An Infinite Family of Integrable Sigma Models Using Auxiliary Fields

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We introduce a class of sigma models in two spacetime dimensions which are parameterized by an interaction function of one real variable. In addition to the physical group-valued field g, these models include an auxiliary vector field v_{α} which mediates interactions in a prescribed way. We prove that every model in this family is weakly integrable, in the sense that the classical equations of motion are equivalent to flatness of a Lax connection for any value of a spectral parameter. We also show that these models are strongly integrable, in the sense that the Poisson bracket of the Lax connection takes the Maillet form, which guarantees the existence of an infinite set of conserved charges in involution. This class of theories includes the principal chiral model (PCM) and all deformations of the PCM by functions of the energy-momentum tensor, such as $T\overline{T}$ and root- $T\overline{T}$.

I. INTRODUCTION

Integrable models in two dimensions offer rare examples of interacting field theories which can nonetheless be analyzed exactly, both at the classical and quantum levels. The dynamics of such models are tightly constrained by the presence of hidden symmetries, which can sometimes be leveraged to "solve" the theories. It is therefore useful to identify many examples of such integrable models, which has been a topic of great recent interest. These integrable 2d theories find applications in many areas of physics, such as in studies of classical and quantum strings at finite tension [1–3] and in condensed matter descriptions of spin chains (see e.g. [4–6]).

A fruitful method for generating new integrable theories is to deform existing ones in a way which preserves integrability. One popular starting point for this enterprise is the principal chiral model (PCM) [7, 8], which is an integrable 2d model that shares certain properties with 4d Yang-Mills theory [9–12]. The PCM admits many integrable deformations, some of which are reviewed in [13–17]; for instance, one can add a Wess-Zumino term [18–21], perform a Yang-Baxter deformation [22, 23], implement a λ -deformation [24], and so on.

In this letter, we introduce a new, infinite class of integrable sigma models by deforming the PCM via the activation of specific interactions that preserve integrability. The key technical ingredient in our construction is the coupling of the physical field in the PCM to an auxiliary field with algebraic equations of motion. There are many examples in field theory where the introduction of auxiliary fields can impose a desirable constraint, such as in the Pasti-Sorokin-Tonin formalism [25–27] which uses an auxiliary vector to implement a chirality condition on a tensor field in 6d, or the Ivanov-Zupnik auxiliary field formalism [28, 29] which imposes electric-magnetic duality invariance in theories of 4d electrodynamics. In our case, the inclusion of an auxiliary field imposes that the model remain integrable in a particularly simple way, while nonetheless allowing for quite general interactions.

An important step in the analysis of a classically integrable field theory is the construction of an infinite set of Poisson-commuting conserved charges. For the PCM and many of its deformations, this procedure can be performed in two steps. The first step is to establish what is called "weak integrability," which means that the equations of motion for the model can be encoded in the flatness of a one-form called the Lax connection. The second step is to show "strong integrability" by computing a particular Poisson bracket involving the spatial component of the Lax connection and demonstrating that it takes a special form considered by Maillet [30, 31]. This Maillet form of the Poisson bracket is parameterized by an r-matrix $r_{12}(z, z')$, which in many cases takes a standard form in terms of a "twist function" $\varphi(z)$. Integrable deformations of the PCM often modify the form of this twist function $\varphi(z)$; see [32] for examples.

In this work, we perform the two steps mentioned above for our new family of models. We find that, unlike some other integrable deformations of the PCM, the twist function is not modified for theories within our class, although the dependence of the Lax connection on the fields is changed. This behavior is similar to that of homogeneous Yang-Baxter deformations [33–35], which can be interpreted via twisted boundary conditions (it would be interesting to explore whether a similar interpretation applies to our models). As we will see, the proofs of weak and strong integrability for our models are nearly as simple as those for the PCM, even though a generic model in this family exhibits highly non-linear interactions.

