Mitigating photon loss in linear optical quantum circuits: classical postprocessing methods outperforming postselection

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Abstract

Photon loss rates set an effective upper limit on the size of computations that can be run on current linear optical quantum devices. We present a family of techniques to mitigate the effects of photon loss on both output probabilities and expectation values derived from noisy linear optical circuits composed of an input of n photons, an m–mode interferometer, and m single photon detectors. Central to these techniques is the construction of objects called *recycled probabilities*. Recycled probabilities are constructed from output statistics affected by loss, and are designed to amplify the signal of the ideal (lossless) probabilities. Classical postprocessing techniques then take recycled probabilities as input and output a set of loss-mitigated probabilities, or expectation values. We provide analytical and numerical evidence that these methods can be applied, up to large sample sizes, to produce more accurate outputs than those obtained from postselection - which is currently the standard method of coping with photon loss when sampling from discrete variable linear optical quantum circuits. In contrast, we provide strong evidence that the popular zero noise extrapolation technique cannot improve on on the performance of postselection for any photon loss rate.

1 Introduction

Discrete variable linear optical quantum computing (DVLOQC) is a framework that uses a discrete number of photons as well as linear optical hardware to store and process quantum information. Many models of universal and fault-tolerant quantum computation tailored to this framework have been developed over the years beginning with the work of [1], and followed by various other models and variants (e.g [2, 3, 4, 5]). Furthermore, promising proposals for the near-term demonstration of quantum-over-classical advantage such as boson sampling [6] can naturally be implemented in this framework. A quantum device capable of performing DVLOQC usually consists of three components: single photon sources [7], multimode interferometers [8], and single photon detectors [9]. We will refer to the collection of these as a *linear optical circuit*. One major impediment to scaling up such a device is photon loss [10]. Postselection, where all *lossy* output statistics with one or more lost photons are discarded and only those where all photons are detected are kept, may be used to obtain the ideal output distribution of a linear optical circuit subject to photon loss [11]. This can be viewed as a form of quantum error mitigation, where the ideal distribution is accessible, but with a sampling cost scaling exponentially with the depth of the circuit. This scaling comes from the fact that the probability of at least one photon being lost approaches one exponentially quickly with increasing circuit depth [12], which results in most outputs being discarded.

In this work we address the question of whether the effects of photon loss in the output distributions of linear optical quantum circuits can be mitigated by classical postprocessing of lossy output statistics. We provide a positive answer to this question by presenting various new techniques to mitigate photon loss in linear optical circuits, and performing a rigorous analysis of their performance. We refer to these techniques collectively as *recycling mitigation*, as they all involve the use of lossy output statistics that otherwise would be discarded. At the heart of recycling mitigation is the construction of the so-called *recycled probabilities*. The recycled probabilities can be thought of as precursors of the mitigated outputs. Mitigated probabilities are

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approximations of the ideal probabilities, and can be obtained from the recycled probabilities by appropriate classical postprocessing. We introduce several techniques by which this classical postprocessing step can be performed. This in turn gives rise to different photon loss mitigation techniques. Fig. 1 shows the main steps underlying our mitigation techniques.

We provide strong analytical and numerical evidence that in the high loss regime where the probability, denoted as η, that a photon is lost in any given mode satisfies $\eta > 0.5$, recycling mitigation outperforms postselection. The intuition being that in this regime, output events where photons are lost are much more likely to occur than events where no photon is lost. Since recycling mitigation uses lossy output statistics, it is natural to expect that the resulting mitigated probabilities are more converged than probabilities computed only from the postselected samples. Furthermore, we provide analytic and numerical evidence showing that mitigation techniques based on artificially increasing noise and Richardson extrapolation, called zero noise extrapolation (ZNE) [13, 14], such as those applied to mitigate photon loss in the continuous variable regime [15], offer no advantage over postselection for the problem of mitigating photon loss in the DVLOQC setting.

Recycling mitigation can be applied to linear optical circuits implementing arbitrary unitaries. In particular, when these unitaries are randomly chosen according to the Haar measure, this gives rise to the non-universal model of computation known as boson sampling [6]. There are a number of classical algorithms for boson sampling that become efficient when photon loss is sufficiently high [16, 12]. In this spirit, we show that recycled probabilities constructed from lossy output statistics where k out of n photons are lost, and $n - k$ is a constant independent of n, can be efficiently computed classically. These correspond to lossy output statistics where the majority of photons are lost. Conversely, and using results from [17], we provide evidence that recycled probabilities constructed from output statistics where k is a constant independent of n are hard to compute classically in worst-case. This corresponds to the case of output statistics where few photons are lost. These results indicate that recycling mitigation provides interesting performances when looking at output statistics with a small number of lost photons, although there is a set of *intermediate* lossy output statistics, where k does not follow either of the above conditions, that we do not explore, and that might also give interesting performances.

Crucially, our developed methods are applicable to current and near-term photonic quantum hardware, and, as will be seen later, provide better performances in the regime of high loss ($\eta > 0.5$). This is in contrast with fault-tolerant photonic quantum computing [3, 4] which typically requires large numbers of photons, as well as very low loss levels ($\eta \ll 0.5$), both of which are beyond the reach of currently available hardware. Furthermore, as discussed in the previous paragraph, the output statistics used in constructing the mitigated probabilities should be those where the number of lost photons k is low. Thus, these conditions define the realm of applicability of recycling mitigation that provides the best performance: *Recycling mitigation should be applied in the $\eta > 0.5$ regime, and the output statistics used in constructing the mitigated probabilities should be those where k is low.*

Recycling mitigation can be used in a variety of applications, namely any DVLOQC algorithm where the desired output is an expectation value of an observable, a set output of probabilities, or a full probability distribution. Applications include boson sampling [6], photonic variational quantum eigensolvers [18, 19], photonic differential equation solving [20], photonic quantum machine learning [21], and graph problems with DVLOQC [22]. In a related article [23], we develop a quantum circuit born machine (QCBM) [24] tailored to DVLOQC. We also demonstrate significant photon loss mitigation, using recycling mitigation, for the training of the QCBM in various scenarios.

This paper is structured as follows. Section 2 gives a high-level overview of our main contributions. Section 3 compares our techniques with those existing in the literature. Section 4 introduces some notation and basic concepts. Sections 5-8 detail the construction of the recycled distributions, the classical postprocessing techniques needed to obtain the mitigated distribution, as well as derive analytical conditions for recycling mitigation techniques to outperform postselection. Sections 9 and 10 contain numerical simulations that aid in understanding how to use our developed mitigation techniques in practice, as well as examples of our techniques in action. Section 11 provides strong evidence that techniques based on ZNE do not in general outperform postselection. Section 12 discusses how our techniques can be used to mitigate both expectation values of observables as well as full distributions. Section 13 discusses ways to improve our performance guarantees, and contains a technical result about sums of permanents of i.i.d. Gaussian matrices [6] that might find use beyond this work. Finally, section 14 presents a set of interesting open questions.

2 Overview of main results

In this section, we give a high-level overview of our key results, which are expanded upon in later parts of the paper.

Our first contribution is the construction of the recycled probabilities, which are the input of all our developed loss mitigation techniques. Consider an ideal n–photon probability $p_{id}(s)$, corresponding to a detection pattern s of n input photons passed through an m-mode linear optical interferometer, and detected in m output modes. Its corresponding recycled probability $p_R^k(s)$ is an n–photon probability computed from output statistics affected by loss. Namely, $p_R^k(s)$ is constructed as a sum of of $n - k$–photon probabilities where k out of the initial n photons are lost. The precise construction and choice of the $n - k$–photon probabilities for each s are detailed in section 5. Using the assumption of uniform loss (η is the same in all modes) to express the $n - k$–photon probabilities as a convex sum of n photon probabilities, we show that for all s the recycled probabilities may be written in the form

$$p_R^k(s) = \frac{1}{\binom{m-n+k}{k}} p_{id}(s) + \left(1 - \frac{1}{\binom{m-n+k}{k}}\right) I_{s,k},$$

where $I_{s,k}$ is a convex combination of unwanted n–photon probabilities.

In section 6, we show that recycled probabilities constructed from $n - k$–photon probabilities where $n - k$ is a constant independent of n are efficiently computable classically. This motivates using $n - k$ output statistics where the value of k is low relative to n to construct recycled probabilities for use in recycling mitigation. One intuition for this is that higher-order photon statistics contain more information on the ideal n–photon distribution than the lower-order statistics. And there are results indicating that the ideal n–photon output distribution is hard to classically simulate, and further that simulating the $n - k$–photon output distribution is hard when k is a constant [25, 17, 6].

The key idea behind all the techniques we develop for recycling mitigation is simple: try to extract $p_{id}(s)$ as accurately as possible from $p_R^k(s)$ via classical postprocessing. Since $p_R^k(s)$ can be obtained from the $n - k$ output photon statistics, if we had access to the value of $I_{s,k}$ we could directly compute $p_{id}(s)$ using the previous equation. Unfortunately it is not straightforward to compute the value of $I_{s,k}$, so instead we resort to deriving statistical inequalities bounding the variation of $I_{s,k}$ around its expected value. We derive these inequalities in the no-collision regime [6]. The details of this are found in section 7. Indeed, our second set of contributions begins with showing that the expected value $\mathbf{E}_s(I_{s,k})$ of $I_{s,k}$ over all s is

$$\mathbf{E}_s(I_{s,k}) = \frac{1}{\binom{m}{n}}.$$

This holds for arbitrary m-mode linear optical interferometers. Then, we use statistical inequalities [26, 27] to show that, for arbitrary m-mode interferometers, and all s,

$$\epsilon_{bias,s} := |I_{s,k} - \mathbf{E}_s(I_{s,k})| \leq \frac{1}{\mathsf{poly}(m)}.$$

The above equation holds, for a given s, with probability $1 - O\left(\frac{\mathsf{poly}(m)}{\binom{m}{n}}\right)$. In addition to this probabilistic bound, for a broad family of m-mode interferometers having a specific structure, and using a result of [28], we manage to deterministically bound $\epsilon_{bias,s}$ as

$$\epsilon_{bias,s} \leq O(e^{-0.000002n}).$$

Bounding the bias error term $\epsilon_{bias,s}$ is essential for the later performance analyses of the techniques.

Our third set of contributions regards the development of classical postprocessing techniques for extracting estimates of $\{p_{id}(s)\}$ from $\{p_R^k(s)\}$, which we call the mitigated values $\{p_{mit}(s)\}$. These classical postprocessing techniques can be used to mitigate a full probability distribution of n photons, a subset of the full distribution, or an expectation value of an observable O, $\langle \mathsf{O} \rangle := \sum_i w_i p_{id}(s_i)$, with $w_i \in \mathbb{R}$, which is a weighted sum over a set of n–photon probabilities. The main cost incurred in these classical postprocessing techniques is a memory cost proportional to the size of the set of probabilities to be mitigated. In the worst-case, when we are mitigating a full probability distribution, this memory cost is $O\left(\binom{m}{n}\right)$. In practice, however, when mitigating an expectation value one usually needs access to a $\mathsf{poly}(n)$ (often of low degree) sized set of mitigated probabilities, making the memory cost manageable, and consequently our methods practical to

apply to near-term DVLOQC hardware. Details about these classical postprocessing techniques can be found in section 8, and pseudocode for their implementation is given in algorithms 2, 3, 4, and 5. Note that since recycling mitigation is applied to the statistics relating to relatively few lost photons, the number of samples (runs of the lossy DVLOQC circuit) required for constructing the recycled probabilities, and computing the mitigated values to a fixed accuracy, scales exponentially with system size. On the other hand, this is a limitation common to all error mitigation protocols involving solely classical postprocessing [29, 30, 31], and not just specific to our technique.

In section 8 we provide performance guarantees for the classical postprocessing techniques. More precisely, for several of the postprocessing methods we bound the deviation of $p_{mit}(s)$ from $p_{id}(s)$ in terms of the bias error $\epsilon_{bias,s}$, as well as the statistical error $\epsilon_{stat,k}$ incurred when constructing the recycled probabilities from a finite number of samples. We observe that, in the $\eta > 0.5$ regime, and up to a certain number of samples N_{max}, the combined worst-case bias and statistical errors of recycling mitigation are lower than the statistical error of postselection. This is strong evidence that our error mitigation methods outperform postselection in this regime. Furthermore, we observe through various numerical simulations in sections 9-10, that N_{max} is large, meaning our methods are expected to outperform postselection for up to large sample sizes.

Our final contribution is in providing strong evidence that photon loss error mitigation techniques based on ZNE [32] cannot outperform postselection. Indeed, in section 11 we provide an upper bound on the error incurred from ZNE, E_{extrap}, which is larger than the worst-case statistical error of postselection (see Thm. 13). In appendix 21, we also provide numerical evidence that for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$, E_{extrap} is always larger than the statistical error of postselection. Our result highlights the subtleties involved in adapting existing error mitigation techniques to mitigate photon loss errors in DVLOQC. Indeed, in DVLOQC there is a natural way to mitigate photon loss errors, postselection, which is not present for other types of errors in other types of hardware [32]. This therefore sets a benchmark that a photon loss mitigation technique in DVLOQC must satisfy to be useful, namely that it must outperform postselection. Our results indicate that recycling mitigation succeeds in mitigating beyond what postselection can offer whereas ZNE does not.

3 Comparison to recent work

Many techniques have been developed to improve the performance of quantum computations run on currently available quantum hardware. These are generally referred to as *quantum error mitigation* (QEM) techniques. Some well-known QEM techniques are zero noise extrapolation (ZNE) [13, 33, 34, 35], probabilistic error cancellation [13, 36, 37, 38], verification-based mitigation [39, 40, 41, 42], virtual distillation and exponential error suppression [43, 44, 45, 46], quantum subspace expansion mitigation [47, 48, 49, 50], and measurement error mitigation [51, 52, 53, 54] (see [32] for a review of QEM techniques). These techniques are tailored to the circuit model of quantum computing, and thus adapting them to DVLOQC can in some cases be complicated by the considerable differences in the computational setup and the types of noise.

Techniques based on ZNE have been applied to mitigate photon loss in Gaussian boson sampling experiments [15]. Interestingly, we find that ZNE-based techniques do not seem to provide an advantage over postselection when adapted to the DVLOQC setting. An important distinction between ZNE-based methods and recycling mitigation, which is presented in this work, is that the former have an associated sampling overhead while recycling mitigation does not. In ZNE, many data points are collected, each corresponding to the expectation value of an observable O computed from a noisy circuit at a given noise level. These points are used to compute, via extrapolation, the noiseless expectation value of O. Computing each of these data points to a desired precision has a sample cost, while recycling mitigation uses information from the lossy output statistics that would otherwise be discarded. This means that recycling mitigation may be applied with no additional sampling overhead relative to postselection.

Finally, recent work [55] has provided a method for mitigating photon loss in the continuous variable (CV) setting, adapting pre-existing methods of probabilistic error cancellation [13, 36, 37, 38] to the CV setting. Although the techniques of [55] are applicable to the DVLOQC setting, these are solely for expectation value mitigation (weak mitigation) [56]. Our methods, by contrast, can perform both strong (full probability distribution mitigation) as well as weak mitigation, with a classical memory cost scaling with the size of the set of probabilities to be mitigated. Furthermore, we present analytical as well as extensive numerical evidence that our methods provably outperform postselection, whereas, to our knowledge, no comparison between the performance of the developed mitigation methods and postselection is presented in [55].
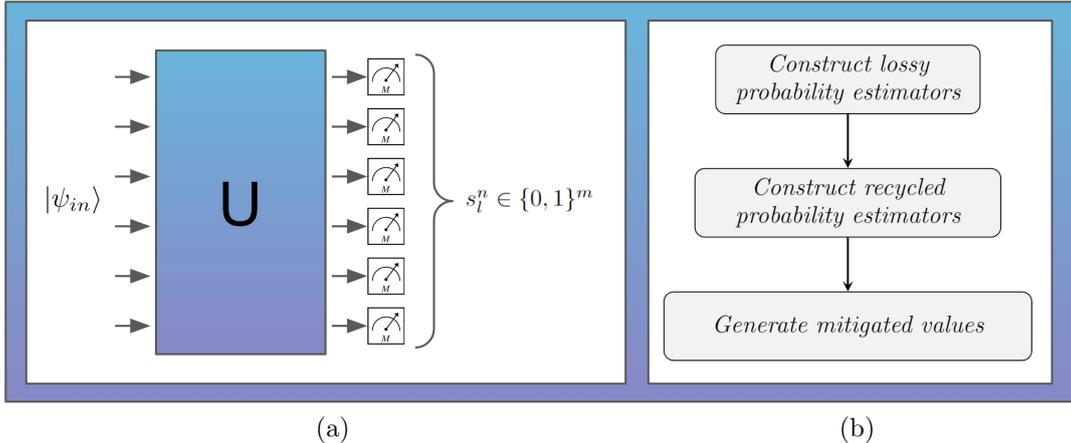
Figure 1: A schematic illustrating the main steps of the recycling mitigation protocol. (a) An input state $|\psi_{in}\rangle$ of n photons is introduced to a lossy m-mode linear optical interferometer implementing a unitary transformation. The classical measurement outcomes of each of the output modes is denoted by the classical $m-$bit string s_l^n. (b) The classical data set generated by repeatedly sampling from the quantum circuit is then used as input for classical postprocessing. This classical postprocessing consists of three stages. First, the data set is used to generate lossy probability estimators. These estimators are then used to construct recycled probability estimators, which are in turn then used to generate mitigated values.

4   Preliminaries

Consider the DVLOQC setting where a photonic quantum device is composed of a single-photon source [7], a universal m-mode linear optical interferometer [8] capable of implementing any unitary transformation $U \in \mathsf{U}(m)$, with $\mathsf{U}(m)$ the group of unitary $m \times m$ matrices, and single photon detectors [9]. Generating a sample using this device proceeds as follows. First, n single photons emitted from the source pass through the linear optical inteferometer in an input configuration $\mathbf{T} := (t_1, \ldots, t_m)$, where t_i is the number of photons in mode i. Let $|\psi_{in}\rangle := |t_1, \ldots, t_m\rangle$ be the input Fock state of single-photons corresponding to the configuration $\mathbf{T}$. The linear optical interferometer implements a unitary transformation $\phi(U)$ on the input state $|\psi_{in}\rangle$ [6], resulting in an output state $|\psi_{out}\rangle := \phi(U)|\psi_{in}\rangle$, where $\phi(U)$ represents the action of the unitary U, implemented by the interferometer, on $|\psi_{in}\rangle$. Note that $\phi(U)$ and U are related by a homomorphism, detailed in [6]. A sample $\mathbf{S} := (s_1, \ldots, s_m)$, with s_i the number of photons in output mode i, is then obtained by measuring the number of photons in each output mode using single-photon detectors. This corresponds to projecting $|\psi_{out}\rangle$ onto the state $|s_1, \ldots, s_m\rangle$. Many computational tasks on photonic quantum devices can be implemented by collecting samples according to the previous procedure, and then performing classical postprocessing [18, 22, 57, 19, 1].

In the absence of any errors affecting the device, $\sum_{i=1,\ldots,m} t_i = \sum_{i=1,\ldots,m} s_i = n$, furthermore, the probability of obtaining the sample $\mathbf{S}$ is proportional to the modulus squared of the permanent of a submatrix $U_{\mathbf{T},\mathbf{S}}$ of U, whose rows and columns are determined by the input and output occupancies $\mathbf{T}$ and $\mathbf{S}$ [6]. By appropriately choosing the unitary transformation U, and performing the above mentioned sampling procedure repeatedly, one can perform both non-universal and universal quantum computing with linear optics. In particular, if U is chosen to be Haar random, one performs boson sampling, a non-universal sampling task which is hard for classical computers to carry out efficiently [6]. Alternatively, choosing specific unitaries U, and postselecting on detecting a specific output configuration, one can perform universal quantum computation [1].

We will now describe our error model as well as the assumptions we will make throughout this paper.

- We consider photon loss as the only source of error affecting our devices. Our error model is the uniform loss model, where a photon is equally likely to be lost in any mode $i \in \{1, \ldots, m\}$ with probability $\eta \in [0,1]$. Following the commutation rules of photon loss [58], we assume without loss of generality that the photons are lost at the output of the interferometer, just before the single-photon detectors which are assumed to be perfect.

- We assume that sampling occurs in the no-collision regime, where at most one photon occupies any

output mode. This is approximately true for $m \geq O(n^2)$ [6]. In this regime the samples are given by $\mathbf{S} = (s_1, \ldots, s_m)$, with $s_i \in \{0, 1\}$, and are therefore bit strings of length m.

The uniform loss model is a widely used error model for simulating photon loss, and is the standard assumption used when constructing loss-tolerant quantum error correcting codes [59]. It is also often assumed when deriving efficient classical algorithms for simulating lossy linear optical setups [60]. Working in the no-collision regime is primarily interesting for two reasons. Firstly, one can prove statements of quantum advantage in boson sampling in this regime [6]. And secondly, the fact that at most one photon occupies each mode allows us to assume the use of the standard and widely available threshold detectors, rather than number resolving ones which are currently challenging to practically implement. As a final note, while the no-collision assumption is a useful one, it is not necessary for our techniques to work. Indeed, as discussed in later parts of this paper, our techniques can be generalised to the case where more than one photon can occupy a mode.

The uniform loss model induces a binomial distribution on the samples, in the sense that sampling N_{tot} times from a uniformly lossy linear optical circuit produces approximately $N_{tot,k} := \binom{n}{k} N_{tot} \eta^k (1 - \eta)^{n-k}$ samples corresponding to k lost photons, for $k \in \{0, \ldots, n\}$. Note that $N_{tot} = \sum_{k=0,\ldots,n} N_{tot,k}$. We will take s_i^{n-k} to mean a bit string of the form $\{s_1, \ldots, s_m\}$ where $\sum_i s_i = n - k$, $s_i \in \{0, 1\}$ is the number of photons in mode i, and $k \in \{0, \ldots, n\}$. This corresponds to a sample drawn from the probability distribution where k of the initial n input photons have been lost. In order to estimate the probability $p(s_i^{n-k})$ from a set $\mathcal{W} := \{s_j^{n-k}\}$ of samples [1] where $|\mathcal{W}| \leq N_{tot,k}$, we perform the following procedure. For each w ranging from 1 to $|\mathcal{W}|$, assign a value 1 to a random variable $X_w \in \{0, 1\}$ if the sample s_w^{n-k} is the bit string s_i^{n-k}, and assign the value 0 to X_w otherwise. The estimate $\tilde{p}(s_i^{n-k})$ is then

$$\tilde{p}(s_i^{n-k}) := \frac{\sum_w X_w}{|\mathcal{W}|}. \tag{1}$$

This estimation therefore induces a statistical error given by

$$\epsilon_{stat}(s_i^{n-k}) := |\tilde{p}(s_i^{n-k}) - p(s_i^{n-k})|. \tag{2}$$

As the the size of the system is increased, the probability of postselecting non-lossy outcomes decays exponentially towards zero. One motivation for the error mitigation techniques we introduce, and which we will detail in subsequent sections, is that in the high loss regime, where $\eta > 0.5$, lossy outputs with fewer than n photons are considerably more likely than n–photon outputs. The errors $\epsilon_{stat}(s_i^{n-k})$ for $k \neq 0$ are therefore smaller in general than $\epsilon_{stat}(s_i^n)$, by standard arguments from statistics such as Hoeffding and Chebyshev inequalities [26]. Consequently, the estimated probabilities for lossy $n - k$–photon outputs are more converged and have lower statistical error than those of (lossless) n–photon outputs. Furthermore, these lossy probabilities can contain information on the probabilities of the n–photon outputs.

Recycling mitigation uses the $n - k$–photon probability estimates $\{\tilde{p}(s_i^{n-k})\}$, potentially for a range of k values, and construct from these a *mitigated* n–photon probability distribution $\{p_{mit}(s_j^n)\}$. To have any utility, a photon loss mitigation technique needs to outperform computing the n–photon probability estimates $\{\tilde{p}(s_j^n)\}$ from the samples $N_{tot,0}$, which we will henceforth refer to as postselection on n–photon outputs, or just postselection. We will therefore use postselection as the benchmark to evaluate the performance of recycling mitigation. Postselection is, to our knowledge, the only technique being used to mitigate the effects of photon loss on current DVLOQC hardware [18]. Another factor motivating recycling mitigation is that it does not increase the overall sample cost relative to postselection. This contrasts favourably with many error mitigation results that have an accompanying sampling overhead [32].

5 Recycled probabilities

The recycled probabilities are constructed from $n - k$ output photon statistics, where $k \in \{1, \ldots, n-1\}$. To explicitly analyse the signal of the ideal probability within the recycled probability, the recycled probabilities may be decomposed into a combination of an ideal n–photon output probability, and an interference term consisting of a mixture of other n–photon output probabilities from the distribution. We first describe the construction of the recycled probabilities from $n - k$ output photon statistics, which generalises for all k. We then describe the analytical decomposition of the recycled probabilities into n–photon output probabilities.

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[1]The set can contain repeated identical bit strings, i.e. there can be $j_1 \neq j_2$, such that $s_{j_1}^{n-k} = s_{j_2}^{n-k}$.

5.1 Construction of recycled probabilities from lossy outputs

We now detail the construction of recycled probabilities from $n-k$–photon output statistics - that is, the output statistics in which exactly k of n photons have been lost. This construction should be applied to obtain the recycled probability distribution in an experiment. The recycled probability for bit string s_l^n computed from $n-k$–photon output statistics is denoted $p_R^k(s_l^n)$, with $k \in \{1, \dots, n-1\}$. Hence there are $n-1$ recycled probabilities one can construct from lossy output statistics for any n–photon output bit string, one for each possible value of k. Performing the construction involves summing over the $n-k$ output photon bit string probabilities that relate to a particular ideal probability. Informally, this relation is that these are the probabilities of $n-k$–photon output bit strings that the ideal output bit string can be mapped to through the loss of k photons. We now provide a formal statement of this relation.

We first define a mapping procedure from each n–photon output bit string to a set of $n-k$–photon output bit strings. Each mapped set of bit strings represents the set of all possible states that the associated n–photon output state could become after losing k photons. Let $\mathcal{M}_{\text{unocc.},k,i}$ be the subset of $\{1, \dots, m\}$ corresponding to the unoccupied modes of the output bit string s_i^{n-k}. That is, the set of indices of the modes $j \in \{1, \dots, m\}$ of the bit string for which $s_j = 0$. The number of unoccupied modes for $n-k$–photon outputs is $|\mathcal{M}_{\text{unocc.},k,i}| = m - n + k$. Let $\overline{\mathcal{M}_{\text{unocc.},k,i}}$ be the complement of $\mathcal{M}_{\text{unocc.},k,i}$ in $\{1, \dots, m\}$, so that $\overline{\mathcal{M}_{\text{unocc.},k,i}} \cup \mathcal{M}_{\text{unocc.},k,i} = \{1, \dots, m\}$. The subset $\overline{\mathcal{M}_{\text{unocc.},k,i}}$ denotes the occupied modes of s_i^{n-k}, consisting of the set of indices of modes $j \in \{1, \dots, m\}$ for which $s_j = 1$. As the bit string s_i^{n-k} represents an $n-k$–photon output the number of occupied modes is $|\overline{\mathcal{M}_{\text{unocc.},k,i}}| = n - k$. We define the set $\mathcal{L}(s_i^n) := \{s_j^{n-k} | s_i^n \Rightarrow s_j^{n-k}\}$. And the set of all size k subsets of $\overline{\mathcal{M}_{\text{unocc.},0,i}}$ is $\mathcal{S}_{0,i} := \{X \subset \overline{\mathcal{M}_{\text{unocc.},0,i}} | \ |X| = k\}$. The symbol '$\Rightarrow$' denotes the operation where, for every size k subset $\{\overline{l_1}, \dots, \overline{l_k}\} \in \mathcal{S}_{0,i}$, the n–photon output bit string s_i^n is mapped to a new $n-k$–photon output bit string s_j^{n-k} by replacing $s_{\overline{l_i}} = 1$ with $s_{\overline{l_i}} = 0$. In this case, the number of k−subsets of $\overline{\mathcal{M}_{\text{unocc.},0,i}}$ is $\binom{n}{k}$, and so $|\mathcal{S}_{0,i}| = \binom{n}{k}$ and the size of the generated set of bit strings is $|\mathcal{L}(s_i^n)| = \binom{n}{k}$.

The recycled probability for the bit string s_l^n can be defined as the sum of probabilities of the $\binom{n}{k}$ bit strings $s_i^{n-k} \in \mathcal{L}(s_l^n)$,

$$p_R^k(s_l^n) := \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k}). \tag{3}$$

To ensure the normalisation of the recycled distribution the above expression is multiplied by a normalisation factor $\mathbf{N} = \frac{1}{\binom{m-n+k}{k}}$, so that in practice it is

$$p_R^k(s_l^n) = \frac{1}{\binom{m-n+k}{k}} \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k}). \tag{4}$$

A derivation of the normalisation factor is provided in the next section.

To illustrate how this construction might work in practice we now provide a small example. If we would like to compute the recycled probability for the bit string: 111000, in an experiment in which there are $m = 6$ modes, $n = 3$ input photons and the construction is being performed for $k = 1$ lost photons. The recycled probability may be computed directly from eqn. 4 as being

$$p_R^1(111000) = \left(p(110000) + p(101000) + p(011000) \right) \binom{4}{1}^{-1}, \tag{5}$$

where the normalising parameter is $\mathbf{N} = \binom{4}{1}^{-1}$.

In practice, rather than using the set of exact probabilities, $\{p(s_i^{n-k})\}$, to compute recycled probabilities in the manner shown in eqn. 4, instead empirical estimates of the exact probabilities, $\{\tilde{p}(s_i^{n-k})\}$, are used. These are calculated from the set of measured experimental output bit strings, as described for eqn. 1, and so include statistical errors due to finite samples. This statistical error is an important consideration when making comparisons with postselection, and is later included in the analysis of the protocol performance. We now provide pseudocode detailing how to construct the recycled probability for a specific n output photon bit string and value of k from a set of N output bit strings sampled from a given circuit.

