{+-}

(and γ_{ij}^{-+} represents the flow curve from p_i^- to p_j^+). Explicitly, we have

$$\begin{aligned}
\partial\overline{\mathcal{M}(p_2^+,p_0^+)} &= \gamma_{21}^{+-} + \gamma_{10}^{-+} - \gamma_{21}^{++} - \gamma_{10}^{++}\,, \\
\partial\overline{\mathcal{M}(p_2^-,p_0^+)} &= \gamma_{21}^{-+} + \gamma_{10}^{++} - \gamma_{21}^{--} - \gamma_{10}^{-+}\,, \\
\partial\overline{\mathcal{M}(p_2^+,p_0^-)} &= \gamma_{21}^{++} + \gamma_{10}^{+-} - \gamma_{21}^{+-} - \gamma_{10}^{--}\,, \\
\partial\overline{\mathcal{M}(p_2^-,p_0^-)} &= \gamma_{21}^{--} + \gamma_{10}^{--} - \gamma_{21}^{-+} - \gamma_{10}^{+-}\,.
\end{aligned}$$

Let us further denote by $a_{21}^{+-} = \int_{\gamma_{21}^{+-}} \alpha$ and similarly for other line integral over α. The $c(d\alpha)$ map acting on index zero points then takes the following form.

$$\begin{pmatrix} p_2^+ \\ p_2^- \end{pmatrix} = \begin{pmatrix} a_{21}^{+-} + a_{10}^{-+} - a_{21}^{++} - a_{10}^{++} & a_{21}^{++} + a_{10}^{+-} - a_{21}^{+-} - a_{10}^{--} \\ a_{21}^{-+} + a_{10}^{++} - a_{21}^{--} - a_{10}^{-+} & a_{21}^{--} + a_{10}^{--} - a_{21}^{-+} - a_{10}^{+-} \end{pmatrix} \begin{pmatrix} p_0^+ \\ p_0^- \end{pmatrix}.$$

This $c(d\alpha)$ matrix has the following determinant:

$$(a_{21}^{++} - a_{21}^{-+} + a_{21}^{--} - a_{21}^{+-})(a_{10}^{++} - a_{10}^{-+} + a_{10}^{--} - a_{10}^{+-})\,.$$

Note that the first factor is the line integral of α over a meridian and the second factor the line integral of α over the equator. Thus, if we work with an one-form α such that both factors are non-zero (such one-forms are abundant, for instance, take α to be a positive one-form localized along γ_{21}^{++} and γ_{10}^{++}), then $v_0 = 2$. From (3.17), we thus find $b_1 \leq m_1 - v_0 = 0$, showing that the first Betti number of the two-sphere must be zero.

A Morse theory conventions and Leibniz rule

We describe here the conventions used to define the differential map ∂ in the Morse cochain complex and also the orientations of the submanifolds which are integrated over in the $c(\psi)$ map of (1.5). The main aim of this appendix is to prove the following:

Lemma A.1 (Leibniz Rule on forms in Morse cohomology)**.** *Let $\psi \in \Omega^\ell(M)$ then*

$$\partial\, c(\psi) + (-1)^{\ell+1} c(\psi)\, \partial = -c(d\psi)\,. \tag{A.1}$$

This formula appeared in Austin-Braam [AB95] and Viterbo [Vit95] though with ambiguous signs. To set our conventions and prove the Lemma, we start with a brief background.