The structure of this letter is as follows. In Section

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II, we introduce the family of sigma models of interest in this work and derive the equations of motion for these theories. In Section III, we establish weak integrability by proving that the equations of motion are equivalent to flatness of a Lax connection. Section IV proves strong integrability by computing the Poisson bracket of the spatial component of this Lax connection with itself, and showing that it takes the Maillet form. In Section V, we show that our family of models includes all deformations of the PCM by functions of the stress tensor, such as $T\overline{T}$ and root- $T\overline{T}$. Finally, Section VI summarizes and presents directions for future research.

II. AUXILIARY FIELD SIGMA MODELS

We now describe the class of models which are of interest in the present work. The physical degree of freedom is a group-valued field $g: \Sigma \to G$ which maps a flat twodimensional spacetime Σ with coordinates $\sigma^{\alpha} = (\tau, \sigma)$ into a Lie group G. We write \mathfrak{g} to denote the Lie algebra of G. The left-invariant Maurer-Cartan form is defined by $j = g^{-1}dg$, and the pullback of this form to Σ is

$$j_{\alpha} = g^{-1} \partial_{\alpha} g \,. \tag{1}$$

It will be convenient to use light-cone coordinates on Σ ,

$$\sigma^{\pm} = \frac{1}{2} \left(\tau \pm \sigma \right) \,, \tag{2}$$

where light-cone indices are lowered or raised with the metric $\eta_{+-} = \eta_{-+} = -2$ or inverse metric $\eta^{+-} = \eta^{-+} = -\frac{1}{2}$. The field j_{α} satisfies the Maurer-Cartan identity, which is written in light-cone coordinates as

$$\partial_{+}j_{-} - \partial_{-}j_{+} + [j_{+}, j_{-}] = 0.$$
(3)

In our notation, the Lagrangian of the PCM is written

$$\mathcal{L}_{\text{PCM}} = \frac{1}{2} \eta^{\alpha\beta} \operatorname{tr}(j_{\alpha} j_{\beta}) = -\frac{1}{2} \operatorname{tr}(j_{+} j_{-}).$$
(4)

We now introduce an additional Lie algebra valued field v_{α} , which is not a physical degree of freedom, but merely an auxiliary field with algebraic equations of motion. One can take traces to build scalars from v_{α} , such as

$$\nu = \operatorname{tr}(v_{+}v_{+})\operatorname{tr}(v_{-}v_{-}).$$
(5)

The family of models which we wish to study are described by Lagrangians of the form

$$\mathcal{L} = \frac{1}{2} \operatorname{tr}(j_+ j_-) + \operatorname{tr}(v_+ v_-) + \operatorname{tr}(j_+ v_- + j_- v_+) + E(\nu),$$
(6)

which are parameterized by an arbitrary interaction function $E(\nu)$ that depends on the single variable ν . The structure of this family of Lagrangians is inspired by the Ivanov-Zupnik formulation of theories of dualityinvariant electrodynamics in four dimensions [28, 29], which also involves an interaction function that depends on a single variable constructed from auxiliary fields.

Let us first consider the equations of motion associated with the Lagrangian (6). Varying the auxiliary field v_{α} gives rise to the Euler-Lagrange equation

$$j_{\pm} = -v_{\pm} - 2E'v_{\mp} \operatorname{tr}(v_{\pm}v_{\pm}), \qquad (7)$$

where we write $E' = \frac{dE}{d\nu}$ for the derivative of E. On the other hand, varying the group-valued field g as

On the other hand, varying the group-valued field g as $\delta g = \epsilon g$, under which the Maurer-Cartan form varies as $\delta j_{\pm} = \partial_{\pm} \epsilon + [j_{\pm}, \epsilon]$, gives the g-field equation of motion

$$\partial_{+}j_{-} + \partial_{-}j_{+} = 2\left([v_{-}, j_{+}] + [v_{+}, j_{-}] - \partial_{+}v_{-} - \partial_{-}v_{+}\right).$$
(8)