Algorithm 1: Construction of a recycled probability estimator from output statistics

input : The n–photon output bit string s_l^n for which the recycled probability estimator is to be constructed, a set of N output sample bit strings from the DVLOQC circuit $\{s_j\}_{j \in \{1,\dots,N\}}$, and the choice of k value indicating that $n-k$ output photon statistics be used for the construction.

1   Initialise variable $\tilde{p}_R^k(s_l^n) \leftarrow 0$ for the recycled probability estimator to be computed.
2   Create a new set of bit strings by discarding all output bit strings from set $\{s_j\}_{j \in \{1,\dots,N\}}$ except those for which the number of measured output photons was $n-k$, with the new list denoted $\{s_l\}_{l \in \{1,\dots,N_{est,k}\}}$ where $N_{est,k} \leq N$.
3   Generate the set of $n-k$–photon lossy output bit strings $\mathcal{L}(s_l^n)$.
4   **for** $l = 1$ **to** $N_{est,k}$ **do**
5   Initialise variable $X_{s_l} \leftarrow 0$.
6   **if** $s_l \in \mathcal{L}(s_l^n)$ **then**
7 | $X_{s_l} \leftarrow X_{s_l} + 1$.
8   **end**
9   **end**
10 Update recycled probability estimator variable as $\tilde{p}_R^k(s_l^n) \leftarrow \dfrac{X_{s_l}}{\binom{m-n+k}{k} N_{est,k}}$.

output: Recycled probability estimator $\tilde{p}_R^k(s_l^n)$

5.2   Decomposition of recycled probabilities into n–photon output bit string probabilities

In later sections, the recycled probabilities are analysed in terms of their decomposition into n–photon output probabilities. For a given output bit string s_l^n, this representation allows explicit treatment of the signal of the ideal n–photon output probability $p(s_l^n)$ within the recycled probability $p_R^k(s_l^n)$. To get the recycled probabilities in this form, the lossy output probabilities within the sum in eqn. 4 are decomposed into n–photon output probabilities from the ideal distribution. The details of this decomposition will now be formalised.

For this purpose we now define another mapping procedure, this time from each lossy $n-k$–photon output bit string from the sum in eqn. 4 to a set of n–photon output bit strings. Where each mapped set of bit strings represents the set of n–photon outputs that the associated lossy output bit string could have been had loss not occurred. The set of all size k subsets of $\mathcal{M}_{\mathrm{unocc.},k,i}$ is defined $\mathcal{S}_{k,i} := \{X \subset \mathcal{M}_{\mathrm{unocc.},k,i} | \, |X| = k\}$. We define the set $\mathcal{G}(s_i^{n-k}) := \{s_j^n | s_i^{n-k} \to s_j^n\}$. The symbol '$\to$' denotes the operation where, for every size k subset $\{l_1, \dots, l_k\} \in \mathcal{S}_{k,i}$, the bit string s_i^{n-k} is mapped to a new bit string by replacing each bit $s_{l_i} = 0$ with $s_{l_i} = 1$. The number of $k-$subsets of $\mathcal{M}_{\mathrm{unocc.},k,i}$ is $\binom{m-n+k}{k}$, and so $|\mathcal{S}_{k,i}| = \binom{m-n+k}{k}$ and the size of the generated set of bit strings is $|\mathcal{G}(s_i^{n-k})| = \binom{m-n+k}{k}$.

From the addition rule of probabilities and the uniformity of the loss, the probability $p(s_i^{n-k})$ of obtaining the output bit string s_i^{n-k} is

$$p(s_i^{n-k}) = \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\binom{n}{k}}, \tag{6}$$

where the sum includes all the bit string outputs $s_j^n \in \mathcal{G}(s_i^{n-k})$ from which the loss of k photons maps to s_i^{n-k}. Each element of the sum is composed of the probability $p(s_j^n)$ that the output is the n–photon bit string s_j^n, multiplied by the uniform probability $\binom{n}{k}^{-1}$ of the loss of k photons from s_j^n resulting in the output s_i^{n-k}.

As in eqn. 6, in the definition of the recycled probabilities in eqn. 4 each of the lossy probabilities can be decomposed into a convex combination of probabilities from the ideal distribution. That is, the $n-k$–photon output probabilities within the sum in eqn. 4 can be replaced with a sum over n–photon output probabilities. For some arbitrary labelling of bit strings s_j^n, and noting that $s_l^n \in \mathcal{G}(s_i^{n-k})$, we can expand the $n-k$ output photon probabilities in the form

$$p(s_i^{n-k}) = p(s_l^n) \frac{1}{\binom{n}{k}} + \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{\binom{n}{k}}. \tag{7}$$

This can then be used to separate the contribution of the ideal bit string probability $p(s_l^n)$ out from the rest of the probabilities which are grouped into a sum we call an *interference term* in the recycled probability. So it becomes

$$p_R^k(s_l^n) = p(s_l^n) + \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{\binom{n}{k}}. \tag{8}$$

The last step is to normalise the distribution $p_R^k(s_l^n)$, namely to compute $\mathbf{N}$ such that $\mathbf{N} \cdot \sum_l p_R^k(s_l^n) = 1$. To do this, note that

$$\begin{aligned}
\sum_{l=1}^{\binom{m}{n}} p_R^k(s_l^n) &= \sum_{l=1}^{\binom{m}{n}} \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k}) \\
&= \sum_{l=1}^{\binom{m}{n}} \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\binom{n}{k}} \\
&= \binom{m-n+k}{k},
\end{aligned} \tag{9}$$

This is because $\sum_l p(s_l^n) = 1$, and, because $|\mathcal{L}(s_i^n)| = \binom{n}{k}$ and $|\mathcal{G}(s_i^{n-k})| = \binom{m-n+k}{k}$, each distinct bit string s_l^n appears exactly $\binom{n}{k}\binom{m-n+k}{k}$ times in the above sum [2]. Therefore $\mathbf{N} = \frac{1}{\binom{m-n+k}{k}}$, and the expression for the normalised recycled probability is

$$p_R^k(s_l^n) = p(s_l^n) \frac{1}{\binom{m-n+k}{k}} + \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{\binom{m-n+k}{k}\binom{n}{k}}. \tag{10}$$

Let $N_k := \binom{m-n+k}{k}\binom{n}{k}$, $N_k' := \left(\binom{m-n+k}{k} - 1\right)\binom{n}{k}$. The recycled probability $p_R^k(s_l^n)$ is composed of the ideal output probability $p(s_l^n)$ and an interference term, defined as

$$I_{s_l^n, k} := \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{N_k'}. \tag{11}$$

The recycled probability may then be written explicitly in this form

$$p_R^k(s_l^n) = p(s_l^n) \frac{1}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k} I_{s_l^n, k}. \tag{12}$$

Importantly, the photon recycled probability $p_R^k(s_l^n)$ contains an amplified signal of the ideal probability $p(s_l^n)$, relative to the other probabilities contained within the interference term.

6 Properties of the recycled distribution

As mentioned in the introduction section, recycling mitigation performs best in the regime of high loss $\eta > 0.5$, and when using $n-k$ output photon statistics where k is low to construct the recycled probabilities. In this section, we show that recycled probabilities constructed from output statistics where most photons were lost (i.e. when $n-k$ is a constant independent of n) are efficiently classically computable, and therefore not useful for obtaining interesting mitigation performance. Furthermore, we provide evidence that recycled probabilities constructed from $n-k$ output statistics where k is a constant independent of n (corresponding to output statistics where a small number of photons have been lost) are hard to compute classically, making these interesting to use for recycling mitigation protocols.

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[2]Another way to see this is to consider the sum over recycled probabilities as $\sum_l p_R(s_l^n) = \sum_l \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k}) = \binom{m-n+k}{k}$. Where the rightmost equality follows from the fact that each $p(s_i^{n-k})$ appears exactly $\binom{m-n+k}{k}$ times in this sum, since there are $|\mathcal{G}(s_i^{n-k})|$ recycled probabilities sharing a single $p(s_i^{n-k})$. Furthermore note that $\sum_i p(s_i^{n-k}) = 1$.

Recall that, up to normalisation, a recycled probability is a sum of the form

$$\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k}),$$

where the number of terms of this sum is $\binom{n}{k}$. For Haar-random interferometers U, in the no-collision regime, we can use the results of [17], and in turn express each $p(s_i^{n-k})$ as

$$p(s_i^{n-k}) = \frac{1}{\binom{n}{k}} \sum_i \frac{|\mathsf{Per}(X_i)|^2}{m^n},$$

where the X_i's are $n - k \times n - k$ matrices with independently distributed Gaussian entries [6], and the number of terms in this sum is also $\binom{n}{k}$. We show that in the high loss regime, where $k = n - r$, and r is a constant independent of n, the sum $\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k})$ is computable efficiently classically. Our result is encompassed in the following lemma proven in appendix 15.

Lemma 1. *Let $k = n - r$, there is a classical algorithm running in time $O(2^{r-1} r (\binom{n}{n-r})^2)$ which exactly computes $\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k})$.*

Notice that when r is a constant independent of n, the runtime of the above algorithm is $O(\binom{n}{n-r}) \leq O(n^r) \leq \mathsf{poly}(n)$. Thus, recycled probabilities corresponding to a high number of lost photons are efficiently computable classically. For low values of loss, in particular when r scales with n, the above efficient classical simulability results break down, this however does not necessarily imply that there is no other, possibly efficient, algorithm for computing $\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k})$.

For the case where k is a constant independent of n, there is evidence that computing $\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k})$ is probably hard (inefficient) to do classically. Indeed, it is known that in worst-case the probabilities $p(s_i^{n-k})$ when k is a constant independent of n are not efficient to compute classically, unless the polynomial hierarchy collapses to its third level [17]. This motivates the application of recycling mitigation in the low k regime, as the $n - k$–photon output statistics are efficiently computable classically when k is high, and therefore any potential quantum advantage is lost. In all our simulations, we apply our error mitigation techniques to statistics where the number of lost photons is a constant independent of system size, and discard all other statistics.

7 Bounding deviation of interference terms

Methods are later presented to extract the ideal probability signals from the recycled probabilities. The precision with which this operation may be achieved is limited by variation of the interference terms from their expected value. As such, upper bounding this variation behaviour is crucial when we later assess the accuracy of the mitigated values. The recycled probabilities have a composite structure, comprising a mixture of the ideal probability and the interference term. While the interference term,

$$I_{s_l^n, k} = \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{N_k'}, \tag{13}$$

is itself a mixture of n–photon output probabilities. It is possible to account for the errors caused by using approximations of the interference terms when generating the mitigated probabilities by upper bounding the deviation of the interference term away from an expected value.

Firstly, we consider the case of Haar random matrices. As we are working in the no-collision regime, the output probabilities generated by sampling from a Haar random matrix is linked to permanents of Gaussian random matrices $X \in \mathcal{G}_{n \times n}$ [17] with $\mathcal{G}_{n \times n}$ the set of all such Gaussian matrices; more precisely the set of complex matrices whose real and imaginary parts are chosen independently from the normal distribution $\mathcal{N}(0, \frac{1}{2})$. Let $p_{unif} := \frac{1}{\binom{m}{n}}$, in appendix 17 we show the following.

Lemma 2. *For all s_l^n, we have*

$$\mathbf{E}_{X \in \mathcal{G}_{n \times n}} \left(I_{s_l^n, k} \right) = p_{unif} \left(1 + O(m^{-1}) \right) \approx p_{unif}. \tag{14}$$

with $\mathbf{E}_{X \in \mathcal{G}_{n \times n}}(.)$ the expectation value over the set $\mathcal{G}_{n \times n}$.

With Lemma 2 in hand, it is possible to bound the deviation of the interference terms around this expected value according to the theorem shown in appendix 17.

Theorem 3. *The deviation of interference terms around p_{unif} for Haar random matrices is bounded*

$$Pr\Big(\big|I_{s_l^n,k} - p_{unif}\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{np_{unif}^2}{\epsilon_{bias,s_l^n}^2}, \tag{15}$$

where ϵ_{bias,s_l^n} is a positive real number.

Secondly, as it is desirable not to be restricted to only mitigating the output of Haar random matrices, we derive an additional bound for arbitrary matrices. Let D_R^k be the uniform distribution over recycled probabilities $\{p_R^k(s_l^n)\}_l$. Since every recycled probability has an associated interference term $I_{s_l^n,k}$, one can equivalently think of D_R^k as a distribution over interference terms. A random variable Y chosen from D_R^k means choosing, with uniform probability p_{unif}, a value from the set $\{p_R^k(s_l^n)\}_l$ (or, equivalently, choosing a value from $\{I_{s_l^n,k}\}_l$ uniformly randomly). In appendix 17, we show the following.

Lemma 4.

$$\mathbf{E}_{D_R^k}\big(I_{s_l^n,k}\big) = p_{unif}, \tag{16}$$

where $\mathbf{E}_{D_R^k}(.)$ denotes the expectation value over D_R^k.

The deviation of interference terms around this expected value is upper bounded according to the following inequality.

Theorem 5. *The deviation of interference terms around p_{unif} for an arbitrary matrix is bounded*

$$Pr\Big(\big|I_{s_l^n,k} - p_{unif}\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}, \tag{17}$$

where ϵ_{bias,s_l^n} is a positive real number.

While both these upper bounds use Chebyshev's inequality [27], the Thm. 3 bound applies only to Haar random matrices, while the Thm. 5 bound applies to arbitrary matrices. However, to upper bound the variance for the Haar random case we use exact moments originally derived in [61]. While in the case of arbitrary matrices the Bhatia-Davis inequality [62] is instead used, which results in a looser bound.

We note that a tighter, although distribution-dependent, bound may be found than the one provided in Thm. 5. This result follows from Lemmas 6 and 7, proven in appendix 17, which will now be stated, that use the definition of the variance that for a set of real values $\{x_i\}_i$ with mean μ and cardinality N, $\mathsf{Var}(\{x_i\}_i) := N^{-1}\sum_i(x_i - \mu)^2$.

Lemma 6. *The variance of the set of recycled probabilities is less than or equal to the variance of the set of ideal probabilities, that is*

$$\mathsf{Var}\big(\{p(s_l^n)\}_l\big) \geq \mathsf{Var}\big(\{p_R^k(s_l^n)\}_l\big). \tag{18}$$

Lemma 7. *The variance of the set of interference terms is less than or equal to the variance of the set of ideal probabilities, that is*

$$\mathsf{Var}\big(\{p(s_l^n)\}_l\big) \geq \mathsf{Var}\big(\{I_{s_l^n,k}\}_l\big). \tag{19}$$

Using these lemmas, an upper bound on the largest probability of the output distribution may be used to derive a tighter upper bound on the variance using the Bhatia-Davis inequality [62]. Let p_{upper} be an experimentally derived upper bound on the largest probability in the ideal n–photon output distribution $p_{\max}$, such that $p_{\text{upper}} \geq p_{\max}$. Following similar steps as for the proof for Thm. 5 results in an upper bound on the confidence of $1 - \frac{p_{unif}p_{upper}}{\epsilon_{bias,s_l^n}^2} - \delta\Big(1 - \frac{p_{unif}p_{upper}}{\epsilon_{bias,s_l^n}^2}\Big)$, where δ is the confidence parameter for the p_{upper} estimator. This result is stated formally in the following.

Theorem 8. *The deviation of interference terms around p_{unif} for an arbitrary matrix is bounded*

$$Pr\Big(\big|I_{s_l^n,k} - p_{unif}\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{p_{unif}p_{upper}}{\epsilon_{bias,s_l^n}^2} + \delta\Big(1 - \frac{p_{unif}p_{upper}}{\epsilon_{bias,s_l^n}^2}\Big),$$

where ϵ_{bias,s_l^n} is a positive real number, and p_{upper} is an empirically computed upper bound on the largest probability of the ideal n output photon probability distribution with confidence $1 - \delta$.

A proof of this result is given in appendix 17. We note also that as $p_{\max} \leq 1$ with confidence 1, so that $\delta = 0$ in Thm. 8, Thm. 5 follows as a corollary.

With assumptions about the structure of the unitaries, like in [28], it is possible to deterministically and exponentially upper bound the interference terms. For a large class of unitary matrices, we show that the bias error scales as an inverse exponential in n. As will be seen later, this shows that our mitigation techniques outperform postselection in estimating output probabilities and expectation values of linear optical circuits for up to inverse exponential precisions. Here we use the the operator 2-norm and the infinity norm. For an $n \times n$ matrix A the operator 2-norm is defined $\|A\|_2 := \sup_{\|\vec{x}\|_2 \leq 1, \vec{x} \in \mathbb{C}^n} \|A\vec{x}\|_2$, where $\|\vec{v}\|_p$ is the l_p norm, that is $\|\vec{v}\|_p = \left(\sum_p |v_i|^p\right)^{1/p}$. And the infinity norm which is defined as $\|\vec{v}\|_\infty := \max_i |v_i|$. Let $h_\infty^A := \frac{1}{n} \sum_{i=1,\dots,n} \|\mathbf{A}_i\|_\infty$, where $\mathbf{A}_i$ is the ith row of A. In appendix 20 we show the following.

Theorem 9. *For the class of unitary matrices U with submatrices A such that $p_{max} := \mathsf{max}_{s_l^n}(p(s_l^n)) = |\mathsf{Per}(A)|^2$, and where these matrices A satisfy $\frac{h_\infty^A}{\|A\|_2} \ll 1$, the bias error ϵ_{bias,s_l^n} is bounded*

$$\epsilon_{bias,s_l^n} \leq\approx O(e^{-0.000002n}). \tag{20}$$

In other words, for the stated class of unitary matrices the bias error is inverse-exponentially (in n) close to zero. For this class of unitary matrices, one can show, as will be seen later on, that our mitigation techniques outperform postselection for up to inverse-exponential additive errors. However the number of photons required for this bound to become significant ($n \approx 10^5$) is beyond what is possible with current technology.

We will now apply the derived bounds to give performance guarantees for several methods which may be applied to extract the signal of the ideal probabilities from the recycled probabilities. In the following, to make the bounds in Thms. 3 and 5 more concrete we will set $\epsilon_{bias,s_l^n} = O\left(\frac{1}{\mathsf{poly}(m)}\right)$.

8 Generating the loss-mitigated outputs

We now present two methods for constructing loss mitigated outputs from the recycled probabilities, these we refer to as *linear solving* and *extrapolation*. In linear solving, the interference term within each recycled probability is substituted for its expected value, and the resulting expressions are then solved to find estimators of the ideal probabilities. While in extrapolation, the decay of the ideal signal in the set of recycled distributions with k is used to compute estimators of the ideal probabilities. We derive inequalities that are conditions for recycling mitigation with the relevant postprocessing techniques to beat postselection.

The output photon statistics where most photons have been lost are not used in recycling mitigation. Indeed, it is clear that if one uses output photon statistics with high k to construct the recycled probabilities, then these statistics may be efficiently classically simulated; as per Lemma 1. This strongly motivates the construction of recycled distributions from output statistics where k is low.

While the postprocessing required to generate each mitigated value may be performed efficiently, the size of the output distribution is exponential in m and n. Meaning the classical postprocessing cost (i.e. the memory cost) is proportional to the number of mitigated probabilities that are to be generated. So that to generate a full mitigated output distribution this scales as m^n, while for a subset of size N_s the postprocessing cost is then proportional to N_s.

As the bias errors in the bounds for Thm. 3 and Thm. 5 have been set to $\epsilon_{bias,s_l^n} = O\left(\frac{1}{\mathsf{poly}(m)}\right)$, the performance guarantees for the Haar random and arbitrary matrix cases are of the same form. Therefore, for the sake of concision we state one performance guarantee for each version of the classical postprocessing. Note, however, that the confidence for the Haar random matrix bias error upper bound is $\approx 1 - O^*(p_{unif}^{-2})$, whereas for the arbitrary matrix bias error upper bound is $\approx 1 - O^*(p_{unif}^{-1})$ [3].

8.1 Linear solving

The linear solving method involves substituting the interference term within each recycled probability for an approximate value, and then solving the resulting expressions for the ideal probabilities. As the expectation of interference terms over the Haar measure and over the recycled distribution are $p_{unif} + O(m^{-n+1})$ and

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[3]The $O^*(f)$ notation means a complexity of $O(f)$ up to polynomial prefactors, that is $O^*(f) := O(\mathsf{poly}(n)f)$. Note that, since $m = O(\mathsf{poly}(n))$ for any useful application, then $O^*(f)$ can equivalently be thought of as $O(f)$ up to a $\mathsf{poly}(m)$ prefactor.

p_{unif}, respectively, see Lemmas 2 and 4, the term $\frac{N'_k}{N_k}p_{unif}$ is used for this substitution. This allows the upper bounding of the error introduced by the substitution using Thms. 3 and 5. Each recycled probability constructed from the $n - k$ output photon statistics may then be written

$$\tilde{p}_R^k(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N'_k}{N_k}\left(p_{unif} + \epsilon_{bias,s_l^n}\right) + \epsilon_{stat.,s_l^n}, \tag{21}$$

where $\frac{N'_k}{N_k}\epsilon_{bias,s_l^n}$ is the bias error introduced by replacing the interference term in recycled probability $p_R^k(s_l^n)$ with p_{unif}. And $\epsilon_{stat.,s_l^n}$ is the statistical error from estimating the recycled probabilities from a finite number of samples. These new expressions can then be solved to generate the mitigated outputs

$$p_{\text{miti}}(s_l^n) = \binom{m-n+k}{k}\left|\tilde{p}_R^k(s_l^n) - \frac{N'_k}{N_k}p_{unif}\right|. \tag{22}$$

We now provide pseudocode with the steps required to use the linear solving method to generate the mitigated output.

Algorithm 2: Linear solving method

    input : Recycled probability estimator $\tilde{p}_R^k(s_l^n)$, and uniform probability p_{unif}.

1 Initialise variable $p_{\text{miti}}(s_l^n) \leftarrow 0$ for the mitigated output to be computed.

2 Update mitigated output variable as $p_{\text{miti}}(s_l^n) \leftarrow \binom{m-n+k}{k}\left|\tilde{p}_R^k(s_l^n) - \frac{N'_k}{N_k}p_{unif}\right|$.

    output: Mitigated output $p_{mit}(s_l^n)$

There exists a regime of recycling mitigation usefulness, where the combined bias and the statistical errors present in the mitigated probabilities are lower than the statistical errors of the postselected distribution. By combining upper bounds for each of these errors we will now provide conditions for recycling mitigation using linear solving to outperform postselection.

These are stated in the form

$$N_k\left(\frac{N'_k}{N_k}\mathsf{M}(\epsilon_{bias,s_l^n}) + \mathsf{M}(\epsilon_{stat.,s_l^n})\right) \leq \mathsf{M}(\epsilon_{stat.,s_l^n}^{post.}), \tag{23}$$

where $\mathsf{M}(\epsilon_{bias,s_l^n})$, $\mathsf{M}(\epsilon_{stat.,s_l^n})$ and $\mathsf{M}(\epsilon_{stat.,s_l^n}^{post.})$ are respectively upper bounds for the bias error of the mitigated probability ϵ_{bias,s_l^n}, the statistical error of the mitigated probability $\epsilon_{stat.,s_l^n}$, and the statistical error of the postselected probability $\epsilon_{stat.,s_l^n}^{post.}$. More explicitly, in the appendix 18 we show the following

Theorem 10. *The condition:*

$$O\left(\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + \left(\binom{m-n+k}{k} - 1\right)O\left(\frac{1}{\text{poly}(m)}\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right), \tag{24}$$

defines a sampling regime (a range of values of N_{tot}) where the sum of the worst-case statistical error and bias error of linear solving recycling mitigation is less than the worst-case statistical error of postselection.

Note that the big O notation here can be interpreted as multiplying by a prefactor, which can be computed by choosing a confidence for the concentration inequalities (as will be seen in section 9). This allows for the direct computation of the bounds for a specific confidence. One can intuitively understand Thm. 10 as follows. For values of loss $\eta > 0.5$, it is expected that the statistical error of recycling mitigation $O\left(\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right)$ is in general smaller than that of postselection $O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right)$. Thus, the sampling regime where recycling outperforms postselection is typically determined by the bias error $2\left(\binom{m-n+k}{k} - 1\right)O\left(\frac{1}{\text{poly}(m)}\right) \approx O\left(\frac{1}{\text{poly}(m)}\right)$, for small values of k independent of n. This means that recycling is expected to outperform postselection for estimating output probabilities to up to inverse polynomial errors, with high confidence. Furthermore, Thm. 9 shows that, for many linear optical circuits, the outperformance is up to inverse exponentially small errors.

We now introduce the notion of dependency. This quantifies the correlation of the interference terms with the ideal probability within the recycled probabilities. A positive correlation means that the signal for the ideal probability is greater than $\binom{m-n+k}{k}^{-1}$, which can be used to improve the performance of linear solving. Each recycled probability can be rewritten to include a dependency term $d_k(s_l^n)$ quantifying this correlation, by reformulating the interference term as a linear function of $p(s_l^n)$ and p_{unif}. The expression

$$I_{s_l^n,k} = \big(1 - d_k(s_l^n)\big)p_{unif} + d_k(s_l^n)p(s_l^n) \tag{25}$$

defines the dependency term $d_k(s_l^n)$ of each recycled probability, and $k \leq n - 1$. An average dependency term over the distribution, denoted d_k, may be calculated from the recycled probabilities (see eqn. 18.2.1 and appendix 18). Each recycled probability constructed from the $n - k$ output photon statistics may then be expressed in the form

$$\tilde{p}_R^k(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}\big(\big((1-d_k)p_{unif} + d_k p(s_l^n) + \epsilon_{bias,s_l^n}\big) + \epsilon_{stat.,s_l^n}, \tag{26}$$

where ϵ_{bias,s_l^n} is the bias error introduced by replacing the interference term in recycled probability $p_R^k(s_l^n)$ with $\big((1-d_k)p_{unif} + d_k p(s_l^n)\big)$. Note that if the computed estimator for d_k is negative or greater than 1 then the dependency approach should be aborted and the original version of linear solving used. We conjecture it is always the case that $1 \geq d_k \geq 0$. The following pseudocode details the steps required to perform the linear solving with dependency method and generate the mitigated output.

Algorithm 3: Linear solving with dependency method

    input : Recycled probability estimator $\tilde{p}_R^k(s_l^n)$, uniform probability p_{unif}, an estimator for the absolute average deviation for the n output photon distribution $\tilde{D}_0$, and an estimator for the absolute average deviation for the $n - k$–photon recycled distribution $\tilde{D}_k$ (eqn. (29)).

1 Initialise variables for the mitigated output $p_{mit}(s_l^n) \leftarrow 0$ and the average dependency term $d_k \leftarrow 0$.

2 Update the average dependency term variable $d_k \leftarrow \frac{1}{\binom{m-n+k}{k}-1}\left(\frac{\binom{m-n+k}{k}\tilde{D}_k}{\tilde{D}_0} - \frac{1}{\binom{m-n+k}{k}}\right)$.

3 Update the mitigated output variable $p_{mit}(s_l^n) \leftarrow \left|\frac{\tilde{p}_R^k(s_l^n) + \frac{N_k'}{N_k}\big((-1+d_k)p_{unif}\big)}{\binom{m-n+k}{k}^{-1} + \frac{N_k'}{N_k}d_k}\right|$

    output: Mitigated output $p_{mit}(s_l^n)$

In appendix 18, we show the following.

Theorem 11. *The condition*

$$O\left(\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + 2\left(\binom{m-n+k}{k} - 1\right)O\left(\frac{1}{\mathsf{poly}(m)}\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right), \tag{27}$$

defines a sampling regime where the sum of the worst-case statistical error and bias error of linear solving with dependency recycling mitigation is less than the worst-case statistical error of postselection.

The performance guarantees for linear solving with and without dependency are too similar to be used to draw inferences on their relative performance. We do note, however, that in numerical simulation linear solving with dependency reliably outperforms linear solving without dependency (see Fig. 3 (a) and (b)).

8.2 Extrapolation

We now present methods by which extrapolation may be used to generate loss-mitigated outputs. From the definition of the recycled probability,

$$p_R^k(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}I_{s_l^n,k}, \tag{28}$$

the magnitude of the ideal probability signal within the recycled probabilities is proportional to $\binom{m-n+k}{k}^{-1}$. As k increases, the ideal signal magnitudes decrease as the recycled distributions converge towards uniform.

The rate of decay of the ideal signal with k can be computed and then used to extrapolate mitigated outputs. We present two variations of extrapolation in which different types of dependence of the ideal probability signal on number of lost photons k are considered. A linear dependence is used for a linear extrapolation method, and an exponential dependence for an exponential extrapolation method. Both the linear and exponential extrapolation methods involve two iterations of optimisation. The first iteration is to compute an average decay parameter using the set of average absolute deviations of the recycled distributions. Where, for the set of recycled probabilities $\{p_R^k(s_l^n)\}_l$ constructed from k photon statistics, the average absolute deviation is defined as

$$D_k := \binom{m}{n}^{-1} \sum_l \left| p_R^k(s_l^n) - p_{unif} \right|. \tag{29}$$

The second iteration then uses the decay parameter to compute the mitigated values.

Linear extrapolation applies a linear model function to compute mitigated outputs. Here the least squares method is used to identify optimal parameters to fit a linear model function to data. Parameter optimisation is performed by minimising the sum of the squared residuals, where a residual is the difference between a data point and the model. A data set of N points is denoted $\{x_i, y_i\}_{i=1}^N$, where $\{x_i\}_{i=1}^N$ are the independent variables and $\{y_i\}_{i=1}^N$ are the dependent variables. The model function $f(x, \boldsymbol{\alpha})$ is optimised by varying the $\boldsymbol{\alpha}$ parameters to approximate the relation between independent and dependent variables found in the data set. The residual for each data point is defined $r_i := y_i - f(x_i, \boldsymbol{\alpha})$. The sum of the squared residuals is minimised to generate the optimal parameters

$$\boldsymbol{\alpha}_{\min} = \arg\min_{\boldsymbol{\alpha}} \sum_{i=1}^N r_i^2, \tag{30}$$

which are used to generate the optimised model function $f(x, \boldsymbol{\alpha}_{\min})$. This can then be used to make predictions about data outside the range of the data set used for optimisation.

