# Generalized van Trees inequality: Local minimax bounds for non-smooth functionals and irregular statistical models 

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#### Abstract

In a decision-theoretic framework, minimax lower bound provides the worst-case performance of estimators relative to a given class of statistical models. For parametric and semiparametric models, the Hájek-Le Cam local asymptotic minimax (LAM) theorem provides the optimal and sharp asymptotic lower bound. Despite its relative generality, this result comes with limitations as it only applies to the estimation of differentiable functionals under regular statistical models. On the other hand, nonasymptotic minimax lower bounds, such as those based on the reduction to hypothesis testing, do not often yield sharp asymptotic constants. Inspired by the recent improvement of the van Trees inequality and related methods in the literature, we provide new non-asymptotic minimax lower bounds under minimal regularity assumptions, which imply sharp asymptotic constants. The proposed lower bounds do not require the differentiability of functionals or regularity of statistical models, extending the efficiency theory to broader situations where standard approaches fail. Additionally, new lower bounds provide non-asymptotic constants, which can shed light on more refined fundamental limits of estimation in finite samples. We demonstrate that new lower bounds recover many classical results, including the LAM theorem and semiparametric efficiency bounds. We also illustrate the use of the new lower bound by deriving the local minimax lower bound for estimating the density at a point and directionally differentiable parameters.


Keywords - van Trees inequality, Hammersley-Chapman-Robbins bound, Local asymptotic minimax theorem, Local minimaxity, Efficiency theory, Nonsmooth functional estimation, Irregular statistical models

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## 1 Introduction

### 1.1 Motivation

In a decision-theoretic framework, the optimality of an estimation procedure is often motivated by the attainability to minimax lower bound. Specifically, a proposed estimator is considered rateoptimal when its convergence rate matches the rate of the corresponding minimax lower bound. A more refined assessment is to examine whether an estimator is asymptotically efficient, meaning the risk of the estimator coincides with the minimax lower bound including the constant. If this is the case, the estimation procedure becomes no longer improvable at least asymptotically. Hence, the construction of precise minimax lower bounds has been a crucial step for evaluating any statistical procedure.

The optimality of functional estimation has been extensively studied in both parametric and nonparametric models (Ibragimov and Has'minskii, 1981; Bickel et al., 1993; Lehmann and Casella, 2006). Semiparametric efficiency theory, in particular, is considered a cornerstone for understanding the asymptotic optimality of functional estimation. The convolution theorem and the local asymptotic minimax theorem are two fundamental results in the development of semiparametric efficiency (Hájek, 1970, 1972; Le Cam, 1972); however, these theorems only apply to smooth functionals and require certain regularity conditions on the underlying statistical models. As optimal estimation of non-smooth functionals has gained growing interest, extending the classical efficiency theory to broader settings is an imminent task.

This manuscript aims to develop general minimax lower bounds under minimal regularity assumptions imposed on functionals, estimators, and underlying statistical models. We consider a (nonparametric) functional $\psi: \mathcal{P} \mapsto \mathbb{R}^{k}$, where $\mathcal{P}$ is a collection of probability measures containing the distribution from which the observation is drawn. The unknown data-generating distribution is denoted by $P_{\theta_{0}}$ and a local statistical model containing $P_{\theta_{0}}$ is given by $\left\{P_{\theta}: \theta \in \Theta_{0} \subseteq \Theta\right\}$, satisfying $\theta_{0} \in \Theta_{0}$. Here, the indexing set $\Theta_{0}$ is a subset of a metric space equipped with an appropriate metric. Our primary focus is on the following general minimax risk for an arbitrary parameter set $\Theta_{0} \subseteq \Theta:$

$$
\begin{equation*}
\inf _{T} \sup _{\theta \in \Theta_{0}} \mathbb{E}_{P_{\theta}}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} \tag{1}
\end{equation*}
$$

where the infimum is taken over mesurable functions and $\|\cdot\|: \mathbb{R}^{k} \mapsto \mathbb{R}_{+}$for $\mathbb{R}_{+}=[0, \infty)$ is any vector norm. We also denote by $\mathbb{E}_{P_{\theta}}$ the expectation under $P_{\theta}$. It should be noted that this framework is not confined to a parametric model, and it can be extended to a nonparametric model using the standard machinery from semiparametric statistics (Bickel et al., 1993; Van der Vaart, 2002).

Classical local asymptotic results investigate when $\Theta_{0}$ forms a shrinking ball around $\theta_{0}$ whose radius converges to zero at a suitable rate, depending on the sample size $n$. For example, the risk displayed by (1) becomes equivalent to the local asymptotic minimax risk under the root- $n$ rate,
which is given by

$$
\begin{equation*}
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left\|\theta-\theta_{0}\right\|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} . \tag{2}
\end{equation*}
$$

Particularly, the local asymptotic minimax (LAM) theorem (Hájek, 1970, 1972; Le Cam, 1972) states that the lower bound to the above display is given by the variance of the efficient influence function under fairly demanding regularity conditions. On the other hand, little is known about the lower bound when these regularity conditions fail to hold.

One immediate limitation of the LAM theorem is its asymptotic nature. To address this limitation, numerous non-asymptotic minimax lower bounds have been proposed in the literature. Many share the following principle: maximizing the separation of functionals evaluated at two, or more, "similar" distributions. The similarity of distributions is quantified by different metrics and divergences such as the Hellinger distance, total variation, or the $f$-divergences, which includes the KL and the chi-squared divergences. While these non-asymptotic lower bounds may provide correct rates, they often fall short of recovering the correct constant implied by the LAM theorem. To the best of our knowledge, existing non-asymptotic lower bounds based on the Hellinger distance or total variation (LeCam, 1973; Ibragimov and Has'minskii, 1981; Assouad, 1983; Donoho and Liu, 1987; Birgé, 1987) cannot imply the LAM theorem even for regular parametric models. Some chi-squared-based lower bounds (Gill and Levit, 1995; Gassiat and Stoltz, 2024), on the other hand, can recover the LAM theorem. Cai and Low (2011) proposed a chi-squared-based lower bound for non-smooth functionals, resulting in a sharp constant for the problem they studied.

The second limitation of the LAM theorem is that it requires the existence of an efficient influence function. This excludes two classes of estimation problems: irregular distributions and nonsmooth functionals. Under irregular statistical models (e.g., Uniform $[0, \theta], \theta \geq 0$ ), the chi-squared divergences are often infinite, implying trivial results from the aforementioned minimax lower bounds using chi-squared divergences (Gill and Levit, 1995; Cai and Low, 2011; Gassiat and Stoltz, 2024). On the other hand, other metrics such as the Hellinger or total variation distances, do not suffer from this problem under irregularity. Thus, one of our main motivations is to understand whether we can obtain lower bounds based on the Hellinger or total variation distances that yield the LAM theorem for regular parametric models. Regarding non-smooth functionals, there is barely any analogous statement to the LAM theorem. A few works have extended local asymptotic minimaxity in the context of plug-in estimators for directionally differentiable functionals (Fang, 2014) or specific non-smooth functionals that are non-differentiable transforms of a regular parameter (Song, 2014a,b). This manuscript also aims to develop new local asymptotic results that apply to any, possibly non-smooth, functionals or estimators.

Before proceeding, we would like to acknowledge that general-purpose minimax lower bounds can only go so far in the sense that their application to specific instances requires additional work. We present some examples to show how our results can be applied.

Summary of main results. Our main results are summarized as follows. On a conceptual level, we demonstrate that the general minimax risk for estimating non-smooth functionals is characterized by the interplay between two non-negative terms in the proceeding display:

$$
\inf _{T} \sup _{\theta \in \Theta_{0}} \mathbb{E}_{P_{\theta}}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} \geq \sup _{\phi}\left[\left\{\operatorname{Term~I}\left(\phi ; \Theta_{0}\right)\right\}^{1 / 2}-\left\{\operatorname{Term~II}\left(\psi, \phi ; \Theta_{0}\right)\right\}^{1 / 2}\right]_{+}^{2}
$$

where $[t]_{+}:=\max (t, 0)$ and $\phi$ is any functional that approximates the original functional $\psi$. The first term, we call it surrogate efficiency, is an efficiency bound for estimating the functional $\phi$.

When $\phi$ provides sufficient structures such as smoothness, this term becomes well-studied in the literature for both parametric and nonparametric models. The second term, we call it approximation bias, quantifies the deviation of the introduced functional $\phi$ from the original functional $\psi$. The overall minimax risk is then obtained by optimizing the choice of $\phi$. Such an interplay between surrogate efficiency and approximation bias resembles the classical bias-variance decomposition of an estimation error. Hence, our result can be seen as the minimax lower bound analog to the bias-variance trade-off.

We provide two concrete instances of the above idea in our main theorems. First, Theorem 8 shows that the minimax risk admits the following lower bound based on the projection onto the set $\Phi_{\mathrm{ac}}$, all absolutely continuous functions of $\theta \in \Theta_{0}$. Let $\psi$ be a real-valued functional of interest and $\phi \in \Phi_{\text {ac }}$ with almost-everywhere derivative $\nabla \phi$. Our formal result in Section 3.3 holds for any vector-valued functional $\psi$. Assume that the statistical model $\left\{P_{\theta}: \theta \in \Theta_{0}\right\}$ is regular such that the Fisher information $\mathcal{I}(\theta)$ at $P_{\theta}$ is well-defined (see our definition in (3)). Then, for all "nice" priors $Q$ on $\Theta_{0}$ (see Definition 1) with Fisher information $\mathcal{I}(Q)$, defined in (5), it follows that:

$$
\sup _{\theta \in \Theta_{0}} \mathbb{E}_{P_{\theta}}|T(X)-\psi(\theta)|^{2} \geq\left[\Gamma_{Q, \phi}^{1 / 2}-\left(\int_{\Theta_{0}}|\psi(\theta)-\phi(\theta)|^{2} d Q(\theta)\right)^{1 / 2}\right]_{+}^{2}
$$

where

$$
\Gamma_{Q, \phi}:=\left(\int_{\Theta_{0}} \nabla \phi(\theta) d Q(\theta)\right)^{\top}\left(\mathcal{I}(Q)+\int_{\Theta_{0}} \mathcal{I}(\theta) d Q(\theta)\right)^{-1}\left(\int_{\Theta_{0}} \nabla \phi(\theta) d Q(\theta)\right) .
$$

The lower bound holds for any choice of $\phi \in \Phi_{\text {ac }}$ and any "nice" prior $Q$, allowing for further optimization. Here, the surrogate efficiency is obtained by the van Trees inequality (Gill and Levit, 1995; Gassiat and Stoltz, 2024).