The interpretations of the two Euler-Lagrange equations (7) and (8) are rather different. While (8) is a true dynamical condition, which implies that the physical field g is on-shell, equation (7) is merely a constraint imposed by the auxiliary field. We introduce the symbol \doteq to denote equality between two quantities which holds when equation (7) is satisfied. For instance, (7) implies

$$[v_{-}, j_{+}] + [v_{+}, j_{-}] \stackrel{\cdot}{=} 0, \qquad (9)$$

so the g-field equation of motion (8) can be written as

$$\partial_+ (j_- + 2v_-) + \partial_- (j_+ + 2v_+) \stackrel{\cdot}{=} 0,$$
 (10)

when the auxiliary field equation of motion (7) is satisfied. It is convenient to define the new quantity

$$\mathfrak{J}_{\pm} = -(j_{\pm} + 2v_{\pm}),$$
(11)

which allows us to express the Euler-Lagrange equation for g as a conservation equation $\partial_{\alpha} \mathfrak{J}^{\alpha} \doteq 0$. Here j_{α} is flat but not conserved, while \mathfrak{J}_{α} is conserved but not flat.

Note that, when the interaction function $E(\nu)$ vanishes, the auxiliary field equation of motion reduces to $j_{\pm} = -v_{\pm}$. In this case, the Lagrangian (6) reduces to the PCM Lagrangian (4), and the quantity \mathfrak{J}_{α} simply becomes j_{α} , which is indeed conserved in the PCM.

More generally, when the auxiliary field v_{α} has been eliminated using its equation of motion, the quantity \mathfrak{J}_{α} becomes the Noether current associated with leftmultiplication of the group-valued field g by an arbitrary element of G. In the theory without auxiliary fields, conservation of this Noether current is equivalent to the equation of motion for the model.

III. WEAK INTEGRABILITY

Ordinarily, we say that a model is weakly integrable if its equations of motion admit a zero-curvature representation, i.e. if they are equivalent to the flatness condition

$$d\mathfrak{L} = 0, \qquad (12)$$

for a Lax connection $\mathfrak{L}(z)$, at any value of a spectral parameter $z \in \mathbb{C}$. The curvature of the Lax connection, $d\mathfrak{L}$, can be expressed in light-cone coordinates as

$$d\mathfrak{L} = \partial_+ \mathfrak{L}_- - \partial_- \mathfrak{L}_+ + [\mathfrak{L}_+, \mathfrak{L}_-].$$
(13)

Let us first define what we mean by integrability of an auxiliary field model, for which we have both the v_{α} -field equation of motion (7) and the *g*-field equation of motion (8). Given a Lagrangian $\mathcal{L}(j_{\alpha}, v_{\alpha})$, one could eliminate the auxiliary field v_{α} using its equation of motion to write

$$\mathcal{L}(j_{\alpha}, v_{\alpha}) \stackrel{\cdot}{=} \mathcal{L}(j_{\alpha}) \,. \tag{14}$$

Likewise, one could eliminate the auxiliary fields in an expression for a Lax connection $\mathfrak{L}(j_{\alpha}, v_{\alpha})$ to obtain

$$\mathfrak{L}(j_{\alpha}, v_{\alpha}) \doteq \mathfrak{L}(j_{\alpha}) \,. \tag{15}$$

If the flatness of the resulting Lax connection $\mathfrak{L}(j_{\alpha})$ for any $z \in \mathbb{C}$ is equivalent to the *g*-field equation of motion arising from $\mathcal{L}(j_{\alpha})$, we say that the original auxiliary field model $\mathcal{L}(j_{\alpha}, v_{\alpha})$ is weakly integrable with Lax connection $\mathfrak{L}(j_{\alpha}, v_{\alpha})$. An equivalent, but more succinct, version of this definition is as follows. If we define

$$\mathcal{E}_g = \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial (\partial_\alpha g)} \right) - \frac{\partial \mathcal{L}}{\partial g} \,, \tag{16}$$

so that $\mathcal{E}_g = 0$ is the g-field equation of motion, then weak integrability of an auxiliary field model is the statement

$$\left(d\mathfrak{L} \doteq 0, \forall z \in \mathbb{C}\right) \iff \left(\mathcal{E}_g \doteq 0\right).$$
 (17)