In linear extrapolation, the linear model function used for the first iteration of linear least squares is

$$f(x, g_{\text{avg}}) = -g_{\text{avg}} x + \tilde{D}_0, \tag{31}$$

where $\tilde{D}_0$ is the average absolute deviation of the n–photon distribution from uniform computed from the statistics corresponding to postselecting on detecting all n photons, and g_{avg} is the optimal global linear decay parameter. The set $\{k, \tilde{D}_k\}_{k=1}^K$ is used as the data set to compute g_{avg}. Where $\{k, \tilde{D}_k\}_{k=1}^K$ is set of average absolute deviations from uniform for the different distributions, for $K \leq n$ and $\tilde{D}_k$ is the estimated (from statistics where k photons were lost) absolute average deviation for the $n - k$–photon recycled distribution. After the optimal decay parameter is identified, another iteration of least squares is performed with updated model functions this time to generate the mitigated values. The data set used in this step is $\{k, |\tilde{p}_R^k(s_l^n) - p_{unif}|\}_{k=1}^K$. For the second iteration of linear least squares, each output bit string is assigned a linear model function of the form

$$f_{s_n}(x, \alpha_{s_l^n}) = \text{sgn}(p_{unif} - y_1) g_{\text{avg}} x + \alpha_{s_l^n}. \tag{32}$$

For each output bit string s_n an optimal $\alpha_{s_l^n}$ is computed, generating the set $\{\alpha_{s_l^n}\}_l$, and the set of mitigated outputs is then $\{\alpha_{s_l^n} + p_{unif}\}_l$. We now provide pseudocode for applying the linear extrapolation method to generate mitigated outputs.

Algorithm 4: Linear extrapolation method

input : The number of data points $n_d \in \{n_d \in \mathbb{Z}^+ | n_d < n\}$ to be used in both iterations of least squares, the data set $\{k, \tilde{D}_k\}_{k=1}^{n_d}$ used to compute the gradient parameter g_{avg} in the first iteration of least squares, $\tilde{D}_0$, and the data set $\{k, |\tilde{p}_R^k(s_l^n) - p_{unif}|\}_{k=1}^{n_d}$ used to compute the mitigated output in the second iteration of least squares.

1 Initialise an average decay parameter variable $\tilde{g}_{\mathrm{avg}} \leftarrow 0$, a prefactor variable $\alpha_{s_l^n} \leftarrow 0$, and a mitigated output variable $p_{mit}(s_l^n) \leftarrow 0$.

2 Use least squares method with model function $f(x_i, g_{\mathrm{avg}}) = -\tilde{g}_{\mathrm{avg}} x_i + \tilde{D}_0$ and data set $\{x_i, y_i\}_{i=1}^{n_d} := \{k, \tilde{D}_k\}_{k=1}^{n_d}$ to compute the value of the average decay parameter (slope), and assign this to variable $\tilde{g}_{\mathrm{avg}}$.

3 Use least squares method with model function $f_{s_l^n}(x_i, \alpha_{s_l^n}) = \mathrm{sgn}(p_{unif} - y_1) \tilde{g}_{\mathrm{avg}} x_i + \alpha_{s_l^n}$ and data set $\{x_i, y_i\}_{i=1}^{n_d} := \{k, |\tilde{p}_R^k(s_l^n) - p_{unif}|\}_{k=1}^{n_d}$ to compute the value of the y-axis intercept, and assign this to variable $\alpha_{s_l^n}$.

4 Update mitigated output variable as $p_{mit}(s_l^n) \leftarrow p_{unif} + \alpha_{s_l^n}$.

output: Mitigated output $p_{mit}(s_l^n)$

The analytic condition for linear extrapolation to outperform postselection is now stated.

Theorem 12. *The condition:*

$$
\frac{n_d + 1}{2} \left(O\left(\left(\frac{1}{(1-\eta)^n N_{tot}} \right)^{1/4} \right) + O\left(\frac{1}{poly(m)} \right) \right)
$$
$$
+ O\left(\frac{n}{m-n+1} \sqrt{ \frac{\binom{m}{n}}{n(1-\eta)^{n-1} \eta N_{tot}} } \right) \leq O\left(\sqrt{ \frac{\binom{m}{n}}{(1-\eta)^n N_{tot}} } \right), \tag{33}
$$

defines a sampling regime where the sum of the worst-case statistical error and bias error of linear extrapolation recycling mitigation using the least squares method is less than the worst-case statistical error of postselection.

Thm. 12 is proven in appendix 19. The left hand side of the above expression is a sum of the upper bounds of the errors for performing linear extrapolation on the recycled probabilities. Using left-to-right ordering, the first term and third terms are due to statistical error and the second term is due to bias error. While the single term on the right side of the inequality is due to the statistical error of the postselected output.

The method for exponential extrapolation is broadly similar. However, non-linear least squares or a non-linear numerical optimisation method (e.g. the Levenberg–Marquardt algorithm [63, 64]) is instead used to compute the decay factor and the mitigated probabilities. The exponential model function used for the first step is

$$
f(x, \alpha_{\mathrm{avg}}) = \tilde{D}_0 e^{-\alpha_{\mathrm{avg}} x}, \tag{34}
$$

where α_{avg} is the optimal global exponential decay parameter. Where again the set $\{k, \tilde{D}_k\}_{k=1}^{K}$ is used as the data, this time to compute the optimal value of α_{avg}. And then each output bit string for which a mitigated probability is to be generated is assigned a model function of the form

$$
f_{s_n}(x, \Lambda_{s_n}) = \Lambda_{s_l^n} e^{-\alpha_{\mathrm{avg}} x} + p_{unif}. \tag{35}
$$

An optimal prefactor, denoted $\Lambda_{s_l^n}^{\mathrm{opt}}$, is computed for each bit string s_l^n using numerical optimisation. This generates the set of prefactors $\{\Lambda_{s_l^n}^{\mathrm{opt}}\}_l$, and the set of mitigated outputs is then $\{\Lambda_{s_l^n}^{\mathrm{opt}} + p_{unif}\}_l$. The following pseudocode gives the steps required to use the exponential extrapolation method to generate the mitigated outputs.

Algorithm 5: Exponential extrapolation method

input : The number of data points $n_d \in \{n_d \in \mathbb{Z}^+ | n_d < n\}$ to be used in both iterations of least squares, the set $\{k, \tilde{D}_k\}_{k=1}^{n_d}$ used to compute the gradient parameter g_{avg} in the first iteration of least squares, $\tilde{D}_0$, and the data set $\{k, |\tilde{p}_R^k(s_l^n) - p_{unif}|\}_{k=1}^{n_d}$ used to compute the prefactor value in the second iteration of least squares.

1 Initialise an average decay parameter variable $\alpha_{avg} \leftarrow 0$, a prefactor variable $\Lambda_{s_l^n} \leftarrow 0$, and a mitigated output variable $p_{mit}(s_l^n) \leftarrow 0$.

2 Use least squares method with model function $f(x, \alpha_{\text{avg}}) = \tilde{D}_0 e^{-\alpha_{\text{avg}} x}$ and data set $\{x_i, y_i\}_{i=1}^{n_d} := \{k, \tilde{D}_k\}_{k=1}^{n_d}$ to compute the value of the average decay parameter and assign this to variable α_{avg}.

3 Use least squares method with model function $f_{s_n}(x, \Lambda_{s_l^n}) = \Lambda_{s_l^n} e^{-\alpha_{\text{avg}} x} + p_{unif}$ and data set $\{x_i, y_i\}_{i=1}^{n_d} := \{k, |\tilde{p}_R^k(s_l^n) - p_{unif}|\}_{k=1}^{n_d}$ to compute the value of the prefactor and assign this to variable $\Lambda_{s_l^n}$.

4 Update mitigated output variable as $p_{mit}(s_l^n) \leftarrow p_{unif} + \Lambda_{s_l^n}$.

output: Mitigated output $p_{mit}(s_l^n)$

Note that the ideal signal magnitude decays proportionally with $\binom{m-n+k}{k}^{-1} \sim m^{-k}$ in eqn. 28, which intuitively motivates the choice of an exponential model function to reflect this decay behaviour. In the next sections, we provide numerical evidence indicating that there exists a non-trivial sampling regime where exponential extrapolation outperforms postselection. We also note that in the numerical simulations, extrapolation using an exponential model function consistently outperforms linear extrapolation (see Fig. 3 (c) and (d)). This may be a consequence of the exponential model function better reflecting the ideal signal decay behaviour.

9 Numerical computation of regime of mitigation usefulness for example experiment

To demonstrate how the performance guarantees may be related to a real experiment, we now show how the performance guarantees for linear solving recycling mitigation may be used to compute a regime of usefulness where the mitigation outperforms postselection. Let the example experiment consist of a computation with size $m = 100$ modes, $n = 10$ photons, that is run on noisy hardware for which the uniform probability of photon loss is $\eta = 0.8$, and for the linear solving method let $k = 1$. We now compute a regime of mitigation usefulness for an arbitrary probability from an arbitrary distribution. The performance guarantee for linear solving recycling mitigation of a single probability is [4]

$$\underbrace{O\left(\sqrt{\frac{1}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right)}_{\text{Mitigation statistical error}} + \underbrace{\left(\binom{m-n+k}{k} - 1\right)O\left(\frac{1}{\text{poly}(m)}\right)}_{\text{Mitigation bias error}} \leq \underbrace{O\left(\sqrt{\frac{1}{(1-\eta)^n N_{tot}}}\right)}_{\text{Postselection statistical error}} . \quad (36)$$

We need to replace the big O notation with prefactors to allow computation of a bound. So first the polynomial interference term bias error is set to $\epsilon_{bias,s_l^n} = \frac{1}{m^{2k}}$. And next the confidence for both mitigated and postselected statistical error bounds is set to $1 - \delta$, $\delta = m^{-n}$. Now from Hoeffding's inequality [26], with $N_{est,k} \approx \binom{n}{k}\eta^k(1-\eta)^{n-k}N_{tot}$, we have

$$2e^{-2N_{est,k}\epsilon_{\text{hoeff}}^2} \geq m^{-n}. \quad (37)$$

Taking the log of both sides

$$-2N_{est,k}\epsilon_{\text{hoeff}}^2 \geq -\log(2) - n\log(m), \quad (38)$$

and rearranging this becomes

$$\epsilon_{\text{hoeff}} \geq \sqrt{\frac{\log(2) + n\log(m)}{2N_{est,k}}}. \quad (39)$$

---

[4]Note that the statistical errors in eqn. 36 and Thm. 10 differ by a numerator $\sqrt{\binom{m}{n}}$, this is because in our proof of Thm. 10 we divided the sample size by $\binom{m}{n}$ to ensure we are computing each recycled probability from an independent sample group (see appendix 18). Here since we are interested in a single probability, we drop this division step.
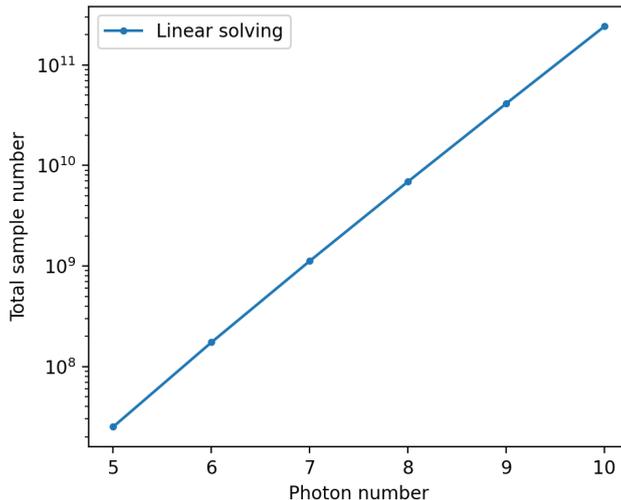
Figure 2: The upper bounds on the total number of samples for which linear solving recycling mitigation will outperform postselection plotted against photon number. These values were computed from the performance guarantees for the parameter values $m = 100$, $\eta = 0.8$, $k = 1$, and with $n \in [5, 10]$. A bias error of $\epsilon_{bias, s_l^n} = m^{-2k}$ and confidence for the statistical error bounds of m^{-n}, which gives rise to a logarithmic prefactor term for the statistical error bounds of $\sqrt{2^{-1} \log 2 + n \log(m)}$. This displays the exponential dependence on the photon number of the upper bound on the total number of samples for which the techniques will outperform postselection.

This leads to a polylogarithmic prefactor term for the statistical error bounds for the mitigated and postselected outputs of $\sqrt{2^{-1}(\log 2 + n \log(m))}$. We then get the following condition for an arbitrary probability in the mitigated distribution to be more accurate than the corresponding probability in the postselected distribution

$$3\sqrt{\frac{\left(\log 2 + n \log(m)\right)}{2\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}} + 3\left(\binom{m-n+k}{k} - 1\right)\frac{1}{m^{2k}} \leq \sqrt{\frac{\left(\log 2 + n \log(m)\right)}{2(1-\eta)^n N_{tot}}}. \tag{40}$$

This can then be rearranged as

$$N_{tot} \leq \frac{\left(\log 2 + n \log(m)\right)m^{4k}}{2\left(\binom{m-n+k}{k} - 1\right)^2}\left(\frac{1}{9(1-\eta)^n} - \frac{1}{\binom{n}{k}(1-\eta)^{n-k}\eta^k}\right). \tag{41}$$

Substituting in the parameters of the example experiment we get that, with exponentially high confidence in terms of n, linear solving recycling mitigation should outperform postselection for at least up to 2.42×10^{11} samples. The upper bound on the usefulness sampling regime for these experimental parameters, only with photon number varying in the range $n \in [5, 10]$, is plotted in Fig. 2.

10   Numerical simulations

In the numerical simulations recycling mitigation is used to mitigate the effects of uniform photon loss error on the output of an otherwise ideal simulation of a linear optical quantum circuit. In the simulations, random matrices are chosen for each experiment which are decomposed and implemented with a linear interferometer, and a uniform photon loss model is applied with a defined loss parameter. The simulations were performed using Perceval [20], a pythonic framework for the simulation of photonic quantum circuits. Uniform photon loss channels commute with the interferometer, and so all loss channels, including loss due to imperfect sources and measurement, can be propagated to the end of the circuit. And so the effects of loss may be modelled by an ideal photon source, an ideal interferometer M, and a combined photon loss channel acting immediately before an ideal measurement operation. This means that rather than an output photon from the circuit incident on a detector being in the state $|1\rangle\langle 1|$, it is instead

$$|1\rangle\langle 1| \rightarrow (1-p)|1\rangle\langle 1| + p|0\rangle\langle 0|.$$
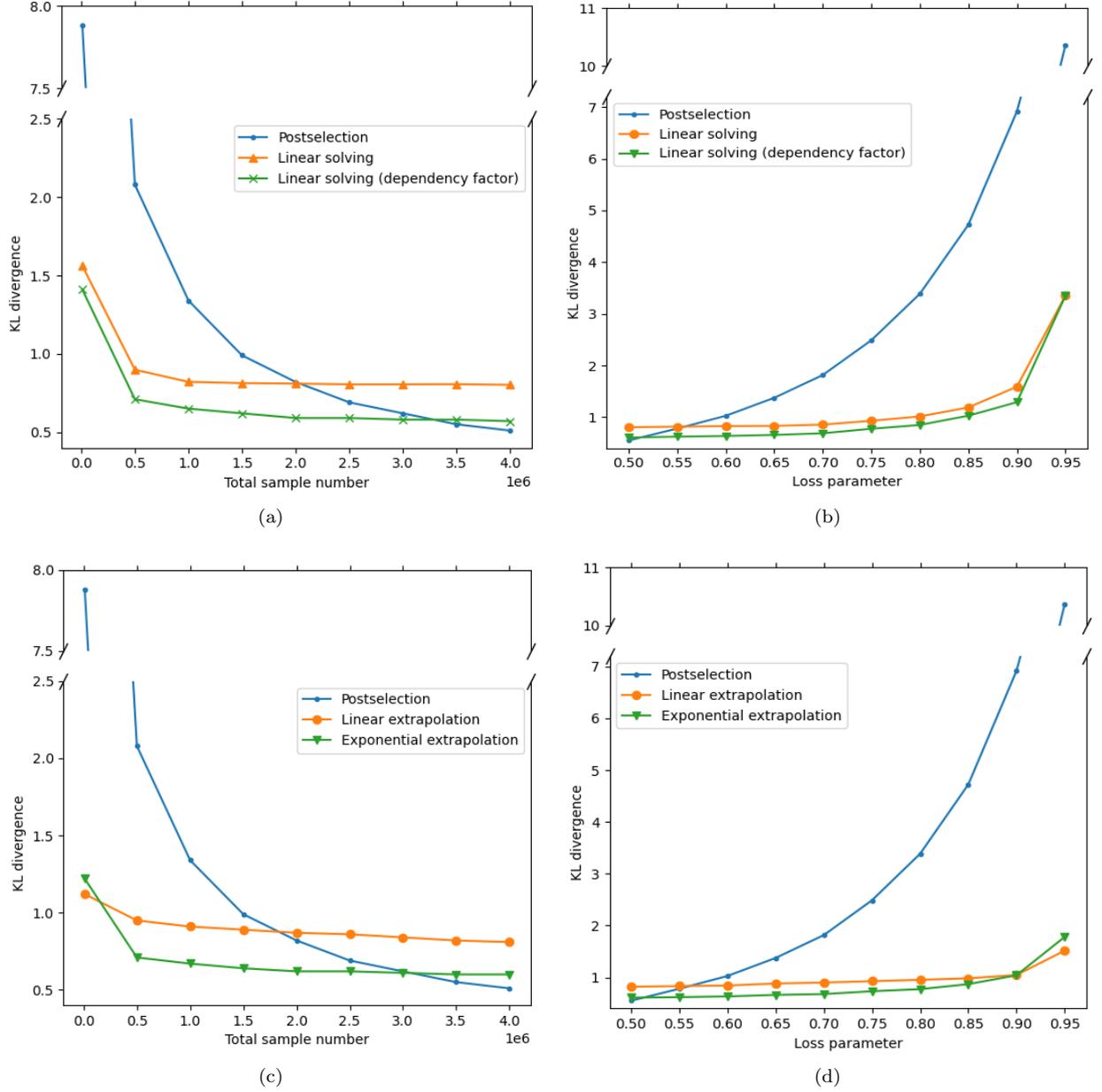
Figure 3: A numerical performance comparison of linear solving and postselection for random unitary circuits with $m = 20$ modes and $n = 4$ photons. (a) The KL divergence of the outputs of linear solving recycling mitigation and postselection from the ideal outputs for a uniform loss parameter of $\eta = 0.8$ against total sample number. (b) The KL divergence for random unitary circuits with a total number of samples of $N_{tot} = 1 \times 10^5$, and with the uniform loss parameter in the range $\eta \in [0.5, 0.9]$. (c) KL divergence for random unitary circuits with 20 modes, 4 photons, a uniform loss parameter of 0.8 against total sample number. (d) KL divergence for random unitary circuits with a total number of samples of $N_{tot} = 1 \times 10^5$, and with the uniform loss parameter in the range $\eta \in [0.5, 0.9]$.
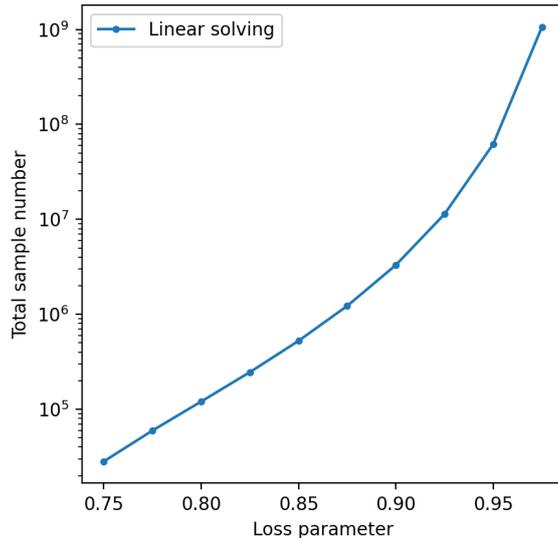
Figure 4: The upper bound for the total number of samples for which linear solving recycling mitigation will outperform postselection according to Thm. 10. The experimental parameters used were $m = 20$, $n = 4$ and $\eta \in [0.75, 0.975]$. The data point for $\eta = 0.8$ may be seen to be in agreement with the data in Fig. 3 (a). Here for an arbitrary single probability the sample upper bound for which the worst case error for linear solving recycling mitigation is lower than that for postselection for $\eta = 0.8$ is $\approx 1.1 \times 10^5$. Whereas in Fig. 3 (a) where the full distribution is mitigated it is $\approx 1.1 \times 10^6$ samples.

With the probability of photon loss, p, the same for all output modes. Noise of this form may be considered analogous to the types of measurement noise commonly considered in circuit model quantum computing, as this is often modelled as an error channel followed by an ideal measurement.

For all the experiments in Fig. 3, parameter settings of 20 modes and 4 photons were used. Fig. 3 (a) plots the performance of linear solving recycling mitigation, both with and without using the dependency factor, and postselection against the total number of samples used for a fixed uniform loss parameter of $\eta = 0.8$. Linear solving outperformed postselection up to $\approx 1.1 \times 10^6$ samples, while linear solving with dependency outperformed postselection up to $\approx 3.1 \times 10^6$ samples. Fig. 3 (b) plots the performance of the linear solving methods and postselection against changing loss parameter for a fixed number of samples of 1×10^5. Linear solving outperformed postselection for loss above ≈ 0.6, and linear solving with the dependency factor above loss of ≈ 0.5. Fig. 3 (c) plots the performance of linear extrapolation, exponential extrapolation and postselection against the total number of samples used for a fixed uniform loss parameter of $\eta = 0.8$. Linear extrapolation outperformed postselection up to $\approx 1.7 \times 10^6$ samples, while exponential extrapolation does so up to $\approx 3.0 \times 10^6$ samples. Fig. 3 (d) plots the performance of the extrapolation methods and postselection against changing loss parameter for a fixed number of samples of 1×10^5. Linear extrapolation outperformed postselection for loss above ≈ 0.5, and linear solving with the dependency factor above loss of ≈ 0.5. In current quantum linear optical devices losses including source, interferometer and measurement are commonly above 50% [18]. Therefore, the evidence from these simulations indicates that recycling mitigation may be usefully applied to the current generation of linear optical devices. Also, in the previous section an upper bound on the number of samples was computed for a particular experiment. Fig. 4 was plotted in order to compare the analytical expression in eqn. 41 to the numerics in 3 (a). In Fig. 4, for experimental parameters $m = 20$, $n = 4$ and $\eta \in [0.75, 0.975]$, the sample upper bound for which the worst-case error for linear solving recycling mitigation for an arbitrary single probability is lower than that for postselection for $\eta = 0.8$ is $\approx 1.1 \times 10^5$. Whereas in Fig. 3 (a) where the full distribution is mitigated it is $\approx 1.1 \times 10^6$ samples. And so the mitigation performance prediction based on eqn. 41 is in agreement with the numerics. Compared with the numerics, eqn. 41 estimates a lower sample threshold up to which the mitigation will outperform postselection. The most likely explanation for this is that the bias error is lower in the numerical experiments than is assumed for the analytics.

An intuitive explanation for the results in Fig. 3 is that the mitigated outputs have a lower statistical error and so converge more quickly than the postselected outputs for increasing sample number and decreasing loss parameter. Furthermore, the bias errors in the mitigated outputs mean that they do not converge to the
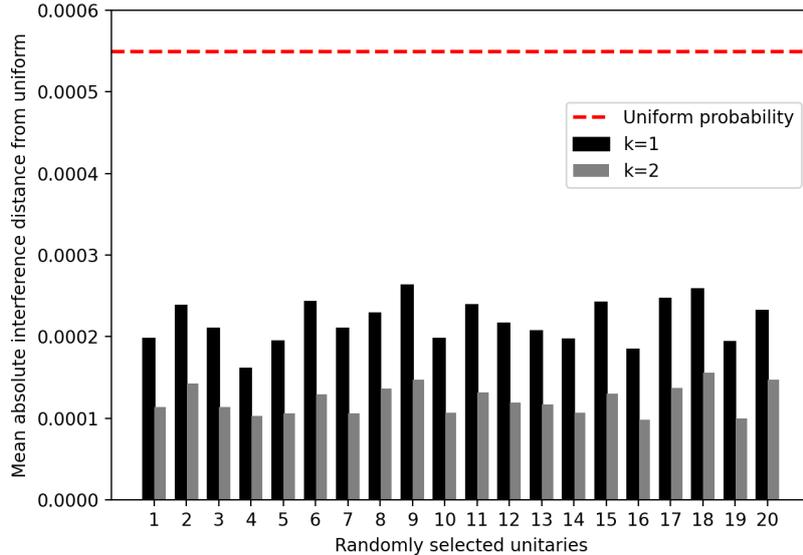
Figure 5: The mean absolute value of the distance of the interference terms from the uniform probability were computed for 20 randomly selected unitaries. For all unitaries, and for $k = 1$ and $k = 2$, the magnitude of the computed values were observed to be exponentially small in terms of m and n (being of the order of $\binom{m}{n}^{-1}$). This indicates that it may be possible to derive tighter analytical bounds than those stated in thm. 3 and thm. 5, and that mitigation performance may be better than indicated by the performance guarantee statements given in section 8.

ideal outputs, whereas the postselected outputs are unbiased estimators. And so for sufficiently high sample number or low loss parameter postselection will eventually outperform recycling mitigation. There are, however, large sampling and loss regimes for which recycling mitigation reliably outperforms postselection.

Theorems 3 and 5 in section 7 provide statistical upper bounds on the deviation of the interference terms from p_{unif}. We have shown that with confidence $\approx 1 - O^*(m^{-n})$ these upper bounds are within a range $[-O\big(\frac{1}{\mathsf{poly}(m)}\big), O\big(\frac{1}{\mathsf{poly}(m)}\big)]$ around p_{unif}. We conjecture that these bounds could be considerably tightened. In support of this we ran numerical experiments for computations of size $m = 16$ modes and $n = 4$ photons, computing the average of $|I_{s_l^n,k} - p_{unif}|$ over all bit strings $\{s_l^n\}_l$. This computation was repeated for 20 randomly selected linear optical intereferometers U. This data is plotted in Fig. 5. It may be observed that for all unitaries, and both when $k = 1$ and when $k = 2$, the computed values were of the order of $\binom{m}{n}^{-1}$, and so were in fact exponentially small in terms of the size of the computation. This indicates that the analytical performance guarantees given in section 8 may under-estimate the size of the sample number up to which recycling mitigation will outperform postselection. Finally, note that $|I_{s_l^n,k} - p_{unif}|$ seems to be generally decreasing with increasing k. However, $\binom{m-n+k}{k}|I_{s_l^n,k} - p_{unif}|$ actually increases with increasing k, as can be verified by a direct calculation using data from Figure 5. Ultimately, since $\binom{m-n+k}{k}|I_{s_l^n,k} - p_{unif}|$ is directly related to the bias error incurred when computing the mitigated probabilities from the recycled probabilities (see section 8), increasing values of k in the recycling mitigation will lead to worse performances, as predicted by Lemma 1. We conjecture that $\binom{m-n+k}{k}|I_{s_l^n,k} - p_{unif}|$ increases monotonically with increasing k (see section 14).

11 Evidence that zero noise extrapolation (ZNE) techniques present no advantage over postselection

We will now provide strong evidence that techniques based on ZNE when applied to mitigating photon loss in DVLOQC in general provide no advantage over postselection. Suppose we are interested in computing a specific marginal probability $p(n_1...n_l|n)$ of observing n_i photons in mode $i \in \{1, \ldots, l\}$, with $l \leq m$, and $\sum_{i=1,\ldots,l} n_i = c$, with $c \leq n$. The notation $|n$ indicates that we are computing the *ideal* marginal probability, when no photon is lost. Let $p(n_1 \ldots n_l \cap j)$ be the probability of observing the output $(n_1, \ldots, n_l)$ and detecting j photons in all m modes, with $j \in \{c, \ldots, n\}$. When postselecting on no photons being lost, we

are computing

$$p(n_1 \dots n_l \cap n) = (1 - \eta)^n p(n_1 \dots n_l | n).$$

However, if we compute $p(n_1 \dots n_l)$ without caring about whether no photon is lost, we end up computing

$$p_\eta(n_1 \dots n_l) = \sum_{i=0,\dots,n-c} (1-\eta)^{n-i} \eta^i p(n_1 \dots n_l | n - i). \tag{42}$$

Extrapolation techniques consist of estimating $p_\eta(n_1 \dots n_l)$ for different values $\{\eta_i\}$ of loss, then deducing from these an estimate of $p(n_1 \dots n_l | n)$. One example of how this can be done is the Richardson extrapolation technique, at the heart of the zero noise extrapolation (ZNE) approach [33]. Interestingly, it is possible to derive a condition indicating that these techniques offer no advantage over postselection in terms of estimating $p(n_1 \dots n_l | n)$. Let $\eta = \min_i \eta_i$, and suppose we collect $O(\frac{1}{\epsilon_{max}^2})$ samples from our device, where $0 < \epsilon_{max} \leq 1$, then $O(\frac{(1-\eta)^n}{\epsilon_{max}^2})$ of these will be samples where no loss has occurred. Using these postselected samples one can compute, with high confidence, an estimate $\tilde{p}(n_1 \dots n_l | n)$ of $p(n_1 \dots n_l | n)$ such that $|\tilde{p}(n_1 \dots n_l | n) - p(n_1 \dots n_l | n)| \leq \frac{\epsilon_{max}}{\sqrt{(1-\eta)^n}}$, by Hoeffding's inequality. For Richardson extrapolation, in appendix 21 we use $O(n \frac{1}{\epsilon_{max}^2})$ samples, and compute an estimate $\tilde{p}_{extrap}(n_1 \dots n_l | n)$, with $E_{extrap} := |\tilde{p}_{extrap}(n_1 \dots n_l | n) - p(n_1 \dots n_l | n)|$ the error incurred. We then explicitly compute $\mathsf{M}(E_{extrap})$, an upper bound on E_{extrap}, in terms of ϵ_{max} and the η_i's, and show the following

Theorem 13. *For all $n \geq n_0$, with n_0 a positive integer, $\mathsf{M}(E_{extrap}) \geq \frac{\epsilon_{max}}{\sqrt{(1-\eta)^n}}$.*

Thm. 13, proven in appendix 21, is strong evidence that techniques based on Richardson extrapolation offer no advantage over postselection. Our main technical contribution in proving Thm. 13 is to link determining the error of ZNE methods to computing the norm of the inverse of a Vandermonde matrix whose entries are determined by η, we then use existing results to upper bound this norm [65]. Furthermore, in appendix 21 we also provide numerical evidence to support this conclusion.