Second, we consider approximating $\psi(t)$ with $\psi(t-h)$ for $h \in \Theta$. Such a local perturbation is well-defined, as long as $t, t-h \in \Theta_{0}$, even when the functional $\psi$ is non-smooth. Then for any probability measure $Q$ over $\mathbb{R}^{d}$, Theorem 10 states that

$$
\begin{aligned}
\sup _{\theta \in \mathbb{R}^{d}} & \mathbb{E}_{P_{\theta}}\|T(X)-\psi(\theta)\|^{2} \\
& \geq \sup _{h \in \mathbb{R}^{d}, Q}\left[\left(\frac{\left\|\int_{\mathbb{R}^{d}} \psi(t)-\psi(t-h) d Q\right\|^{2}}{4 H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)}\right)^{1 / 2}-\left(\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

where $H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)$ is the Hellinger distance between two mixture distributions $\mathbb{P}_{0}$ and $\mathbb{P}_{h}$ indexed by $h \in \mathbb{R}^{d}$ (see (7)). Informally, $\mathbb{P}_{h}$ is a probability measure "slightly perturbed" from $\mathbb{P}_{0}$ by $h$, which is a mixture between $P_{0}$ and $Q$. The sharpest minimax lower bound is obtained by maximizing the difference of functionals $\psi\left(P_{t}\right)-\psi\left(P_{t-h}\right)$ while minimizing the divergence between probability measures. This approach conceptually aligns with classical methods for deriving minimax lower bounds such as the Hellinger modulus of continuity (Donoho and Liu, 1987) or the general reduction to hypothesis testing. See, for instance, the list of approaches in Chapter 2 of Tsybakov (2008).

The lower bounds given by Theorems 8 and 10 are non-asymptotic and require no additional assumptions as long as all terms in the expression are well-defined. In particular, both lower bounds are obtained from the worse Bayes risk, optimized over the choice of a prior $Q$. This manuscript demonstrates that the introduction of the prior $Q$ enables us to recover not only the optimal rate but also the constant. Hence, these bounds possibly provide non-asymptotic efficiency lower
bounds, which generalize well-known asymptotic results such as the Cramér-Rao and semiparametric efficiency bounds. Theorem 10 in particular holds under the irregularity exhibited through the underlying statistical model, which heuristically results in the undefined Fisher information and the failure of the LAM theorem. We address this irregularity and derive the asymptotic minimax rate for irregular problems. Our conclusions are consistent with the literature on irregular estimation (Ibragimov and Has'minskii, 1981; Shemyakin, 2014; Lin et al., 2019) and the asymptotic theory developed by Hájek and Le Cam (Hájek, 1970, 1972; Le Cam, 1972; Pollard, 2023).

In summary, we focus on the construction of minimax lower bounds that satisfy the following criteria: (i) they yield the general minimax risk defined in equation (1) for an arbitrary $\Theta_{0}$, which can depend on the sample size or the data-generating distribution; (ii) they require minimal regularity conditions, allowing for the application to non-smooth functionals, irregular estimators and irregular statistical models; and (iii) they provide a non-asymptotic constant, which converges to the asymptotically optimal constant for regular problems. To the best of our knowledge, there is limited work in the minimax theory literature that considers all three requirements simultaneously.

### 1.2 Literature review

There has been a growing interest in investigating the minimax optimality of non-smooth functional estimation. These problems include the $L_{r}$-norm estimation in a Gaussian sequence model (Lepski et al., 1999; Cai and Low, 2011; Collier et al., 2020; Han et al., 2020), the property estimation of high-dimensional discrete distributions (Valiant and Valiant, 2011; Jiao et al., 2015; Wu and Yang, 2019), the hypothesis testing for shape constraints (Juditsky and Nemirovski, 2002), the estimation of conditional variance of a nonparametric regression function (Shen et al., 2020), many parameter estimation problems such as the maximum of average potential outcomes (Fang, 2014) and conditional average treatment effect (Kennedy et al., 2022), among others. For many such problems, minimax optimal rates are not well understood, let alone the optimal constants.

One popular method for deriving minimax lower bounds involves reducing the original estimation problem to a set of carefully designed testing problems. It has been observed that non-smooth functional estimation often requires the reduction to composite versus composite hypothesis testing, which is known as the fuzzy hypothesis approach (Ibragimov et al., 1987; Lepski et al., 1999; Nemirovski, 2000; Tsybakov, 2008). This reduction often leads to the study of the divergence metrics between two mixture distributions. In view of the Neyman-Pearson Lemma, a uniformly most powerful test is based on the likelihood ratio, which has a natural connection to the total variation distance between two hypotheses. Minimax lower bounds can then be obtained in terms of various metrics, including the Hellinger distance, the chi-squared divergence, the Kullback-Leibler divergence, and others, by invoking a series of inequalities between metrics such as Le Cam's and Pinsker's inequalities. See, for instance, Section 2.7.4 of Tsybakov (2008). However, it is important to note that in the process of applying different inequalities between metrics, the precision of the constant may be lost. As a result, the minimax optimality of non-smooth functional estimation is often considered in terms of rate, with no specific focus on the constant.

In a slightly different approach, Cai and Low (2011) extends the notion of fuzzy hypotheses to the constrained risk inequality (Brown and Low, 1996), deriving a precise constant for the $L_{1-}{ }^{-}$ norm estimation under a Gaussian sequence model. In this manuscript, we aim to explore similar extensions of classical two-point risk lower bounds to mixture distributions in order to obtain a sharp constant for the minimax lower bounds. Towards this goal, we refer to the well-established literature on parametric estimation, which has extensively investigated risk lower bounds with accurate non-asymptotic constants. Although these results are not typically presented as minimax risk, they are essential references regardless. Among the commonly used bounds are the

Cramér-Rao bound (Fisher, 1922; Radhakrishna Rao, 1945; Cramér, 1999), which was improved by Barankin (1949), the higher-order analogue given by Bhattacharyya bound (Bhattacharyya, 1946), the Hammersley-Chapman-Robbins bounds (Hammersley, 1950; Chapman and Robbins, 1951), and the Weiss-Weinstein bound (Weiss and Weinstein, 1985). Each of these bounds requires varying degrees of regularity conditions and is a variant of the others, as unified by Weinstein and Weiss (1988). They all asymptotically imply the Cramér-Rao bound under different levels of regularity conditions, and they offer sharper non-asymptotic constants for different settings. Notably, all of these bounds, including Cai and Low (2011), define the lower bounds in terms of the likelihood ratio between local models, or some analogous objects. This can be understood as a variant of the chi-squared divergence between local models, which requires strong regularity assumptions for its asymptotic behavior. This will be the focus of Section 2.1 of this manuscript.

The lower bound presented here is also closely related to the technique of obtaining a sharp minimax lower bound by exploring the least-favorable priors of the Bayes risk such as the van Trees inequality (Van Trees, 1968). Traditionally, the literature on the van Trees inequality (Van Trees, 1968; Gill and Levit, 1995; Jupp, 2010) requires strong regularity assumptions to guarantee the limiting behavior of the chi-squared divergence. Taking inspiration from the recent development by Gassiat and Stoltz (2024), we provide extensions of the van Trees inequality for non-smooth functionals based on the Hellinger distance. The asymptotic behavior of the Hellinger distance requires far weaker regularity conditions compared to the analogous result under the chi-squared divergence (Pollard, 2023). This property has gathered a growing interest in the use of the Hellinger distance as the desirable metric for studying minimax lower bounds (Ibragimov and Has'minskii, 1981; Donoho and Liu, 1987; Chen, 1997; Shemyakin, 2014; Lin et al., 2019; Pollard, 2023). Our lower bound is consistent with the message of these proponents.

Finally, we discuss connections between the presented lower bounds and the classical result in functional estimation, namely the modulus of continuity, which was investigated by Donoho and Liu (1987). As a consequence of the new minimax lower bounds for non-smooth functional estimation, we resolve one of the open problems since Donoho and Liu (1987) to characterize the nonasymptotic minimax lower bound that implies a precise limiting constant. For linear functional estimation, the modulus of continuity has been analyzed with an attempt to establish non-asymptotic efficiency theory (Mou et al., 2022). Our result can be seen as one of the first steps towards understanding non-asymptotics for non-smooth functional estimation. The modulus of continuity is also considered in the context of impossibility results, often known as ill-posedness in econometrics literature (Pötscher, 2002; Forchini and Hillier, 2005). The presented results generalize their asymptotic statements to the non-asymptotic context.

Organization. The remaining manuscript is organized as follows: Section 2 provides necessary backgrounds and defines several existing lower bounds that largely inspired this work. Section 3 presents new minimax lower bounds. Section 4 investigates asymptotic properties of the proposed lower bounds with particular attention to the preservation of sharp constants. Section 5 presents the application to several estimation problems in the presence of non-smoothness or irregularity and Section 6 provides a visual comparison of the new non-asymptotic lower bounds to upper bounds exhibited by different estimators. Finally, concluding remarks and a discussion of open problems are provided in Section 7 .

Notation. Throughout the manuscript, we adopt the following convention for notation. We denote by $\|\cdot\|$ a general vector norm. Given $x \in \mathbb{R}$, we write $[x]_{+}=\max (x, 0)$. For $x \in \mathbb{R}^{d}$, $\|x\|_{2}$ denotes the Euclidean norm in $\mathbb{R}^{d}$. For a univariate function $f$ from $S \subseteq \mathbb{R}$ to $\mathbb{R}$, the
supremum norm is defined as $\|f\|_{\infty}=\sup _{x \in S}|f(x)|$. The open $\mathbb{R}^{d}$-ball centered at $y_{0} \in \mathbb{R}^{d}$ with radius $\delta>0$ is denoted as $B\left(y_{0}, \delta\right):=\left\{y:\left\|y_{0}-y\right\|_{2}<\delta\right\}$. The unit sphere in $\mathbb{R}^{d}$ is denoted by $\mathbb{S}^{d-1}:=\left\{u \in \mathbb{R}^{d}:\|u\|_{2}=1\right\}$. We let $\mathbb{E}_{P}$ represent the expectation under $P$, whereas $\mathbb{E}_{\theta}$ denotes the expectation under $P_{\theta}$. Furthermore, $L_{2}(P)$ is the set of $P$-measurable functions that satisfy the condition

$$
L_{2}(P):=\left\{f: \mathcal{X} \mapsto \mathbb{R}^{k} \mid \int\|f(x)\|_{2}^{2} d P(x)<\infty\right\} .
$$

## 2 Setup and background

Throughout this section, we consider probability measures defined on a shared measurable space $(\mathcal{X}, \mathcal{A})$ with a $\sigma$-finite measure $\nu$. We assume that any probability measure $P$ we consider is absolutely continuous with respect to $\nu$ and has a well-defined density function, denoted by $d P \equiv$ $d P / d \nu$. We omit the specification of the base measure $\nu$ when it is clear from the context. Suppose we observe $X$ from an unknown distribution $P_{0}$, belonging to a possibly nonparametric model $\mathcal{P}$. The parametric submodel $\mathcal{P}_{\Theta}:=\left\{P_{\theta}: \theta \in \Theta\right\} \subset \mathcal{P}$ is defined as a set of probability measures indexed by a parameter space $\Theta \subseteq \mathbb{R}^{d}$, which contains the data-generating distribution $P_{0}$. Without loss of generality, we assume that the true parameter corresponds to $\theta=0$ in $\Theta$.