Let ϕ_t be the flow of the vector field $-\nabla f$. For a critical point $r \in Crit(f)$, the stable S_r and unstable U_r submanifolds are defined to be

$$S_r = \{x \in M : \lim_{t\to\infty} \phi_t(x) = r\}\,, \qquad U_r = \{x \in M : \lim_{t\to-\infty} \phi_t(x) = r\}\,,$$

and the moduli spaces of gradient lines between two critical points, $q, r \in Crit(f)$,

$$\mathcal{M}(r,q) = S_q \cap U_r\,, \qquad \widetilde{\mathcal{M}}(r,q) = \frac{S_q \cap U_r}{\{x \sim y : \phi_t(x) = y \text{ for some } t \in \mathbb{R}\}}.$$

We define the orientation of the moduli spaces similar to that in Austin-Braam [AB95, Section 2.2]. For an oriented manifold M, we first specify an orientation for either the stable submanifolds, or equivalently, the unstable ones. The orientation of one type determines the other by the relation

$$[S_r][U_r] = [M]\,. \tag{A.2}$$

The orientation of the moduli space is then just the orientation of the transversal intersection which can be expressed as

$$[\mathcal{M}(r,q)] = [U_r][M]^{-1}[S_q] = [U_r][U_q]^{-1}\,. \tag{A.3}$$

We will also take as convention

$$[\mathcal{M}(r,q)] = [\widetilde{\mathcal{M}}(r,q)][\nabla f]\,. \tag{A.4}$$

In the special case when $ind(r) = ind(q) + 1$, $\mathcal{M}(r,q)$ is an oriented one-dimensional submanifold of gradient flow lines and $\widetilde{\mathcal{M}}(r,q)$ is an oriented collection of points. Also, recall that the Morse differential is defined by $\partial q = \sum_r n(r,q)\, r$ where

$$n(r,q) = \#\widetilde{\mathcal{M}}(r,q)\,. \tag{A.5}$$

It follows from (A.4) that $n(r,q)$ is equal to the number of gradient lines flowing in the direction of ∇f minus the number flowing in the direction of $-\nabla f$.

As an example of why (A.1) has the correct signs, we first prove the zero-form case with $\psi = h$, a function.

Corollary A.2. *If $h \in C^\infty(M)$, then $-c(dh) = \partial c(h) - c(h)\partial$.*

Proof. Evaluating $c(dh)$ by integrating over the gradient curves with orientation, we have

$$\begin{aligned} c(dh)q_k &= \sum_{r_{k+1}} \left(\int_{\overline{\mathcal{M}(r_{k+1},q_k)}} dh \right) r_{k+1} \\ &= \sum_{r_{k+1}} \left(n(r_{k+1},q_k)(h(r_{k+1}) - h(q_k)) \right) r_{k+1} \\ &= \sum_{r_{k+1}} h(r_{r+1}) n(r_{k+1},q_k) r_{k+1} - \sum_{r_{k+1}} n(r_{k+1},q_k) h(q_k) r_{k+1} \\ &= c(h)\partial q_k - \partial c(h) q_k = (c(h)\partial - \partial c(h)) q_k \end{aligned}$$

where $c(h)q_k = (\int_{\overline{\mathcal{M}(q_k,q_k)}} h) q_k = h(q_k)q_k$. Thus, having taken into account our orientation convention, we find that $-c(dh) = \partial c(h) - c(h)\partial$. $\square$

To prove (A.1) in general, we re-express the right-hand side by Stokes' theorem

$$c(d\psi)q_k = \sum_{r_{k+\ell+1}} \left(\int_{\overline{\mathcal{M}(r_{k+\ell+1},q_k)}} d\psi \right) r_{k+\ell+1} = \sum_{r_{k+\ell+1}} \left(\int_{\partial \mathcal{M}(r_{k+\ell+1},q_k)} \psi \right) r_{k+\ell+1} \,.$$

The relevant components of $\partial \mathcal{M}(r_{k+\ell+1},q_k)$ for integrating ψ consists of

$$\left\{ \bigcup_{p_{k+\ell}} \mathcal{M}(p_{k+\ell},q_k) \times \widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell}) \right\} \bigcup \left\{ \bigcup_{p_{k+1}} \mathcal{M}(r_{k+\ell+1},p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1},q_k) \right\} .$$