We will now demonstrate that every Lagrangian of the form (6) satisfies this weak integrability condition (17), where the Lax connection is

$$\mathfrak{L}_{\pm} = \frac{j_{\pm} \pm z \mathfrak{J}_{\pm}}{1 - z^2} \,. \tag{18}$$

First note that the auxiliary field equation (7) implies

$$[\mathfrak{J}_+,\mathfrak{J}_-] \doteq [j_+,j_-], \qquad [\mathfrak{J}_+,j_-] \doteq [j_+,\mathfrak{J}_-], \qquad (19)$$

which agree with relations that were found in the study of the root- $T\overline{T}$ -deformed PCM without auxiliary fields [36]. These formulas allow us to simplify the commutator

$$[\mathfrak{L}_{+},\mathfrak{L}_{-}] = \frac{[j_{+},j_{-}] - z\left([j_{+},\mathfrak{J}_{-}] - [\mathfrak{J}_{+},j_{-}]\right) - z^{2}[\mathfrak{J}_{+},\mathfrak{J}_{-}]}{(1-z^{2})^{2}}$$
$$\stackrel{:}{=} \frac{[j_{+},j_{-}]}{1-z^{2}}.$$
(20)

The curvature (13) of the Lax connection (18) is then

$$d\mathfrak{L} \doteq \frac{\partial_{+}j_{-} - \partial_{-}j_{+} + [j_{+}, j_{-}] - z \left(\partial_{+}\mathfrak{J}_{-} + \partial_{-}\mathfrak{J}_{+}\right)}{1 - z^{2}}.$$
(21)

The first three terms in the numerator of (21) vanish due to the Maurer-Cartan identity (3), and the final two terms vanish if and only if $\partial_{\alpha} \mathfrak{J}^{\alpha} \doteq 0$, which is the equation of motion for g. We conclude that every model in this family obeys the weak integrability condition (17).

IV. STRONG INTEGRABILITY

We say that a field theory is strongly integrable if the theory possesses an infinite set of conserved charges which are in involution, or mutually Poisson-commuting. Demonstrating weak integrability, i.e. that the equations of motion for a model admit a zero-curvature representation, is a useful step in this direction, since one can then construct an infinite set of conserved charges using the monodromy matrix. However, the Poissoncommutativity of these charges is not guaranteed unless the Poisson bracket of the Lax connection,

$$\{\mathfrak{L}_{\sigma,1}(\sigma,z),\mathfrak{L}_{\sigma,2}(\sigma',z')\},\qquad(22)$$

takes a special form. Here we have introduced, for any Lie algebra valued quantity X, the subscript notation

$$X_1 = X \otimes 1, \qquad X_2 = 1 \otimes X, \tag{23}$$

which tensors X with the identity on either side. We do not carefully distinguish between \mathfrak{g} and $U(\mathfrak{g})$, the universal enveloping algebra of \mathfrak{g} , but strictly speaking $1 \in U(\mathfrak{g})$ and $X_1, X_2 \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. If (22) takes a Sklyanin [37] or Maillet [30, 31] form, then one can appeal to general theorems to conclude that the theory possesses an infinite set of conserved charges in involution.

We will now prove that, for every model in the family (6), the Poisson bracket (22) takes the non-ultralocal Maillet form, which establishes strong integrability. Remarkably, this argument proceeds almost exactly as in the analogous proof for the PCM [31], except replacing every instance of j_{τ} with \mathfrak{J}_{τ} ; see [38] for a review.