The local behavior of a path $t \mapsto P_{t} \in \mathcal{P}_{\Theta}$ as it approaches $P_{0}$ is often of interest. A local path is said to be Hellinger differentible at $t=0$ (Pollard, 2023) if there exists a measurable vector-valued function $\dot{\xi}_{0}: \mathcal{X} \mapsto \mathbb{R}^{d}$ that is square integrable with respect to the $\sigma$-finite measure $\nu$, and as $\|t\|_{2} \longrightarrow 0$,

$$
\int\left[d P_{t}^{1 / 2}-d P_{0}^{1 / 2}-t^{\top} \dot{\xi}_{0}\right]^{2} d \nu=o\left(\|t\|_{2}^{2}\right)
$$

The Fisher information matrix of $P_{t}$ at $t=0$ is defined as

$$
\begin{equation*}
\mathcal{I}(0):=4 \int \dot{\xi}_{0} \dot{\xi}_{0}^{\top} d \nu \tag{3}
\end{equation*}
$$

under the Hellinger differentiability. We also define a slightly stronger but more commonly used notion of differentiability for parametric paths, called quadratic mean differentiability (QMD). A local path is differentiable in quadratic mean, or QMD, at $t=0$ if there exists a $P_{0}$-square integrable vector-valued function $g_{0}: \mathcal{X} \mapsto \mathbb{R}^{d}$ that satisfies, as $\|t\|_{2} \longrightarrow 0$,

$$
\begin{equation*}
\int\left[d P_{t}^{1 / 2}-d P_{0}^{1 / 2}-\frac{1}{2} t^{\top} g_{0} d P_{0}^{1 / 2}\right]^{2} d \nu=o\left(\|t\|_{2}^{2}\right) . \tag{4}
\end{equation*}
$$

The function $g_{0}$ is commonly referred to as the score function of the path at $t=0$. When $P_{t}$ is QMD at $t=0$, it implies the Hellinger differentiability with $\dot{\xi}_{0}=\frac{1}{2} g_{0} d P_{0}^{1 / 2}$ (See Theorem 20 of Pollard (2023)) however the converse is not true. The corresponding Fisher information under the QMD assumption is given by

$$
\mathcal{I}(0):=\int g_{0} g_{0}^{\top} d P_{0} .
$$

As the Hellinger differentiability is a weaker condition, we only use the QMD assumption when necessary. Both definitions allow for the Fisher information to exist for distributions with pointwise non-differentiable densities, which is a crucial aspect of the asymptotic theory developed by Hàjek and Le Cam. For instance, consider the double-exponential density $t \mapsto \frac{1}{2} \exp (-|t-\theta|)$ with a
parameter $\theta \in \Theta$. This density function is not differentiable at $t=\theta$, yet its Fisher information matrix exists.

When $\Theta \subseteq \mathbb{R}^{d}$ with some base measure $\mu$, we define a location family induced by a probability measure $Q$ on $\Theta$ as $\{Q(\cdot-h): h \in \Theta\}$. If we assume that $Q$ has an absolutely continuous density function $q$ with an almost-everywhere derivative $\nabla q$, the corresponding location family is differentiable in quadratic mean with the Fisher information given by

$$
\begin{equation*}
\mathcal{I}(Q)=\int \frac{\nabla q(t) \nabla q(t)^{\top}}{q(t)} I(q(t)>0) d \mu . \tag{5}
\end{equation*}
$$

See Example 7.8 of Van der Vaart (2000). The Fisher information $\mathcal{I}(Q)$ does not depend on the location parameter $h$, making this notation a convenient reference to the Fisher information of the location families. Finally, we impose the following regularity conditions on $Q$, under which the priors are called "nice".

Definition 1 (Nice priors). Given an absolutely continuous mapping $\phi: \Theta_{0} \rightarrow \mathbb{R}^{k}$ and a vector norm $\|\cdot\|: \mathbb{R}^{k} \mapsto \mathbb{R}_{+}$, a probability measure $Q$ on $\Theta_{0}$ is "nice" if it satisfies the following conditions:
(1) It has a Lebesgue density $q$ and $q$ is an absolutely continuous function with a positive definite Fisher information $\mathcal{I}(Q)$,
(2) Both $\|\phi\|$ and $\|\nabla \phi\|$ is $Q$-integrable and $\int_{\Theta_{0}} \operatorname{tr}(\mathcal{I}(t)) d Q<\infty$, and
(3) It holds that $q(\theta) \rightarrow 0$ as $\theta$ approaches any boundary point of $\Theta_{0}$ with finite norm along some canonical direction.

We note that this property of priors is defined for each particular functional $\phi$.

### 2.1 The local behavior of divergence metrics

This manuscript primarily focuses on two divergence metrics, namely the chi-squared divergence and the Hellinger distance. Consider two probability measures, $P_{0}$ and $P_{1}$, defined on a common measurable space. Further, assume that there exist densities $d P_{0}=d P_{0} / d \nu$ and $d P_{1}=d P_{1} / d \nu$ with respect to a common $\sigma$-finite measure $\nu$. The chi-squared divergence is defined as

$$
\chi^{2}\left(P_{1} \| P_{0}\right):= \begin{cases}\int\left(d P_{1} / d P_{0}-1\right)^{2} d P_{0} & \text { if } P_{1} \text { is dominated by } P_{0} \\ \infty & \text { otherwise }\end{cases}
$$

and the Hellinger distance is defined as

$$
H^{2}\left(P_{0}, P_{1}\right):=\int\left(d P_{0}^{1 / 2}-d P_{1}^{1 / 2}\right)^{2}
$$

Below, we introduce conditions that imply certain limiting behaviors of these metrics over a parametric path as it passes through $P_{0}$. We first extend the standard notion of absolute continuity of a univariate function to a multivariate function as follows:

Definition 2 (Multivariate absolute continuity). A function $\omega: \mathbb{R}^{d} \mapsto \mathbb{R}$ is absolutely continuous over an open $\mathbb{R}^{d}$-ball $B(\theta, \delta)$ if for all $u \in \mathbb{S}^{d-1}$, the induced univariate function $t \mapsto \omega(\theta+t u)$ is absolutely continuous over $0<t<\delta$.

The concept of multivariate absolute continuity has been explored in various real analysis literature. For instance, Malỳ (1999) and Hencl (2004) extend the classical $\delta-\varepsilon$ definition of absolute continuity by considering the oscillation of the functions over $d$-dimensional balls. Additionally, Šremr (2010) presents a similar extension using $d$-dimensional hyper-cubes. We do not claim that Definition 2 is the most general definition of multivariate absolutely continuous functions for our purpose. Definition 2 is introduced solely for our complementary result (Theorem 9), and it is not a necessary condition for our main results (Theorem 7 or Theorem 10).

We then introduce the following regularity conditions:
(A1) There exists $\delta>0$ such that
(a) for $\nu$-almost everywhere, the mapping $t \mapsto d P_{t}$ is absolutely continuous over an open $\mathbb{R}^{d}$-ball $B([0], \delta)$ with the gradient $t \mapsto u^{\top} \dot{\rho}_{t, u}$ for each $u \in \mathbb{S}^{d-1}$,
(b) for all $u \in \mathbb{S}^{d-1}$, the gradient is continuous such that $\lim _{t \rightarrow 0} u^{\top} \dot{\rho}_{t, u}=u^{\top} \dot{\rho}_{0}$ for $\nu$-almost everywhere, and
(c) for all $0<t<\delta$ and $u \in \mathbb{S}^{d-1}, d P_{0}(x)=0$ implies $u^{\top} \dot{\rho}_{t, u}(x)=0$, and

$$
\int \sup _{0<t_{1}, t_{2}<\delta} \frac{\dot{\rho}_{t_{1}, u} \dot{\rho}_{t_{2}, u}^{\top}}{d P_{0}} d \nu<\infty
$$

With this regularity condition in place, we obtain the following local expansion of the chi-squared divergence:

Lemma 1 (Theorem 7.20 of Polyanskiy and Wu (2022)). Assuming that the local path $\left\{P_{t}: t \in \Theta\right\}$ satisfies (A1) and $t \mapsto \mathcal{I}(t)$ is continuous at $t=0$, then as $\|t\|_{2} \longrightarrow 0$,

$$
\chi^{2}\left(P_{t} \| P_{0}\right)=t^{\top} \mathcal{I}(0) t+o\left(\|t\|_{2}^{2}\right) .
$$

The regularity condition (A1) is stronger than the Hellinger differentiability. Gassiat and Stoltz (2024) shows that an analogous (but weaker) condition to (A1) implies the Hellinger differentiability. Furthermore, Example 7.1 of Polyanskiy and Wu (2022) provides a concrete example where (A1) fails while Hellinger differentiability still holds. The local expansion of the chi-squared divergence can hold under the following weaker conditions:
(A2) There exits $\delta>0$ such that
(a) for $\nu$-almost everywhere, the mapping $t \mapsto d P_{t}^{1 / 2}$ is absolutely continuous over an open $\mathbb{R}^{d}$-ball $B([0], \delta)$ with the gradient $t \mapsto u^{\top} \dot{\gamma}_{t, u}$ for each $u \in \mathbb{S}^{d-1}$,
(b) for all $u \in \mathbb{S}^{d-1}$, the gradient is continuous such that $\lim _{t \rightarrow 0} u^{\top} \dot{\gamma}_{t, u}=u^{\top} \dot{\gamma}_{0}$ for $\nu$-almost everywhere, and

$$
\int \sup _{0<t_{1}, t_{2}<\delta} \dot{\gamma}_{t_{1}, u} \dot{\gamma}_{t_{2}, u}^{\top} d \nu<\infty
$$