This implies up to signs

$$
\begin{aligned}
&c(d\psi)q_k \\
&=\sum_{r_{k+\ell+1}}\left[\pm\sum_{p_{k+\ell}}\int_{\overline{\mathcal{M}(p_{k+\ell},q_k)}\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})}\psi \;\pm\; \sum_{p_{k+1}}\int_{\overline{\mathcal{M}(r_{k+\ell+1},p_{k+1})}\times\widetilde{\mathcal{M}}(p_{k+1},q_k)}\psi\right] r_{k+\ell+1} \\
&=\sum_{r_{k+\ell+1}}\left[\pm\sum_{p_{k+\ell}}\left(\int_{\overline{\mathcal{M}(p_{k+\ell},q_k)}}\psi\right) n(r_{k+\ell+1},p_{k+\ell}) \pm \sum_{p_{k+1}} n(p_{k+1},q_k)\left(\int_{\overline{\mathcal{M}(r_{k+\ell+1},p_{k+1})}}\psi\right)\right] r_{k+\ell+1} \\
&= \pm\partial c(\psi)q_k \pm c(\psi)\partial q_k
\end{aligned}
\tag{A.6}
$$

To fix the signs, we will proceed in two steps. First, we make a choice of the orientation of the stable and unstable manifolds at the critical points $\{q_k, p_{k+1}, p_{k+l}, r_{k+l+1}\}$. By (A.3), this determines the orientation of the various moduli spaces that arise in the Stokes' theorem calculation above. Then in step two, we compare the orientation of the relevant boundary components, $\mathcal{M}(p_{k+\ell}, q_k)\times\widetilde{\mathcal{M}}(r_{k+l+1}, p_{k+l})$ and $\mathcal{M}(r_{k+\ell+1}, p_{k+\ell})\times\widetilde{\mathcal{M}}(p_{k+1}, q_k)$, with the orientation needed to satisfy Stokes' theorem. The relative difference in the orientations will determine the signs in (A.6).

Step 1: Computing the orientation of the moduli spaces.

By (A.3), the orientation of a moduli space $\mathcal{M}(r, q)$ can be determined by the orientation of the unstable submanifolds U_r and U_q. Hence, we will write below our choice for the orientation for the relevant unstable submanifolds explicitly. (The orientation of the stable submanifolds of a critical point are then fixed by (A.2).) Similar to [AB95, Section 2.2], we will express the orientations in terms of orthonormal frame vectors grouped together by Clifford multiplication.

Let $e_1, \ldots, e_k$ be an orthonormal set of frame vectors that are shared by both U_{q_k} and $U_{r_{k+\ell+1}}$. Let $e_{k+1}, \ldots, e_{k+\ell+1}$ be the additional frame vectors in $U_{r_{k+\ell+1}}$ defined such that they point in the direction away from q_k towards r_{k+l+1}, i.e. in the direction of ∇f. Then, for p_{k+1}, there is a vector $e_{i_{p_{k+1}}}$ that points along the gradient curve $\mathcal{M}(p_{k+1}, q_k)$ from q_k to p_{k+1}, and for $p_{k+\ell}$, there is a vector $e_{i_{p_{k+\ell}}}$ that points along the gradient curve $\mathcal{M}(r_{k+l+1}, p_{k+l})$ from $p_{k+\ell}$ to $r_{k+\ell+1}$. Note both $e_{i_{p_{k+1}}}$ and $e_{i_{p_{k+\ell}}}$ are defined to point in the direction of ∇f. See Figure 3 below.

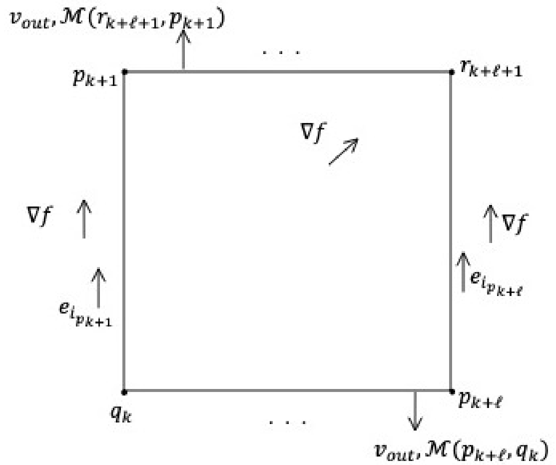

Figure 3: $\mathcal{M}(r_{k+\ell+1}, q_k)$ with orientations.