To study the canonical structure of models with groupvalued fields [39], it is convenient to introduce local coordinates ϕ^{μ} on the Lie group G, not to be confused with $\varphi(z)$, so that $g = g(\phi^{\mu}(\sigma^{\alpha}))$. We use early Greek letters like α , β for indices on Σ and middle Greek letters like μ , ν for coordinates on G. We also introduce capital early Latin letters (e.g. A, B) which label the generators T_A of the Lie algebra \mathfrak{g} . For any Lie algebra valued quantity X, one can expand in generators as $X = X^A T_A$. Furthermore, we may use the pull-back map $\partial_{\alpha} \phi^{\mu}$ to convert between indices on Σ and indices on G. For instance,

$$j_{\alpha} = j_{\mu}^{A} \frac{\partial \phi^{\mu}}{\partial \sigma^{\alpha}} T_{A} \,. \tag{24}$$

We write γ_{AB} for the Killing form on \mathfrak{g} , which we assume to be non-degenerate with inverse γ^{AB} , and we denote the structure constants by $f_{AB}{}^C$. In our conventions, these two objects are defined by the relations

$$\gamma_{AB} = \operatorname{Tr}\left[T_A T_B\right], \qquad \left[T_A, T_B\right] = f_{AB}{}^C T_C. \qquad (25)$$

In terms of these quantities, the canonical momentum π_{μ} which is conjugate to the coordinates ϕ^{μ} on G is

$$\pi_{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\tau} \phi^{\mu})} = j^{A}_{\mu} \left(j^{B}_{\tau} + 2v^{B}_{\tau} \right) \gamma_{AB} \,. \tag{26}$$

This means that the quantity $\mathfrak{J}_{\tau}^{A} = -(j_{\tau}^{B}+2v_{\tau}^{A})$ is simply related to the canonical momentum as

$$\mathfrak{J}_{\tau}^{A} = -\gamma^{AB} \pi_{\mu} j_{B}^{\mu} \,, \qquad (27)$$

where we have defined the inverse field j_B^{μ} which satisfies $j_{\mu}^{A} j_B^{\mu} = \delta^A_{\ B}$. The fundamental Poisson brackets are then

$$\{\pi_{\mu}(\sigma), \phi^{\nu}(\sigma')\} = \delta_{\mu}{}^{\nu}\delta(\sigma - \sigma'), \qquad (28)$$

which hold at equal times τ . Because no time derivatives of the auxiliary field v_{α} appear in the Lagrangian, the momentum \mathfrak{p}^{α} which is conjugate to v_{α} vanishes:

$$\mathfrak{p}^{\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_{\tau} v_{\alpha})} = 0.$$
 (29)

In the canonical formulation, our model (6) is therefore a constrained Hamiltonian system, for which (29) is a primary constraint. This structure is reminiscent of the Hamiltonian formulation of Maxwell theory, where the temporal gauge field A^0 also has vanishing conjugate momentum. A systematic treatment of the Hamiltonian structure of our models would proceed using the method of Dirac [40], which implements the primary constraint in the Hamiltonian via a Lagrange multiplier, and then imposes that the primary constraint is preserved under time evolution. However, we will not undertake a detailed discussion of this procedure here, since it is not necessary for our present purposes. Because we are only interested in computing Poisson brackets involving j_{α} and \mathfrak{J}_{α} , where the latter is related to the canonical momentum by (27), it suffices to compute brackets using the fundamental relations (28) and ignore the fact that \mathfrak{J}_{α} is itself composed from a constrained auxiliary field. In fact, the auxiliary field equation of motion will play no role whatsoever in the following calculations.

Using the fundamental brackets (28), one can show

$$\{\mathfrak{J}_{\tau}^{A}(\sigma), \mathfrak{J}_{\sigma}^{B}(\sigma')\} = f^{AB}{}_{C} \mathfrak{J}_{\tau}^{C} \delta(\sigma - \sigma'), \{\mathfrak{J}_{\tau}^{A}(\sigma), j_{\sigma}^{B}(\sigma')\} = f^{AB}{}_{C} j_{\sigma}^{C}(\sigma) \delta(\sigma - \sigma') - \gamma^{AB} \delta'(\sigma - \sigma'), \{j_{\sigma}^{A}(\sigma), j_{\sigma}^{B}(\sigma')\} = 0,$$

$$\{30\}$$

where $\delta'(\sigma - \sigma') = \partial^{(\sigma)}\delta(\sigma - \sigma')$. To obtain these results, we have used the Maurer-Cartan identity (3) in the form

$$\partial_{\mu}j_{\nu}^{A} - \partial_{\nu}j_{\mu}^{A} + j_{\mu}^{B}j_{\nu}^{C}f_{BC}^{A} = 0, \qquad (31)$$

along with δ -function identities such as

$$f(y)\partial^{(x)}\delta(x-y) = f(x)\partial^{(x)}\delta(x-y) + f'(x)\delta(x-y).$$
(32)