12 On normalising the mitigated distribution

The postprocessing techniques we have presented thus far take as input a set of recycled probabilities $\{p_R(s_l^n)\}_l$ and output a set of mitigated values $\{p_{mit}(s_l^n)\}_l$, or a subset of these values. We have made the distinction between *values* and *probabilities*, as the outputs $\{p_{mit}(s_l^n)\}_l$ satisfy $p_{mit}(s_l^n) > 0$ but are not normalised in general. That is, $\sum_l p_{mit}(s_l^n) = N$ with no guarantee that $N = 1$. Analytical and numerical calculations performed in previous sections have shown that, with high probability,

$$||\vec{p}_{mit} - \vec{p}_{id}||_1 \leq ||\vec{p}_{post} - \vec{p}_{id}||_1,$$

and $|p_{mit}(s_l^n) - p(s_l^n)| \leq |p_{post}(s_l^n) - p(s_l^n)|$ as long as we are in a sampling regime defined by eqn. (23). $\vec{p}_{mit}$, $\vec{p}_{id}$, and $\vec{p}_{post}$, are vectors containing respectively the mitigated values, the ideal n–photon probabilities, and the probabilities obtained by postselection, and $||.||_1$ is the usual l_1 norm, $||\vec{v}||_1 := \sum_i |v_i|$.

In practice, one is usually interested in computing the expectation value of some observable O, defined as $\langle \mathsf{O} \rangle = \sum_i p_i w_i$, where w_i are some weights, and $\{p_i\}$ a subset of probabilities of the quantum circuit. In this case we can replace the p_i's by the unnormalised mitigated values and obtain guarantees similar to those stated above. However, if we are interested in mitigating the entire distribution, then we would need to normalise $\vec{p}_{mit}$. The easiest way to do this, and what we do for our numerical simulations, is to define, $\vec{p}_{mit,nor} := \frac{1}{N} \vec{p}_{mit}$. One can easily check that $||\vec{p}_{mit,nor}||_1 = 1$, and therefore that it is a vector of normalised probabilities.

We will show that the normalised probabilities $\vec{p}_{mit,nor}$ exhibit similar guarantees to the unnormalised mitigated values $\vec{p}_{mit}$, albeit in a sampling regime which is slightly more restrictive than eqn. (23). We believe, however, that this restriction is just an artifact of the proof technique. Let $\vec{q}_m$ be a vector of positive values which is a *closest* (in l_1-norm) normalised vector to $\vec{p}_{mit}$. More precisely, $\vec{q}_m$ satisfies

$$||\vec{q}_m - \vec{p}_{mit}||_1 = \min_{\vec{q} | ||\vec{q}||_1 = 1} ||\vec{q} - \vec{p}_{mit}||_1.$$

An immediate observation, by definition, is that

$$||\vec{q}_m - \vec{p}_{mit}||_1 \leq ||\vec{p}_{id} - \vec{p}_{mit}||_1.$$

We now prove

Lemma 14. *A valid choice of $\vec{q}_m$ is $\vec{q}_m = \vec{p}_{mit,nor}$.*

Proof. $||\vec{p}_{mit,nor} - \vec{p}_{mit}||_1 = ||\vec{p}_{mit,nor} - \vec{p}_{mit,nor}N||_1 = |1 - N| = \left|1 - ||\vec{p}_{mit}||_1\right|$.

For any normalized vector of positive values $\vec{q}$, we can use a reverse triangle inequality to show $||\vec{q} - \vec{p}_{mit}||_1 \geq \left|||\vec{q}||_1 - ||\vec{p}_{mit}||_1\right| \geq \left|1 - ||\vec{p}_{mit}||_1\right| \geq ||\vec{p}_{mit,nor} - \vec{p}_{mit}||_1$. Thus $\vec{p}_{mit,nor}$ is a valid choice of $\vec{q}_m$, by definition of $\vec{q}_m$. $\qquad\square$

This lemma, together with the previous observations can be used to show

$$||\vec{p}_{mit,nor} - \vec{p}_{id}||_1 \leq ||\vec{p}_{mit} - \vec{p}_{id}||_1 + ||\vec{p}_{mit,nor} - \vec{p}_{mit}||_1 \leq 2||\vec{p}_{mit} - \vec{p}_{id}||_1.$$

We demand that the normalised mitigated probabilities be closer to the ideal probabilities than their postselected counterparts, this imposes the constraint

$$||\vec{p}_{mit} - \vec{p}_{id}||_1 \leq \frac{1}{2}||\vec{p}_{post} - \vec{p}_{id}||_1.$$

Using similar analysis as for the linear solving technique in section 8 allows us to derive a condition for the above constraint to hold. Namely, the following should hold with high probability

$$N_k\left(\frac{N_k'}{N_k}\mathsf{M}\left(\epsilon_{bias,s_l^n}\right) + \mathsf{M}\left(\epsilon_{stat.,s_l^n}\right)\right) \leq \frac{1}{2}\mathsf{M}\left(\epsilon_{stat.,s_l^n}^{post.}\right).$$

This is a slightly more restrictive condition than eqn. (23), since the postselection error is divided by a factor of 2. Nevertheless, this spans a sampling regime where we expect the normalised mitigated distribution to be closer, in l_1-norm, than the postselected probabilities to the ideal probabilities.

13 Prospects of improving bounds on the interference terms

A natural question is whether one can tighten the bounds, beyond what is guaranteed from Thm. 3, on the error term ϵ_{bias,s_l^n} incurred by replacing the interference term $I_{s_l^n,k}$ with its average over the Haar measure. Indeed, Thm. 3 guarantees with high confidence that for Haar random matrices $\epsilon_{bias,s_l^n} = O(\frac{1}{\mathsf{poly(n)}})$. It would be interesting to get high confidence guarantees that $\epsilon_{bias,s_l^n} = O(\frac{1}{\mathsf{exp(n)}})$. Ultimately, because ϵ_{bias,s_l^n} determines the precision up to which recycling mitigation can still outperform postselection (see linear solving section), showing that $\epsilon_{bias,s_l^n} = O(\frac{1}{\mathsf{exp(n)}})$ would guarantee that recycling mitigation outperforms postselection for up to exponentially small additive errors. This additive error on estimating output probabilities of linear optical circuits is beyond what can be simulated efficiently classically.

In this section, we explore one attempt to tighten the bound on ϵ_{bias,s_l^n}. We will work in the no-collision regime where $m \gg n^2$, so that we can approximate output probabilities as moduli squared of permanents of i.i.d. Gaussian matrices, with appropriate rescaling. In this regime, the interference term $I_{s_l^n,k}$ can be thought of as a sum of N_k' random variables

$$X_i := \frac{|\mathsf{Per}(G_i)|^2}{m^n}, \tag{43}$$

for $i \in \{1, \ldots, N_k'\}$, where G_i is an i.i.d. Gaussian $n \times n$ matrix with entries chosen independently from $\mathcal{N}_\mathbb{C}(0,1)$. Note that for some i and j, it is possible that G_i or G_j share some rows in common, or are even equal. This corresponds to the fact that the interference term is in general a sum of probabilities of output bit strings sharing some overlap (meaning their associated permanents have common rows [6]). This means that the random variables X_i need not all be independent. Nevertheless, we will look at a sum of independent random variables X_i having the form of eqn. (43). Our reason for working with an independent sum is that it simplifies the analysis, while giving an intuition about what to expect in the more general case of possibly dependent X_i's. We will further comment on this point below.

In the remainder of this section, we will show that a $\mathsf{poly}(n)$ sized sum of independent X_i's distributed according to eqn. (43) *does not* verify a sufficient condition, the Lyapunov condition [66], for the central limit theorem (CLT) to hold as $n \to \infty$. Our proof relies on a conjecture of [61] on the expectation value $\mathbf{E}(X_i^t)$ over the set of Gaussian matrices G_i, where $t \in \mathbb{N}$ and $t > 2$. We also provide numerical evidence that the distribution of this sum is indeed not a normal distribution. Our result shows that it is non-trivial to improve the bound on ϵ_{bias,s_l^n} by trying to link $I_{s_l^n,k}$ to some known probability distribution, which was our initial motivation for trying to prove a CLT convergence result.

Let Y_i, $i \in \{1, \ldots, N\}$ be real, independent and identically distributed, random variables satisfying $\mathbf{E}(Y_i) = 0$, where $\mathbf{E}(.)$ denotes expectation value over the distribution of the Y_i's. Let

$$S_N := \sum_{i=1,\ldots,N} Y_i, \tag{44}$$

and

$$\sigma_N := \sqrt{\mathbf{E}(S_N^2)}. \tag{45}$$

Furthermore, for any $r > 2$ and $r \in \mathbb{N}$ let

$$\mathbf{E}(|Y|^r) := \mathbf{E}(|Y_i|^r), \tag{46}$$

for all $i \in \{1, \ldots, N\}$. The Lyapunov condition can be stated in this case as [66]

Theorem 15. (Lyapunov condition) *If for some fixed $r > 2$,*

$$N \frac{\mathbf{E}(|Y|^r)}{\sigma_N^r} \to 0, \tag{47}$$

then as $N \to \infty$

$$\frac{S_N}{\sigma_N} \to^d \mathcal{N}(0,1). \tag{48}$$

Where $\to^d$ denotes convergence of the distribution of $\frac{S_N}{\sigma_N}$.

For dependent identically distributed random variables, which correspond to the probabilities constituting the interference term, the Lyapunov condition becomes stricter to verify. In particular, for a specific type of dependence, $M(n)$-dependent random variables [66], the numerator in eqn. (47) is multiplied by $M(n)^{r-1}$, where $M(n) > 1$ and $M(n) \in \mathbb{N}$ is an integer whose value can depend on n [66]. One would expect, as mentioned earlier, that the non-convergence results established here for the case of independent random variables hold as well for the dependent case, although we do not formally prove this.

A final ingredient we will use is the following conjecture appearing in [61] (Section 4.8, Conjecture 4.12), and whose truth is supported by numerical simulations performed in [61].

Conjecture 16. *For $n, t > 2$, $n, t \in \mathbb{N}$*

$$\mathbf{E}_{G \in \mathcal{G}_{n \times n}}(|\mathsf{Per}(G)|^{2t}) = O\left(\frac{(n!)^{2t}(t!)^{2n}}{(nt)!}\right). \tag{49}$$

Where $\mathbf{E}_{G \in \mathcal{G}_{n \times n}}(.)$ is the expectation value over the set of $n \times n$ Gaussian matrices with entries from $\mathcal{N}_{\mathbb{C}}(0,1)$. For simplicity, we will henceforth denote $\mathbf{E}_{G \in \mathcal{G}_{n \times n}}(.)$ as $\mathbf{E}(.)$. Let

$$Y_i := X_i - \frac{n!}{m^n}, \tag{50}$$

where X_i are as defined in eqn. (43), and are independent, identically distributed random variables, $i \in \{1, \ldots, N\}$. It is immediate to observe that $\mathbf{E}(Y_i) = 0$. We show the following in appendix 22.

Theorem 17. *For all $r > 2$, $n \gg 1$, and for the independent, identically distributed random variables Y_i defined in eqn. (50), we have that*

$$N \frac{\mathbf{E}(|Y|^r)}{\sigma_N^r} \geq O\left(\frac{\beta(r)^n}{N^{\frac{r}{2}-1}}\right), \tag{51}$$

where $\beta(r) > 1$ is a positive real number dependent on r.

When $N = \mathsf{poly}(n)$, Thm. 17 shows that Lyapunov condition is not satisfied. Although this condition is sufficient, but not necessary, for the CLT to hold we provide numerical evidence that $N = n^2$, $N = n^3$, and $N = n^4$ sized sums of i.i.d. Gaussian matrices do not converge to a normal distribution for $n \in \{2, 3, 4, 5\}$. We plot our results for $N = n^3$ and $n \in \{2, 3, 4, 5\}$ in Figures 6 (a)-(d).

It is interesting to note that in Thm. 17, when $N = O(\mathsf{exp}(n))$, the lower bound on the Lyapunov condition can converge to 0 as $n \to \infty$. Marginal probabilities corresponding to a large number of lost photons are sums of, possibly exponential, numbers of probabilities having the form of eqn. (43). Our result provides evidence that these marginals are asymptotically normally distributed, and therefore efficient to sample from. Although low order as well as high loss marginals of boson sampling are known to be easy to compute and sample from [58, 67, 17], our result might provide a new perspective on simulating lossy boson sampling marginals by linking these to normally distributed random variables.
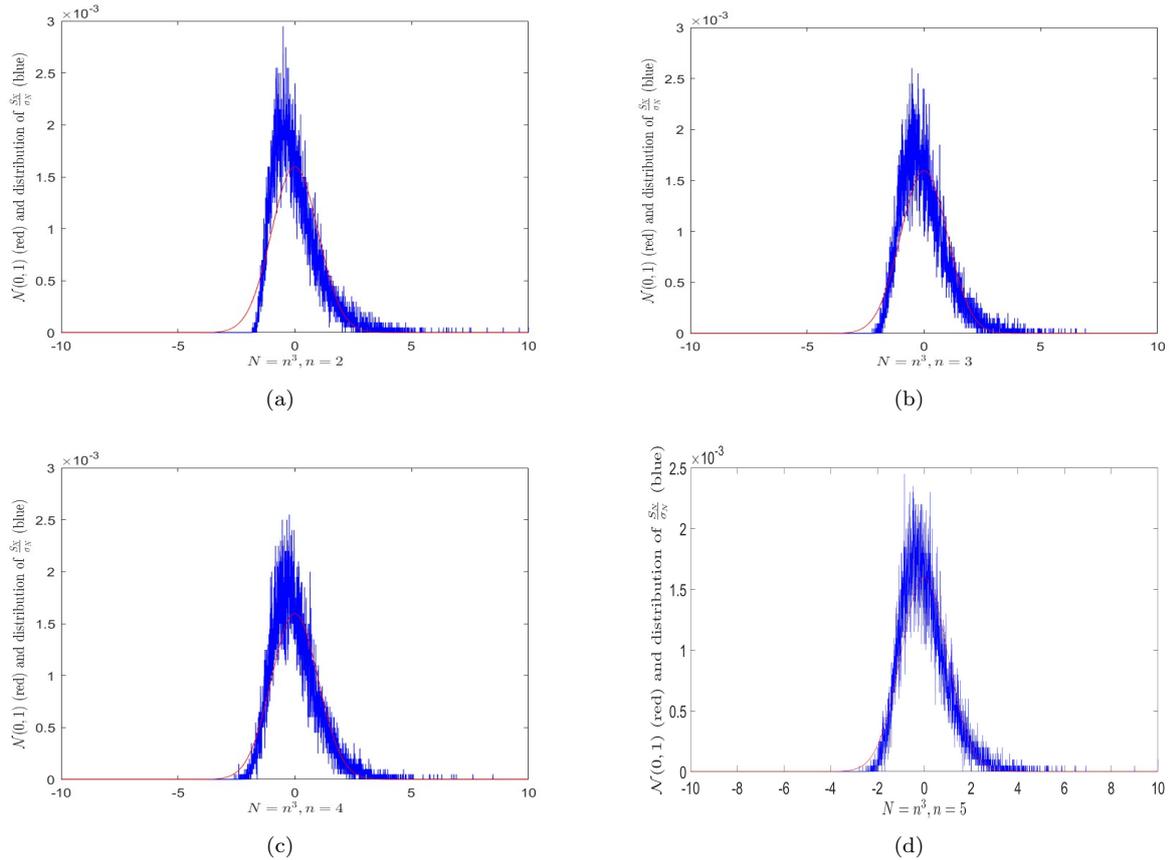
(a)   (b)

(c)   (d)

Figure 6: (a)-(b) *Comparison of distributions of $\frac{S_N}{\sigma_N}$ and the normal $\mathcal{N}(0,1)$ distribution.* (a) Distribution of $\frac{S_N}{\sigma_N}$ compared to the normal $\mathcal{N}(0,1)$ distribution. $n = 2$, $N = n^3$, and 20000 samples of i.i.d. Gaussian matrices were used to construct the distribution of $\frac{S_N}{\sigma_N}$. (b) Distribution of $\frac{S_N}{\sigma_N}$ compared to the normal $\mathcal{N}(0,1)$ distribution. $n = 3$, $N = n^3$, and 20000 samples of i.i.d. Gaussian matrices were used to construct the distribution of $\frac{S_N}{\sigma_N}$. (c) Distribution of $\frac{S_N}{\sigma_N}$ compared to the normal $\mathcal{N}(0,1)$ distribution. $n = 4$, $N = n^3$, and 20000 samples of i.i.d. Gaussian matrices were used to construct the distribution of $\frac{S_N}{\sigma_N}$. (d) Distribution of $\frac{S_N}{\sigma_N}$ compared to the normal $\mathcal{N}(0,1)$ distribution. $n = 5$, $N = n^3$, and 20000 samples of i.i.d. Gaussian matrices were used to construct the distribution of $\frac{S_N}{\sigma_N}$.

14   Discussion and open questions

In summary, we have presented a family of techniques for mitigating the effects of photon loss on the outputs of linear optical quantum circuits in the discrete variable setting. We provide analytical and numerical evidence that these techniques outperform postselection - currently the standard method of dealing with loss in linear optical circuits.

We first described a novel construction of objects that we refer to as recycled probabilities, each of which may be decomposed into a convex combination of ideal probability and an interference term. The deviation of interference terms around their expected values (both for circuits implementing arbitrary unitary matrices and unitary matrices chosen at random over the Haar measure) was then bounded with high confidence. This allowed the derivation of analytical bounds on the accuracy with which the ideal probability signal may be extracted from recycled probabilities. A number of methods for classically postprocessing the recycled probabilities to generate loss-mitigated outputs were presented, namely linear solving, linear solving with dependency, linear extrapolation, and exponential extrapolation. Analytical worst-case performance guarantees were given, indicating that with high confidence and in the regime of high photon loss rate ($\eta > 0.5$) these methods can outperform postselection. Furthermore, numerical evidence was provided which also strongly indicates that there is a regime where the loss mitigation techniques outperform postselection.

To demonstrate how the analytical statements may be related to a real experiment, numerical experiments were performed for a computation of size $m = 100$ modes, $n = 10$ photons with a uniform probability of photon loss with rate $\eta = 0.8$ and the results were found to be in agreement with the analytical performance guarantee. Next, evidence was provided that, in the DVLOQC setting, photon loss mitigation techniques based on zero noise extrapolation will not outperform postselection for any loss rate. Finally we discussed how these techniques may be applied to mitigate both expectation values of observables as well as full distributions, and the question of whether the interference term bounds for Haar random unitaries may be improved upon.

There are many potential avenues and open questions that would further develop the work presented here. The first of these is extending our techniques to the collision regime (where more than one photon can occupy each mode). The construction of the recycled probabilities, and consequently the mitigated probabilities extends straightforwardly to this regime. However, a number of the proof techniques, such as those used in computing expectation values of the interference terms, do not. One of the main technical reasons are that in the collision regime, the probabilities of Haar random circuits are no longer given by permanents of i.i.d. Gaussians [6]. Although [61] gives a closed-form expression of the expectation (over the Haar measure) of permanents minors of Haar random matrices, thereby overcoming the previously mentioned issue, however this expression does not take into account repeated rows in the minors, which one typically encounters in the collision regime [6]. Another reason is because $\mathcal{L}(s_l^n)$ has a cardinality which varies when s_l^n is a bit string with collisions. For example, if $s_l^n = (n, 0, \dots, 0)$, then $|\mathcal{L}(s_l^n)| = 1$, whereas when s_l^n is a no-collision bit string $|\mathcal{L}(s_l^n)| = \binom{n}{k}$, for any such bit string. We conjecture that the proofs derived here, or similar versions of them, should hold in the collision regime as well.

Another set of questions arises when considering linear solving with dependency. Firstly, as information about the dependency of interference terms on ideal probabilities within recycled probabilities can be used to enhance the numerical performance of linear solving, it would be interesting to see whether this can be formally linked to the phenomenon of non-uniformity of the marginal probabilities in distributions generated by linear optical circuits. As this might explain why the d_k term is always observed to be positive in our numerics. Secondly, the bias error bound for linear solving with dependency is very similar to the bound for linear solving without dependency, however, we expect linear solving with dependency to have a lower bias error than without including dependency. The better performance of linear solving with dependency term over linear solving without dependency can be seen in our simulations, as for $d_k > 0$ the set of interference terms is positively correlated with the set of ideal probabilities with a magnitude quantified by d_k. Hence the following conjecture.

Conjecture 18. *The following inequality holds for distributions generated by linear optical circuits constructed from both Haar random and non-Haar random matrices for $d_k \geq 0$ and $k \leq n$,*

$$
\mathbf{E}_{D_R^k}\left(\left|I_{s_l^n,k} - \frac{\left(\binom{m-n+k}{k} - 1\right)}{\binom{m-n+k}{k}}p_{unif}\right|\right)
$$
$$
\geq \mathbf{E}_{D_R^k}\left(\left|I_{s_l^n,k} - \frac{\left(\binom{m-n+k}{k} - 1\right)}{\binom{m-n+k}{k}}\left((1 - d_k)p_{unif} + d_k p(s_l^n)\right)\right|\right). \tag{52}
$$

We believe this is true due to the strong numerical evidence that linear solving with dependency outperforms linear solving without dependency. Furthermore, as mentioned previously, we conjecture that $d_k \geq 0$ for any linear optical circuit.

In all numerical simulations it was observed that low loss statistics give better results than high loss ones, this fact is also implied by Lemma 1. This leads us to the two following conjectures.

Conjecture 19. $\binom{m-n+k}{k}|\epsilon_{bias,s_l^n}|$ *is a monotonically increasing function with k. Similarly, the total variation distance of the mitigated output distribution from the ideal distribution monotonically increases with increasing k.*

Conjecture 20. *The dependency term d_k is monotonically decreasing with increasing k.*

Another interesting question, motivated by finding ways to eliminate the bias error in our introduced mitigation techniques is whether one can *exactly* represent an n–photon probability as a sum of $n-k$–photon probabilities. This can be done for the case of minors of unitary matrices with real and positive entries, such as those used to solve graph problems in DVLOQC [22]. As an example, consider a Laplace expansion of

the permanent of an $n \times n$ minor $U_{\mathsf{s},\mathsf{t}} := (u_{ij})_{i,j \in \{1,\dots,n\}}$ of a linear optical unitary U, this reads

$$\mathsf{Per}(U_{\mathsf{s},\mathsf{t}}) = u_{11}\mathsf{Per}\begin{pmatrix} u_{22} & \dots & u_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ u_{n2} & \dots & u_{nn} \end{pmatrix} + \dots + u_{1n}\mathsf{Per}\begin{pmatrix} u_{21} & \dots & u_{2,n-1} \\ . & . & . \\ . & . & . \\ . & . & . \\ u_{n1} & \dots & u_{n,n-1} \end{pmatrix}.$$

If $U_{\mathsf{s},\mathsf{t}}$ is composed of positive real entries, then $\mathsf{Per}(U_{\mathsf{s},\mathsf{t}})$ is proportional to the square root of a probability $\sqrt{p(s|t)}$ of a certain output of linear optical circuit U with n input photons [6]. In turn, each of the permanents of $n-1 \times n-1$ matrices on the right hand side of the above equation is proportional to the square root of a probability of a certain output of a linear optical circuit with $n-1$ input photons. Thus, in the case of minors with positive real entries, there is an exact way to represent an n–photon probability as a sum of $n-1$ photon probabilities. In this case as well, the process can be generalised, by successive Laplace expansions, to expressing n–photon probabilities exactly as a sum of $n-k$–photon probabilities.

This idea of exactly representing n–photon probabilities as sums of $n-k$–photon probabilities is interesting since, in the presence of photon loss, $n-k$–photon experiments in general produce more converged statistics than n–photon experiments, for a comparable number of runs of the experiments in both cases. Furthermore, since the n–photon probabilities are exactly expressible as a sum of $n-k$–photon probabilities, one can use the $n-k$–photon experiments to compute the n–photon probabilities and *indefinitely* (because of lower statistical error) outperform postselection. It is an interesting question to determine whether similar results hold for broader classes of matrices.

Acknowledgments

References

[1] E. Knill, R. Laflamme, and G. J. Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, 409(6816):46–52, January 2001. Number: 6816 Publisher: Nature Publishing Group.

[2] R. Raussendorf, J. Harrington, and K. Goyal. Topological fault-tolerance in cluster state quantum computation. *New Journal of Physics*, 9(6):199, June 2007.

[3] Grégoire de Gliniasty, Paul Hilaire, Pierre-Emmanuel Emeriau, Stephen C. Wein, Alexia Salavrakos, and Shane Mansfield. A Spin-Optical Quantum Computing Architecture. November 2023. arXiv:2311.05605 [quant-ph].

[4] Sara Bartolucci, Patrick Birchall, Hector Bombín, Hugo Cable, Chris Dawson, Mercedes Gimeno-Segovia, Eric Johnston, Konrad Kieling, Naomi Nickerson, Mihir Pant, Fernando Pastawski, Terry Rudolph, and Chris Sparrow. Fusion-based quantum computation. *Nature Communications*, 14(1):912, February 2023. Number: 1 Publisher: Nature Publishing Group.

[5] Robert Raussendorf and Hans J. Briegel. A One-Way Quantum Computer. *Physical Review Letters*, 86(22):5188–5191, May 2001. Publisher: American Physical Society.

[6] Scott Aaronson and Alex Arkhipov. The computational complexity of linear optics. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, STOC '11, pages 333–342, New York, NY, USA, June 2011. Association for Computing Machinery.

[7] Pascale Senellart, Glenn Solomon, and Andrew White. High-performance semiconductor quantum-dot single-photon sources. *Nature Nanotechnology*, 12(11):1026–1039, November 2017.

[8] Michael Reck, Anton Zeilinger, Herbert J. Bernstein, and Philip Bertani. Experimental realization of any discrete unitary operator. *Physical Review Letters*, 73(1):58–61, July 1994.

[9] Robert H. Hadfield. Single-photon detectors for optical quantum information applications. *Nature Photonics*, 3(12):696–705, December 2009.

[10] Hui Wang, Wei Li, Xiao Jiang, Y.-M. He, Y.-H. Li, X. Ding, M.-C. Chen, J. Qin, C.-Z. Peng, C. Schneider, M. Kamp, W.-J. Zhang, H. Li, L.-X. You, Z. Wang, J.P. Dowling, S. Höfling, Chao-Yang Lu, and Jian-Wei Pan. Toward Scalable Boson Sampling with Photon Loss. *Physical Review Letters*, 120(23):230502, June 2018. Publisher: American Physical Society.

[11] Daniel J. Brod, Ernesto F. Galvão, Andrea Crespi, Roberto Osellame, Nicolò Spagnolo, and Fabio Sciarrino. Photonic implementation of boson sampling: a review. *Advanced Photonics*, 1(3):034001, May 2019. Publisher: SPIE.

[12] Jelmer Renema, Valery Shchesnovich, and Raul Garcia-Patron. Classical simulability of noisy boson sampling. April 2019. arXiv:1809.01953 [cond-mat, physics:physics, physics:quant-ph].

[13] Kristan Temme, Sergey Bravyi, and Jay M. Gambetta. Error Mitigation for Short-Depth Quantum Circuits. *Physical Review Letters*, 119(18):180509, November 2017. Publisher: American Physical Society.

[14] Ying Li and Simon C. Benjamin. Efficient Variational Quantum Simulator Incorporating Active Error Minimization. *Physical Review X*, 7(2):021050, June 2017. Publisher: American Physical Society.

[15] Daiqin Su, Robert Israel, Kunal Sharma, Haoyu Qi, Ish Dhand, and Kamil Brádler. Error mitigation on a near-term quantum photonic device. *Quantum*, 5:452, May 2021. Publisher: Verein zur Förderung des Open Access Publizierens in den Quantenwissenschaften.

[16] Alexandra E. Moylett, Raúl García-Patrón, Jelmer J. Renema, and Peter S. Turner. Classically simulating near-term partially-distinguishable and lossy boson sampling. *Quantum Science and Technology*, 5(1):015001, November 2019. Publisher: IOP Publishing.

[17] Scott Aaronson and Daniel J. Brod. BosonSampling with lost photons. *Physical Review A*, 93(1):012335, January 2016.

[18] Nicolas Maring, Andreas Fyrillas, Mathias Pont, Edouard Ivanov, Petr Stepanov, Nico Margaria, William Hease, Anton Pishchagin, Thi Huong Au, Sébastien Boissier, Eric Bertasi, Aurélien Baert, Mario Valdivia, Marie Billard, Ozan Acar, Alexandre Brieussel, Rawad Mezher, Stephen C. Wein, Alexia Salavrakos, Patrick Sinnott, Dario A. Fioretto, Pierre-Emmanuel Emeriau, Nadia Belabas, Shane Mansfield, Pascale Senellart, Jean Senellart, and Niccolo Somaschi. A general-purpose single-photon-based quantum computing platform. June 2023. arXiv:2306.00874 [quant-ph].