Our choice for the orientation of the relevant unstable submanifolds are

$$[U_{q_k}] = e_k \dots e_1 \,, \qquad [U_{p_{k+\ell}}] = e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_k \dots e_1 \,,$$
$$[U_{p_{k+1}}] = e_{i_{p_{k+1}}} e_k \dots e_1 \,, \qquad [U_{r_{k+\ell+1}}] = e_{k+\ell+1} \dots e_k \dots e_1 \,.$$

Then by (A.3), $[\mathcal{M}(r,q)] = [U_r][U_q]^{-1}$, we find the orientations of the moduli spaces:

$$\begin{aligned} [\mathcal{M}(r_{k+\ell+1}, q_k)] &= (e_{k+\ell+1} \dots e_k \dots e_1)(e_1 \dots e_k) = e_{k+\ell+1} \dots e_{k+1} \,, \\ [\mathcal{M}(p_{k+\ell}, q_k)] &= (e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_k \dots e_1)(e_1 \dots e_k) = e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_{k+1} \,, \\ [\mathcal{M}(r_{k+\ell+1}, p_{k+1})] &= (e_{k+\ell+1} \dots e_k \dots e_1)(e_1 \dots e_k e_{i_{p_{k+1}}}) \\ &= (-1)^{i_{p_{k+1}}-k-1} \, e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+1}}}} \dots e_{k+1} \,. \end{aligned} \tag{A.7}$$

And by (A.4), we also have

$$\begin{aligned} [\widetilde{\mathcal{M}}(r_{k+\ell+1}, p_{k+\ell})] &= [\mathcal{M}(r_{k+\ell+1}, p_{k+\ell})][\nabla f]^{-1} \\ &= (e_{k+\ell+1} \dots e_k \dots e_1)(e_1 \dots e_k \dots \widehat{e_{i_{p_{k+\ell}}}} \dots e_{k+\ell+1})(e_{i_{p_{k+\ell}}}) \\ &= (-1)^{k+\ell+1-i_{p_{k+\ell}}} \,, \\ [\widetilde{\mathcal{M}}(p_{k+1}, q_k)] &= [\mathcal{M}(p_{k+1}, q_k)][\nabla f]^{-1} \\ &= (e_{i_{p_{k+1}}} e_k \dots e_1)(e_1 \dots e_k)(e_{i_{p_{k+1}}}) = 1 \,. \end{aligned}$$

Hence, we find

$$[\mathcal{M}(p_{k+\ell},q_k)\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})]=(-1)^{k+\ell+1-i_{p_{k+\ell}}}\,e_{k+\ell+1}\ldots\widehat{e_{i_{p_{k+\ell}}}}\ldots e_{k+1}\,, \tag{A.8}$$

$$[\mathcal{M}(r_{k+\ell+1},p_{k+1})\times\widetilde{\mathcal{M}}(p_{k+1},q_k)]=(-1)^{i_{p_{k+1}}-k-1}\,e_{k+\ell+1}\ldots\widehat{e_{i_{p_{k+1}}}}\ldots e_{k+1}\,. \tag{A.9}$$

Step 2: Orientation of the boundary components, $\mathcal{M}(p_{k+\ell},q_k)\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})$ *and* $\mathcal{M}(r_{k+\ell+1},p_{k+1})\times\widetilde{\mathcal{M}}(p_{k+1},q_k)$*, as specified by Stokes' theorem.*

For a manifold N with boundary ∂N, Stokes' theorem holds only if the orientation of the boundary ∂N is chosen such that

$$[v_{out}][\partial N]=[N] \tag{A.10}$$

where v_{out} is the outward pointing normal on the boundary.

