SHORT PROOFS
OF TVERBERG-TYPE THEOREMS FOR CELL COMPLEXES

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Abstract. We present short proofs of two Tverberg-type theorems for cell complexes by S. Hasui, D. Kishimoto, M. Takeda, and M. Tsutaya. One of them states that for any prime power r, any simplicial sphere X of dimension $(d+1)(r-1)-1$, and any continuous map $f : X \to \mathbb{R}^d$ there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of X such that $f(\sigma_1) \cap \ldots f(\sigma_r) \neq \emptyset$.

We present short proofs of two Tverberg-type theorems for cell complexes ([HKTT, Theorem 1.4 and Corollary 1.5] quoted as Corollary 1.b and Theorem 3 below).

We abbreviate 'finite simplicial complex' to 'complex'. We mostly omit 'continuous' for maps and group actions. Denote by Δ_n the simplex of dimension n.

A map $f\colon X \to \mathbb{R}^d$ from a complex X is said to be an *almost r-embedding* if $f(\sigma_1) \cap \ldots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces of X.

The topological Tverberg theorem states that *if r is a prime power, then the simplex $\Delta_{(d+1)(r-1)}$ (or, equivalently, its boundary) has no almost r-embeddings to $\mathbb{R}^d$*. This was proved by Bárány–Schlossman–Szűcs and Özaydin–Volovikov, see surveys [BZ16, Sk16] and the references therein. (A counterexample for r not a prime power follows from results of Özaydin, Mabillard–Wagner, and Gromov–Blagojević–Frick–Ziegler; for the details, see the surveys [BZ16, Sk16] and the references therein; cf. Remark 5.c.) The topological Tverberg theorem was generalized as follows.

Corollary 1. *(a)* [HKTT, before question 1.2] *If r is a prime power, then no boundary of a convex $(d+1)(r-1)$-polytope has an almost r-embeddings to $\mathbb{R}^d$.*

(b) [HKTT, Corollary 1.5] *If r is a prime power, then no complex homeomorphic to $\partial\Delta_{(d+1)(r-1)}$ has an almost r-embeddings to $\mathbb{R}^d$.*

These results immediately follow from the topological Tverberg theorem and Lemmas 2.a,b below, respectively.[1] Corollary 1.a proves for prime powers the Tverberg–Bárány–Kalai conjecture stated in [HKTT, before question 1.2], [SZ24, Conjecture 1.2]. For non-prime-powers that conjecture concerns maps *linear* on faces, so it remains open.

A complex A is said to be a *refinement* of a complex B if there is a homeomorphism $|A| \to |B|$ (called refining homeomorphism) between their bodies such that every face of B is the union of images of some faces of A.

Lemma 2. *(a)* [Gr03, p. 200] *The boundary of a convex $(n+1)$-polytope is a refinement of $\partial\Delta_{n+1}$.*

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[1]This was noted in [HKTT, before question 1.2]. Lemma 2.b was not present in [HKTT], so that paper provided a more complicated proof of Corollary 1.b relying on Theorem 3 below. Cf. arXiv version [HKTT, §1, after corollary 1.5].

(b) Any complex homeomorphic to the boundary $\partial\Delta_{n+1}$ of the simplex is a refinement of $\partial\Delta_{n+1}$.

(b') For any complex D homeomorphic to Δ_n any refining homeomorphism $h:\partial D\to\partial\Delta_n$ extends to a homeomorphism $D\to\Delta_n$.

Proof. (b') Take a homeomorphism $F:D\to\Delta_n$. Denote by cX the cone over a complex X. Take a homeomorphism $C:\Delta_n\to c\partial\Delta_n$. Then $C^{-1}\circ ch\circ cF|_{\partial D}\circ C\circ F$ is the required extension:

$$D\overset{F}{\to}\Delta_n\overset{C}{\to}c\partial\Delta_n\overset{cF|_{\partial D}}{\to}c\partial D\overset{ch}{\to}c\partial\Delta_n\overset{C^{-1}}{\to}\Delta_n.$$