[19] Donghwa Lee, Jinil Lee, Seongjin Hong, Hyang-Tag Lim, Young-Wook Cho, Sang-Wook Han, Hyundong Shin, Junaid ur Rehman, and Yong-Su Kim. Error-mitigated photonic variational quantum eigensolver using a single-photon ququart. *Optica*, 9(1):88–95, January 2022. Publisher: Optica Publishing Group.

[20] Nicolas Heurtel, Andreas Fyrillas, Grégoire de Gliniasty, Raphaël Le Bihan, Sébastien Malherbe, Marceau Pailhas, Eric Bertasi, Boris Bourdoncle, Pierre-Emmanuel Emeriau, Rawad Mezher, Luka Music, Nadia Belabas, Benoît Valiron, Pascale Senellart, Shane Mansfield, and Jean Senellart. Perceval: A Software Platform for Discrete Variable Photonic Quantum Computing. *Quantum*, 7:931, February 2023. Publisher: Verein zur Förderung des Open Access Publizierens in den Quantenwissenschaften.

[21] Beng Yee Gan, Daniel Leykam, and Dimitris G. Angelakis. Fock State-enhanced Expressivity of Quantum Machine Learning Models. *EPJ Quantum Technology*, 9(1):16, December 2022. arXiv:2107.05224 [physics, physics:quant-ph, stat].

[22] Rawad Mezher, Ana Filipa Carvalho, and Shane Mansfield. Solving graph problems with single photons and linear optics. *Physical Review A*, 108(3):032405, September 2023. Publisher: American Physical Society.

[23] Alexia Salavrakos, Tigran Sedrakyan, James Mills, and Rawad Mezher. An error-mitigated photonic quantum circuit Born machine. May 2024. arXiv:2405.02277 [quant-ph].

[24] Marcello Benedetti, Erika Lloyd, Stefan Sack, and Mattia Fiorentini. Parameterized quantum circuits as machine learning models. *Quantum Science and Technology*, 4(4):043001, November 2019. Publisher: IOP Publishing.

[25] Scott Aaronson and Alex Arkhipov. Bosonsampling is far from uniform. *Quantum Information & Computation*, 14(15-16):1383–1423, November 2014.

[26] The Collected Works of Wassily Hoeffding - Wassily Hoeffding - Google Books.

[27] Zhengyan Lin and Zhidong Bai. *Probability Inequalities*. Springer, Berlin, Heidelberg, 2011.

[28] Ross Berkowitz and Pat Devlin. A stability result using the matrix norm to bound the permanent. *Israel Journal of Mathematics*, 224(1):437–454, April 2018.

[29] Giacomo De Palma, Milad Marvian, Cambyse Rouzé, and Daniel Stilck França. Limitations of Variational Quantum Algorithms: A Quantum Optimal Transport Approach. *PRX Quantum*, 4(1):010309, January 2023. Publisher: American Physical Society.

[30] Ryuji Takagi, Hiroyasu Tajima, and Mile Gu. Universal Sampling Lower Bounds for Quantum Error Mitigation. *Physical Review Letters*, 131(21):210602, November 2023. Publisher: American Physical Society.

[31] Yihui Quek, Daniel Stilck França, Sumeet Khatri, Johannes Jakob Meyer, and Jens Eisert. Exponentially tighter bounds on limitations of quantum error mitigation. February 2023. arXiv:2210.11505 [math-ph, physics:quant-ph].

[32] Suguru Endo, Zhenyu Cai, Simon C. Benjamin, and Xiao Yuan. Hybrid Quantum-Classical Algorithms and Quantum Error Mitigation. *Journal of the Physical Society of Japan*, 90(3):032001, March 2021. Publisher: The Physical Society of Japan.

[33] Suguru Endo, Simon C. Benjamin, and Ying Li. Practical Quantum Error Mitigation for Near-Future Applications. *Physical Review X*, 8(3):031027, July 2018. Publisher: American Physical Society.

[34] Tudor Giurgica-Tiron, Yousef Hindy, Ryan LaRose, Andrea Mari, and William J. Zeng. Digital zero noise extrapolation for quantum error mitigation. In *2020 IEEE International Conference on Quantum Computing and Engineering (QCE)*, pages 306–316, October 2020.

[35] Andre He, Benjamin Nachman, Wibe A. de Jong, and Christian W. Bauer. Zero-noise extrapolation for quantum-gate error mitigation with identity insertions. *Physical Review A*, 102(1):012426, July 2020. Publisher: American Physical Society.

[36] Armands Strikis, Dayue Qin, Yanzhu Chen, Simon C. Benjamin, and Ying Li. Learning-Based Quantum Error Mitigation. *PRX Quantum*, 2(4):040330, November 2021. Publisher: American Physical Society.

[37] Andrea Mari, Nathan Shammah, and William J. Zeng. Extending quantum probabilistic error cancellation by noise scaling. *Physical Review A*, 104(5):052607, November 2021. Publisher: American Physical Society.

[38] Ewout van den Berg, Zlatko K. Minev, Abhinav Kandala, and Kristan Temme. Probabilistic error cancellation with sparse Pauli–Lindblad models on noisy quantum processors. *Nature Physics*, 19(8):1116–1121, August 2023. Number: 8 Publisher: Nature Publishing Group.

[39] X. Bonet-Monroig, R. Sagastizabal, M. Singh, and T. E. O'Brien. Low-cost error mitigation by symmetry verification. *Physical Review A*, 98(6):062339, December 2018. Publisher: American Physical Society.

[40] R. Sagastizabal, X. Bonet-Monroig, M. Singh, M. A. Rol, C. C. Bultink, X. Fu, C. H. Price, V. P. Ostroukh, N. Muthusubramanian, A. Bruno, M. Beekman, N. Haider, T. E. O'Brien, and L. DiCarlo. Experimental error mitigation via symmetry verification in a variational quantum eigensolver. *Physical Review A*, 100(1):010302, July 2019. Publisher: American Physical Society.

[41] Thomas E. O'Brien, Stefano Polla, Nicholas C. Rubin, William J. Huggins, Sam McArdle, Sergio Boixo, Jarrod R. McClean, and Ryan Babbush. Error Mitigation via Verified Phase Estimation. *PRX Quantum*, 2(2):020317, May 2021. Publisher: American Physical Society.

[42] Rawad Mezher, James Mills, and Elham Kashefi. Mitigating errors by quantum verification and postselection. *Physical Review A*, 105(5):052608, May 2022. Publisher: American Physical Society.

[43] Bálint Koczor. Exponential Error Suppression for Near-Term Quantum Devices. *Physical Review X*, 11(3):031057, September 2021.

[44] Bálint Koczor. The dominant eigenvector of a noisy quantum state. *New Journal of Physics*, 23(12):123047, December 2021. Publisher: IOP Publishing.

[45] William J. Huggins, Sam McArdle, Thomas E. O'Brien, Joonho Lee, Nicholas C. Rubin, Sergio Boixo, K. Birgitta Whaley, Ryan Babbush, and Jarrod R. McClean. Virtual Distillation for Quantum Error Mitigation. *Physical Review X*, 11(4):041036, November 2021. Publisher: American Physical Society.

[46] Kaoru Yamamoto, Suguru Endo, Hideaki Hakoshima, Yuichiro Matsuzaki, and Yuuki Tokunaga. Error-Mitigated Quantum Metrology via Virtual Purification. *Physical Review Letters*, 129(25):250503, December 2022. Publisher: American Physical Society.

[47] Jarrod R. McClean, Zhang Jiang, Nicholas C. Rubin, Ryan Babbush, and Hartmut Neven. Decoding quantum errors with subspace expansions. *Nature Communications*, 11(1):636, January 2020. Number: 1 Publisher: Nature Publishing Group.

[48] Nobuyuki Yoshioka, Hideaki Hakoshima, Yuichiro Matsuzaki, Yuuki Tokunaga, Yasunari Suzuki, and Suguru Endo. Generalized Quantum Subspace Expansion. *Physical Review Letters*, 129(2):020502, July 2022. Publisher: American Physical Society.

[49] Bo Yang, Nobuyuki Yoshioka, Hiroyuki Harada, Shigeo Hakkaku, Yuuki Tokunaga, Hideaki Hakoshima, Kaoru Yamamoto, and Suguru Endo. Dual-GSE: Resource-efficient Generalized Quantum Subspace Expansion. September 2023. arXiv:2309.14171 [quant-ph].

[50] Yasuhiro Ohkura, Suguru Endo, Takahiko Satoh, Rodney Van Meter, and Nobuyuki Yoshioka. Leveraging hardware-control imperfections for error mitigation via generalized quantum subspace. March 2023. arXiv:2303.07660 [quant-ph].

[51] Filip B. Maciejewski, Zoltán Zimborás, and Michał Oszmaniec. Mitigation of readout noise in near-term quantum devices by classical post-processing based on detector tomography. *Quantum*, 4:257, April 2020. arXiv:1907.08518 [quant-ph].

[52] Andrew Arrasmith, Andrew Patterson, Alice Boughton, and Marco Paini. Development and Demonstration of an Efficient Readout Error Mitigation Technique for use in NISQ Algorithms. April 2023. arXiv:2303.17741 [quant-ph].

[53] James Mills, Debasis Sadhukhan, and Elham Kashefi. Simplifying errors by symmetry and randomisation. May 2023. arXiv:2303.02712 [quant-ph].

[54] Sergey Bravyi, Sarah Sheldon, Abhinav Kandala, David C. Mckay, and Jay M. Gambetta. Mitigating measurement errors in multiqubit experiments. *Physical Review A*, 103(4):042605, April 2021. Publisher: American Physical Society.

[55] Adam Taylor, Gabriele Bressanini, Hyukjoon Kwon, and M. S. Kim. Quantum error cancellation in photonic systems – undoing photon losses. March 2024. arXiv:2403.05252 [quant-ph].

[56] Yihui Quek, Daniel Stilck França, Sumeet Khatri, Johannes Jakob Meyer, and Jens Eisert. Exponentially tighter bounds on limitations of quantum error mitigation. February 2023. arXiv:2210.11505 [math-ph, physics:quant-ph].

[57] Deepesh Singh, Boxiang Fu, Gopikrishnan Muraleedharan, Chen-Mou Cheng, Nicolas Roussy Newton, Peter P. Rohde, and Gavin K. Brennen. Proof-of-work consensus by quantum sampling. May 2023. arXiv:2305.19865 [quant-ph, q-fin].

[58] Michał Oszmaniec and Daniel J. Brod. Classical simulation of photonic linear optics with lost particles. *New Journal of Physics*, 20(9):092002, September 2018. Publisher: IOP Publishing.

[59] Thomas M. Stace and Sean D. Barrett. Error correction and degeneracy in surface codes suffering loss. *Physical Review A*, 81(2):022317, February 2010. Publisher: American Physical Society.

[60] Raúl García-Patrón, Jelmer J. Renema, and Valery Shchesnovich. Simulating boson sampling in lossy architectures. *Quantum*, 3:169, August 2019. Publisher: Verein zur Förderung des Open Access Publizierens in den Quantenwissenschaften.

[61] Sepehr Nezami. Permanent of random matrices from representation theory: moments, numerics, concentration, and comments on hardness of boson-sampling. April 2021. arXiv:2104.06423 [quant-ph].

[62] Rajendra Bhatia and Chandler Davis. A Better Bound on the Variance. *The American Mathematical Monthly*, 107(4):353–357, April 2000.

[63] Kenneth Levenberg. A method for the solution of certain non-linear problems in least squares. *Quarterly of Applied Mathematics*, 2(2):164–168, 1944.

[64] Donald W. Marquardt. An Algorithm for Least-Squares Estimation of Nonlinear Parameters. *Journal of the Society for Industrial and Applied Mathematics*, 11(2):431–441, June 1963. Publisher: Society for Industrial and Applied Mathematics.

[65] Walter Gautschi. On inverses of Vandermonde and confluent Vandermonde matrices III. *Numerische Mathematik*, 29(4):445–450, April 1978.

[66] Svante Janson. A central limit theorem for m-dependent variables. August 2021. arXiv:2108.12263 [math].

[67] Benjamin Villalonga, Murphy Yuezhen Niu, Li Li, Hartmut Neven, John C. Platt, Vadim N. Smelyanskiy, and Sergio Boixo. Efficient approximation of experimental Gaussian boson sampling. February 2022. arXiv:2109.11525 [quant-ph].

[68] Herbert John Ryser. *Combinatorial Mathematics*. American Mathematical Soc., December 1963.

[69] J. L. W. V. Jensen. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Mathematica*, 30(none):175–193, January 1906. Publisher: Institut Mittag-Leffler.

[70] Scott Aaronson and Travis Hance. Generalizing and Derandomizing Gurvits's Approximation Algorithm for the Permanent. December 2012. arXiv:1212.0025 [quant-ph].

Appendix

Table of Contents

15 Results on useful regimes of photon loss

15.1 Proof of Lemma 1

We now prove Lemma 1 from the main text:

Let $k = n - r$, there is a classical algorithm running in time $O(2^{r-1} r (\binom{n}{n-r})^2)$ which exactly computes $\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k})$.

Proof. Each permanent $\mathsf{Per}(X_i)$ in $p(s_i^{n-k})$ can be computed exactly via Ryser's algorithm [68] in time $O(2^{r-1} r)$. Thus, each $p(s_i^{n-k})$ can be computed in $O(\binom{n}{n-r} 2^{r-1} r)$ time. Further, we need $O(\binom{n}{n-r}^2 2^{r-1} r)$ time to compute all the $p(s_i^{n-k})$'s, and therefore to compute $\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} p(s_i^{n-k})$, by an iterative approach where we first set a variable $\mathsf{sum} = 0$, then iteratively add each computed $p(s_i^{n-k})$ to sum as soon as we compute it. $\qquad\square$

16 Statistical error bounds for probability estimators

We now give upper bounds for the statistical error of the estimators of the ideal probabilities from postselection, and for the estimators of the recycled probabilities. These are used in later proofs.

16.1 Proof of Lemma 21

Lemma 21. *The statistical error of each postselected probability, $\epsilon_{hoeff,\tilde{p}^0(s_l^n)}$, is upper bounded*

$$|\epsilon_{hoeff,\tilde{p}^0(s_l^n)}| \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right).$$

Proof. For each sample w ranging from 1 to $N_{est,0}$ assign a value 1 to a random variable $X_w \in \{0,1\}$ if sample $w \in \mathcal{L}(s_l^n)$, and assign the value 0 to X_w otherwise. The estimator $\tilde{p}^0(s_i^n)$ of $p^0(s_i^n)$ is then

$$\tilde{p}^0(s_l^n) = \frac{\sum_w X_w}{N_{est,0}}.$$

with

$$|\tilde{p}^0(s_l^n) - p^0(s_l^n)| \leq |\epsilon_{\text{hoeff},\tilde{p}^0(s_l^n)}|.$$

There are $\binom{m}{n}$ postselected probabilities, and so if we use $N_{est,0} \approx \frac{\eta^0 (1-\eta)^{n-0} N_{tot}}{\binom{m}{n}}$ samples to estimate each $p^0(s_l^n)$, Hoeffding's inequality [26] guarantees with high confidence

$$|\epsilon_{\text{hoeff},\tilde{p}^0(s_l^n)}| \leq O\left(\frac{1}{\sqrt{N_{est,0}}}\right) = O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right).$$

$\qquad\square$

16.2 Proof of Lemma 22

Lemma 22. *The statistical error of each recycled probability, $\epsilon_{hoeff,\tilde{p}_R^k(s_l^n)}$, is upper bounded*

$$|\epsilon_{hoeff,\tilde{p}_R^k(s_l^n)}| \leq O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right).$$

Proof. The probability of losing k out of n photons is

$$\Pr(k) = \binom{n}{k}\eta^k(1-\eta)^{n-k},$$

and so the number of samples from $n-k$–photon outputs is $N_{tot,k} \approx \binom{n}{k} N_{tot} \eta^k (1-\eta)^{n-k}$, for $k \in \{0, \dots, n\}$. There are $\binom{m}{n}$ recycled probabilities, and to guarantee the independence required for Hoeffding's inequality we use $N_{est,k} \approx \frac{\binom{n}{k} \eta^k (1-\eta)^{n-k} N_{tot}}{\binom{m}{n}}$ samples to estimate each $p_R^k(s_l^n)$. For each sample w ranging from 1 to $N_{est,k}$ assign the value 1 to a random variable $X_w \in \{0,1\}$ if sample $w \in \mathcal{L}(s_l^n)$, and assign the value 0 to X_w otherwise. The estimator $\tilde{p}_R^k(s_i^n)$ of $p_R^k(s_i^n)$ is then

$$\tilde{p}_R^k(s_l^n) = \frac{\sum_w X_w}{\binom{m-n+k}{k} N_{est,k}},$$

where the $\binom{m-n+k}{k}^{-1}$ term is the normalisation factor detailed in eqn. 9. Finally, Hoeffding's inequality guarantees with high confidence

$$|\epsilon_{\text{hoeff}, \tilde{p}_R^k(s_l^n)}| \leq O\left(\frac{1}{\binom{m+n-k}{k} \sqrt{N_{est,k}}} \right) = O\left(\frac{1}{\binom{m-n+k}{k}} \sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k} \eta^k N_{tot}}} \right),$$

where

$$|\tilde{p}_R^k(s_l^n) - p_R^k(s_l^n)| \leq |\epsilon_{\text{hoeff}, \tilde{p}_R^k(s_l^n)}|.$$

$\square$

17 Interference deviation bounds

17.1 Proof of Lemma 2

We now prove Lemma 2 from the main text:

For all s_l^n, we have

$$\mathbf{E}_{X \in \mathcal{G}_{n \times n}}\left(I_{s_l^n, k}\right) = p_{unif}\left(1 + O(m^{-1})\right) \approx p_{unif}.$$

with $\mathbf{E}_{X \in \mathcal{G}_{n \times n}}(.)$ the expectation value over the set $\mathcal{G}_{n \times n}$.

Proof. Let $N_k' := \left(\binom{m-n+k}{k} - 1\right)\binom{n}{k}$. For $k > 0$, $I_{s_l^n, k}$ is a sum of $N_k' := N_k - \binom{n}{k}$ terms of the form $p(s_j^n)\frac{1}{N_k}$. We can thus rewrite

$$I_{s_l^n, k} = \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n)\frac{1}{N_k'}.$$

From [25], when $m \gg n^2$, and U is Haar random, which corresponds to Boson sampling unitaries, each $p(s_j^n) \approx \frac{|\text{Per}(X_i)|^2}{m^n}$, where X_i is an $n \times n$ matrix of i.i.d. Gaussian random variables whose entries chosen from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(0,1)$ of mean 0 and variance 1. $\text{Per}(.)$ denotes the matrix permanent. Furthermore, from [25, 61], we know that

$$\mathbf{E}_{X \in \mathcal{G}_{n \times n}}(|\text{Per}(X)|^2) = n!,$$

where $\mathbf{E}_{X \in \mathcal{G}_{n \times n}}(.)$ is the expectation value over the set of $n \times n$ Gaussian matrices with entries from $\mathcal{N}_{\mathbb{C}}(0,1)$. We can reexpress

$$I_{s_l^n, k} = \frac{1}{N_k'} \sum_i \frac{|\text{Per}(X_i)|^2}{m^n}.$$

Now, from the linearity of the expected value

$$\mathbf{E}_{X \in \mathcal{G}_{n \times n}}(I_{s_l^n, k}) = \mathbf{E}_{X \in \mathcal{G}_{n \times n}}\left(\frac{1}{N_k'} \sum_i \frac{|\text{Per}(X_i)|^2}{m^n} \right) = \frac{1}{m^n N_k'} \sum_i \mathbf{E}_{X_i \in \mathcal{G}_{n \times n}}(|\text{Per}(X_i)|^2) = \frac{n!}{m^n},$$

where the rightmost part follows from the fact that $\mathbf{E}_{X_i \in \mathcal{G}_{n \times n}}(|\text{Per}(X_i)|^2) = n!$, for all i.

Now $p_{unif} = \binom{m}{n}^{-1}$, and

$$
\begin{aligned}
\binom{m}{n}^{-1} &= \frac{n!(m-n)!}{m!} \\
&= \frac{n!}{(m)(m-1)\dots(m-n+1)} \\
&= \frac{n!}{(m^n + O(m^{n-1}))}
\end{aligned}
$$

This leads to

$$
\frac{n!}{m^n} = \binom{m}{n}^{-1}\frac{(m^n + O(m^{n-1}))}{m^n},
$$

which then becomes

$$
\frac{n!}{m^n} = p_{unif}\big(1 + O(m^{-1})\big).
$$

Which was the result to be proved. $\square$

17.2 Proof of Thm. 3

We now prove Lemma 3 from the main text:

The deviation of interference terms around p_{unif} for Haar random matrices is bounded

$$
Pr\Big(\big|I_{s_l^n,k} - p_{unif}\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{np_{unif}^2}{\epsilon_{bias,s_l^n}^2},
$$

where ϵ_{bias,s_l^n} is a positive real number.

Proof. Let $X := \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}),j\neq l} p(s_j^n)$, $M := \binom{n}{k}\big(\binom{m-n+k}{k}-1\big)$. For simplicity, relabel $p(s_j^n) := p_i$, so that X can be rewritten as $X = \sum_{i=1,\dots,M} p_i$ is a sum of, possibly dependent, random variables p_i and $\mathsf{E}_U(p_i) := \mathsf{E}_U(p) = \frac{n!}{m^n} \approx p_{unif}$, where $\mathsf{E}_U(p_i)$ is the expectation value over the Haar measure of the m-mode unitary group $\mathsf{U}(m)$. From [6], we know that this expectation value is given by $\frac{n!}{m^n}$ in the no-collision regime. Let $\sigma^2 = \mathsf{Var}(X) = \mathsf{E}_U(X^2) - (\mathsf{E}_U(X))^2$. Expanding out, we get that $\sigma^2 = M\mathsf{E}_U(p^2) - M(\mathsf{E}_U(p))^2 + 2\sum_i \sum_{j>i} \mathsf{Cov}(p_i,p_j)$, where $\mathsf{Cov}(p_i,p_j) = \mathsf{E}_U(p_ip_j) - \mathsf{E}_U(p_i)\mathsf{E}_U(p_j)$, and $\mathsf{E}_U(p^2) = \mathsf{E}_U(p_i^2) = \frac{(n+1)!n!}{m^{2n}} \approx (n+1)p_{unif}^2$, for all i, and this is from [6]. Using the Cauchy-Schwarz inequality,

$$
|\mathsf{Cov}(p_i,p_j)| \leq \sqrt{\mathsf{Var}(p_i)\mathsf{Var}(p_j)} \leq \sqrt{(\mathsf{Var}(p))^2} \leq \mathsf{Var}(p) \leq \mathsf{E}_U(p^2) - (\mathsf{E}_U(p))^2
$$

Thus,

$$
\sigma^2 \leq M(\mathsf{E}_U(p^2)-(\mathsf{E}_U(p))^2)+2\sum_i \sum_{j>i}|\mathsf{Cov}(p_i,p_j)| \leq M(\mathsf{E}_U(p^2)-(\mathsf{E}_U(p))^2)+M(M-1)(\mathsf{E}_U(p^2)-(\mathsf{E}_U(p))^2) \leq
$$

$$
M^2(\mathsf{E}_U(p^2) - (\mathsf{E}_U(p))^2) \leq M^2 np_{unif}^2.
$$

Finally, using Chebyshev's inequality we have

$$
Pr\Big(\big|X - \mathsf{E}_U(X)\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{\sigma^2}{\epsilon_{bias,s_l^n}^2},
$$

and noting that $\mathsf{E}_U(X) = Mp_{unif} = \sum_i \mathsf{E}_U(p)$ from linearity of expectation value, then dividing both sides of $|X - \mathsf{E}_U(X)| \geq \epsilon_{bias,s_l^n}$ by M and redefining $\epsilon_{bias,s_l^n} := \frac{\epsilon_{bias,s_l^n}}{M}$, and finally using $\sigma^2 \leq M^2 np_{unif}^2$, we get

$$
Pr\Big(\big|I_{s_l^n,k} - p_{unif}\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{np_{unif}^2}{\epsilon_{bias,s_l^n}^2}.
$$

This completes the proof. $\square$

17.3   Proof of Lemma 4

We now prove Lemma 4 from the main text:

$$\mathbf{E}_{D_R^k}\left(I_{s_l^n,k}\right) = p_{unif},$$

where $\mathbf{E}_{D_R^k}(.)$ denotes the expectation value over D_R^k.

Proof. Recalling that

$$p_R(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k} I_{s_l^n,k},$$

and

$$I_{s_l^n,k} = \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}),j \neq l} p(s_j^n) \frac{1}{\left(\binom{m-n+k}{k}-1\right)\binom{n}{k}}.$$

$$\mathbf{E}_{D_R^k}\left(I_{s_l^n,k}\right) = \frac{1}{\binom{m}{n}} \sum_{l=1}^{\binom{m}{n}} \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}),j \neq l} p(s_j^n) \frac{1}{\left(\binom{m-n+k}{k}-1\right)\binom{n}{k}}$$

$$= \frac{1}{\binom{m}{n}\left(\binom{m-n+k}{k}-1\right)\binom{n}{k}} \sum_{l=1}^{\binom{m}{n}} \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \left(\sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) - p(s_l^n) \right)$$

$$= \frac{1}{\binom{m}{n}\left(\binom{m-n+k}{k}-1\right)\binom{n}{k}} \left(\sum_{l=1}^{\binom{m}{n}} \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) - \sum_{l=1}^{\binom{m}{n}} \binom{n}{k} p(s_l^n) \right)$$

$$= \frac{1}{\binom{m}{n}\left(\binom{m-n+k}{k}-1\right)\binom{n}{k}} \left(\sum_{l=1}^{\binom{m}{n}} \binom{m-n+k}{k}\binom{n}{k} p(s_l^n) - \sum_{l=1}^{\binom{m}{n}} \binom{n}{k} p(s_l^n) \right)$$

$$= \frac{1}{\binom{m}{n}\left(\binom{m-n+k}{k}-1\right)\binom{n}{k}} \left(\binom{m-n+k}{k}\binom{n}{k} - \binom{n}{k} \right)$$

$$= p_{unif}.$$

Where between the third and fourth lines we have used the cardinality of the sets $|\mathcal{L}(s_l^n)| = \binom{n}{k}$ and $|\mathcal{G}(s_i^{n-k})| = \binom{m-n+k}{k}$, and between the fourth and fifth lines that $\sum_{l=1}^{\binom{m}{n}} p(s_l^n) = 1$. $\qquad\square$

17.4   Proof of Thm. 5

We now prove Thm. 5 from the main text:

The deviation of interference terms around p_{unif} for an arbitrary matrix is bounded

$$\Pr\left(\left|I_{s_l^n,k} - p_{unif}\right| \geq \epsilon_{bias,s_l^n}\right) \leq \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2},$$

where ϵ_{bias,s_l^n} is a positive real number.

Proof. Let $X := I_{s_l^n,k}$ be a random variable defined over the uniform distribution of bit strings s_l^n that returns interference terms as values, $\mu := \mathsf{E}_{s_l^n}(X) = p_{unif} = \frac{1}{\binom{m}{n}}$, with $\mathsf{E}_{s_l^n}(.)$ denoting the expectation value of X over the uniform distribution of s_l^n, it is equal to p_{unif} from lemma 4. Let $\sigma^2 := \mathsf{Var}(X)$. From Chebyshev's inequality,

$$Pr(|X - \mu| \geq \epsilon_{bias,s_l^n}) \leq \frac{\sigma^2}{\epsilon_{bias,s_l^n}^2}$$

Let $M := \max_{s_l^n}(X)$, $m := \min_{s_l^n}(X)$, and note that $M \leq 1$ and $m \geq 0$. We now use the Bhatia-Davis inequality [62]

$$\sigma^2 \leq (M - \mu)(\mu - m)$$

with the upper and lower bounds on M and m to obtain

$$\sigma^2 \leq (1 - p_{unif})(p_{unif}) \leq p_{unif}.$$

Replacing this in Chebyshev's inequality completes the proof. $\square$

17.5   Proofs of Lemmas 6 and 7

Both proofs will make use of Jensen's inequality [69]:

Theorem 23. (Jensen's inequality) *Let $f(x)$ be a convex function on (a, b) and suppose $a < x_1 \leq x_2 \leq \ldots \leq x_n < b$. Then*

$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right).$$

Equality holds if, and only if, $x_1 = x_2 = \ldots = x_n$.