Roughly speaking, assumption (A1) pertains to the smoothness of density functions, whereas assumption (A2) pertains to a similar smoothness of the square roots of density functions. In fact, we can demonstrate that (A1) implies (A2) (the proof is provided in Section H of Supplementary Material). However, the converse is not true. For instance, location families with compact support fail to satisfy (A1)(c), while they may satisfy (A2) under certain conditions. See Example 7.1 of Polyanskiy and Wu (2022) for more details. Under (A2), the following result holds:

Lemma 2 (Theorem 7.21 of Polyanskiy and Wu (2022)). Assuming that the local path $\left\{P_{t}: t \in \Theta\right\}$ satisfies (A2) and $t \mapsto \mathcal{I}(t)$ is continuous at $t=0$, then as $\|t\|_{2} \longrightarrow 0$,

$$
\chi^{2}\left(P_{t} \| \lambda P_{0}+(1-\lambda) P_{t}\right)=(1-\lambda)^{2} t^{\top}\left(\mathcal{I}(0)-\frac{1-4 \lambda}{4 \lambda} \mathcal{I}^{\dagger}(0)\right) t+o\left(\|t\|_{2}^{2}\right) \quad \text { for all } \lambda \in(0,1)
$$

where $\mathcal{I}^{\dagger}(0):=4 \int \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top} I\left(d P_{0}=0\right) d \nu$ is known as the Fisher defect.
Remark 1 (On the Fisher defect $\mathcal{I}^{\dagger}$ ). If the parameter $t=0$ is an interior point of the parameter space $\Theta$, then $\dot{\gamma}_{0}=0$ for $\nu$-almost all $x$ in $\left\{x: d P_{0}(x)=0\right\}$. Consequently, the Fisher defect must be zero. However, even for a Hellinger differentiable statistical model, it is still possible for the Fisher defect to be non-zero on the boundary. An illustrative example is the Bernoulli distribution with parameter $p^{2}$ at $p=0$. Example 7.2 of Polyanskiy and $W u$ (2022) provides a formal derivation, and Example 18 of Pollard (2023) offers an additional example.

As shown by Lemma 23 in Supplementary Material, the regularity condition (A2) implies the Hellinger differentiability with $\dot{\gamma}_{0}=\dot{\xi}_{0}$ for $\nu$-almost everywhere and hence two definitions of the Fisher information coincide under (A2). In contrast, building directly from the Hellinger differentiability, we obtain the following result:

Lemma 3. Assuming that the local path $\left\{P_{t}: t \in \Theta\right\}$ is Hellinger differentiable and $t \mapsto \mathcal{I}(t)$ is continuous at $t=0$, then as $\|t\|_{2} \longrightarrow 0$,

$$
H^{2}\left(P_{t}, P_{0}\right)=\frac{1}{4} t^{\top} \mathcal{I}(0) t+o\left(\|t\|_{2}^{2}\right)
$$

Slightly generalized statements of Lemmas 1-3 are provided with proofs as Lemmas 19-21 in Supplementary Material. The main takeaway of this section is as follows: The local behavior of the Hellinger distance is easily understood under weaker conditions, while an analogous result under the chi-squared divergence requires more unpleasant regularity conditions. This is one of many reasons why the asymptotic theory according to Hàjek and Le Cam promotes the square roots of density functions as the primary object to investigate.

### 2.2 Minimax lower bounds via parametric paths

In this section, we introduce the local asymptotic minimax (LAM) theorem, a fundamental result in the efficiency theory. We then provide several non-asymptotic minimax lower bounds that asymptotically imply the best constant in view of the LAM theorem. For clarity and ease of illustration, we focus on a real-valued function $\psi: \Theta \mapsto \mathbb{R}$ where $\Theta \subseteq \mathbb{R}^{d}$ under parametric models. Although the result applies to nonparametric models, we defer a detailed discussion to Section 4.2. We assume that $X_{1}, \ldots, X_{n} \in \mathcal{X}^{n}$ are independent and identically distributed (IID) observations from $P_{\theta_{0}} \in\left\{P_{\theta}: \theta \in \Theta\right\}$ and $t \mapsto P_{t}$ is Hellinger differentiable at all $\theta \in \Theta$. The joint distribution of $n$ IID observations is denoted by $P_{\theta_{0}}^{n}$. We now state the LAM theorem.

Theorem 4 (Local asymptotic minimax theorem (Hájek, 1970, 1972; Le Cam, 1972)). Assuming that the mappings $t \mapsto \psi(t)$ and $t \mapsto \mathcal{I}(t)$ are continuously differentiable at $t=\theta_{0}$, then for any measurable function $T: \mathcal{X}^{n} \mapsto \mathbb{R}$,

$$
\begin{equation*}
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left\|\theta-\theta_{0}\right\|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \geq \nabla \psi\left(\theta_{0}\right)^{\top} \mathcal{I}\left(\theta_{0}\right)^{-1} \nabla \psi\left(\theta_{0}\right) . \tag{6}
\end{equation*}
$$

When the risk of an estimator matches the constant provided by the LAM theorem, the estimation becomes no longer improvable asymptotically. This, for instance, suggests the asymptotic efficiency of maximum likelihood estimators (MLEs) for this specific estimation task. This feature of the precise constant sets it apart from the minimax theory through the testing reduction, which typically claims the optimality in terms of the rates alone. It is essential to note that the first supremum over a neighborhood around $\theta_{0}$ is a crucial feature that cannot be removed. This supremum over a small neighborhood eliminates any estimator that performs exceptionally well on a Lebesgue measure-zero set, known as superefficient estimators. Without the supremum, the LAM theorem only applies to a family of regular estimators, which is restrictive in the context of non-smooth functional estimation (Hirano and Porter, 2012). See Section 2.3 of Fang (2014) for further discussion on the motivation behind the local asymptotic minimax risk.

One of our results is particularly inspired by the Hammersley-Chapman-Robbins (HCR) bound, initially introduced by Hammersley (1950) and Chapman and Robbins (1951). Although we present the results within the context of parametric estimation, it is possible to establish analogous results for nonparametric functionals using the argument to be discussed in Section 4.2. We now present the HCR bound:

Theorem 5 (HCR bound (Hammersley, 1950; Chapman and Robbins, 1951)). Let $\psi: \Theta \mapsto \mathbb{R}$ be a real-valued mapping for $\Theta \subseteq \mathbb{R}^{d}$.

$$
\inf _{T: \text { unbiased }} \sup _{\theta \in \Theta} \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq \sup _{\theta_{0}, \theta_{1} \in \Theta} \frac{\left|\psi\left(\theta_{1}\right)-\psi\left(\theta_{0}\right)\right|^{2}}{\chi^{2}\left(P_{\theta_{1}} \| P_{\theta_{0}}\right)} .
$$

The HCR bound is an elegant lower bound as it expresses the separation between functions and the divergence between two underlying probability measures in a concise manner. Assuming that $P_{\theta_{0}}$ is a fixed data generating distribution and taking the limit as $\theta_{1} \rightarrow \theta_{0}$, the HCR lower bound implies the Cramér-Rao bound under the regularity conditions required for Lemma 1. The regularity conditions can be weakened to the Hellinger differentiability by deriving an analogous lower bound in terms of the Hellinger distance (see Simons and Woodroofe (1983) and Exercise VI. 5 of Polyanskiy and $\mathrm{Wu}(2022)$ ). However, one major limitation of the HCR bound is that it only holds for unbiased estimators, which can be overly restrictive when analyzing minimax risks. In particular, if the function $\psi$ is non-differentiable, no sequences of unbiased estimators exist (Hirano and Porter, 2012), indicating that the HCR lower bound cannot be directly applied in our context.

The van Trees inequality differs from other approaches as it considers the minimax lower bound in terms of the worst Bayes risk. It achieves a sharp constant by seeking the supremum over all possible priors, known as the least-favorable prior. Let $\Theta \subseteq \mathbb{R}^{d}$ and $Q$ be a probability measure defined on $\Theta$ with a density function $d Q$. Further assuming that $Q$ is "nice", we state the following:
Theorem 6 (The van Trees inequality (Gassiat and Stoltz, 2024)). Suppose $\left\{P_{\theta}: \theta \in \Theta\right\}$ for $\Theta \subseteq \mathbb{R}^{d}$ is Hellinger differentiable for all $\Theta$ and the mapping $\psi$ is absolutely continuous on $\Theta$. Then for any measurable function $T$ and a "nice" prior distribution $Q$ (see Definition 1),

$$
\inf _{T} \sup _{\theta \in \Theta} \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq\left(\int_{\Theta} \nabla \psi(\theta) d Q(\theta)\right)^{\top}\left(\mathcal{I}(Q)+\int_{\Theta} \mathcal{I}(\theta) d Q(\theta)\right)^{-1}\left(\int_{\Theta} \nabla \psi(\theta) d Q(\theta)\right)
$$

where $\mathcal{I}(Q)$ is the Fisher information of the location family induced by the prior $Q$ defined in (5).
Theorem 6 is stated under the Hellinger differentiability, and this is due to the recent refinement by Gassiat and Stoltz (2024). The classical literature on the van Trees inequality (Van Trees, 1968;

Gill and Levit, 1995; Jupp, 2010) requires stronger regularity conditions to assert the local behavior of the chi-squared divergence, as shown in Lemma 1. The van Trees inequality is particularly desirable since it does not confine the choice of estimators to unbiased estimators. Furthermore, it has been shown that by assuming $\psi$ is continuously differentiable at $\theta_{0}$ and selecting a prior distribution $Q$ that concentrates at the true parameter $\theta_{0}$ with the rate $n^{-1 / 2}$, the exact constant for the LAM theorem can be obtained (Gassiat and Stoltz, 2024). However, the van Trees inequality gives a trivial lower bound for an irregular statistical model where the Fisher information is undefined. Our main results, presented below, aim to remove these limitations of the existing results.

## 3 General minimax lower bounds under weaker regularities

This section provides the main results of this manuscript. Section 3.1 provides a general minimax lower bound via smooth approximation. Sections 3.2 and 3.3 provide two extensions of the van Trees inequality that do not require the differentiability of functionals. We first present our results for parametric models with $\Theta=\mathbb{R}^{d}$ for ease of exposition, with an extension to more general sets $\Theta \subseteq \mathbb{R}^{d}$ discussed in Section 3.4. These results hold for nonparametric functionals and are achieved through the standard least-favorable parametric paths argument from semiparametric statistics, which is postponed to Section 4.2. Proofs of all theorems in this section are provided in Supplementary Material.

### 3.1 Lower bound based on the approximating functionals

When the target functional $\psi$ does not possess certain properties, such as smoothness, it is often useful to approximate $\psi$ with an alternative functional $\phi$ at the expense of the bias introduced by approximation. This is a common approach for non-smooth functional estimation where the target functional is first smoothed by, for instance, convolution. We formalize this idea in our first main result.