For the boundary component $\mathcal{M}(p_{k+\ell},q_k)\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})$, the outward pointing normal at for instance $p_{k+\ell}$ can be expressed as (see Figure 3)

$$v_{out,\mathcal{M}(p_{k+\ell},q_k)}=-e_{i_{p_{k+\ell}}}+\sum_{k+j\neq i_{p_{k+\ell}}}a_je_{k+j}\,.$$

Therefore, the specified orientation from Stokes' theorem (denoted with a subscript 'S') is

$$\begin{aligned}[\mathcal{M}(p_{k+\ell},q_k)\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})]_S&=[v_{out,\mathcal{M}(p_{k+\ell},q_k)}]^{-1}[\mathcal{M}(r_{k+\ell+1},q_k)]\\&=(-e_{i_{p_{k+\ell}}})(e_{k+\ell+1}\ldots e_{k+1})\\&=(-1)^{k+\ell+i_{p_{k+\ell}}}\,e_{k+\ell+1}\ldots\widehat{e_{i_{p_{k+\ell}}}}\ldots e_{k+1}\\&=-[\mathcal{M}(p_{k+\ell},q_k)\times\widetilde{\mathcal{M}}(r_{k+\ell+1},p_{k+\ell})]\end{aligned} \tag{A.11}$$

having used (A.7) in the first line and (A.8) in the last line.

Similarly, for the boundary component $\mathcal{M}(r_{k+\ell+1},p_{k+1})\times\widetilde{\mathcal{M}}(p_{k+1},q_k)$, the outward pointing normal at for instance p_{k+1} can be expressed as (see Figure 3)

$$v_{out,\mathcal{M}(r_{k+\ell+1},p_{k+1})}=e_{i_{p_{k+1}}}+\sum_{k+j\neq i_{p_{k+1}}}a_je_{k+j}\,.$$

This gives for the specified orientation from Stokes' theorem

$$\begin{aligned}
[\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)]_S &= [v_{out,\mathcal{M}(r_{k+\ell+1},p_{k+1})}]^{-1}[\mathcal{M}(r_{k+\ell+1}, q_k)] \\
&= (e_{i_{p_{k+1}}})(e_{k+\ell+1} \dots e_{k+1}) \\
&= (-1)^{k+\ell+1-i_{p_{k+1}}}\, e_{k+\ell+1} \dots \widehat{e_{i_{p_{k+1}}}} \dots e_{k+1} \\
&= (-1)^{\ell}[\mathcal{M}(r_{k+\ell+1}, p_{k+1}) \times \widetilde{\mathcal{M}}(p_{k+1}, q_k)] \qquad \text{(A.12)}
\end{aligned}$$

having used (A.7) in the first line and (A.9) in the last line.

Finally, with (A.11)-(A.12) and matching up with the corresponding terms in (A.6), we have

$$\begin{aligned}
c(d\psi)q_k = \sum_{r_{k+\ell+1}} \Bigg[&- \sum_{p_{k+\ell}} \left(\int_{\overline{\mathcal{M}(p_{k+\ell},q_k)}} \psi \right) n(r_{k+\ell+1}, p_{k+\ell}) \\
&+ \sum_{p_{k+1}} (-1)^{\ell} n(p_{k+1}, q_k) \left(\int_{\overline{\mathcal{M}(r_{k+\ell+1},p_{k+1})}} \psi \right) \Bigg] r_{k+\ell+1} \\
= -\partial c(\psi) q_k + (-1)^{\ell} c(\psi)\partial q_k &
\end{aligned}$$

or equivalently, $-c(d\psi) = \partial c(\psi) + (-1)^{\ell+1} c(\psi)\partial$.