(b) Induction on n. The case $n=0$ is trivial. Let us prove the inductive step $n-1\to n$. Take any vertex v of the complex. The link $\operatorname{lk}v$ is homeomorphic to $\partial\Delta_n$. So by the induction hypothesis $\operatorname{lk}v$ is a refinement of $\partial\Delta_n$. Then $\operatorname{st}v$ is the conical refinement of the complement in $\partial\Delta_{n+1}$ to the interior of Δ_n. Denote by D the complement in the given complex to the open star of v. By [RS72, 3.13] D is homeomorphic to Δ_n. So by (b) the refining homeomorphism $\operatorname{lk}v\to\partial\Delta_n$ extends to a homeomorphism $D\to\Delta_n$, which is automatically refining.[2] □

Although the proof of Corollary 1.b in [HKTT] is much more complicated than above, let us describe the approach of that paper, because it leads to a result of independent interest ([HKTT, Theorem 1.4] quoted as Theorem 3 below). The idea is to use analogy to the Özaydin–Volovikov proof of the topological Tverberg theorem. Namely, to use a lemma of Özaydin–Volovikov (quoted below, [Vo96, Lemma]), and mimick the spectral sequence argument proving high enough acyclicity of the total space of a bundle, in the situation when a map is not a bundle (this terminology is not used in the statements below). There naturally appears a property of a simplicial sphere (complementary acyclicity defined in [HKTT, §1] and below) allowing to carry this argument. Thus the proof of [HKTT, Theorem 1.4] is an easy combination of known results and methods, see details in Remark 5.ab and footnote 3.

Let X be a complex. Let $X-\sigma$ be the subcomplex of X consisting of faces which do not intersect σ. This is not to be confused with $X\setminus\sigma$. We take a prime a and $r=a^m$; we omit coefficients $\mathbb{Z}_a$ from the notation of homology groups.

For integers $n\ge -1$ and $s\ge 1$ a complex X is said to be

• *n-acyclic* if X is non-empty and $\widetilde{H}_j(X)=0$ for every $j\le n$.

• *s-complementary n-acyclic* if for every $i=0,1,\dots,s$, and pairwise disjoint faces $\sigma_1,\dots,\sigma_i$ of X, the complex $X-\sigma_1-\ldots-\sigma_i$ is $(n-\dim\sigma_1-\ldots-\dim\sigma_i)$-acyclic.

Set
$$n:=d(r-1)-1.$$

Theorem 3. *If r is a power of a prime, then no $(r-1)$-complementary n-acyclic complex has an almost r-embedding to $\mathbb{R}^d$.*

Set
$$X^{\underline{r}}:=\cup\{\sigma_1\times\dots\times\sigma_r\ :\sigma_i\text{ is a face of }X,\ \sigma_i\cap\sigma_j=\emptyset\text{ for every }i\ne j\}.$$

[2]The above proofs of (b,b') are analogous to [Gr03, p. 200] (they use [RS72, 3.13] instead of central projection).
Each one of (b), (b') implies that *any complex D PL homeomorphic to Δ_n is a refinement of Δ_n* (absolute version of (b')).
In the above proof of (b) we need (b'), not its absolute version.

This is a topological analogue of the set of arrangements, so we use the notation analogous to $x^{\underline{r}} = x(x-1)\dots(x-r+1)$.

Let $\pi : X^{\underline{r}} \to X$ be the restriction of the projection $X^r \to X$.

Proof of Theorem 3. The group $G := \mathbb{Z}_a^m$ acts on $X^{\underline{r}}$ by permutations of the r points, the permutations corresponding to the action of G on itself by left shifts. Thus G can be considered as a subgroup of the group of permutation of $r = |G|$ elements. Hence it is clear and well-known (see e.g. surveys [BZ16, Lemma 3.9], [Sk16, Lemma 2.3]) that

if there is an almost r-embedding $X \to \mathbb{R}^d$, then there are a fixed points free action of G on S^n and a G-equivariant map $X^{\underline{r}} \to S^n$.

Apply the following result [Vo96, Lemma] for $k = n$ and $Y = X^{\underline{r}}$.

Suppose that G acts on Y and on S^k without fixed points, and Y is k-acyclic. Then there does not exist a G-equivariant map $Y \to S^k$.