Suppose we jointly observe a random sequence of observations $X$ from an unknown distribution $P_{\theta}$, which belongs to a statistical model $\mathcal{P}:=\left\{P_{t}: t \in \Theta\right\}$ defined on a measurable space $(\mathcal{X}, \mathcal{A})$ with each possessing a density with respect to $\sigma$-finite measure $\nu$. Let $\psi: \mathcal{P} \mapsto \mathbb{R}^{k}$ denote a vectorvalued functional where the estimand of interest is the evaluation of the functional at the population parameter $\psi\left(P_{\theta_{0}}\right)$. We state the following as a consequence of the reverse triangle inequality:

Lemma 7. Given a measure space $(\Theta, \mathcal{T}, \mu)$, let $\Theta_{0} \subseteq \Theta$ be any subset of $\Theta$ and let $\mathcal{Q} \equiv \mathcal{Q}\left(\Theta_{0}\right)$ be a collection of probability measures on $\Theta_{0}$ equipped with a density function with respect to the base measure $\mu$, denoted as $d Q:=d Q / d \mu$. Let $\Phi$ be any collection of functionals $\Phi:=\left\{\phi: \mathcal{P} \mapsto \mathbb{R}^{k}\right\}$. For any measurable function $T: \mathcal{X} \mapsto \mathbb{R}^{k}$ and vector norm $\|\cdot\|: \mathbb{R}^{k} \mapsto \mathbb{R}_{+}$, it holds that

$$
\begin{aligned}
\sup _{\theta \in \Theta_{0}} & \mathbb{E}_{\theta}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} \\
& \geq \sup _{\phi \in \Phi, Q \in \mathcal{Q}}\left[\left(\int \mathbb{E}_{\theta}\left\|T(X)-\phi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2}-\left(\int\|\psi(\theta)-\phi(\theta)\|^{2} d Q(\theta)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

This result states that the minimax lower bound for any functional estimation can be expressed as the trade-off between the Bayes risk of estimating $\phi$ and the approximation error of $\psi$ by $\phi$. For
a real-valued functional, the approximation error is expressed as $L_{2}(Q)$-norm, hence the functional $\phi$ needs not to approximate $\psi$ uniformly.

We present the application of this result by considering the projection of $\psi$ onto the space of absolutely continuous functions. The Bayes risk for absolutely continuous functionals can be characterized by the van Trees inequality. We define a collection of probability measures $\mathcal{Q}^{\dagger}$ supported on $\Theta_{0}$, satisfying Definition 1. Note that $\mathcal{Q}^{\dagger}$ depends on $\phi$. We then state the following result:

Theorem 8. Suppose $\left\{P_{\theta}: \theta \in \Theta_{0}\right\}$ for any open subset $\Theta_{0} \subseteq \mathbb{R}^{d}$ is Hellinger differentiable for all $\Theta_{0}$ and $\phi: \mathbb{R}^{d} \mapsto \mathbb{R}^{k}$ is an absolutely continuous vector-valued functions on $\Theta_{0}$ with almosteverywhere derivative $\nabla \phi$. Then for any measurable function $T: \mathcal{X} \mapsto \mathbb{R}^{k}$, vector norm $\|\cdot\|: \mathbb{R}^{k} \mapsto$ $\mathbb{R}_{+}$, its dual $\|\cdot\|_{*}$, and $Q \in \mathcal{Q}^{\dagger}$,

$$
\inf _{T} \sup _{\theta \in \Theta_{0}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \geq\left[\sup _{\|u\|_{*} \leq 1}\left\|\Gamma_{Q, \phi}^{1 / 2} u\right\|-\left(\int\|\psi(\theta)-\phi(\theta)\|^{2} d Q(\theta)\right)^{1 / 2}\right]_{+}^{2}
$$

where

$$
\Gamma_{Q, \phi}:=\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right)^{\top}\left(\mathcal{I}(Q)+\int_{\Theta_{0}} \mathcal{I}(t) d Q\right)^{-1}\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right) .
$$

Theorem 8 is a straightforward application of the multivariate van Trees inequality (Theorem 12 of Gassiat and Stoltz (2024)) in conjunction with Lemma 7. Here, $\mathcal{I}(t)$ is the Fisher information associated with the data-generating distribution $P_{t}$. It is well-known that the Fisher information for an $n$-fold product measure is given by $n \mathcal{I}(t)$ where $\mathcal{I}(t)$ is the Fisher information associated with a single observation under $P_{t}$. Hence, under $n$ IID observations from $P_{t}$, the theorem above yields the identical lower bound but replaces $\Gamma_{Q, \phi}$ with

$$
\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right)^{\top}\left(\mathcal{I}(Q)+n \int_{\Theta_{0}} \mathcal{I}(t) d Q\right)^{-1}\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right) .
$$

### 3.2 Lower bound based on the chi-squared divergence

We now move on to the direct extension of the HCR inequality without the earlier projection argument. Throughout, we assume that $\Theta=\mathbb{R}^{d}$. To establish the extension of the van Trees inequality, we define two probability measures on a product space $\left(\mathcal{X} \times \mathbb{R}^{d}\right)$. Specifically, we let

$$
\begin{equation*}
d \mathbb{P}_{0}(x, \theta):=d P_{\theta}(x) d Q(\theta) \quad \text { and } \quad d \mathbb{P}_{h}(x, \theta):=d P_{\theta+h}(x) d Q(\theta+h) . \tag{7}
\end{equation*}
$$

The measure under the translation $Q(\theta+h)$ is well-defined for all $h \in \mathbb{R}^{d}$. We now provide an extension of the HCR inequality.

Theorem 9 (Mixture HCR inequality with the chi-squared divergence). For any probability measure $Q$ on $\mathbb{R}^{d}$, define

$$
A_{\lambda, \psi, Q, h}:=\frac{\left\|\int_{\mathbb{R}^{d}} \psi(t)-\psi(t-h) d Q(t)\right\|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \lambda \mathbb{P}_{h}+(1-\lambda) \mathbb{P}_{0}\right)} \quad \text { and } \quad B_{\psi, Q, h}:=\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t) .
$$

Then for any measurable function $T: \mathcal{X} \mapsto \mathbb{R}^{k}$, vector norm $\|\cdot\|: \mathbb{R}^{k} \mapsto \mathbb{R}_{+}$, and $\lambda \in[0,1]$,

$$
\inf _{T} \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \geq \sup _{h \in \mathbb{R}^{d}}\left[\sqrt{(1-\lambda)^{2} A_{\lambda, \psi, Q, h}}-\sqrt{\lambda B_{\psi, Q, h}}\right]_{+}^{2}
$$

The above result holds for any choice of $Q$ and $\lambda$ allowing for the derivation of the sharpest constant by taking supremum over them. The additional parameter $\lambda$ is introduced to prevent a trivial constant of zero under the chi-squared divergence. This might happen, for instance, when the statistical model $\mathcal{P}$ exhibits some irregularity. Under additional regularity conditions, the theorem can be significantly simplified by choosing $\lambda=0$. This provides a non-trivial lower bound when the necessary conditions for Lemma 1 hold.

Corollary 9.1. When $\lambda=0$, Theorem 9 implies that

$$
\begin{equation*}
\inf _{T} \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \geq \sup _{h \in \mathbb{R}^{d}} \frac{\left\|\int_{\mathbb{R}^{d}} \psi(t)-\psi(t-h) d Q(t)\right\|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \mathbb{P}_{0}\right)} \tag{8}
\end{equation*}
$$

This inequality can be seen as the modification of the classical HCR lower bound from a twopoint risk to mixture distributions $\mathbb{P}_{h}$ and $\mathbb{P}_{0}$. We now provide a heuristic argument that Theorem 9 implies the van Trees inequality as $\|h\|_{2} \longrightarrow 0$. Assuming the conditions for Lemma 1 hold and for a suitably selected prior $Q$, the following expansion of the chi-squared divergence can be obtained:

$$
\begin{equation*}
\chi^{2}\left(\mathbb{P}_{h} \| \mathbb{P}_{0}\right)=\chi^{2}\left(Q_{h} \| Q\right)+\int_{\mathbb{R}^{d}} \chi^{2}\left(P_{t+h} \| P_{t}\right) \frac{d Q_{h}^{2}}{d Q}=h^{\top}\left(\mathcal{I}(Q)+\int_{\mathbb{R}^{d}} \mathcal{I}(t) d Q\right) h+o\left(\|h\|_{2}^{2}\right) \tag{9}
\end{equation*}
$$

provided that we can exchange the limit and the integral under the dominated convergence theorem. Hence, if $\psi$ is a continuously differentiable function, the inequality (8) and the limit of its denominator given by (9) together imply the van Trees inequality as $\|h\|_{2} \longrightarrow 0$. A similar result holds for $\lambda>0$ as an application of Lemma 2 assuming that the Fisher defect is zero. As discussed earlier, the local behavior of the chi-squared divergence requires stronger assumptions than the Hellinger distance. The additional parameter $\lambda \in[0,1]$ can be removed under a similar result based on the Hellinger distance. Before presenting the corresponding result, we provide a short discussion on the role of $\lambda$.

Remark 2 (The role of the mixing weight $\lambda$ ). The mixing weight $\lambda$ is introduced to take advantage of Lemma 2, which states the convergence of the chi-squared divergence under the Hellinger differentiability. It also prevents a trivial lower bound of zero when the denominator of the lower bound (8) diverge, in other words, $P_{\theta}$ and $P_{\theta+h}$ are not absolutely continuous. This happens, for instance, considering any location family induced by a density function with compact support. By choosing $\lambda>0, P_{\theta}$ is absolutely continuous with respect to $(1-\lambda) P_{\theta}+\lambda P_{\theta+h}$ and thus, it prevents the chi-squared divergence from diverging. If the local expansion of the chi-squared divergence is characterized by (9), we can always take $\lambda=0$.

### 3.3 Lower bound based on the Hellinger distance

This section presents our second main result: the mixture extension of the HCR bound under the Hellinger distance. This extension is desirable since it does not require additional regularity conditions, unlike the analogous statement under the chi-squared divergence. Additionally, we discuss the connections between the presented result and the classical minimax lower bound in terms of the Hellinger distance by Donoho and Liu (1987), which has been frequently considered for non-smooth functionals, irregular estimation, or non-asymptotic minimax lower bound. Similar to Theorem 9 , we first introduce the result for $\Theta=\mathbb{R}^{d}$ and then discuss more generalized settings in the proceeding section.