One can then contract each of the relations (30) with the tensor product $T_A \otimes T_B$ to find

$$\{\mathfrak{J}_{\tau,1}(\sigma),\mathfrak{J}_{\tau,2}(\sigma')\} = [\mathfrak{J}_{\tau,2}, C_{12}]\delta(\sigma - \sigma'), \{\mathfrak{J}_{\tau,1}(\sigma), j_{\sigma,2}(\sigma')\} = [j_{\sigma,2}, C_{12}]\delta(\sigma - \sigma') - C_{12}\delta'(\sigma - \sigma'), \{j_{\sigma,1}(\sigma), j_{\sigma,2}(\sigma')\} = 0,$$
(33)

where we have defined the Casimir $C_{12} = \gamma^{AB} T_A \otimes T_B$. From (33) one can show that

$$\{\mathfrak{L}_{\sigma,1}(\sigma,z),\mathfrak{L}_{\sigma,2}(\sigma',z')\} = \frac{1}{(1-z^2)(1-z'^2)} \Big([zz'\mathfrak{J}_{\tau,2} + (z+z')j_{\sigma,2},C_{12}] \,\delta(\sigma-\sigma') - (z+z')C_{12}\delta'(\sigma-\sigma') \Big) \\ = [r_{12}(z,z'),\mathfrak{L}_{\sigma,1}(\sigma,z)] \,\delta(\sigma-\sigma') - [r_{21}(z',z),\mathfrak{L}_{\sigma,2}(\sigma,z')] \,\delta(\sigma-\sigma') - s_{12}(z,z')\delta'(\sigma-\sigma') \,, \tag{34}$$

where $s_{12}(z, z') = r_{12}(z, z') + r_{21}(z', z)$, the *r*-matrix is

$$r_{12}(z,z') = \frac{C_{12}}{z-z'}\varphi^{-1}(z'), \qquad (35)$$

which solves the classical Yang-Baxter equation, and

$$\varphi(z) = \frac{z^2 - 1}{z^2} \tag{36}$$

is the same twist function as for the undeformed PCM. The structure in the second line of (34) is precisely the desired non-ultralocal Maillet form of the Poisson bracket.

Perhaps surprisingly, the result (34) takes exactly the same form as in the PCM. This calculation has been completely blind to the form of the interaction function $E(\nu)$, and the auxiliary field has entered only through the combination \mathfrak{J}_{τ} of equation (27). We stress that, although

the auxiliary field equation of motion was needed in the proof of weak integrability, this equation is not used anywhere in the computation of the bracket (34). We therefore conclude that every theory within the class (6) is strongly integrable, in the sense that it possesses an infinite collection of Poisson-commuting conserved charges.

V. STRESS TENSOR FLOWS

The family of sigma models (6) is classically equivalent to the set of all deformations of the PCM by functions of the stress tensor, which we define by

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} \,. \tag{37}$$

In components, $T^{\alpha}_{\ \beta}$ is a 2 × 2 matrix, which means that there are two functionally independent Lorentz scalars that can be constructed from it. One is the trace,

$$T^{\alpha}_{\ \alpha} \doteq 2\left(E + 2E'\nu\right), \qquad (38)$$

where we have simplified using the auxiliary field equation of motion (7). The other is $T^{\alpha}_{\ \beta}T^{\beta}_{\ \alpha}$, which satisfies

$$T^{\alpha}_{\ \beta}T^{\beta}_{\ \alpha} - \frac{1}{2}\left(T^{\alpha}_{\ \alpha}\right)^{2} \doteq \frac{1}{2}\nu\left(1 - 4(E')^{2}\nu\right)^{2}.$$
 (39)