We see that it suffices to show that $X^{\underline{r}}$ *is n-acyclic.*

We show this by induction on r. The base $r = 1$ follows because $X^{\underline{1}} = X$. Let us prove the inductive step. Since X is $(r-1)$-complementary n-acyclic, for each face σ of X the complex $X - \sigma$ is $(r-2)$-complementary $(n - \dim\sigma)$-acyclic. Hence by the inductive hypothesis $(X - \sigma)^{\underline{r-1}}$ is $(n - \dim\sigma)$-acyclic. Now the inductive step holds by the following lemma because X is n-acyclic. □

Lemma 4. *If $F_\sigma := (X - \sigma)^{\underline{r-1}}$ is $(n - \dim\sigma)$-acyclic for every face σ of a complex X, then $H_s(X^{\underline{r}}) \cong H_s(X)$ for every $s \le n$.*

Proof. Consider [3] the π-preimage of the skeletal filtration $X^{(0)} \subset X^{(1)} \subset \dots$ of X. One obtains the spectral sequence [FF89, §20.2] with

$$E^1_{p,q} = H_{p+q}(\pi^{-1}(X^{(p)}, \pi^{-1}(X^{(p-1)})) \cong \oplus H_{p+q}(\sigma \times F_\sigma \cup R, \pi^{-1}(\partial\sigma)) \cong$$

$$\oplus H_{p+q}(\sigma \times F_\sigma, \partial\sigma \times F_\sigma) \cong \oplus H_q(F_\sigma).$$

Here the sums are over all p-faces σ of X, and $R = \pi^{-1}(\partial\sigma) \setminus (\partial\sigma \times F_\sigma)$. By the acyclicity we have

$$E^1_{p,0} = C_p(X) \quad \text{for} \quad p \le n, \quad E^1_{p,q} = 0 \quad \text{for} \quad p + q \le n \quad \text{and} \quad q > 0.$$

Since being (-1)-acyclic means being non-empty, the group $E^1_{n+1,0}$ has a quotient group $C_{n+1}(X)$. The differential $E^1_{p,0} \to E^1_{p-1,0}$ is the boundary map $C_p(X) \to C_{p-1}(X)$ for $p \le n$, and is the composition $E^1_{n+1,0} \to C_{n+1}(X) \to C_n(X)$ of the quotient and the boundary maps for $p = n + 1$. Hence

$$E^2_{p,0} = H_p(X) \quad \text{for} \quad p \le n, \quad \text{and} \quad E^2_{p,q} = 0 \quad \text{for} \quad p + q \le n \quad \text{and} \quad q > 0.$$

Then $E^2_{p,q} = E^\infty_{p,q}$ for $p + q \le n$. Therefore $H_s(X^{\underline{r}}) \cong H_s(X)$ for every $s \le n$. □

[3] This proof is an exercise on spectral sequences analogous to [FF89, §21.1B]. In particular, construction of E^1 mimicks construction of E^1 for the spectral sequence of a bundle, although π need not be even the Serre fibration.

Lemma 4 and the construction of E^1 in its proof are clarified versions of [HKTT, Lemma 4.4 Proposition 4.3] (omitting unnecessary hocolimits). In [HKTT, §4] the construction of E^1 (sufficient for the main result) was implicitly stated (before Proposition 4.3), and was used to deduce Proposition 4.3 (stated in the hocolimits language unnecessary for the main result).

Remark 5. (a) In spite of being much shorter, the above proof of Theorem 3 is not an alternative proof compared to [HKTT] but is just a different exposition avoiding artificially sophisticated language, and applying [Vo96, Lemma] instead of rewriting its proof in a slightly more sophisticated way. Analogously, the proofs of [KM21] could be simplified (except possibly for the chirality result).

(b) Corollary 1.b can be deduced from Theorem 3 using Alexander duality and the following known result:

If any intersection of n complexes (including intersections involving only one complex) is contractible (including empty), then H^{n-1} of the union is zero.

See the details in [HKTT, §2] (where this known result was not explicitly stated).

(c) In [HKTT, end of the paragraph afer Theorem 1.1] the phrase *'Frick proved [13] that the condition that r is a prime power is necessary'* is misleading as explained in [Sk16, Remark 1.9, §3.1, §5]. It is also misleading not to cite [Sk16] (for accurate description of references on the counterexample to topological Tverberg Conjecture), and [MW15], [Sk16, §3] (for the detailed version of the original proof, and for a published simplified survey exposition of the main ingredient to the counterexample).

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