Theorem 10 (Mixture HCR inequality with the Hellinger distance). For any probability measure $Q$ on $\mathbb{R}^{d}$, define

$$
A_{\psi, Q, h}:=\frac{\left\|\int_{\mathbb{R}^{d}} \psi(t)-\psi(t-h) d Q(t)\right\|^{2}}{4 H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)} \quad \text { and } \quad B_{\psi, Q, h}:=\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t) .
$$

Then for any measurable function $T: \mathcal{X} \mapsto \mathbb{R}^{k}$ and vector norm $\|\cdot\|: \mathbb{R}^{k} \mapsto \mathbb{R}_{+}$,

$$
\inf _{T} \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \geq \sup _{h \in \mathbb{R}^{d}}\left[\sqrt{A_{\psi, Q, h}}-\sqrt{B_{\psi, Q, h}}\right]_{+}^{2} .
$$

Theorem 10 can be compared to Lemma 1 of Simons and Woodroofe (1983) which restricts to unbiased estimators and does not consider priors. Similar to Theorem 9, the sharpest constant can be achieved by taking the supremum over all prior distributions $Q$. Under the Hellinger differentiability and suitably selected $Q$, the Hellinger distance between two mixture distributions admits the following expansion:

$$
\begin{aligned}
H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right) & =H^{2}\left(Q_{h}, Q\right)+\int_{\mathbb{R}^{d}} H^{2}\left(P_{t+h}, P_{t}\right) d Q_{h}^{1 / 2} d Q^{1 / 2} \\
& =\frac{1}{4} h^{\top}\left(\mathcal{I}(Q)+\int_{\mathbb{R}^{d}} \mathcal{I}(t) d Q\right) h+o\left(\|h\|^{2}\right)
\end{aligned}
$$

as $\|h\|_{2} \longrightarrow 0$. Therefore, assuming $\psi$ is continuously differentiable, Theorem 10 implies the van Trees inequality under weaker conditions than Theorem 9.

### 3.4 The extension from $\mathbb{R}^{d}$ to a general parameter space

Thus far, Theorems 9 and 10 are restricted to the case of $\Theta_{0}=\Theta=\mathbb{R}^{d}$. These results may still be useful for deriving a global minimax risk. However, it cannot be applied directly to a local minimax risk, for instance, when $\Theta_{0} \subset \Theta$ is a shrinking neighborhood of the parameter space around a particular value $\theta_{0}$. In this section, Theorems 9 and 10 are extended to the general space $\Theta_{0} \subseteq \mathbb{R}^{d}$ by constricting a diffeomorphism ${ }^{1} \varphi: \mathbb{R}^{d} \mapsto \Theta_{0}$. Although we only provide the illustration using Theorem 10, an identical argument can be applied to Theorem 9 .

Given a functional $\psi: \Theta_{0} \mapsto \mathbb{R}^{k}$, we define $\widetilde{\psi}: \mathbb{R}^{d} \mapsto \mathbb{R}^{k}$, a composite function $(\psi \circ \varphi)(t)$ for $t \in \mathbb{R}^{d}$ and also define respectively

$$
d \widetilde{\mathbb{P}}_{0}(x, t):=d P_{\varphi(t)}(x) d Q(t) \quad \text { and } \quad d \widetilde{\mathbb{P}}_{h}(x, t):=d P_{\varphi(t+h)}(x) d Q(t+h),
$$

where $Q(\cdot)$ is a probability measure on $\mathbb{R}^{d}$. The direct application of Theorem 10 leads to the following corollary:

Corollary 10.1. For any probability measure $Q$ on $\mathbb{R}^{d}$ and a diffeomorphism $\varphi: \mathbb{R}^{d} \mapsto \Theta_{0}$, define a composite function $\widetilde{\psi}:=(\psi \circ \varphi)$,

$$
A_{\widetilde{\psi}, Q, h}:=\frac{\left\|\int_{\mathbb{R}^{d}} \widetilde{\psi}(t)-\widetilde{\psi}(t-h) d Q(t)\right\|^{2}}{4 H^{2}\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)} \quad \text { and } \quad B_{\widetilde{\psi}, Q, h}:=\int_{\mathbb{R}^{d}}\|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)\|^{2} d Q(t) \text {. }
$$

[^0]Then for any measurable function $T: \mathcal{X} \mapsto \mathbb{R}^{k}$ and vector norm $\|\cdot\|: \mathbb{R}^{k} \mapsto \mathbb{R}_{+}$,

$$
\inf _{T} \sup _{\theta \in \Theta_{0}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \geq \sup _{\varphi} \sup _{h \in \mathbb{R}^{d}}\left[\sqrt{A_{\widetilde{\psi}, Q, h}}-\sqrt{B_{\widetilde{\psi}, Q, h}}\right]_{+}^{2}
$$

where the first supremum is over the diffeomorphism $\varphi$ between $\mathbb{R}^{d}$ and $\Theta_{0}$.
We provide the intuition behind the role of the diffeomorphism with an example. Suppose, for instance, the local parameter space $\Theta_{0}$ is an open $\mathbb{R}^{d}$-ball around $\theta_{0}$ whose radius shrinks at the rate $n^{-r}$ for $r>0$. For such $\Theta_{0}$, we may consider diffeomorphism $\varphi$ in the form of

$$
\varphi(t):=\theta_{0}+n^{-r} \varphi_{0}(t)
$$

where $\varphi_{0}: \mathbb{R}^{d} \mapsto B([0], 1)$ and $B([0], 1)$ is an open unit ball in $\mathbb{R}^{d}$. Without loss of generality, we assume that $\varphi_{0}(0)=0$ so $\varphi(0)=\theta_{0}$. Then $\varphi$ admits the following local expansion:

$$
\begin{equation*}
\varphi(\delta)-\varphi(0)=n^{-r} \varphi_{0}(\delta) \approx n^{-r} \nabla \varphi_{0}(\delta) \delta . \tag{10}
\end{equation*}
$$

Hence as $\|\delta\| \longrightarrow 0$, the functional $\psi(\varphi(\delta))$ approaches $\psi(\varphi(0))=\psi\left(\theta_{0}\right)$ through a nonlinear path uniquely defined by the diffeomorphism $\varphi_{0}$. We frequently use this result in the later sections where we demonstrate the application of Theorems 9 and 10 to recover the constant implied by the LAM theorem.

Remark 3. Another way to apply Theorems 9 and 10 is to use the fact that the supremum over $\theta \in \Theta_{0}$ can be obtained by considering any two points $\theta_{0}, \theta_{1} \in \Theta_{0}$ and any smooth path $\gamma:=\gamma_{\theta_{0}, \theta_{1}}$ : $(-\infty, \infty) \mapsto \Theta_{0}$ such that $\lim _{u \rightarrow-\infty} \gamma(u)=\theta_{0}, \lim _{u \rightarrow \infty} \gamma(u)=\theta_{1}$. Let $\widetilde{\psi}(u):=\psi(\gamma(u))$ for $u \in \mathbb{R}$,

$$
d \mathbb{P}_{0}(x, u):=d P_{\gamma(u)}(x) d Q(u) \quad \text { and } \quad d \mathbb{P}_{h}(x, u):=d P_{\gamma(u+h)}(x) d Q(u+h),
$$

where $Q$ is a probability measure on $\mathbb{R}$. Using this notation, we obtain

$$
\begin{aligned}
& \inf _{T} \sup _{\theta \in \Theta_{0}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \\
& \quad=\inf _{T} \sup _{\theta_{0}, \theta_{1}} \sup _{\gamma} \sup _{u \in \mathbb{R}} \mathbb{E}_{\gamma(u)}\|T(X)-\widetilde{\psi}(u)\|^{2} \\
& \quad \geq \sup _{\theta_{0}, \theta_{1}} \sup _{\gamma} \sup _{u \in \mathbb{R}}\left[\left(\frac{\left\|\int_{\mathbb{R}}(\widetilde{\psi}(u)-\widetilde{\psi}(u-h)) d Q(u)\right\|^{2}}{4 H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)}\right)^{1 / 2}-\left(\int_{\mathbb{R}}\|\widetilde{\psi}(u)-\widetilde{\psi}(u-h)\|^{2} d Q(u)\right)^{1 / 2}\right]_{+}^{2} .
\end{aligned}
$$

As a concrete example, suppose we are interested in finding a lower bound when $\Theta_{0}:=\{\theta: \rho(\theta-$ $\left.\left.\theta_{0}\right) \leq \delta\right\}$ for some positive homogeneous norm function $\rho(\cdot)$, then one can take $\theta_{1}=\theta_{0}+\delta h$ such that $\rho(h) \leq 1$ and consider $\gamma(u)=\theta_{0}+\Phi(u) \delta h$ where $\Phi(\cdot)$ is the CDF of the standard normal distribution. This approach of taking a smooth path between two arbitrary points allows for the application of the results above to non-convex, but continuous, parameter spaces $\Theta_{0}$.

### 3.5 Connection to the modulus of continuity

We now discuss the connections between Theorem 10 and existing minimax lower bounds in terms of the Hellinger distance. Previous works by Donoho and Liu (1987, 1991) provide a geometric
interpretation of the minimax rate through the modulus of continuity. We define the modulus of continuity of a real-valued functional $\psi$ with respect to the Helligner distance as

$$
\omega(\varepsilon):=\left\{\sup _{\theta_{1}, \theta_{2} \in \Theta_{0}}\left|\psi\left(P_{\theta_{1}}\right)-\psi\left(P_{\theta_{2}}\right)\right|: H^{2}\left(P_{\theta_{1}}, P_{\theta_{2}}\right) \leq \varepsilon^{2}\right\} .
$$

This quantity captures the maximum fluctuation of functionals evaluated at sufficiently "similar" distributions. We momentarily focus on estimating $\psi\left(P_{0}\right)$ based on $n$ IID observations $X_{1}, \ldots, X_{n}$ drawn from $P_{0}$. The minimax risk is considered for the supremum over a local model $\mathcal{P}_{n}:=\{P$ : $\left.H^{2}\left(P_{0}, P\right) \leq n^{-1}\right\}$, which is a set of distributions concentrated at $P_{0}$, analogous to the setting of the LAM theorem. Section 9.4 of Donoho and Liu (1987) proves that, under additional assumptions, for any $n$ sufficiently large,

$$
\begin{equation*}
\inf _{T} \sup _{P \in \mathcal{P}_{n}} n \mathbb{E}_{P}|T(X)-\psi(P)|^{2}>\frac{1}{16}\left(\frac{\omega\left(n^{-1 / 2}\right)}{n^{-1 / 2}}\right)^{2} . \tag{11}
\end{equation*}
$$

Donoho and Liu (1987) also shows that, under the setting of the LAM theorem (Theorem 4) and $\psi\left(P_{\theta}\right)=\theta$, it holds

$$
\begin{equation*}
\liminf _{n \longrightarrow \infty} \frac{1}{16}\left(\frac{\omega\left(n^{-1 / 2}\right)}{n^{-1 / 2}}\right)^{2}=\frac{1}{4 \mathcal{I}\left(\theta_{0}\right)} \tag{12}
\end{equation*}
$$

Hence, the non-asymptotic minimax lower bound given by (11) converges to $1 / 4$ the optimal constant according to the LAM theorem. It may be tempting to multiply the lower bound of (11) by 4 , that is, to consider the sequence $4^{-1}\left\{n^{1 / 2} \omega\left(n^{-1 / 2}\right)\right\}^{2}$. Although this sequence converges to the correct constant as $n \longrightarrow \infty$, it is an invalid lower bound as there exists an estimator that violates this inequality, provided in Section 8.4 of Donoho and Liu (1987). Donoho and Liu (1987) conjectured that a different approach is necessary to develop a sequence of non-asymptotic minimax lower bounds that converge to the correct limit. This result is provided by Theorem 10. Our result is similar to Donoho and Liu (1987) in the sense that both define local models relative to the Hellinger distance; however, they differ as we consider the mixture of distribution over the Hellinger ball instead of two distributions. It is also worth noting that an analogous non-asymptotic result in terms of the two-point modulus of continuity is studied by Chen (1997) without particular focus on a sharp constant.