We now prove Lemma 6 from the main text:

The variance of the set of recycled probabilities is less than or equal to the variance of the set of ideal probabilities, that is

$$\mathsf{Var}\big(\{p(s_l^n)\}_l\big) \geq \mathsf{Var}\big(\{p_R^k(s_l^n)\}_l\big).$$

Proof. Let $\mu = p_{unif}$ and $f(x) = (x - \mu)^2$. Defining the variance of the ideal n output photon distribution as

$$\mathsf{Var}\big(\{p(s_l^n)\}_l\big) := \binom{m}{n}^{-1} \sum_l (p(s_l^n) - \mu)^2,$$

and the variance of the $n - k$ output photon recycled distribution as

$$\mathsf{Var}\big(\{p_R^k(s_l^n)\}_l\big) := \binom{m}{n}^{-1} \sum_l (p_R^k(s_l^n) - \mu)^2.$$

These definitions are equivalent to the variance of random variables uniformly distributed over the set of bit strings $\{s_l^n\}$ that return ideal n output photon probabilities in the first case, and $n - k$ output photon recycled probabilities in the second case. Recall that the recycled probabilities are defined by the expression

$$p_R^k(s_l^n) := \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\binom{m-n+k}{k}\binom{n}{k}}.$$

As $f(x)$ is concave, by applying Jensen's inequality it is true that

$$\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} \frac{f(p(s_j^n))}{\binom{m-n+k}{k}\binom{n}{k}} \geq f\left(\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\binom{m-n+k}{k}\binom{n}{k}}\right).$$

From which it follows

$$\binom{m}{n}^{-1} \sum_l \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} \frac{f(p(s_j^n))}{\binom{m-n+k}{k}\binom{n}{k}} \geq \binom{m}{n}^{-1} \sum_l f\left(\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\binom{m-n+k}{k}\binom{n}{k}}\right),$$

and after simplifying

$$\binom{m}{n}^{-1} \sum_l f(p(s_j^n)) \geq \binom{m}{n}^{-1} \sum_l f\left(\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\binom{m-n+k}{k}\binom{n}{k}}\right).$$

And this is just: $\mathsf{Var}\big(\{p(s_l^n)\}_l\big) \geq \mathsf{Var}\big(\{p_R^k(s_l^n)\}_l\big)$, the result to be shown. $\square$

We now prove Lemma 7 from the main text:

The variance of the set of interference terms is less than or equal to the variance of the set of ideal probabilities, that is

$$\mathsf{Var}\big(\{p(s_l^n)\}_l\big) \geq \mathsf{Var}\big(\{I_{s_l^n,k}\}_l\big).$$

Proof. Let $\mu = p_{unif}$ and $f(x) = (x - \mu)^2$. Defining the variance of the ideal n output photon distribution as

$$\mathsf{Var}\big(\{p(s_l^n)\}_l\big) := \binom{m}{n}^{-1} \sum_l (p(s_l^n) - \mu)^2,$$

and the variance of the interference terms for the $n - k$ output photon recycled distribution as

$$\mathsf{Var}\big(\{I_{s_l^n,k}\}_l\big) := \binom{m}{n}^{-1} \sum_l \big(I_{s_l^n,k} - \mu\big)^2.$$

These definitions are equivalent to the variance of random variables uniformly distributed over the set of bit strings $\{s_l^n\}$ that return ideal n output photon probabilities in the first case, and interference terms from the $n - k$ output photon recycled distribution in the second case. Recall that the interference terms of a recycled distribution are defined by the expression

$$I_{s_l^n,k} = \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{\big(\binom{m-n+k}{k} - 1\big)\binom{n}{k}}.$$

As $f(x)$ is concave, by applying Jensen's inequality it is true that

$$\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} \frac{f(p(s_j^n))}{\big(\binom{m-n+k}{k} - 1\big)\binom{n}{k}} \geq f\bigg(\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{\big(\binom{m-n+k}{k} - 1\big)\binom{n}{k}} \bigg).$$

From which it follows

$$\binom{m}{n}^{-1} \sum_l \sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} \frac{f(p(s_j^n))}{\big(\binom{m-n+k}{k} - 1\big)\binom{n}{k}}$$
$$\geq \binom{m}{n}^{-1} \sum_l f\bigg(\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k}), j \neq l} p(s_j^n) \frac{1}{\big(\binom{m-n+k}{k} - 1\big)\binom{n}{k}} \bigg),$$

and after simplifying

$$\binom{m}{n}^{-1} \sum_l f(p(s_j^n)) \geq \binom{m}{n}^{-1} \sum_l f\bigg(\sum_{s_i^{n-k} \in \mathcal{L}(s_l^n)} \sum_{s_j^n \in \mathcal{G}(s_i^{n-k})} p(s_j^n) \frac{1}{\big(\binom{m-n+k}{k} - 1\big)\binom{n}{k}} \bigg).$$

And this is just: $\mathsf{Var}\big(\{p(s_l^n)\}_l\big) \geq \mathsf{Var}\big(\{I_{s_l^n,k}\}_l\big)\}_l\big)$, the result to be shown. $\qquad\square$

17.6 Proof of Thm. 8

We now prove Thm. 8 from the main text:

The deviation of interference terms around p_{unif} for an arbitrary matrix is bounded

$$\Pr\Big(\big|I_{s_l^n,k} - p_{unif}\big| \geq \epsilon_{bias,s_l^n}\Big) \leq \frac{p_{unif}p_{upper}}{\epsilon_{bias,s_l^n}^2} + \delta\Big(1 - \frac{p_{unif}p_{upper}}{\epsilon_{bias,s_l^n}^2}\Big),$$

where ϵ_{bias,s_l^n} is a positive real number, and p_{upper} is an empirically computed upper bound on the largest probability of the ideal n output photon probability distribution with confidence $1 - \delta$.

Proof. Let $X := I_{s_l^n, k}$ be a random variable defined over the uniform distribution of bit strings s_l^n that returns interference terms as values. $\mathsf{E}_{s_l^n}(X)$ denotes the expectation value of random variable X over the uniform distribution of s_l^n, and let $\mu_X := \mathsf{E}_{s_l^n}(X) = p_{unif}$, the latter equality comes from lemma 4. Let $Y := p(s_l^n)$ be a random variable defined over the uniform distribution of bit strings s_l^n that returns n–photon output state probabilities as values, and $\mu_Y := \mathsf{E}_{s_l^n}(Y) = p_{unif}$, where $\mathsf{E}_{s_l^n}(Y)$ denotes the expectation value of random variable Y over the uniform distribution of s_l^n. Using Chebyshev's inequality we have that

$$Pr(|X - \mu_X| \geq \epsilon_{bias, s_l^n}) \leq \frac{\mathsf{Var}(X)}{\epsilon_{bias, s_l^n}^2}$$

By lemma 7 we have that $\mathsf{Var}(X) \leq \mathsf{Var}(Y)$. Let $M := \max_{s_l^n}(Y)$, $m := \min_{s_l^n}(Y)$. An empirical upper bound on the largest probability on the largest probability, p_{upper}, may be computed from sample data such that $p_{upper} \geq M$. A Hoeffding inequality can be used to bound the statistical error of the empirical estimator of M, $\tilde{M}$, so that

$$|\epsilon_{hoeff, M}| \leq \sqrt{\frac{\log\left(\frac{2}{\delta}\right)}{2N_{est, M}}},$$

where $N_{est, M}$ is the number of samples used to compute the estimator $\tilde{M}$, and $1 - \delta$ is the confidence. Defining $\epsilon_{hoeff, M}^{max} := \sqrt{\frac{\log\left(\frac{2}{\delta}\right)}{2N_{est, M}}}$ and $p_{upper} := \tilde{M} + \epsilon_{hoeff, M}^{max}$. Then $p_{upper} > M$ with confidence $1 - \delta$. The smallest probability is lower bounded $m \geq 0$. We now use the Bhatia-Davis inequality [62]

$$\mathsf{Var}(Y) \leq (M - \mu_Y)(\mu_Y - m) = (M - p_{unif})(p_{unif} - m)$$

the upper and lower bounds on M and m then lead to

$$(M - p_{unif})(p_{unif} - m) \leq (p_{upper} - p_{unif})(p_{unif}) \leq p_{upper} p_{unif}.$$

The confidence that $|I_{s_l^n, k} - p_{unif}| < \epsilon_{bias, s_l^n}$ would be $1 - \frac{p_{unif} p_{upper}}{\epsilon_{bias, s_l^n}^2}$, however this must be multiplied by the independent confidence of the statement $p_{upper} > M$, which is $1 - \delta$, to get an overall confidence of $1 - \frac{p_{unif} p_{upper}}{\epsilon_{bias, s_l^n}^2} - \delta + \frac{p_{unif} p_{upper} \delta}{\epsilon_{bias, s_l^n}^2}$. This means the Chebyshev inequality becomes

$$\Pr\left(\left|I_{s_l^n, k} - p_{unif}\right| \geq \epsilon_{bias, s_l^n}\right) \leq \frac{p_{unif} p_{upper}}{\epsilon_{bias, s_l^n}^2} + \delta\left(1 - \frac{p_{unif} p_{upper}}{\epsilon_{bias, s_l^n}^2}\right),$$

and the result follows. $\qquad\square$

18 Linear solving

18.1 Performance guarantee inequality for linear solving

Before giving a proof of Thm. 10 we first state and prove two results that we will use. These are upper bounds for the statistical error and the bias error of the mitigated value respectively, these we state as lemmas 24 and 25.

Lemma 24. $\epsilon_{mit, bias} \leq 3\binom{m-n+k}{k}|\epsilon_{bias, s_l^n}|$.

Proof. Without statistical error the mitigated value may be written $p_{mit}(s_l^n) = \left|\binom{m-n+k}{k} p_R(s_l^n) - \binom{m-n+k}{k}\frac{N_k'}{N_k} p_{unif}\right|$. Let $\epsilon_{mit, bias} := |p_{mit}(s_l^n) - p(s_l^n)|$ and $A_k := \binom{m-n+k}{k} p_R(s_l^n) - \binom{m-n+k}{k}\frac{N_k'}{N_k} p_{unif}$, so that $\epsilon_{mit, bias} = ||A_k| - p(s_l^n)|$, then $p_{mit}(s_l^n) = |A_k|$, and $p(s_l^n) = A_k - \binom{m-n+k}{k}\frac{N_k'}{N_k}\epsilon_{bias, s_l^n}$ (see main text).

Consider the first possible case where $A_k < 0$, this means that

$$\epsilon_{mit, bias} = |-A_k - p(s_l^n)|$$

$$= |-A_k - A_k + \binom{m-n+k}{k}\frac{N_k'}{N_k}\epsilon_{bias, s_l^n}|$$

$$\leq 2|A_k| + \binom{m-n+k}{k}\frac{N_k'}{N_k}|\epsilon_{bias, s_l^n}|,$$

by triangle inequality. Since $p(s_l^n) \geq 0$ by definition, then $\binom{m-n+k}{k}\frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}| = -\binom{m-n+k}{k}\epsilon_{bias,s_l^n} \geq -A_k$. Since $|A_k| = -A_k$, it follows that $|A_k| \leq \binom{m-n+k}{k}|\frac{N_k'}{N_k}\epsilon_{bias,s_l^n}|$, and therefore that $\epsilon_{mit,bias} \leq 2|A_k| + \binom{m-n+k}{k}\frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}| \leq 3\binom{m-n+k}{k}\frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}|$.

Now, consider the second possible case where $A_k \geq 0$, then $|A_k| = A_k$, and therefore $\epsilon_{mit,bias} = \binom{m-n+k}{k}\frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}| \leq 3\binom{m-n+k}{k}\frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}|$. This completes the proof. $\qquad\square$

Lemma 25. $|\tilde{p}_{mit}(s_l^n) - p_{mit}(s_l^n)| \leq 3\binom{m-n+k}{k}|\epsilon_{hoeff,\tilde{p}_R^k(s_l^n)}|$.

Proof. First, note that the estimator of the mitigated value may be written $\tilde{p}_{mit} = |A_k + \epsilon_{mit,stat}|$, where A_k is defined in the proof of Thm. 2, and $\epsilon_{mit,stat} = \binom{m-n+k}{k}\tilde{p}_R(s_l^n) - \binom{m-n+k}{k}p_R(s_l^n)$, and $|\epsilon_{mit,stat}| \leq \binom{m-n+k}{k}|\epsilon_{hoeff,\tilde{p}_R(s_l^n)}|$. And therefore $|\tilde{p}_{mit} - p_{mit}| = ||A_k + \epsilon_{mit,stat}| - |A_k||$.
Consider the case where
$$|A_k + \epsilon_{mit,stat}| = A_k + \epsilon_{mit,stat}.$$

If $A_k > 0$, then $|A_k| = A_k$ and therefore $|\tilde{p}_{mit} - p_{mit}| = |\epsilon_{mit,stat}| \leq \binom{m-n+k}{k}|\epsilon_{hoeff,\tilde{p}_R^k(s_l^n)}|$. If $A_k < 0$, then $A_k = -|A_k|$, and furthermore, since $A_k + \epsilon_{mit,stat} \geq 0$, then $\epsilon_{mit,stat} \geq |A_k|$. In this sub-case, we have that $|\tilde{p}_{mit} - p_{mit}| = |-2|A_k| - \epsilon_{mit,stat}| \leq 2|A_k| + |\epsilon_{mit,stat}| \leq 3\epsilon_{mit,stat} \leq 3\binom{m-n+k}{k}|\epsilon_{hoeff}|$.
Now, consider the case where
$$|A_k + \epsilon_{mit,stat}| = -A_k - \epsilon_{mit,stat}.$$

If $A_k < 0$, then $-A_k = |A_k|$, in this sub-case we have $|\tilde{p}_{mit} - p_{mit}| = |\epsilon_{mit,stat}| \leq \binom{m-n+k}{k}\epsilon_{hoeff}$. Now, if $A_k > 0$, then $-A_k = -|A_k|$ and furthermore, since $-A_k - \epsilon_{mit,stat} \geq 0$, then $-\epsilon_{mit,stat} \geq |A_k|$. In this sub-case, $|\tilde{p}_{mit} - p_{mit}| = |-2|A_k| - \epsilon_{mit,stat}| \leq 3|\epsilon_{mit,stat}| \leq 3\binom{m-n+k}{k}|\epsilon_{hoeff,\tilde{p}_R^k(s_l^n)}|$.
In all possible cases,
$$|\tilde{p}_{mit} - p_{mit}| \leq 3\binom{m-n+k}{k}|\epsilon_{hoeff,\tilde{p}_R^k(s_l^n)}|,$$

which was the result to be proved. $\qquad\square$

We now prove Thm. 10 from the main text:

The condition:

$$O\left(\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + \left(\binom{m-n+k}{k} - 1\right)O\left(\frac{1}{\text{poly}(m)}\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right), \quad (53)$$

defines a sampling regime (a range of values of N_{tot}) where the sum of the worst-case statistical error and bias error of linear solving recycling mitigation is less than the worst-case statistical error of postselection.

Proof.

$$p_{mit}(s_l^n) = \binom{m-n+k}{k}\left|\tilde{p}_R^k(s_l^n) - \frac{N_k'}{N_k}p_{unif}\right|.$$

Using Lemmas 24 and 25 the total error of the mitigated probability, $\epsilon_{p_{mit}(s_l^n)} := |\tilde{p}_{mit}(s_l^n) - p(s_l^n)|$, can be upper bounded

$$\left|\epsilon_{p_{mit}(s_l^n)}\right| \leq 3\binom{m-n+k}{k}\left(\left|\epsilon_{hoeff,p_R^k}\right| + \frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}|\right).$$

This follows from the fact that $\epsilon_{pmiti} \leq |p_{mit}(s_l^n) - p(s_l^n)| + |\tilde{p}_{mit}(s_l^n) - p_{mit}(s_l^n)|$. Now, using the error $\epsilon_{bias,s_l^n} = O\left(\frac{1}{\text{poly}(m)}\right)$, which holds with high confidence, and

$$|\epsilon_{hoeff,p_R^k}| \leq O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right),$$

the overall error for linear solving with dependency can be upper bounded

$$\left|\epsilon_{p_{mit}(s_l^n)}\right| \leq \binom{m-n+k}{k}\left(O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + \frac{N_k'}{N_k}O\left(\frac{1}{\text{poly}(m)}\right)\right).$$

And so the condition to beat postselection is

$$\binom{m-n+k}{k}\left(O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + \frac{N_k'}{N_k}O\left(\frac{1}{\text{poly}(m)}\right)\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right),$$

which is rearranged for the result. Where the rightmost part is just the statistical error of postselection.

□

18.2 Linear solving with dependency

We now consider the case where there is positive correlation of the interference on ideal probabilities within the recycled probabilities. This effect can be modelled as a linear dependence of interference terms on respective ideal probabilities. For a given recycled probability, a dependency term $d_k(s_l^n)$ can be used to express the interference term as a linear function of $p(s_l^n)$ and p_{unif}. Then

$$\left(1 - d_k(s_l^n)\right)p_{unif} + d_k(s_l^n)p(s_l^n) = I_{s_l^n,k}$$

defines the value $d_k(s_l^n)$, where $k \leq n - 1$. For each interference term, $I_{s_l^n,k}$, there is a corresponding dependency term, $d_k(s_l^n)$. The dependency term, $d_k(s_l^n)$, encodes the dependence of interference term $I_{s_l^n,k}$ on ideal probability $p(s_l^n)$ for the recycled probability $p_R^k(s_l^n)$. The set of dependency terms is then denoted $\{d_k(s_l^n)\}_l$.

The original linear solving method involves approximating the interference term as p_{unif}, and solving to find the mitigated probability. Now we propose first extracting an average dependency term from the distribution, which we will call the *average dependency term d_k*, and then approximating the interference terms as $\left(1 - d_k\right)p_{unif} + d_k p(s_l^n)$ and solving as before. The motivation for this variation on the original protocol is to improve mitigation performance by capturing the enhanced ideal probability signal caused by this correlation behaviour. In the original solving method, excluding statistical error, the recycled probability is decomposed as

$$p_R(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}(p_{unif} + \epsilon_{bias,s_l^n}),$$

where ϵ_{bias,s_l^n} is the bias error from approximating $I_{s_l^n,k}$ as $\frac{N_k'}{N_k}p_{unif}$. If the interference term $I_{s_l^n,k}$ is instead approximated as $\frac{N_k'}{N_k}\left(\left(1 - d_k\right)p_{unif} + d_k p(s_l^n)\right)$, then the expression becomes

$$p_R^k(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}\left(\left(\left(1 - d_k\right)p_{unif} + d_k p(s_l^n)\right) + \epsilon_b\right)$$

$$= p(s_l^n)\left(\frac{1}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}d_k\right) + \frac{N_k'}{N_k}\left(\left(1 - d_k\right)p_{unif} + \epsilon_b\right),$$

where ϵ_b is the bias error in the interference terms away from the linear model. From the positivity of the interference terms

$$\epsilon_b \geq -\left(\left(1 - d_k\right)p_{unif} + d_k p(s_l^n)\right),$$

where $k \leq n - 1$. One factor motivating the inclusion of a dependency term is that positive correlation of the interference terms with the ideal probabilities means that rather than the signal of the ideal probability having magnitude $\frac{1}{\binom{m-n+k}{k}}$, its magnitude is instead $\frac{1}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}d_k$. And this stronger signal may be used to improve mitigation performance. We now show how to compute the dependency term d_k from the absolute average deviation. Then compute bounds on the bias error and statistical error for the use of linear dependency in linear solving, which are then used to provide performance guarantees relative to postselection.

18.2.1 Computing the average dependency term d_k

One consequence of a general correlation of interference terms with ideal probabilities within recycled probabilities is that $D_k^{\text{no dep.}} \leq D_k$. The average dependency term d_k is a weighted average of dependency terms, it can be computed from $D_k^{\text{no dep.}}$ and D_k. We know by definition that

$$D_0 = \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} |p(s_l^n) - p_{unif}|.$$

For the moment ignoring statistical error, the dependency factor can be calculated from the average distance of probabilities from the uniform probability for the $n - k$–photon recycled distribution

$$
\begin{aligned}
D_k &= \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \frac{\binom{n}{k} p(s_l^n)}{N_k} + \frac{N_k'}{N_k}\left((1 - d_k(s_l^n)) p_{unif} + d_k(s_l^n) p(s_l^n) \right) - p_{unif} \right| \\
&= \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \frac{\binom{n}{k} p(s_l^n)}{N_k} + \frac{N_k'}{N_k} p_{unif} - \frac{N_k'}{N_k} d_k(s_l^n) p_{unif} + \frac{N_k'}{N_k} d_k(s_l^n) p(s_l^n) - p_{unif} \right| \\
&= \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(\frac{\binom{n}{k}}{N_k} \right)\left(p(s_l^n) - p_{unif} \right) + d_k(s_l^n) \frac{N_k'}{N_k}\left(p(s_l^n) - p_{unif} \right) \right| \\
&= \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(\frac{\binom{n}{k}}{N_k} + d_k(s_l^n) \frac{N_k'}{N_k} \right)\left(p(s_l^n) - p_{unif} \right) \right|
\end{aligned}
$$

The average dependency term, d_k, for the recycled distribution is defined by the expression

$$\left(\frac{\binom{n}{k}}{N_k} + d_k \frac{N_k'}{N_k} \right) \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(p(s_l^n) - p_{unif} \right) \right| = \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(\frac{\binom{n}{k}}{N_k} + d_k\left(p(s_l^n) \right) \frac{N_k'}{N_k} \right)\left(p(s_l^n) - p_{unif} \right) \right|.$$

This can then be used to rewrite the previous expression in terms of the dependency

$$
\begin{aligned}
D_k &= \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(\frac{\binom{n}{k}}{N_k} + d_k(s_l^n) \frac{N_k'}{N_k} \right)\left(p(s_l^n) - p_{unif} \right) \right| \\
&= \left(\frac{\binom{n}{k}}{N_k} + d_k \frac{N_k'}{N_k} \right) \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(p(s_l^n) - p_{unif} \right) \right| \\
&= \left(1 + d_k\left(\binom{m-n+k}{k} - 1 \right) \right) \frac{1}{\binom{m-n+k}{k}} \frac{1}{|\boldsymbol{S}|} \sum_{s_l^n \in \boldsymbol{S}} \left| \left(p(s_l^n) - p_{unif} \right) \right| \\
&= \left(\frac{1 + d_k \binom{m-n+k}{k} - d_k}{\binom{m-n+k}{k}} \right) D_0 \\
&= \left(\frac{1}{\binom{m-n+k}{k}} + d_k \frac{\binom{m-n+k}{k} - 1}{\binom{m-n+k}{k}} \right) D_0.
\end{aligned}
$$

If there is no correlation of interference terms with ideal probabilities then $d_k = 0$, and then

$$D_k^{d_k = 0} = \frac{1}{\binom{m-n+k}{k}} D_0.$$

Meaning that if there is no dependence the average absolute deviation decays with k proportionally to $\frac{1}{\binom{m-n+k}{k}}$. Whereas with dependency the decay is proportional to $\left(\frac{1 + d_k \binom{m-n+k}{k} - d_k}{\binom{m-n+k}{k}} \right)$. The average dependency term may be directly computed from the previous expression as

$$d_k = \frac{1}{\binom{m-n+k}{k} - 1}\left(\frac{\binom{m-n+k}{k} D_k}{D_0} - \frac{1}{\binom{m-n+k}{k}} \right).$$

18.3 Bias error bound with average dependency term d_k

We now prove Lemma 26, which will be used in the next section.

Lemma 26. *The bias error from substituting $I_{s_l^n,k}$ with $(1-d_k)p_{unif} + d_k p(s_l^n)$ in the recycled probabilities is upper bounded*

$$Pr\left[\left|I_{s_l^n,k} - \left((1-d_k)p_{unif} + d_k p(s_l^n)\right)\right| \geq 2\epsilon_{bias,s_l^n}\right] \leq \frac{2p_{unif}}{\epsilon_{bias,s_l^n}^2}.$$

Proof. Rather than bounding $I_{s_l^n,k}$ from p_{unif}, we would now like to bound deviation of $I_{s_l^n,k}$ from $\left(1 - d_k\right)p_{unif} + d_k p(s_l^n)$. As it is required that $1 \geq d_k \geq 0$, terms of the form $\left((1-d_k)p_{unif} + d_k p(s_l^n)\right)$ are bounded

$$\left|p(s_l^n) - \left((1-d_k)p_{unif} + d_k p(s_l^n)\right)\right| \geq \left|\left((1-d_k)p_{unif} + d_k p(s_l^n)\right) - p_{unif}\right|,$$

or

$$\left|p(s_l^n) - \left((1-d_k)p_{unif} + d_k p(s_l^n)\right)\right| \leq \left|\left((1-d_k)p_{unif} + d_k p(s_l^n)\right) - p_{unif}\right|,$$

with both statements true only with equidistance. The mean of the set $\{p(s_l^n)\}_l$ is p_{unif}, and so also the mean of the set $\{(1-d_k)p_{unif} + d_k p(s_l^n)\}_l$ is p_{unif}. If $d_k = 0$ then $\mathsf{Var}(\{(1-d_k)p_{unif} + d_k p(s_l^n)\}_l) = 0$, while if $d_k = 1$ then $\mathsf{Var}(\{(1-d_k)p_{unif}+d_k p(s_l^n)\}_l) = \mathsf{Var}(\{p(s_l^n)\}_l)$. As $d_k \leq 1$ the variance of the set $\{(1-d_k)p_{unif}+d_k p(s_l^n)\}_l$ is upper bounded by the variance of $\{p(s_l^n)\}_l$, so $\mathsf{Var}\left((1-d_k)p_{unif}+d_k p(s_l^n)\right) \leq p_{unif}$, where we have used the Bhatia-Davis inequality. We can then bound the distance of $\left((1-d_k)p_{unif}+d_k p(s_l^n)\right)$ from p_{unif} using Chebyshev's inequality so that

$$\Pr\left[\left|\left((1-d_k)p_{unif} + d_k p(s_l^n) - p_{unif}\right)\right| \geq \epsilon_{bias,s_l^n}\right] \leq \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}.$$

And in reverse form this inequality is

$$\Pr\left[\left|\left((1-d_k)p_{unif} + d_k p(s_l^n) - p_{unif}\right)\right| < \epsilon_{bias,s_l^n}\right] \geq 1 - \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}. \tag{54}$$

The Thm. 5 bound on the bias of the interference term for arbitrary matrices is

$$\Pr\left[\left|I_{s_l^n,k} - p_{unif}\right| \geq \epsilon_{bias,s_l^n}\right] \leq \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}.$$

Which in reverse form is

$$\Pr\left[\left|I_{s_l^n,k} - p_{unif}\right| < \epsilon_{bias,s_l^n}\right] \geq 1 - \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}. \tag{55}$$

Now, we use triangle inequality to combine the reverse forms of the above two inequalities, so that

$$\begin{aligned} \left|I_{s_l^n,k} - \left((1-d_k)p_{unif} + d_k p(s_l^n)\right)\right| &\leq \left|I_{s_l^n,k} - p_{unif}\right| + \left| -\left((1-d_k)p_{unif} + d_k p(s_l^n)\right) + p_{unif}\right| \\ &\leq 2\epsilon_{bias,s_l^n}. \end{aligned} \tag{56}$$

Assuming independence of the inequalities in eqn. 54 and eqn. 55, the the inequality in eqn. 56 holds with confidence $(1 - \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2})^2$. Using a Bernoulli approximation $(1 - \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2})^2 \approx 1 - 2\frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}$, and the result follows.

$\square$

18.4 Performance guarantee inequality for linear solving with dependency

We now prove Thm. 11 from the main text:

The condition

$$O\left(\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + 2\left(\binom{m-n+k}{k} - 1\right)O\left(\frac{1}{\mathsf{poly}(m)}\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right),$$

defines a sampling regime where the sum of the worst-case statistical error and bias error of linear solving with dependency recycling mitigation is less than the worst-case statistical error of postselection.

Proof. The estimator of the recycled probability may be rewritten in terms of the dependency term d_k as

$$p_R^k(s_l^n) = \frac{p(s_l^n)}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}\left(\left((1 - d_k - \epsilon_{\text{hoeff},d_k})p_{unif} + (d_k + \epsilon_{\text{hoeff},d_k})p(s_l^n)\right) + \epsilon_b\right) + \epsilon_{\text{hoeff},p_R^k}$$

$$= p(s_l^n)\left(\frac{1}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}(d_k + \epsilon_{\text{hoeff},d_k})\right) + \frac{N_k'}{N_k}\left((1 - d_k - \epsilon_{\text{hoeff},d_k})p_{unif} + \epsilon_{bias,s_l^n}\right) + \epsilon_{\text{hoeff},p_R^k}.$$

Solving to find the mitigated value in the ideal case one obtains

$$p_{mit,dep}(s_l^n) = \left|\frac{p_R^k(s_l^n) + \frac{N_k'}{N_k}\left((-1 + d_k)p_{unif}\right)}{\frac{1}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}d_k}\right|.$$

However, the estimator of the mitigated value including error is

$$\tilde{p}_{mit,dep}(s_l^n) = \frac{p_R^k(s_l^n) - \frac{N_k'}{N_k}\left((1 - (d_k + \epsilon_{\text{hoeff},d_k}))p_{unif} + \epsilon_{bias,s_l^n}\right) + \epsilon_{\text{hoeff},p_R^k}}{\frac{1}{\binom{m-n+k}{k}} + \frac{N_k'}{N_k}(d_k + \epsilon_{\text{hoeff},d_k})}.$$

As, by definition, $1 \geq d_k \geq 0$, this means that $-d_k \leq \epsilon_{\text{hoeff},d_k} \leq 1$. And the value of $\epsilon_{\text{hoeff},d_k}$ that maximally increases the error of the above quotient is then $\epsilon_{\text{hoeff},d_k} = -d_k$, this substitution removes the d_k terms. From this point we can use Lemma 24 and Lemma 25, with the one difference in the latter that the bias error ϵ_{bias,s_l^n} from Lemma 26 is used instead of ϵ_{bias,s_l^n}. And then the total error of the mitigated probability, $\epsilon_{p_{mit,dep}(s_l^n)}$, can be upper bounded

$$\left|\epsilon_{p_{mit,dep}(s_l^n)}\right| \leq 3\binom{m-n+k}{k}\left(\left|\epsilon_{\text{hoeff},p_R^k}\right| + \frac{N_k'}{N_k}|\epsilon_{bias,s_l^n}|\right).$$

Now we set $\epsilon_{bias,s_l^n} = O\left(\frac{1}{\text{poly}(m)}\right)$, so the error is polynomially small with a confidence of $1 - \frac{\text{poly}(m)^2}{\binom{m}{n}}$, which is exponentially close to 1. And we use that

$$|\epsilon_{\text{hoeff},p_R^k}| \leq O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right).$$

So that the overall error for linear solving with dependency is upper bounded

$$\left|\epsilon_{p_{mit,dep}(s_l^n)}\right| \leq \binom{m-n+k}{k}\left(O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + \frac{2N_k'}{N_k}O\left(\frac{1}{\text{poly}(m)}\right)\right).$$

And so the condition to beat postselection is

$$\binom{m-n+k}{k}\left(O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right) + \frac{N_k'}{N_k}O\left(\frac{1}{\text{poly}(m)}\right)\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right),$$

which is rearranged for the result. $\qquad\square$

19 Extrapolation

19.1 Statistical error for average absolute deviation estimator $\tilde{D}_k$

The following lemma upper bounds the statistical error of the empirically computed absolute average deviation terms $\{\tilde{D}_k\}_{k=1}^K$, and is used in deriving the condition for linear extrapolation to outperform postselection in the next section.