B Cochain complexes of a chain map and their relations

We here review some relations between cochain complexes that arise from a chain map. For a reference, see [Wei94].

Let $\varphi : (B, d_B) \to (A, d_A)$ be a degree ℓ chain map between two cochain complexes,

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d_A} & A^{n-1} & \xrightarrow{d_A} & A^n & \xrightarrow{d_A} & A^{n+1} & \xrightarrow{d_A} & \dots \\
 & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} & & \\
\dots & \xrightarrow{d_B} & B^{n-\ell-1} & \xrightarrow{d_B} & B^{n-\ell} & \xrightarrow{d_B} & B^{n-\ell+1} & \xrightarrow{d_B} & \dots
\end{array} \qquad \text{(B.1)}$$

that is, the map $\varphi : B^n \to A^{n+\ell}$ satisfy the chain map condition

$$\varphi\, d_B = d_A\, \varphi\,. \qquad \text{(B.2)}$$

Associated to such a map are the following cochain complexes.

(1) The *kernel* complex $(\ker\varphi, d_B)$. This is a subcomplex of (B, d_B) where $\ker^n\varphi = \ker\varphi \,\cap\, B^n$. To see that $d_B : \ker^n\varphi \to \ker^{n+1}\varphi$, consider an element $b_n \in \ker^n\varphi$, i.e. $\varphi\, b_n = 0$. By (B.2), we have $\varphi(d_B b_n) = d_A(\varphi\, b_n) = 0$; hence, $d_B b_n \in \ker^{n+1}\varphi$.

(2) The *image* complex $(\operatorname{im}\varphi, d_A)$. This is a subcomplex of (A, d_A) where $\operatorname{im}^n\varphi = \operatorname{im}\varphi \,\cap\, A^n$. Specifically, if $a_n \in \operatorname{im}^n\varphi$, then there exists an $b_{n-\ell} \in B^{n-\ell}$ such that $a_n = \varphi\, b_{n-\ell}$. Again, it follows directly from (B.2) that $d_A : \operatorname{im}^n\varphi \to \operatorname{im}^{n+1}\varphi$.

(3) The *cokernel* complex $(\operatorname{coker}\varphi, d_A^\pi)$. This is also a subcomplex of (A, d_A) where $\operatorname{coker}^n\varphi = A^n/\operatorname{im}\varphi$ and $d_A^\pi = \pi\, d_A$ is the composition of d_A with the quotient map $\pi : A^n \to \operatorname{coker}^n\varphi$. To denote elements of the cokernel complex, we shall use a bracket, i.e. $[a_n] := \{a_n + \varphi\, b_{n-\ell} \mid b_{n-\ell} \in B^{n-\ell}\} \in \operatorname{coker}^n\varphi$. Note that $d_A^\pi[a_n] = [d_A a_n]$, and therefore, $d_A^\pi\, d_A^\pi = 0$.

(4) The (mapping) *cone* complex $(\operatorname{Cone}(\varphi), d_C)$, the main focus of this paper, involves both (B, d_B) and (A, d_A). Here,

$$\operatorname{Cone}^n(\varphi) = A^n \oplus B^{n-\ell+1}\,, \qquad d_C = \begin{pmatrix} d_A & \varphi \\ 0 & -d_B \end{pmatrix}.$$

with $d_C : \operatorname{Cone}^n(\varphi) \to \operatorname{Cone}^{n+1}(\varphi)$. Note that the chain map relation (B.2) ensures that $d_C\, d_C = 0$.