The local behavior of the Hellinger distance has been a frequent tool in analyzing irregular statistical models (Ibragimov and Has'minskii, 1981; Donoho and Liu, 1987; Shemyakin, 2014; Duchi and Ruan, 2018; Lin et al., 2019). One of the fundamental two-point risk bounds in terms of the Hellinger distance is provided by Theorem 6.1 of Ibragimov and Has'minskii (1981); however, the original proof does not provide the optimal constant, leaving room for a simple improvement. Here, we present a refined two-point risk lower bound with an optimal constant. Although this improvement still fails to recover the asymptotic constant of the LAM theorem as we later demonstrate, it may still be of independent interest for its simplicity.

Lemma 11 (Refinement of Theorem 6.1 of Ibragimov and Has'minskii (1981) for real-valued functionals). For any real-valued functional $\psi: \Theta \mapsto \mathbb{R}$, a measurable function $T$ and $\theta_{1}, \theta_{2} \in \Theta$, we have

$$
\frac{1}{2}\left\{\mathbb{E}_{\theta_{1}}\left|T(X)-\psi\left(\theta_{1}\right)\right|^{2}+\mathbb{E}_{\theta_{2}}\left|T(X)-\psi\left(\theta_{2}\right)\right|^{2}\right\} \geq\left[\frac{1-H^{2}\left(P_{\theta_{1}}, P_{\theta_{2}}\right)}{4}\right]_{+}\left|\psi\left(\theta_{1}\right)-\psi\left(\theta_{2}\right)\right|^{2}
$$

This implies that

$$
\inf _{T} \sup _{\theta \in \Theta} \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq \sup _{\theta_{1}, \theta_{2} \in \Theta}\left[\frac{1-H^{2}\left(P_{\theta_{1}}, P_{\theta_{2}}\right)}{4}\right]_{+}\left|\psi\left(\theta_{1}\right)-\psi\left(\theta_{2}\right)\right|^{2} .
$$

Lemma 11 can be compared to Lemma 1 of Simons and Woodroofe (1983). It should be noted that analogous results as Lemma 11 have reported in the literature as the optimal constant for two-point risk has been explored as early as Donoho and Liu (1987). This result is presented to underscore the insufficiency of the two-point risk inequality in recovering the asymptotic constant a conclusion also reached by Donoho and Liu (1987). Although Lemma 11 is stated for a real-valued functional, these results can be extended to any vector-valued functional, with the leading constant naturally dependent on the vector norm $\|\cdot\|$.

Finally, similar objects as the Hellinger modulus of continuity have been considered in the econometrics literature. When the local asymptotic minimax lower bound, such as the righthand term of Theorem 10, is bounded away from zero, the functional estimation is called ill-posed (Pötscher, 2002; Forchini and Hillier, 2005). This precludes the existence of a (locally) uniformly consistent estimator of $\psi\left(P_{0}\right)$ in the nonparametric model. Pötscher (2002) and Forchini and Hillier (2005) define the modulus of continuity in the total variation distance as

$$
\omega(\varepsilon):=\left\{\sup _{\theta_{1}, \theta_{2} \in \Theta}\left|\psi\left(P_{\theta_{1}}\right)-\psi\left(P_{\theta_{2}}\right)\right|: \operatorname{TV}\left(P_{\theta_{1}}, P_{\theta_{2}}\right) \leq \varepsilon\right\}
$$

where the total variation distance is defined as

$$
\operatorname{TV}\left(P_{0}, P_{1}\right):=\sup _{A \in \mathcal{A}}\left|P_{0}(A)-P_{1}(A)\right|,
$$

with the supremum over all Borel measurable sets $A$. Theorem 2.1 of Pötscher (2002) provides the sufficient condition under which the following holds:

$$
\lim _{\varepsilon \rightarrow 0} \inf _{T} \sup _{P \in \mathcal{P}(\varepsilon)} \mathbb{E}_{P}|T(X)-\psi(P)|^{2} \geq \lim _{\varepsilon \longrightarrow 0} \frac{1}{4} \omega(\varepsilon)^{2}
$$

where $\mathcal{P}(\varepsilon):=\left\{P: \operatorname{TV}\left(P_{0}, P\right) \leq \varepsilon\right\}$. Pötscher (2002) and Forchini and Hillier (2005) analyze concrete problems where the lower bound is bounded away from zero. We note that there is no significant loss in considering the TV- or Hellinger-moduli as both distances define the same topology of the space of probability measures. However, our result also differs from Pötscher (2002) and Forchini and Hillier (2005) as we focus on the non-asymptotic results as well as the optimal constant. When the asymptotic lower bound of Theorem 10 is bounded away from zero, it can be considered as the measure of ill-posedness. We defer the corresponding analysis to future works.

## 4 Asymptotic properties

In this section, we examine the asymptotic properties of the general minimax lower bounds, presented in Section 3. Specifically, we investigate whether these bounds can recover established asymptotic constants, such as the LAM theorem for both parametric and nonparametric models as well as the local minimax rates for irregular estimation. These findings further reinforce our general understanding that (1) the local behavior of the Hellinger distance is easier to assert than that of the chi-squared divergence and (2) refined constants may be obtained using the mixture-based method instead of the two-point risk method.

The following two subsections are dedicated to recovering the classical LAM theorem for parametric and semiparametric models. Interestingly, they can imply a whole spectrum of such results; in classical LAM results, the neighborhood around a parameter shrinks at an $n^{-1 / 2}$ rate while we can let that neighborhood shrink at an arbitrary rate given the finite sample nature of our results. This includes the superefficiency phenomenon as well. For example, if the neighborhood is a singleton, then the lower bound should be zero. See Section 6 for more details.

### 4.1 Local asymptotic minimax lower bound

We first investigate whether the proposed lower bounds can recover the exact asymptotic constant provided by the LAM theorem for a parametric model. The setting is identical to the LAM theorem, that is, $X_{1}, \ldots, X_{n} \in \mathcal{X}$ are IID observations from $P_{\theta_{0}} \in\left\{P_{\theta}: \theta \in \Theta \subseteq \mathbb{R}^{d}\right\}$ where $P_{t}$ is Hellinger differentiable at $t$. Let $\psi: \Theta \mapsto \mathbb{R}$ be continuously differentiable at $\theta_{0}$ and $T: \mathcal{X}^{n} \mapsto \mathbb{R}$ be any sequence of measurable functions. We can then state the following result:

Proposition 1. Assuming the setting of Theorem 4, the following statements hold:
(i) Theorem 8 implies the LAM theorem,
(ii) if (A2) holds and the Fisher defect is zero at $\theta_{0}$, then Theorem 9 implies the LAM theorem,
(iii) Theorem 10 implies the LAM theorem, and
(iv) Lemma 11 implies the LAM theorem with its lower bound multiplied by the constant $C \approx$ 0.28953 .

We provide proofs for (i)-(iv) in Supplementary Material. In particular, Theorems 8 and 10 do not require additional conditions to imply the LAM theorem, even though they hold for a broader class of estimation problems. The LAM theorem remains valid for distributions with non-zero Fisher defect, which can occur, for instance, when the parameter of interests lies on the boundary (see Remark 1). Theorem 9 requires additional regularity conditions in order to rule out this scenario since it leads to possible misbehavior of the chi-squared divergence in its local limit. Although Proposition 1 (iv) slightly improves the leading constant from $1 / 4$, given by (12), it falls short of recovering the optimal constant. This result highlights that the minimax lower bound based on two distributions (i.e., Lemma 11) may not be sufficient to recover the precise asymptotic constant.

### 4.2 Semiparametric efficiency bound

Although the results thus far may appear to be confined to parametric settings, they can be extended to nonparametric functionals using the standard argument in semiparametric statistics: the supremum over the collection of parametric submodels. To begin, we introduce additional notation. We consider the estimation of nonparametric functional $\psi: \mathcal{P} \mapsto \mathbb{R}$, where $\mathcal{P}$ is a collection of probability measures belonging to an infinite-dimensional set. We reduce the study of nonparametric functionals to their behavior along parametric paths $P_{t}$. Although a univariate path suffices for our purpose, all statements in this section can be extended to a multivariate path.