Lemma 27. *The statistical error of the average absolute deviation estimator $\tilde{D}_k$ is upper bounded*

$$\left|\epsilon_{hoeff,\tilde{D}_k}\right| \leq O\left(\left(\frac{1}{\binom{m-n+k}{k}^2\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}\right)^{1/4}\right). \tag{57}$$

Proof. Let X be a random variable defined over the uniform distribution of bit strings s_l^n that returns terms from the set $\{\tilde{p}_R^k(s_l^n)\}_l$ as values, $\mu := \mathsf{E}_{s_l^n}(X) = p_{unif} = \frac{1}{\binom{m}{n}}$, with $\mathsf{E}_{s_l^n}(.)$ denoting the expectation value of X over the uniform distribution of s_l^n, it is equal to p_{unif}. The average absolute deviation estimator, denoted $\{\tilde{D}_k\}_{k=0}^n$, is defined

$$\tilde{D}_k := \binom{m}{n}^{-1} \sum_{s_l^n \in \boldsymbol{S}} \left| \tilde{p}_R^k(s_l^n) - p_{unif} \right| = D_k + \epsilon_{\mathrm{hoeff}, \tilde{D}_k},$$

where $\epsilon_{\mathrm{hoeff}, \tilde{D}_k}$ is the absolute average deviation statistical error. Let $M := \mathsf{max}_{s_l^n}(X)$, $m := \mathsf{min}_{s_l^n}(X)$, and note that $M \leq 1$ and $m \geq 0$. We can apply the Bhatia-Davis inequality [62]

$$\mathsf{Var}(\{X\}) \leq (M - \mu)(\mu - m),$$

with an upper bound of $M = 1$, a lower bound of $m = 0$, and the expected value $\mu = p_{unif}$, to obtain an upper bound on the variance of the random variable X so that

$$\begin{aligned} \mathsf{Var}(X) &\leq (1 - p_{unif})(p_{unif}) \\ &\leq p_{unif}. \end{aligned} \tag{58}$$

Now, by Jensen's inequality [69] we know that $\mathsf{E}_{s_l^n}(|X - \mu|^2) \geq (\mathsf{E}_{s_l^n}|X - \mu|)^2$, therefore

$$\mathsf{Var}(X) \geq \left(\binom{m}{n}^{-1} \sum_{s_l^n \in \boldsymbol{S}} \left| \tilde{p}_R^k(s_l^n) - p_{unif} \right| \right)^2,$$

and as both sides of this inequality are positive and the square root operation is a monotonically increasing function for positive reals

$$\mathsf{Var}(X)^{1/2} \geq \binom{m}{n}^{-1} \sum_{s_l^n \in \boldsymbol{S}} \left| \tilde{p}_R^k(s_l^n) - p_{unif} \right|.$$

And using the result from eqn. 58 we have

$$p_{unif}^{1/2} \geq \binom{m}{n}^{-1} \sum_{s_l^n \in \boldsymbol{S}} \left| \tilde{p}_R^k(s_l^n) - p_{unif} \right|$$

$$\geq D_k + \epsilon_{\mathrm{hoeff}, \tilde{D}_k}.$$

Now because $\tilde{D}_k \geq 0$ then $\epsilon_{\mathrm{hoeff}, \tilde{D}_k} \geq -D_k \geq -p_{unif}^{1/2}$. Also $p_{unif}^{1/2} \geq D_k$ and $p_{unif}^{1/2} \geq D_k + \epsilon_{\mathrm{hoeff}, \tilde{D}_k}$ mean that $2(p_{unif}^{1/2} - D_k) \geq \epsilon_{\mathrm{hoeff}, \tilde{D}_k}$. And as $D_k \geq 0$ this means $2p_{unif}^{1/2} \geq \epsilon_{\mathrm{hoeff}, \tilde{D}_k}$. Then we have that $2p_{unif}^{1/2} \geq \epsilon_{\mathrm{hoeff}, \tilde{D}_k} \geq -p_{unif}^{1/2}$ which gives

$$|\epsilon_{\mathrm{hoeff}, \tilde{D}_k}| \leq 2p_{unif}^{1/2}. \tag{59}$$

The absolute error for $\tilde{D}_k$ can also be upper bounded in terms of the statistical error of the recycled probabilities

$$\begin{aligned} |\epsilon_{\mathrm{hoeff}, \tilde{D}_k}| &= |\tilde{D}_k - D_k| \\ &= \left| \binom{m}{n}^{-1} \left(\sum_{s_l^n \in \boldsymbol{S}} \left| p_R^k(s_l^n) + \epsilon_{\mathrm{hoeff}, \tilde{p}_R^k(s_l^n)} - p_{unif} \right| - \sum_{s_l^n \in \boldsymbol{S}} \left| p_R^k(s_l^n) - p_{unif} \right| \right) \right| \\ &\leq \binom{m}{n}^{-1} \left(\sum_{s_l^n \in \boldsymbol{S}} \left| \left| p_R^k(s_l^n) + \epsilon_{\mathrm{hoeff}, \tilde{p}_R^k(s_l^n)} - p_{unif} \right| - \left| p_R^k(s_l^n) - p_{unif} \right| \right| \right) \\ &\leq \binom{m}{n}^{-1} \sum_{s_l^n \in \boldsymbol{S}} \left| \epsilon_{\mathrm{hoeff}, \tilde{p}_R^k(s_l^n)} \right|. \end{aligned} \tag{60}$$

Where the inequalities follow by using a triangle, then a reverse triangle inequality. All terms in the inequalities from eqns. 59 and 60 are positive reals, and so the inequalities may be combined to give

$$|\epsilon_{\mathrm{hoeff}, \tilde{D}_k}|^2 \leq 2p_{unif}^{1/2} \binom{m}{n}^{-1} \sum_{s_l^n \in \boldsymbol{S}} \left| \epsilon_{\mathrm{hoeff}, \tilde{p}_R^k(s_l^n)} \right|.$$

We now define $\mathcal{E}_{\mathrm{hoeff},\tilde{p}_R^k}$ as the magnitude of the upper bound for the statistical error $|\epsilon_{\mathrm{hoeff},\tilde{p}_R^k(s_l^n)}|$ which holds with high confidence from Hoeffdings inequality.

$$\mathcal{E}_{\mathrm{hoeff},\tilde{p}_R^k} := O\left(\frac{1}{\binom{m-n+k}{k}}\sqrt{\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}}\right).$$

As $|\epsilon_{\mathrm{hoeff},\tilde{p}_R^k(s_l^n)}| \leq \mathcal{E}_{\mathrm{hoeff},\tilde{p}_R^k}$ we can now write

$$|\epsilon_{\mathrm{hoeff},\tilde{D}_k}|^2 \leq 2p_{unif}^{1/2}\mathcal{E}_{\mathrm{hoeff},\tilde{p}_R^k},$$

and, again using that the square root operation is a monotonically increasing function for positive reals, it follows that

$$\begin{aligned}
|\epsilon_{\mathrm{hoeff},\tilde{D}_k}| &\leq (2\mathcal{E}_{\mathrm{hoeff},\tilde{p}_R^k})^{1/2}p_{unif}^{1/4} \\
&\leq O\left(\mathcal{E}_{\mathrm{hoeff},\tilde{p}_R^k}^{1/2}p_{unif}^{1/4}\right) \\
&\leq O\left(\left(\frac{1}{\binom{m-n+k}{k}}\right)^{1/2}\left(\frac{\binom{m}{n}}{\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}\right)^{1/4}\left(\frac{1}{\binom{m}{n}}\right)^{1/4}\right) \\
&\leq O\left(\left(\frac{1}{\binom{m-n+k}{k}^2\binom{n}{k}(1-\eta)^{n-k}\eta^k N_{tot}}\right)^{1/4}\right).
\end{aligned}$$

Which was the result to be proved. $\qquad\square$

19.2 Performance guarantee inequality for extrapolation with linear least squares

Before giving a proof of Thm. 12 we first state and prove two results that we will use. These are an upper bound for the statistical error for the gradient parameter. And an upper bound on the bias error introduce by using the average gradient parameter rather than the ideal gradient parameter. These we state as lemmas 28 and 29.

Lemma 28. *The average gradient parameter statistical error $\epsilon_{hoeff,g}$ is upper bounded*

$$|\epsilon_{hoeff,g}| \leq O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right). \tag{61}$$

Proof. First note that using the linear least squares method the average gradient parameter estimator may be written

$$\tilde{g}_{avg} = \frac{3}{2n_d+2}\tilde{D}_0 - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\tilde{D}_i x_i,$$

and the average gradient parameter without statistical error

$$g_{avg} = \frac{3}{2n_d+2}D_0 - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}D_i x_i.$$

The statistical error for the gradient term is then

$$
\begin{aligned}
\left|\epsilon_{\text{hoeff},g}\right| &= |\tilde{g}_{avg} - g_{avg}| \\
&= \Big|\frac{3}{2n_d+2}\tilde{D}_0 - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\tilde{D}_i x_i - \frac{3}{2n_d+2}D_0 + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}D_i x_i\Big| \\
&= \Big|\frac{3}{2n_d+2}(D_0 + \epsilon_{\text{hoeff},\tilde{D}_0}) - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}(D_i + \epsilon_{\text{hoeff},\tilde{D}_i})x_i - \frac{3}{2n_d+2}D_0 \\
&\qquad + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}D_i x_i\Big| \\
&= \Big|\frac{3}{2n_d+2}\epsilon_{\text{hoeff},\tilde{D}_0} + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\epsilon_{\text{hoeff},\tilde{D}_i}x_i\Big| \\
&\leq \frac{3}{2n_d+2}\left|\epsilon_{\text{hoeff},\tilde{D}_0}\right| + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\left|\epsilon_{\text{hoeff},\tilde{D}_i}\right|x_i \\
&\leq \frac{3}{2n_d+2}O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right) + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right)x_i \\
&\leq O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right).
\end{aligned}
$$

Where a triangle inequality was used for the fourth line, and that $\left|\epsilon_{\text{hoeff},\tilde{D}_i}\right| \leq O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right)$ for $i \in \{0,\dots,n_d\}$ from lemma 27 was used to get the sixth line.

$\square$

Lemma 29. *With confidence* $1 - O*(m^{-n})$, *the bias error* $\epsilon_{bias,g}$ *from substituting the ideal gradient parameter,* g_{ideal}, *for the average gradient parameter,* g_{avg}, *is upper bounded*

$$
\left|\epsilon_{bias,g}\right| \leq O\Big(\frac{1}{poly(m)}\Big). \tag{62}
$$

Proof. Linear least squares with the stated model function gives the optimal average gradient parameter as

$$
g_{\text{avg}} = \frac{3}{2n_d+2}D_0 - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}D_i x_i,
$$

and the ideal gradient parameter as

$$
g_{ideal} = \frac{3}{2n_d+2}|p(s_l^n) - p_{unif}| - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}|p_i^R(s_l^n) - p_{unif}|x_i.
$$

The magnitude of the gradient bias error is then

$$
\begin{aligned}
\left|\epsilon_{\mathrm{bias},g}\right| &= \left|g_{ideal} - g_{avg}\right| \\
&= \Bigg| \frac{3}{2n_d+2}\left|p(s_l^n) - p_{unif}\right| - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\left|p_R^i(s_l^n) - p_{unif}\right|x_i \\
&\qquad - \frac{3}{2n_d+2}D_0 + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}D_i x_i \Bigg| \\
&\le \frac{3}{2n_d+2}\left|\left|p(s_l^n) - p_{unif}\right| - D_0\right| + \frac{6}{n_d(n_d+1)(2n_d+2)}\left|\sum_{i=1}^{n_d}\left(-\left|p_R^i(s_l^n) - p_{unif}\right|x_i + D_i x_i\right)\right| \\
&\le \frac{3}{2n_d+2}\left|\left|p(s_l^n) - p_{unif}\right| - D_0\right| + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\left|\left|p_R^i(s_l^n) - p_{unif}\right|x_i - D_i x_i\right| \\
&\le \frac{3}{2n_d+2}\left|\left|p(s_l^n) - p_{unif}\right| - D_0\right| + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}x_i\left|\left|p_R^i(s_l^n) - p_{unif}\right| - D_i\right| \\
&\le \frac{3}{2n_d+2}\left|\left|p(s_l^n) - p_{unif}\right| - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}\left|p(s_l^n) - p_{unif}\right|\right| \\
&\qquad + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}x_i\left|\left|p_R^i(s_l^n) - p_{unif}\right| - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}\left|p_R^i(s_l^n) - p_{unif}\right|\right| \\
&\le \frac{3}{2n_d+2}\left|p(s_l^n) - p_{unif} - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}\left(p(s_l^n) - p_{unif}\right)\right| \\
&\qquad + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}x_i\left|p_R^i(s_l^n) - p_{unif} - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}\left(p_R^i(s_l^n) - p_{unif}\right)\right| \\
&\le \frac{3}{2n_d+2}\left|p(s_l^n) - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}p(s_l^n)\right| \\
&\qquad + \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}x_i\left|p_R^i(s_l^n) - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}p_R^i(s_l^n)\right|.
\end{aligned}
$$

Where a triangle inequality was used to get line three and line four, and reverse triangle inequalities were used to get line seven. Now from Thm. 8 we have that

$$
Pr\left(\left|p(s_l^n) - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}p(s_l^n)\right| \le \epsilon_{bias,s_l^n}\right) \ge 1 - \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}.
$$

Setting $\epsilon_{bias,s_l^n} = O\left(\frac{1}{\mathrm{poly}(m)}\right)$, the error is then polynomially small with a confidence of $1 - \frac{\mathrm{poly}(m)^2}{\binom{m}{n}}$. Now by lemma 6 $\mathsf{Var}\left(\{p(s_l^n)\}_l\right) \ge \mathsf{Var}\left(\{p_R^k(s_l^n)\}_l\right)$, which allows the derivation of the same Chebyshev bound for the $\left|p_R^k(s_l^n) - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}p_R^k(s_l^n)\right|$ terms with $k \in \{1,\ldots,n_d\}$, so that

$$
Pr\left(\left|p_R^k(s_l^n) - \binom{m}{n}^{-1}\sum_{s_l^n \in \boldsymbol{S}}p_R^k(s_l^n)\right| \le \epsilon_{bias,s_l^n}\right) \ge 1 - \frac{p_{unif}}{\epsilon_{bias,s_l^n}^2}.
$$

Therefore with confidence $\left(1 - \frac{p_{unif}}{\epsilon^2_{bias,s^n_l}}\right)^{n_d} \approx 1 - n_d \frac{p_{unif}}{\epsilon^2_{bias,s^n_l}}$, using a Bernoulli approximation, we have that

$$\frac{3}{2n_d + 2}\left|p_0 - \binom{m}{n}^{-1} \sum_{s^n_l \in \boldsymbol{S}} p(s^n_l)\right| + \frac{6}{n_d(n_d + 1)(2n_d + 2)} \sum_{i=1}^{n_d} x_i\left|p^i_R - \binom{m}{n}^{-1} \sum_{s^n_l \in \boldsymbol{S}} p^i_R(s^n_l)\right|$$

$$\leq \frac{3}{2n_d + 2}O\left(\frac{1}{\text{poly}(m)}\right) + \frac{6}{n_d(n_d + 1)(2n_d + 2)} \sum_{i=1}^{n_d} x_i O\left(\frac{1}{\text{poly}(m)}\right)$$

$$\leq O\left(\frac{1}{\text{poly}(m)}\right).$$

The upper bound on the gradient bias error is then

$$\left|\epsilon_{\text{bias},g}\right| \leq O\left(\frac{1}{\text{poly}(m)}\right),$$

with confidence $1 - n_d \frac{p_{unif}}{\epsilon^2_{bias,s^n_l}}$. $\qquad\qquad\square$

We now prove Thm. 12 from the main text:

The condition:

$$\frac{n_d + 1}{2}\left(O\left(\left(\frac{1}{(1 - \eta)^n N_{tot}}\right)^{1/4}\right) + O\left(\frac{1}{\text{poly}(m)}\right)\right) + O\left(\frac{n}{m - n + 1}\sqrt{\frac{\binom{m}{n}}{n(1 - \eta)^{n-1}\eta N_{tot}}}\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1 - \eta)^n N_{tot}}}\right).$$

defines a sampling regime where the sum of the worst-case statistical error and bias error of linear extrapolation recycling mitigation using the least squares method is less than the worst-case statistical error of postselection.

Proof. The linear extrapolation method consists of two iterations of linear least squares. The first iteration calculates an average gradient parameter, g_{avg}, this is parameter then used to generate the mitigated outputs in the second iteration of least squares. There are $n_d \in \{n_d \in \mathbb{Z}^+ | n_d < n\}$ data points used in both iterations of least squares. The data set $\{k, \tilde{D}_k\}_{k=1}^{n_d}$ is used to compute the gradient parameter g_{avg}. Where $n_d \leq n$ and $\tilde{D}_k$ is the estimated absolute average deviation for the $n - k$–photon recycled distribution from statistics where k photons were lost. We will initially use the notation that $\{x_i, y_i\}_{i=1}^{n_d} := \{k, \tilde{D}_k\}_{k=1}^{n_d}$, this will be redefined in the second iteration of least squares. In the first iteration of least squares this data set is used to find the average gradient parameter estimator, $\tilde{g}_{\text{avg}}$, and the linear model function used is

$$f(x_i, g_{\text{avg}}) = -\tilde{g}_{\text{avg}}x_i + \tilde{D}_0.$$

For which the residuals are of the form

$$r_i = y_i - (-\tilde{g}_{\text{avg}}x_i + \tilde{D}_0),$$

The sum of the squared residuals is a function of α_{s_n}, and may be written

$$S(\tilde{g}_{\text{avg}}) = \sum_{i=1}^{n_d} r^2_i$$

$$= \sum_{i=1}^{n_d}(y_i - (-\tilde{g}_{\text{avg}}x_i + \tilde{D}_0))^2$$

The optimal solution may be found by taking the derivative of S with respect to $\tilde{g}_{\text{avg}}$

$$0 = \frac{\text{d}S}{\text{d}\tilde{g}_{\text{avg}}} = 2\sum_{i=1}^{n_d}(y_i - (-\tilde{g}_{\text{avg}}x_i + \tilde{D}_0))(x_i),$$

and then solving

$$\sum_{i=1}^{n_d} \tilde{g}_{\text{avg}}x^2_i = \sum_{i=1}^{n_d} \tilde{D}_0 x_i - \sum_{i=1}^{n_d} y_i x_i.$$

Simplifying and substituting $\tilde{D}_i$ terms, this gives the optimal average gradient parameter as

$$\tilde{g}_{\text{avg}} = \frac{3}{2n_d + 2}\tilde{D}_0 - \frac{6}{n_d(n_d+1)(2n_d+2)}\sum_{i=1}^{n_d}\tilde{D}_i x_i.$$

Using lemma 28, the statistical error of the average gradient parameter estimator, $\epsilon_{\text{hoeff},g}$, can be upper bounded

$$\left|\epsilon_{\text{hoeff},g}\right| \leq |\tilde{g}_{\text{avg}} - g_{\text{avg}}|$$

$$\leq O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right).$$

The average gradient parameter estimator, $\tilde{g}_{\text{avg}}$, is used in a second iteration of least squares to generate the mitigated outputs. The data set used to generate the mitigated output for each s_l^n is defined as $\{x_i, y_i\}_{i=1}^{n_d} := \{k, |\tilde{p}_R^k(s_l^n) - p_{unif}|\}_{k=1}^{n_d}$. Each output string is assigned an individual model function of the form

$$f_{s_l^n}(x_i, \alpha_{s_l^n}) = \text{sgn}(p_{unif} - y_1)\tilde{g}_{\text{avg}}x_i + \alpha_{s_l^n}.$$

We will assume the sign is negative ($\text{sgn}(p_{unif} - y_1) = -1$) so that the model function is

$$f_{s_l^n}(x_i, \alpha_{s_l^n}) = -\tilde{g}_{\text{avg}}x_i + \alpha_{s_l^n},$$

noting that the analysis is identical for the case of positive sign. The optimal prefactor variable $\alpha_{s_l^n}$ to fit the model function to the data set is computed using the residuals

$$r_i = y_i - (-\tilde{g}_{\text{avg}}x_i + \alpha_{s_l^n}),$$

The sum of the squared residuals is a function of $\alpha_{s_l^n}$, and may be written

$$S(\alpha_{s_l^n}) = \sum_{i=1}^{n_d} r_i^2$$

$$= \sum_{i=1}^{n_d}(y_i - (-\tilde{g}_{\text{avg}}x_i + \alpha_{s_l^n}))^2$$

The value of $\alpha_{s_l^n}$ that minimises S may be found by taking the derivative of S with respect to $\alpha_{s_l^n}$, and solving

$$0 = \frac{\mathrm{d}S}{\mathrm{d}\alpha_{s_l^n}} = -2\sum_{i=1}^{n_d}(y_i - (-g_{\text{avg}}x_i + \alpha_{s_l^n}))$$

Which gives the optimal value as

$$\alpha_{s_l^n} = \frac{n_d+1}{2}\tilde{g}_{\text{avg}} + \frac{1}{n_d}\sum_{i=1}^{n_d} y_i. \tag{63}$$

And substituting in the data values this is then

$$\alpha_{s_l^n} = \frac{n_d+1}{2}\tilde{g}_{\text{avg}} + \frac{1}{n_d}\sum_{k=1}^{n_d} |\tilde{p}_R^k(s_l^n) - p_{unif}|. \tag{64}$$

Now $\tilde{g}_{\text{avg}}$ can be decomposed into an ideal gradient parameter, g_{ideal}, an average gradient parameter statistical error, $\epsilon_{\text{hoeff},g}$, and a bias error from substituting the ideal gradient parameter for the average gradient parameter, $\epsilon_{\text{bias},g}$. And the recycled probability estimators $\tilde{p}_R^k(s_l^n)$ can be decomposed into recycled probabilities, $p_R^k(s_l^n)$, and their associated statistical errors, $\epsilon_{\text{hoeff},\tilde{p}_R^k(s_l^n)}$. So that

$$\alpha_{s_l^n} = \frac{n_d+1}{2}\big(g_{ideal} + \epsilon_{\text{hoeff},g} + \epsilon_{\text{bias},g}\big) + \frac{1}{n_d}\sum_{i=1}^{n_d} |p_R^k(s_l^n) + \epsilon_{\text{hoeff},\tilde{p}_R^k(s_l^n)} - p_{unif}|. \tag{65}$$

While the ideal prefactor variable can be stated as

$$\alpha_{ideal,s_l^n} = \frac{n_d + 1}{2}g_{ideal} + \frac{1}{n_d}\sum_{i=1}^{n_d}|p_R^k(s_l^n) - p_{unif}|. \tag{66}$$

And the magnitude of the error for the mitigated output is then

$$
\begin{aligned}
|\epsilon_\alpha| &= |\alpha_{s_l^n} - \alpha_{ideal,s_l^n}| \\
&\leq \frac{n_d + 1}{2}(|\epsilon_{\text{hoeff},g}| + |\epsilon_{\text{bias},g}|) + |\epsilon_{\text{hoeff},\tilde{p}_R^1(s_l^n)}| \\
&\leq \frac{n_d + 1}{2}\left(O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right) + O\left(\frac{1}{\text{poly}(m)}\right)\right) + O\left(\frac{n}{m - n + 1}\sqrt{\frac{\binom{m}{n}}{n(1-\eta)^{n-1}\eta N_{tot}}}\right).
\end{aligned}
\tag{67}
$$

Where to get the second line a triangle inequality and that in the high loss regime $|\epsilon_{\text{hoeff},\tilde{p}_R^1(s_l^n)}| \geq \frac{1}{n_d}\sum_{i=1}^{n_d}|\epsilon_{\text{hoeff},\tilde{p}_R^k(s_l^n)}|$ were used, and lemmas 28, 29 and 22 were used to get the fourth line. And so the condition to beat postselection is

$$\frac{n_d+1}{2}\left(O\left(\left(\frac{1}{(1-\eta)^n N_{tot}}\right)^{1/4}\right) + O\left(\frac{1}{\text{poly}(m)}\right)\right) + O\left(\frac{n}{m-n+1}\sqrt{\frac{\binom{m}{n}}{n(1-\eta)^{n-1}\eta N_{tot}}}\right) \leq O\left(\sqrt{\frac{\binom{m}{n}}{(1-\eta)^n N_{tot}}}\right).$$

$\square$

20 Breaking the exponential barrier

20.1 Proof of Thm. 9

We now prove Thm. 9 from the main text:

For the class of unitary matrices U with submatrices A such that $p_{max} := \mathsf{max}_{s_l^n}(p(s_l^n) = |\mathsf{Per}(A)|^2$, and where these matrices A satisfy $\frac{h_\infty^A}{\|A\|_2} \ll 1$, the bias error ϵ_{bias,s_l^n} is bounded

$$\epsilon_{bias,s_l^n} \leq\approx O(e^{-0.000002n}).$$

Proof. Let $p_{max} = \mathsf{max}_{s_l^n}(p(s_l^n))$, where $\mathsf{max}_{s_l^n}(.)$ denotes the maximum over the n–photon probability distribution. It is immediate to see that the interference term satisfies $I_{s_l^n,k} \leq p_{max}$. Furthermore, $p_{max} = |\mathsf{Per}(A)|^2$, where $A := (a_{ij})_{i,j\in\{1,...,n\}}$ is a submatrix of the linear optical unitary U [25]. Note that $\|A\|_2 \leq \|U\|_2 \leq 1$, where $\|.\|_2$ denotes the spectral norm [70]. Let $h_\infty^A := \frac{1}{n}\sum_{i=1,...,n}\|\mathbf{A}_i\|_\infty$, where $\mathbf{A}_i$ is the ith row of A, and $\|\mathbf{A_i}\|_\infty := \mathsf{max}_j(|a_{ij}|)$. Let $\|A\|_2 \leq T$. Using Thm. 2 in [28], we have that

$$|\mathsf{Per}(A)| \leq 2T^n e^{-0.000001(1-\frac{h_\infty^A}{\|A\|_2})^2 n}.$$

Since $\|A\|_2 \leq 1$, we immediately have

$$|\mathsf{Per}(A)| \leq 2e^{-0.000001(1-\frac{h_\infty^A}{\|A\|_2})^2 n}.$$

For classes of matrices where $\frac{h_\infty^A}{\|A\|_2} \ll 1$, we have that

$$|\mathsf{Per}(A)| \leq 2e^{-0.000001n},$$

and therefore that

$$p_{max} \leq 4e^{-0.000002n}.$$

Now, we have that $-p_{unif} \leq I_{s_l^n,k} - p_{unif} \leq p_{max} - p_{unif}$, and therefore the bias error is given by

$$\epsilon_1 = |I_{s_l^n,k} - p_{unif}| \leq \mathsf{max}\{|p_{max} - p_{unif}|, p_{unif}\}.$$

In the case where $|p_{max} - p_{unif}| < p_{unif}$, then we have an exponentially small bias error $\epsilon_{bias,s_l^n} \leq p_{unif} \leq \frac{1}{\binom{m}{n}} \approx \frac{1}{m^n} \approx e^{-O(n log(n))}$, when $m \approx n^{5.1}$ which is the boson sampling regime. Now, when $|p_{max} - p_{unif}| > p_{unif}$, we can use the bound for p_{max} we obtained, and show that

$$\epsilon_{bias,s_l^n} \leq |p_{max} - p_{unif}| \leq\approx |4^{-0.000002n} - e^{-O(n log(n))}| \approx O(e^{-0.000002n}).$$

$\square$

21 Richardson extrapolation methods for photon loss mitigation

In this section, we prove Thm. 13 as well as provide similar evidence (that the methods present no advantage over postselection) for different methods of performing extrapolation with increasing rates of photon loss.

First method of extrapolation at various noise rates

Let m be the number of modes of a linear optical circuit which can implement any $m \times m$ unitary. Into this circuit we input n photons in the first n modes. The notation $|n_1, \ldots, n_m\rangle$ denotes a state with n_i photons in the ith mode, where $i \in \{1, \ldots, m\}$. $\eta \in]0, 1[$ is the probability to lose a photon in any given mode, and is the same for all modes. We want to compute a specific marginal probability $p(n_1 ... n_l | n)$ with $l \leq m$, and $\sum_{i=1,\ldots,l} n_i = c$, with $c \leq n$. The $|n$ indicates that we are computing the *ideal* marginal probability, when no photon is lost. Let $p(n_1 \ldots n_l \cap j)$ be the probability of observing the output $(n_1, \ldots, n_l)$ and detecting j photons in all m modes, with $j \in \{c, \ldots, n\}$. When postselecting on no photons being lost, we are computing

$$p(n_1 \ldots n_l \cap n) = (1 - \eta)^n p(n_1 \ldots n_l | n).$$

However, if we compute $p(n_1 \ldots n_l)$ without caring about whether no photon is lost, we end up computing

$$p_\eta(n_1 ... n_l) = \sum_{i=0,\ldots,n-c} (1 - \eta)^{n-i} \eta^i p(n_1 \ldots n_l | n - i). \tag{68}$$

Extrapolation techniques consist of estimating $p_\eta(n_1 \ldots n_l)$ for different values of η, then deducing from these an estimate of $p(n_1 \ldots n_l | n)$. One example of how this can be done is the Richardson extrapolation technique, at the heart of the zero noise extrapolation (ZNE) approach. Rather interestingly, we will show that these techniques offer no advantage over post-selection in terms of estimating $p(n_1 \ldots n_l | n)$. Let $\alpha_i := p(n_1 \ldots n_l | n - i)$, we can then write

$$p_\eta(n_1 \ldots n_l) := \sum_{i=0,\ldots,n-c} (1 - \eta)^{n-i} \eta^i \alpha_i. \tag{69}$$

A natural way to estimate α_0 through extrapolation would be to compute an estimate $\tilde{p}_\eta(n_1 \ldots n_l)$ of $p_\eta(n_1 \ldots n_l)$ for $n - c + 1$ values η_i of η, with $i \in \{0, \ldots, n - c\}$, and (by convention) $\eta_{i+1} > \eta_i$. We will deal with additive error estimates, that is, $\tilde{p}_\eta(n_1 \ldots n_l) = p_\eta(n_1 \ldots n_l) + \epsilon$, with $|\epsilon| \leq \epsilon_{max}$, and $\epsilon_{max} \in [0, 1]$ is the additive error estimate. Note that an ϵ_{max} additive error estimate $\tilde{p}_\eta(n_1 \ldots n_l)$ can be obtained with high probability from $O(\frac{1}{\epsilon_{max}^2})$ runs of the boson sampling device, by Hoeffding's inequality [26].