Importantly, both of these contractions can be expressed entirely in terms of ν , $E(\nu)$, and $E'(\nu)$, when the auxiliary field equation of motion is satisfied. This means that any classical flow equation of the form

$$\frac{\partial \mathcal{L}}{\partial \lambda} \doteq f\left(T^{\alpha}_{\ \alpha}, T^{\alpha}_{\ \beta}T^{\beta}_{\ \alpha}, \lambda\right) , \qquad (40)$$

gives rise to an ordinary differential equation for the interaction function $E(\nu)$, whose solution is another model within the same class of theories. This structure is similar to that of classical stress tensor deformations of theories of duality-invariant electrodynamics in 4d [41], chiral tensor theories in 6d [42], or chiral bosons in 2d [43].

Examples of deformations (40) include the $T\overline{T}$ [44, 45] and root- $T\overline{T}$ [46–48] flows, driven by the functions

$$\mathcal{O}_{T\overline{T}} = T^{\alpha}_{\ \beta} T^{\beta}_{\ \alpha} - (T^{\alpha}_{\ \alpha})^2 , \qquad (41)$$

$$\mathcal{R} = \frac{1}{\sqrt{2}} \sqrt{T^{\alpha}_{\ \beta} T^{\beta}_{\ \alpha} - \frac{1}{2} \left(T^{\alpha}_{\ \alpha}\right)^2}. \tag{42}$$

The solution for the interaction function $E(\nu)$ deformed by the root- $T\overline{T}$ flow driven by \mathcal{R} , subject to the initial condition $E(\lambda = 0, \nu) = 0$, is

$$E(\lambda,\nu) = \tanh\left(\frac{\lambda}{2}\right)\sqrt{\nu},$$
 (43)

which is the same as the interaction function for the Modified Maxwell theory [49] in the 4d Ivanov-Zupnik formulation. The solution to the $T\overline{T}$ flow equation driven by (41) with the same initial condition can be written in terms of a hypergeometric function [42].

It is well-known that applying the $T\overline{T}$ deformation to a classically integrable theory produces another classically integrable theory [50], and the Lax connection for the $T\overline{T}$ -deformed PCM was obtained in [51]. Likewise, the Lax connection for the PCM deformed by both $T\overline{T}$ and root- $T\overline{T}$ appeared in [36]. Both of these results can be viewed as special cases of the general Lax connection (18), which is appropriate for a deformation of the PCM by an arbitrary function of the energy-momentum tensor.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have introduced a new auxiliary field technique for proving integrability of 2d sigma models. We used this machinery to establish both weak and strong integrability for an infinite class of interacting PCM-like theories, which includes all deformations of the PCM by functions of the energy-momentum tensor.

We emphasize that, by virtue of the auxiliary fields in our model, the proofs of weak and strong integrability for this entire infinite class are hardly more difficult than those for the PCM. It would be very interesting to see whether similar techniques could be used to streamline proofs of integrability for other theories, or to construct larger families of integrable models. For instance, it seems likely that stress tensor deformations of symmetric and semi-symmetric space sigma models, with or without Wess-Zumino term, may also be addressed within our framework; weak integrability of $T\overline{T}$ and root- $T\overline{T}$ deformations of these models was established in [36].

Our discussion has been entirely classical. It would be interesting to see whether our auxiliary field framework aids in the study of integrability for models at the quantum level. At least in the case of the $T\overline{T}$ -deformed PCM, it is believed that the theory is quantum-mechanically well-defined due to the point-splitting definition of the $T\overline{T}$ operator [44]. Perhaps other models within this class also admit quantum definitions, and auxiliary fields may be useful for studying quantum integrability.

A third avenue for future research is to investigate whether there is a connection between these deformed integrable sigma models and 4d Chern-Simons theory [52– 54]; see [55] for a recent review. The conventional TT deformation admits an interpretation via 4d Chern-Simons, at least to leading order in the deformation parameter [56], and it might be that a similar narrative applies to the general family of models constructed here.

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