Each of the above cochain complexes results in a cohomology, denoted by $H^n(\ker\varphi)$, $H^n(\operatorname{im}\varphi)$, $H^n(\operatorname{coker}\varphi)$, and $H^n(\operatorname{Cone}(\varphi))$, respectively. We are interested in the relations amongst these cohomologies and also their relations with $H^n(A)$ and $H^n(B)$. A first basic relation used throughout the paper follows from the following short exact sequence of cochain complexes

$$0 \longrightarrow (A^n, d_A) \xrightarrow{\iota_A} (\operatorname{Cone}^n(\varphi), d_C) \xrightarrow{\pi_B} (B^{n-\ell+1}, d_B) \longrightarrow 0$$

where the chain map ι_A is the inclusion into the first component of $\operatorname{Cone}(\varphi)$ and π_B is the projection of the second component. The short exact sequence gives the long exact sequence

$$\cdots\ H^{n-\ell}(B) \xrightarrow{\varphi} H^n(A) \xrightarrow{\iota_A} H^n(\mathrm{Cone}(\varphi)) \xrightarrow{\pi_B} H^{n-\ell+1}(B) \xrightarrow{\varphi} H^{n+1}(A)\ \cdots \tag{B.3}$$

which implies the following:

Lemma B.1. *Given a degree ℓ chain map $\varphi : (B, d_B) \to (A, d_A)$ between two cochain complexes, the resulting cone cohomology has the following relation:*

$$H^n(\mathrm{Cone}(\varphi)) \cong \mathrm{coker}(\,\varphi\colon H^{n-\ell}(B) \to H^n(A)\,) \,\oplus\, \ker(\,\varphi\colon H^{n-\ell+1}(B) \to H^{n+1}(A)\,)\,.$$

To relate the other cohomologies, it is useful to introduce another cone complex defined by the inclusion map $\iota : \mathrm{im}^n\varphi \to A^n$, which is a degree $\ell = 0$ map. We shall denote this cone complex with a tilde:

$$\widetilde{\mathrm{Cone}}^n(\iota) = A^n \oplus \mathrm{im}^{n+1}\varphi\,, \qquad d_{\widetilde{C}} = \begin{pmatrix} d_A & \iota \\ 0 & -d_A \end{pmatrix}.$$

Of note, the cohomology of this complex, $H^n(\widetilde{\mathrm{Cone}}(\iota))$ is isomorphic to $H^n(\mathrm{coker}\,\varphi)$.

Lemma B.2. *The map $\pi_1 : \widetilde{\mathrm{Cone}}^n(\iota) \to \mathrm{coker}^n\varphi$ given by $\pi_1 \begin{pmatrix} a \\ \tilde{a} \end{pmatrix} = \pi\, a$, where $\pi : A^n \to \mathrm{coker}^n\varphi$, induces an isomorphism on cohomology: $H^n(\widetilde{\mathrm{Cone}}(\iota)) \cong H^n(\mathrm{coker}\,\varphi)$.*

Proof. That the π_1 map is a chain map follows straightforwardly from the definition. To prove the isomorphism, we will show that $\pi_1 : H^n(\widetilde{\mathrm{Cone}}(\iota)) \to H^n(\mathrm{coker}\,\varphi)$ is bijective.

Let $[a] \in \mathrm{coker}^n\varphi$. To show surjectivity, assume $[a] \in H^n(\mathrm{coker}\,\varphi)$, that is, $[a]$ is closed under $d^\pi_A = \pi\, d_A$, or equivalently, that the representative $a \in A^n$ satisfies

$$d_A a + \varphi\, b = 0 \tag{B.4}$$

for some $b \in B^{n-\ell+1}$. Now let $\tilde{a} = \varphi\, b$. Then since $d_A d_A = 0$, (B.4) implies $d_A\tilde{a} = 0$. Therefore, the pair $\begin{pmatrix} a \\ \tilde{a} \end{pmatrix}$ is $d_{\widetilde{C}}$-closed, i.e. it is an element of $H^n(\widetilde{\mathrm{Cone}}(\iota))$, and moreover, $\pi_1 : \begin{pmatrix} a \\ \tilde{a} \end{pmatrix} \to [a]$ as desired.