Performing the above mentioned extrapolation strategy, we obtain the following set of equations, written in matrix form, to be solved for obtaining an estimate of α_0

$$\begin{pmatrix} (1 - \eta_0)^n & \eta_0 (1 - \eta_0)^{n-1} & \ldots & \eta_0^{n-c}(1 - \eta_0)^k \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (1 - \eta_{n-c})^n & \eta_{n-c}(1 - \eta_{n-c})^{n-1} & \ldots & \eta_{n-c}^{n-c}(1 - \eta_{n-c})^c \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_0 \\ . \\ . \\ \tilde{\alpha}_{n-c} \end{pmatrix} = \begin{pmatrix} p_{\eta_0}(n_1 \ldots n_l) \\ . \\ . \\ p_{\eta_{n-c}}(n_1 \ldots n_l) \end{pmatrix} + \begin{pmatrix} \epsilon_0 \\ . \\ . \\ \epsilon_{n-c} \end{pmatrix}, \tag{70}$$

with $|\epsilon_i| \leq \epsilon_{max}$, and $\tilde{\alpha}_i$ is an estimate of α_i obtained from using the estimates $\tilde{p}_{\eta_i}(n_1 \ldots n_l) = p_{\eta_i}(n_1 \ldots n_l) + \epsilon_i$ Let,

$$L := \begin{pmatrix} (1 - \eta_0)^n & \eta_0 (1 - \eta_0)^{n-1} & \ldots & \eta_0^{n-c}(1 - \eta_0)^c \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (1 - \eta_{n-c})^n & \eta_{n-c}(1 - \eta_{n-c})^{n-1} & \ldots & \eta_{n-c}^{n-c}(1 - \eta_{n-k})^c \end{pmatrix}.$$

We can rewrite L as

$$L = DW, \tag{71}$$

with

$$D = \begin{pmatrix} (1 - \eta_0)^n & 0 & 0 & \ldots 0 \\ 0 & (1 - \eta_1)^n & 0 & \ldots 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & 0 & \ldots (1 - \eta_{n-c})^n \end{pmatrix} \tag{72}$$

a diagonal matrix, and

$$
W = \begin{pmatrix}
1 & \frac{\eta_0}{1-\eta_0} & \cdots & \left(\frac{\eta_0}{1-\eta_0}\right)^{n-c} \\
1 & \frac{\eta_1}{1-\eta_1} & \cdots & \left(\frac{\eta_1}{1-\eta_1}\right)^{n-c} \\
. & . & . & . \\
. & . & . & . \\
1 & \frac{\eta_{n-c}}{1-\eta_{n-c}} & \cdots & \left(\frac{\eta_{n-c}}{1-\eta_{n-c}}\right)^{n-c}
\end{pmatrix}
\tag{73}
$$

a Vandermonde matrix [65]. Multiplying both sides of eqn. 70 by L^{-1} and invoking standard matrix multiplication rules, we obtain

$$
\tilde{\alpha}_0 = \sum_{i=0,n-c} L_{1i}^{-1} p_{\eta_i}(n_1 \ldots n_l) + \sum_{i=0,\ldots,n-c} L_{1i}^{-1} \epsilon_i,
\tag{74}
$$

where L_{1i}^{-1} is the element of the first row and ith column of L^{-1}. Note that, by construction of our method, $\eta_i \neq \eta_{i+1}$, then D is invertible, and so is W [65], thus L^{-1} always exists. Furthermore, $\alpha_0 = \sum_{i=0,n-k} L_{1i}^{-1} p_{\eta_i}(n_1 \ldots n_l)$, since exactly computing the probabilities $p_{\eta_i}(n_1 \ldots n_l)$ will lead to an exact computation of α_0. Therefore, the error associated to our extrapolation technique is given by

$$
E_{extrap} := \left| \sum_{i=0,\ldots,n-k} L_{1i}^{-1} \epsilon_i \right|.
\tag{75}
$$

Note that the overall sample complexity of the extrapolation protocol is $O(\frac{n-c+1}{\epsilon_{max}^2})$. For post-selection, an ϵ_{max} additive error estimate $\tilde{p}_\eta(n_1 \ldots n_l \cap n)$ requires $O(\frac{1}{\epsilon_{max}^2})$ samples, and can be performed for $\eta = \eta_0$, that is without artificially increasing loss. From eqn. 68, we see that post-selection induces an error of

$$
E_{post} := \left| \frac{\epsilon}{\sqrt{(1-\eta_0)^n}} \right| \leq \frac{\epsilon_{max}}{\sqrt{(1-\eta_0)^n}},
\tag{76}
$$

with $|\epsilon| \leq \epsilon_{max}$.

In the remainder of this section, we will compute an upper bound for E_{extrap}, and show that this upper bound is greater than the corresponding upper bound for E_{post} shown in eqn. 76. This gives strong evidence that the error induced by extrapolation is higher than that of postselection for a comparable number of samples, and therefore that extrapolation offers no advantage over postselection. Although the upper bound argument we show gives strong evidence that extrapolation techniques are not advantageous when compared to postselection, we will provide further evidence that this is the case. In particular, for a random distribution of errors $\{\epsilon_i\}$ with $|\epsilon_i| \leq \epsilon_{max}$, we numerically show that the condition

$$
E_{extrap} > \frac{\epsilon_{max}}{(1-\eta_0)^n}
\tag{77}
$$

is never violated after some value of n, confirming our analytical results. We start by the analytical upper bound argument. By a triangle inequality,

$$
E_{extrap} \leq ||L^{-1}||_\infty \epsilon_{max},
$$

with $||L^{-1}||_\infty := \mathsf{max}_i \sum_j |L_{ij}^{-1}|$. Also,

$$
||L^{-1}||_\infty \leq ||D^{-1}||_\infty ||W^{-1}||_\infty.
$$

From the definition of $||.||_\infty$, we can directly show

$$
||D^{-1}||_\infty = \frac{1}{(1-\eta_{n-c})^n}.
$$

For $||W^{-1}||_\infty$, we can use an upper bound on the norm of Vandermonde matrices shown in [65]

$$
||W^{-1}||_\infty \leq \mathsf{max}_i \prod_{j \neq i=0,\ldots,n-c} \frac{1 + \frac{\eta_i}{1-\eta_i}}{\left| \frac{\eta_i}{1-\eta_i} - \frac{\eta_j}{1-\eta_j} \right|}.
$$

Thus,

$$
||L^{-1}||_\infty \leq \frac{1}{(1-\eta_{n-c})^n} \mathsf{max}_i \prod_{j \neq i=0,\ldots,n-c} \frac{1 + \frac{\eta_i}{1-\eta_i}}{\left| \frac{\eta_i}{1-\eta_i} - \frac{\eta_j}{1-\eta_j} \right|}.
$$

and

$$E_{extrap} \leq \epsilon_{max} \frac{1}{(1-\eta_{n-c})^n} \mathsf{max}_i \prod_{j\neq i=0,\ldots,n-c} \frac{1+\frac{\eta_i}{1-\eta_i}}{|\frac{\eta_i}{1-\eta_i} - \frac{\eta_j}{1-\eta_j}|}.$$

We will now show that the upper bound on E_{extrap} is greater than that of E_{post}. This is quantified through the following theorem .

Theorem 30. *For $\eta_0, \ldots, \eta_{n-c}$ with $\eta_{i+1} > \eta_i$ and $\eta_0 \geq 0$, $\eta_{n-c} < 1$ the following holds*

$$\frac{1}{(1-\eta_{n-c})^n} \mathsf{max}_i \prod_{j\neq i=0,\ldots,n-c} \frac{1+\frac{\eta_i}{1-\eta_i}}{|\frac{\eta_i}{1-\eta_i} - \frac{\eta_j}{1-\eta_j}|} \geq \frac{1}{(1-\eta_0)^n}. \tag{78}$$

Proof. First, note that $\frac{\eta}{1-\eta}$ is a monotonically increasing function of η, thus $\frac{\eta_i}{1-\eta_i} < \frac{\eta_{i+1}}{1-\eta_{i+1}}$. This allows us to lower bound $\mathsf{max}_i \prod_{j\neq i=0,\ldots,n-k} \frac{1+\frac{\eta_i}{1-\eta_i}}{|\frac{\eta_i}{1-\eta_i} - \frac{\eta_j}{1-\eta_j}|}$ as

$$\mathsf{max}_i \prod_{j\neq i=0,\ldots,n-c} \frac{1+\frac{\eta_i}{1-\eta_i}}{|\frac{\eta_i}{1-\eta_i} - \frac{\eta_j}{1-\eta_j}|} \geq \frac{(1+\frac{\eta_0}{1-\eta_0})^{n-c}}{(\frac{\eta_{n-c}}{1-\eta_{n-c}} - \frac{\eta_0}{1-\eta_0})^{n-c}}.$$

Our strategy is to show that the following holds

$$\frac{1}{(1-\eta_{n-c})^n} \frac{(1+\frac{\eta_0}{1-\eta_0})^{n-c}}{(\frac{\eta_{n-c}}{1-\eta_{n-c}} - \frac{\eta_0}{1-\eta_0})^{n-c}} \geq \frac{1}{(1-\eta_0)^n} \tag{79}$$

eqn. 79 being true implies that eqn. 78 is also true, and thus is sufficient for proving Thm. 30. eqn. 79 can be rewritten as

$$\frac{(1-\eta_0)^n}{(1-\eta_{n-c})^n} \frac{(1-\eta_0)^{n-c}}{(\frac{\eta_{n-c}}{1-\eta_{n-c}} - \frac{\eta_0}{1-\eta_0})^{n-c}} \geq 1. \tag{80}$$

rewriting the left hand side of the above eqn.

$$\frac{(1-\eta_0)^n}{(1-\eta_{n-c})^n} \frac{(1+\frac{\eta_0}{1-\eta_0})^{n-c}}{(\frac{\eta_{n-c}}{1-\eta_{n-c}} - \frac{\eta_0}{1-\eta_0})^{n-c}} = \frac{(1-\eta_0)^c}{(1-\eta_{n-c})^c} \Big(\frac{(1-\eta_0)(1+\frac{\eta_0}{1-\eta_0})}{(1-\eta_{n-c})(\frac{\eta_{n-c}}{1-\eta_{n-c}} - \frac{\eta_0}{1-\eta_0})} \Big)^{n-c},$$

and observing that

$$\frac{(1-\eta_0)(1+\frac{\eta_0}{1-\eta_0})}{(1-\eta_{n-c})(\frac{\eta_{n-c}}{1-\eta_{n-c}} - \frac{\eta_0}{1-\eta_0})} = \frac{1-\eta_0}{\eta_{n-c} - \eta_0},$$

and plugging this into eqn. (80) we obtain

$$\frac{(1-\eta_0)^c}{(1-\eta_{n-c})^c} \Big(\frac{1-\eta_0}{\eta_{n-c} - \eta_0} \Big)^{n-c} \geq 1. \tag{81}$$

Now, $1-\eta_0 \geq 1-\eta_{n-c}$ and $1-\eta_0 \geq \eta_{n-c} - \eta_0$ since $\eta_0 < \eta_{n-c} < 1$. This implies that eqn. 81 is true, and thus eqn. s 80 and 79 hold, and therefore Thm. 30 is proved. □

We also numerically compute the number of violations of the inequality $E_{extrap} \geq \frac{\epsilon_{max}}{(1-\eta_0)^n}$ and plot these in Figure 7. We took $\epsilon_{max} = 0.01$, $\eta_0 = 0.01$, $\eta_{n-c} = 0.95$, and η_i for $1 < i < n-c$ equally spaced. We varied the value of $n-c$ between 3 and 14, with $c = \mathsf{ceil}(n/3)$ and $\mathsf{ceil}(.)$ is the ceiling function. For each value of $n-c$ we performed 3000 runs, where at each run we took $n-c+1$ values of $\{\epsilon_i\}$ chosen uniformly randomly from $[-\epsilon_{max}, \epsilon_{max}]$. As can be observed in Figure 7, the number of violations approaches zero with increasing n, confirming that extrapolation performs worse than post-selection after some value of n. A similar behaviour is observed for different values of ϵ_{max}, η_0 and η_{n-c}.
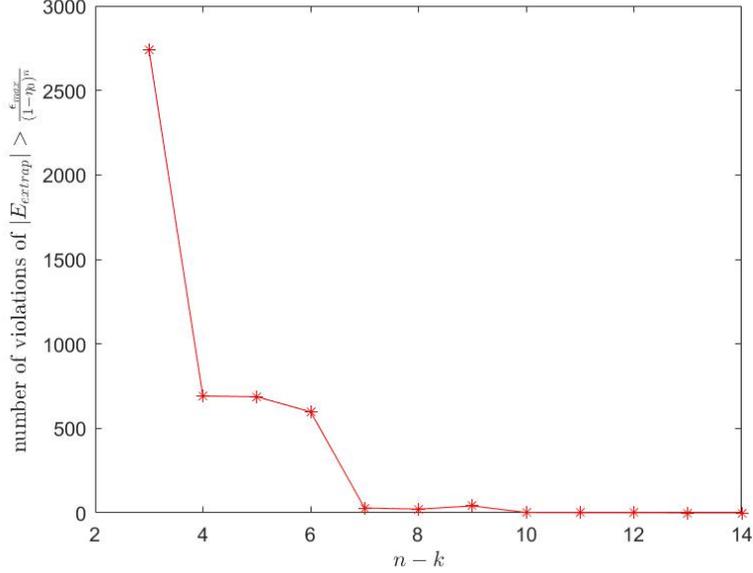
Figure 7: Number of violations of $|E_{extrap}| \geq \frac{\epsilon_{max}}{(1-\eta_0)^n}$ plotted versus $n - c$ (see main text).

Second method of extrapolation at various noise rates

Another possible extrapolation technique can be performed by considering eqn. (68) where the $(1 - \eta_i)^{n-i}$ terms are expanded in order to obtain

$$p_\eta(n_1 \ldots n_l) = \sum_{i=0,\ldots,n} \beta_i \eta^i. \tag{82}$$

Where $\beta_0 = \alpha_0$, and β_is are linear combinations of the α_is defined previously. The extrapolation procedure proceeds in a similar manner to that described above, but now we compute $p_\eta(n_1 \ldots n_l)$ for $n + 1$ values of loss $\eta_n > \eta_{n-1} \cdots > \eta_0$ to solve for the coefficients $\{\beta_i\}$, with the matrix L in this case given by

$$L = \begin{pmatrix} 1 & \eta_0 & \ldots & \eta_0^n \\ 1 & \eta_1 & \ldots & \eta_1^n \\ . & . & . & . \\ . & . & . & . \\ 1 & \eta_n & \ldots & \eta_n^n \end{pmatrix}. \tag{83}$$

L is a Vandermonde matrix, and we can directly use the result of [65] to upper bound $||L^{-1}||_\infty$ as

$$||L^{-1}||_\infty \leq \mathsf{max}_i \prod_{j \neq i=0,\ldots,n} \frac{1 + \eta_i}{|\eta_i - \eta_j|}.$$

Therefore,

$$E_{extrap} \leq \epsilon_{max} \mathsf{max}_i \prod_{j \neq i=0,\ldots,n} \frac{1 + \eta_i}{|\eta_i - \eta_j|}$$

We will now prove that the upper bound on E_{extrap} is larger than that of E_{post} for this method of extrapolation, giving strong evidence that this technique offers no advantage over post-selection. This amounts to proving the following.

Theorem 31. *For $\eta_0, \ldots, \eta_n$ with $\eta_{i+1} > \eta_i$ and $\eta_0 \geq 0$, $\eta_n < 1$ the following holds*

$$\mathsf{max}_i \prod_{j \neq i=0,\ldots,n} \frac{1 + \eta_i}{|\eta_i - \eta_j|} \geq \frac{1}{(1 - \eta_0)^n}. \tag{84}$$

Proof. We note that

$$\mathsf{max}_i \prod_{j \neq i = 0,\ldots,n} \frac{1 + \eta_i}{|\eta_i - \eta_j|} \geq \frac{(1 + \eta_0)^n}{(\eta_n - \eta_0)^n},$$

and that $\frac{(1+\eta_0)^n}{(\eta_n-\eta_0)^n} \geq \frac{1}{(1-\eta_0)^n}$ since $\frac{1}{\eta_n-\eta_0} > \frac{1}{1-\eta_0}$ and $1 + \eta_0 > 1$. This completes the proof. $\square$

For this technique as well we numerically compute the number of violations of $E_{extrap} > \frac{\epsilon_{max}}{(1-\eta_0)^n}$ and plot these in Figure 8. We took $\epsilon_{max} = 0.01$, $\eta_0 = 0.01$, $\eta_n = 0.95$, and η_i for $0 < i < n$ equally spaced. We varied the value of n between 3 and 16. For each value of n we performed 3000 runs, where at each run we took $n+1$ values of $\{\epsilon_i\}$ chosen uniformly randomly from $[-\epsilon_{max}, \epsilon_{max}]$. As can be observed in Figure 8, the number of violations approaches zero with increasing n, confirming that this extrapolation performs worse than post-selection after some value of n. A similar behaviour is observed for different values of ϵ_{max}, η_0 and η_n.
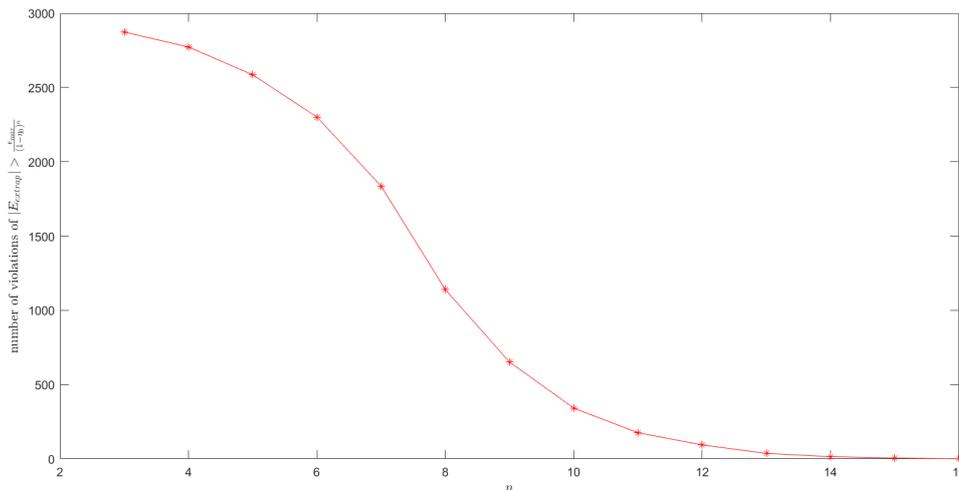

Figure 8: Number of violations of $|E_{extrap}| \geq \frac{\epsilon_{max}}{(1-\eta_0)^n}$ plotted versus n (see main text).

Proof of Thm. 13

We now prove Thm. 13 from the main text:

For all $n \geq n_0$, with n_0 a positive integer, $\mathsf{M}(E_{extrap}) \geq \frac{\epsilon_{max}}{\sqrt{(1-\eta)^n}}$.

A third extrapolation technique which can be used is the Richardson extrapolation technique, at the heart of the zero noise extrapolation method for mitigating qubit errors, which has gained widespread use and is at the heart of ZNE [32]. This method uses the expansion of eqn. (82) with $\eta_i = c_i\eta$, for $i \in \{0, \ldots, n\}$ with $\eta \in [0, 1]$, $c_0 = 1$, and c_i are positive reals satisfying $c_{i+1} > c_i$. The Richardson extrapolation method consists of computing an estimate $\tilde{p}(n_1 \ldots n_l | n)$ of $p(n_1 \ldots n_l | n)$ as follows

$$\tilde{p}(n_1 \ldots n_l | n) = \sum_{i=0\ldots n} \gamma_i \tilde{p}_{\eta_i}(n_1 \ldots n_l),$$

where, $\gamma_i := (-1)^n \prod_{j \neq i} \frac{c_j}{c_i - c_j}$. It can be shown that [13] $\sum_{i=0,\ldots,n} \gamma_i = 1$ and $\sum_{j=0,\ldots,n} \gamma_i c_i^j = 0$ for $j = 1, \ldots, n$. Furthermore, it can also be shown that $p(n_1 \ldots n_l | n) = \sum_{i=0,\ldots n} \gamma_i p_{\eta_i}(n_1 \ldots n_l)$. Thus, the error associated with this technique is given by

$$E_{extrap} := \big| \sum_{j=0,\ldots n} \gamma_i \epsilon_i \big|.$$

In the remainder of this paragraph, we will show that $\gamma_{i-1} = L_{1i}^{-1}$ for $i \in \{1, \ldots, n+1\}$ where L is the Vandermonde matrix of eqn. (83) with $\eta_i = c_i\eta$. This means that the results of the previous section follow through, and therefore that Richardson extrapolation offers no advantage over post-selection.

Let $\mathbf{B} := \begin{pmatrix} \beta_0 \\ \beta_1 \\ . \\ . \\ . \\ \beta_n \end{pmatrix}$, and $\mathbf{P} = \begin{pmatrix} p_{\eta_0}(n_1 \ldots n_l) \\ p_{\eta_1}(n_1 \ldots n_l) \\ . \\ . \\ . \\ p_{\eta_n}(n_1 \ldots n_l) \end{pmatrix}$. In the case of a perfect computation of the probabilities $p_{\eta_i}(n_1 \ldots n_l)$ (i.e $\epsilon_i = 0$), the extrapolation technique based on the matrix L of eqn. (83) amounts to computing β_0 from the following system of eqn. s

$$\mathbf{B} = L^{-1}\mathbf{P}.$$

Invoking standard multiplication rules for the above equation, one obtains

$$p(n_1 \ldots n_l) = \beta_0 = \sum_{i=1,\ldots n+1} L_{1i}^{-1} p_{\eta_{i-1}}(n_1 \ldots n_l).$$

Plugging the expansion $p_{\eta_j}(n_1 \ldots n_l) = \sum_{i=0,\ldots n} \beta_i \eta_i^j$ and $\eta_i = c_i\eta$ into the above equation, one obtains

$$\beta_0 = \sum_{i=1,\ldots n+1} L_{1i}^{-1}\beta_0 + \sum_{j=1,\ldots,n} \sum_{i=1\ldots n+1} L_{1i}^{-1} c_{i-1}^j \beta_j \eta^j.$$

By a direct identification of the left hand side of the above equation with its right hand side, we find that $\sum_{i=1\ldots n+1} L_{1i}^{-1} = 1$, and $\sum_{i=1,\ldots,n+1} L_{1i}^{-1} c_{i-1}^j = 0$ for $j \in \{1 \ldots n\}$. These sets of equations are exactly those defining the coefficients γ_i (as seen previously) and allowing to uniquely determine them, thus we can make the identification $L_{1i}^{-1} = \gamma_{i-1}$ for $i \in \{1, \ldots, n\}$ and our result is demonstrated.

22 Lyapunov bound

22.1 Proof of Thm. 17

We now prove Thm. 17 from the main text:

For all $r > 2$, $n \gg 1$, and for the independent, identically distributed random variables Y_i defined in eqn. (50), we have that

$$N\frac{\mathbf{E}(|Y|^r)}{\sigma_N^r} \geq O\Big(\frac{\beta(r)^n}{N^{\frac{r}{2}-1}}\Big), \tag{85}$$

where $\beta(r) > 1$ is a positive real number dependent on r.

Proof. Fix an $r > 2$. We begin by evaluating σ_N^r.

$$\sigma_N^2 = (\mathbf{E}(S_N^2)) = \mathbf{E}((\sum_{i=1,\ldots,N} Y_i)^2) = \sum_{i=1,\ldots,N} \mathbf{E}(Y_i^2) + 2 \sum_{i=1,\ldots,N} \sum_{j>i} \mathbf{E}(Y_iY_j).$$

The $Y_i's$ are independent identically distributed, and $\mathbf{E}(Y_i) = 0$ by construction, therefore $\mathbf{E}(Y_iY_j) = \mathbf{E}(Y_i)\mathbf{E}(Y_j) = 0, \forall i \neq j$, and $\sum_{i=1,\ldots,N} \mathbf{E}(Y_i^2) = N\mathbf{E}(Y^2) = N\mathbf{E}(X - \frac{n!}{m^n})^2 = N\big(\mathbf{E}(X^2) - 2\mathbf{E}(X)\frac{n!}{m^n} + (\frac{n!}{m^n})^2\big)$. Now, $\mathbf{E}(X^2) = \frac{(n!)^2(n+1)}{m^{2n}}$ and $\mathbf{E}(X) = \frac{n!}{m^n}$ [61]. Thus

$$\sigma_N^2 = N(\frac{n(n!)^2}{m^{2n}}),$$

and then

$$\sigma_N^r = (N(\frac{n(n!)^2}{m^{2n}}))^{\frac{r}{2}}.$$

We will now compute a lower bound $\mathbf{E}(|Y|^r)$. First, we will use a triangle inequality,

$$\mathbf{E}(|Y|^r) \geq |\mathbf{E}(Y^r)|.$$

We now focus on computing $|\mathbf{E}(Y^r)|$. Let $\mu := \frac{n!}{m^n}$

$$\mathbf{E}(Y^r) = \mathbf{E}(X-\mu)^r = \sum_{i=0,\dots,r} \binom{r}{i} \mathbf{E}(X^i)\mu^{r-i}(-1)^{r-i} = \sum_{i=0,\dots,r} \binom{r}{i} \frac{\mathbf{E}(|\mathsf{Per}(G)|^{2i})\mu^{r-i}(-1)^{r-i}}{m^{ni}}.$$

Plugging conjecture 16 into $\mathbf{E}(Y^r)$, we obtain

$$\mathbf{E}(Y^r) = \sum_{i=0,\dots,r} \binom{r}{i} \frac{O(\frac{(n!)^{2i}(i!)^{2n}}{(ni)!})(\frac{n!}{m^n})^{r-i}(-1)^{r-i}}{m^{ni}} = \frac{1}{m^{nr}}\sum_{i=0,\dots,r} (-1)^{r-i}O(\frac{\binom{r}{i}(n!)^{r+i}(i!)^{2n}}{(ni)!}).$$

Since $n \gg 1$, we can use a Stirling approximation for $i > 0$

$$\frac{(n!)^{r+i}}{(ni)!} \approx \frac{(\frac{n}{e})^{nr+ni}(\sqrt{2\pi n})^{r+i}}{(\frac{ni}{e})^{ni}\sqrt{2\pi ni}} = \left(\frac{1}{i}\right)^{ni}\left(\frac{n}{e}\right)^{nr}\frac{(2\pi n)^{\frac{i+r}{2}}}{\sqrt{2\pi ni}}.$$

Plugging this into $\mathbf{E}(Y^r)$, we get

$$\mathbf{E}(Y^r) = \frac{(\frac{n}{e})^{nr}}{m^{nr}}\left((-1)^r O(1)(2\pi n)^{\frac{r}{2}} + \sum_{i=1,\dots,r}(-1)^{r-i}O\left(\binom{r}{i}(i!)^{2n}\frac{1}{i^{ni}}\frac{(2\pi n)^{\frac{i+r}{2}}}{\sqrt{2\pi ni}}\right)\right).$$

For any $i \le r$, $\frac{(2\pi n)^{\frac{i+r}{2}}}{\sqrt{2\pi ni}} = O(n^r)$, we can simplify the above to

$$\mathbf{E}(Y^r) = \frac{O(n^r)(\frac{n}{e})^{nr}}{m^{nr}}\left((-1)^r O(1) + \sum_{i=1,\dots,r}(-1)^{r-i}O\left(\binom{r}{i}(i!)^{2n}\frac{1}{i^{ni}}\right)\right).$$

Now, we will look at $|(-1)^r O(1) + \sum_{i=1,\dots,r}(-1)^{r-i}O(\binom{r}{i}(i!)^{2n}\frac{1}{i^{ni}}))| = |(-1)^r O(1) + \sum_{i=1,\dots,r}(-1)^{r-i}O(\binom{r}{i}(\frac{(i!)^{\frac{2}{i}}}{i})^{ni})|$. $\alpha(i) := \frac{(i!)^{\frac{2}{i}}}{i}$ is a monotonically increasing function of i for $i > 2$, this can be directly verified using Stirling's approximation for large i, since $\frac{(i!)^{\frac{2}{i}}}{i} \approx \frac{O(i^2)}{i} = O(i)$. For small i, we verified this by numerical simulation. Furthermore, $\alpha(i) \ge 1$ for $i \ge 1$. Therefore $|(-1)^r O(1) + \sum_{i=1,\dots,r}(-1)^{r-i}O(\binom{r}{i}(\frac{(i!)^{\frac{2}{i}}}{i})^{ni})| = O((\alpha(r))^{nr}) = O(\beta(r)^n)$, where $\beta(r) := \alpha(r)^r = \frac{(r!)^2}{r^r}$. Note that $\beta(r) > 1$ for $r > 2$. Therefore

$$|\mathbf{E}(Y^r)| = \frac{O(n^r(\frac{n}{e})^{nr}\beta(r)^n)}{m^{nr}}.$$

Replacing $n! \approx (\frac{n}{e})^n\sqrt{2\pi n}$ in σ_N^r, then performing straightforward simplifications, we obtain

$$\frac{N|\mathbf{E}(Y^r)|}{\sigma_N^r} = O\left(\frac{\beta(r)^n}{N^{\frac{r}{2}-1}}\right).$$

Noting, as previously mentioned, that $\frac{N\mathbf{E}(|Y|^r)}{\sigma_N^r} \ge \frac{N|\mathbf{E}(Y^r)|}{\sigma_N^r}$ completes the proof. $\qquad\square$