To show that π_1 is also injective, let now $[a] = \pi d_A[a']$ representing the trivial class in

$H^n(\operatorname{coker}\varphi)$. This implies that $a = d_A a' + \varphi\, b'$ for some $b' \in B^{n-\ell}$. But this also means,

$$\pi_1 \left\{ d_{\widetilde{C}} \begin{pmatrix} a' \\ \varphi\, b' \end{pmatrix} \right\} = \pi_1 \begin{pmatrix} d_A a' + \varphi\, b' \\ -d_A\, \varphi\, b' \end{pmatrix} = \pi(d_A a' + \varphi\, b') = [a]\,.$$

Hence, π_1 maps trivial class to trivial class. $\square$

Now applying Lemma B.1 to $H^n(\widetilde{\mathrm{Cone}}(\varphi))$ and using Lemma B.2, we find the following:

Lemma B.3. *For the cohomology of the cokernel complex, we have*

$$H^n(\operatorname{coker}\varphi) \cong \operatorname{coker}(\,\iota\colon H^n(\operatorname{im}\varphi) \to H^n(A)\,) \,\oplus\, \ker(\,\iota\colon H^{n+1}(\operatorname{im}\varphi) \to H^{n+1}(A)\,)\,,$$

where $\iota : \operatorname{im}^n\varphi \to A^n$ *is the inclusion map.*

Finally, we give a relation that links $H(\mathrm{Cone}(\varphi))$ with $H(\ker\varphi)$ and $H(\operatorname{coker}\varphi)$. At the cochain level, we can write down the following short exact sequence of cochain complexes:

$$0 \longrightarrow (\ker^{n-\ell+1}\varphi, d_B) \xrightarrow{\iota_2} (\mathrm{Cone}^n(\varphi), d_C) \xrightarrow{\varphi_2} (\widetilde{\mathrm{Cone}}^n(\iota), d_{\widetilde{C}}) \longrightarrow 0 \qquad \text{(B.5)}$$

where the maps ι_2 and φ_2 are defined by

$$\begin{aligned} \iota_2 : \ker^{n-\ell+1}\varphi &\longrightarrow \mathrm{Cone}^n(\varphi). & \qquad \varphi_2 : \mathrm{Cone}^n(\varphi) &\longrightarrow \widetilde{\mathrm{Cone}}^n(\iota) \\ b &\longmapsto \begin{pmatrix} 0 \\ b \end{pmatrix} & \begin{pmatrix} a \\ b \end{pmatrix} &\longmapsto \begin{pmatrix} a \\ \varphi\, b \end{pmatrix} \end{aligned}$$

The short exact sequence (B.5) implies the following long exact sequence of cohomology:

$$\ldots \xrightarrow{\delta} H^{n-\ell+1}(\ker\varphi) \xrightarrow{\iota_2} H^n(\mathrm{Cone}(\varphi)) \xrightarrow{\varphi_2} H^n(\widetilde{\mathrm{Cone}}(\iota)) \xrightarrow{\delta} H^{n-\ell+2}(\ker\varphi) \xrightarrow{\iota_2} \ldots$$

where the connecting homomorphism δ can be obtained by standard diagram chasing. Now using Lemma B.2 to replace $H^n(\widetilde{\mathrm{Cone}}(\iota))$ by $H^n(\operatorname{coker}\varphi)$, we have derived the below long exact sequence.

Lemma B.4. *Let* $\varphi : (B, d_B) \to (A, d_A)$ *be a degree* ℓ *chain map between cochain com-*

plexes. Then there exists a connecting homomorphism δ' such that

$$\ldots \xrightarrow{\delta'} H^{n-\ell+1}(\ker\varphi) \xrightarrow{\iota_2} H^n(\mathrm{Cone}(\varphi)) \xrightarrow{\pi_1\circ\varphi_2} H^n(\mathrm{coker}\,\varphi) \xrightarrow{\delta'} H^{n-\ell+2}(\ker\varphi) \xrightarrow{\iota_2} \ldots \tag{B.6}$$

is a long exact sequence.

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