In what follows, we define the notion of the smoothness of infinite-dimensional functionals along the QMD path, which is defined as (4). We emphasize that the theory remains intact under the weaker condition of the Hellinger differentiability, and we only use the QMD since many objects in semiparametric statistics are defined in terms of the score function $g_{0}$. This allows for consistent use of terminology without modifying definitions. A functional is called pathwise differentiable given a

QMD path with the score function $g_{0}$ if there exists a real-valued measurable function $\dot{\psi}_{0}: \mathcal{X} \mapsto \mathbb{R}$ such that

$$
\begin{equation*}
\left|\psi\left(P_{t}\right)-\psi\left(P_{0}\right)+t \int \dot{\psi}_{0} g_{0} d P_{0}\right|=o(t), \quad \text { as } \quad t \longrightarrow 0 \tag{13}
\end{equation*}
$$

The definition of pathwise differentiability implies that the local behavior of $\psi$ only depends on each path through the linear functional of $\dot{\psi}_{0}$. Below, we denote each path by $P_{t, g} \in \mathcal{P}$ for a given generic score function $g$ to make the dependence explicit. Then, we observe that the minimax risk of functional estimation in a nonparametric model must be at least larger than the supremum of the minimax risk along any parametric paths indexed by $g$. Here, the supremum is taken over the entire space of $g$, called the tangent set $\mathcal{T}_{P_{0}}$ of $\mathcal{P}$ at $P_{0}$. A functional is pathwise differentiable relative to $\mathcal{T}_{P_{0}}$ if equation (13) holds for all $g \in \mathcal{T}_{P_{0}}$. Although the function $\dot{\psi}_{0}$ is not generally unique, there is a unique projection of $\psi_{0}$ onto the closed linear span of the tangent set. The projected function is called the efficient influence function and plays an important role in the semiparametric efficiency theory. We consider the nonparametric model, containing all distributions on the shared measurable space. The tangent space associated with this model at $P_{0}$ corresponds to the collection of meanzero functions in $L_{2}\left(P_{0}\right)$, i.e., the entire Hilbert space of mean-zero, finite-variance functions. This is the maximal tangent space and we denote by $L_{2}^{0}\left(P_{0}\right)$. With this terminology in place, we present the following nonparametric analog of the LAM theorem,

Theorem 12 (Local asymptotic minimax theorem II (Theorem 5.2 of Van der Vaart (2002))). Let the functional $\psi: \mathcal{P} \mapsto \mathbb{R}$ be pathwise differentiable at $P_{0}$ relative to the maximal tangent space $L_{2}^{0}\left(P_{0}\right)$ with an efficient influence function $\dot{\psi}_{0}$. Assuming that the tangent set is a linear closure, then for any measurable function $T$ of the $n I I D$ observations from $P_{t, g}$,

$$
\begin{equation*}
\sup _{g \in L_{2}^{0}\left(P_{0}\right)} \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|t|<c n^{-1 / 2}} n \mathbb{E}_{P_{t, g}^{n}}\left|T(X)-\psi\left(P_{t, g}\right)\right|^{2} \geq \int \dot{\psi}_{0}^{2} d P_{0} \tag{14}
\end{equation*}
$$

The resulting lower bound is often called the semiparametric efficiency bound and it extends the LAM theorem to nonparametric contexts. We now demonstrate the application of Theorems 810 and Lemma 11 to derive the semiparametric efficiency bound. Before applying Theorem 9 in particular, we need to introduce an additional condition. Following Example 25.16 of Van der Vaart (2000), we define smooth and bounded parametric paths as follows:

$$
\begin{equation*}
d P_{t, g}(x)=\frac{1}{C_{t}} \kappa(t g(x)) d P_{0}(x) \tag{15}
\end{equation*}
$$

where $C_{t}:=\int \kappa(\operatorname{tg}(x)) d P_{0}$. We assume that $\kappa(0)=\kappa^{\prime}(0)=1$ and $\left\|\kappa^{\prime}\right\|_{\infty} \leq K$ and $\left\|\kappa^{\prime \prime}\right\|_{\infty} \leq K$ for some constant $K$. For instance, the function $\kappa(t):=2 /(1+\exp (-2 t))$ satisfies this condition. This choice of parametric paths allows the score function to be unbounded but the paths are bounded themselves. Crucially, this path asserts the local behavior of the chi-squared divergence, as in Lemma 1, uniformly over $g \in \mathcal{T}_{P_{0}} \subseteq L_{2}^{0}\left(P_{0}\right)$ (See Lemma 1 of Duchi and Ruan (2021)). We then state the following result.

Proposition 2. Assuming the identical setting as Theorem 12, the following statements hold:
(i) Theorem 8 implies the semiparametric efficiency bound for any QMD parametric path,
(ii) if each parametric path is defined as (15), then Theorem 9 implies the semiparametric efficiency bound,
(iii) Theorem 10 implies the semiparametric efficiency bound for any $Q M D$ parametric path, and
(iv) Lemma 11 implies the semiparametric efficiency bound for any QMD parametric path, multiplied by the constant $C \approx 0.28953$.

We provide proofs for (i)-(iv) in Supplementary Material for completeness although they follow naturally from Proposition 1. The proof proceeds roughly as follows: We apply the results of Proposition 1 on each parametric path with a fixed score function $g$, and then take the supremum of the score function over the linear closure of the tangent set $\mathcal{T}_{P_{0}}$. The statement (ii) of Proposition 2 shows that the lower bound based on the chi-squared divergence restricts the choice of parametric paths beyond the Hellinger differentiability. Such restriction can be undesirable for certain cases, as we discuss in the following remark.

Remark 4 (The implication from restricting parametric paths). When the linear closure of the tangent set spans the entire $L_{2}\left(P_{0}\right)$, then the choice of parametric path does not impact the lower bound. Therefore, there is no loss in selecting a specific path such as the one given by (15). However, if the tangent set is a strict subset of $L_{2}\left(P_{0}\right)$, such as a tangent cone, the constraint on the path becomes undesirable. In particular, this limitation on paths can have significant implications for semiparametric inference under shape constraints or parameters on bounded domains. For instance, Kuchibhotla et al. (2021) considers the projection of arbitrary QMD parametric paths onto a working statistical model in order to satisfy certain shape constraints. The statement (ii) of Proposition 2 may not be generally applicable in such cases.

Remark 5 (Non-pathwise differentiable functionals). While we demonstrate the application of the proposed lower bounds to derive the semiparametric efficiency bound for pathwise differentiable functionals, the main results from Section 3 still hold for general non-pathwise differentiable functionals. First, by taking $\Phi$ as a collection of pathwise differentiable functionals, Lemma 7 implies

$$
\begin{aligned}
& \sup _{g \in \mathcal{T}_{P_{0}}} \inf _{T} \sup _{\theta \in \Theta} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta, g}\right)\right|^{2} \\
& \geq \sup _{g \in \mathcal{T}_{P_{0}}} \sup _{\phi \in \Phi, Q \in \mathcal{Q}(\Theta)}\left[\left(\int \mathbb{E}_{\theta}\left|T(X)-\phi\left(P_{\theta, g}\right)\right|^{2} d Q\right)^{1 / 2}-\left(\int\left\|\psi\left(P_{\theta, g}\right)-\phi\left(P_{\theta, g}\right)\right\|^{2} d Q(\theta)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

This suggests that the efficiency bound for non-pathwise differentiable functionals can be analyzed by first placing a prior over the local parametric paths and then deriving the Bayes risk for estimating $\phi$ at the expense of the approximation error.

Alternatively, we can apply Theorem 10 to each parametric path and take the supremum over any generic tangent set $\mathcal{T}_{P_{0}} \subseteq L_{2}^{0}\left(P_{0}\right)$. This implies

$$
\begin{aligned}
& \sup _{g \in \mathcal{T}_{P_{0}}} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta, g}\right)\right|^{2} \\
& \geq \sup _{g \in \mathcal{T}_{P_{0}}} \sup _{h \in \mathbb{R}^{d}}\left[\left(\frac{\mid \int_{\mathbb{R}^{d}}\left(\psi\left(P_{t, g}\right)-\left.\psi\left(P_{t-h, g}\right) d Q(t)\right|^{2}\right.}{4 H^{2}\left(\mathbb{P}_{0, g}, \mathbb{P}_{h, g}\right)}\right)^{1 / 2}-\left(\int_{\mathbb{R}^{d}}\left|\psi\left(P_{t, g}\right)-\psi\left(P_{t-h, g}\right)\right|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

where

$$
d \mathbb{P}_{0, g}(x, \theta):=d P_{\theta, g}(x) d Q(\theta) \quad \text { and } \quad d \mathbb{P}_{h, g}(x, \theta):=d P_{\theta+h, g}(x) d Q(\theta+h)
$$

In the above non-asymptotic lower bound, the limiting behavior of $\psi\left(P_{t, g}\right)-\psi\left(P_{t-h, g}\right)$ as $h \longrightarrow 0$ is unspecified, allowing $\psi$ to be non-smooth at $t$.

### 4.3 Local minimax rate for irregular estimation

The use of minimax lower bounds based on the Hellinger distance by Theorem 10 is also attractive in the context of irregular problems. Recently, Lin et al. (2019) has proposed using the Hölder smoothness of the local Hellinger distance, relative to $P_{0}$, as the degrees of irregularity for the corresponding estimation problem. This proposal is motivated by the classical result in the literature, namely, Theorem 6.1 of Ibragimov and Has'minskii (1981). This result provides a two-point risk inequality in terms of the Hölder smoothness of the functional and that of the local Hellinger distance between two distributions in a model. In what follows, we demonstrate that Theorem 10 also recovers the asymptotic minimax rate given by Theorem 6.1 of Ibragimov and Has'minskii (1981).

Lemma 13 (Theorem 6.1 of Ibragimov and Has'minskii (1981)). Consider the estimation of a realvalued functional $\psi(\theta)$ based on the $n$ IID observations from $P_{\theta}$. We define $\operatorname{sign}(x):=x /\|x\|$ for $x \in \mathbb{R}^{d}$. We assume that there exits a constant $\delta>0$ such that for all $t, t+h \in\left\{\theta:\left\|\theta_{0}-\theta\right\|<\delta\right\}$ and as $\|h\| \longrightarrow 0$,

$$
\begin{aligned}
H^{2}\left(P_{t}, P_{t+h}\right) & =C_{1}\|h\|^{\alpha}+o\left(\|h\|^{\alpha}\right), & & \alpha \in(0,2], \\
\psi(t+h)-\psi(t) & =C_{2}\|h\|^{\beta}+o\left(\|h\|^{\beta}\right), & & \beta>0,
\end{aligned}
$$

where $C_{1} \equiv C_{1, t, \operatorname{sign}(h)}$ and $C_{2} \equiv C_{2, t, \operatorname{sign}(h)}$. Futhermore, these constants are assumed uniformly bounded such that $\sup _{t, h} C_{1, t, \operatorname{sign}(h)} \leq \bar{C}_{1}$ and $\sup _{t, h} C_{2, t, \operatorname{sign}(h)} \leq \bar{C}_{2}$ for some $\bar{C}_{1}, \bar{C}_{2} \in(0, \infty)$. Theorem 10 then implies that there exists a constant $C$, depending on $\bar{C}_{1}, \bar{C}_{2}, \alpha$, and $\beta$, satisfying

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left\|\theta-\theta_{0}\right\|<c n^{-1 / \alpha}} n^{2 \beta / \alpha} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \geq C>0
$$

The proof of this lemma can be found in Supplementary Material. We note that $\alpha$ is defined on the interval $(0,2]$ with $\alpha=2$ corresponding to the standard Hellinger differentiable models where the Fisher information is well-defined. Hence the most regular setting corresponds to $\alpha=2$ and $\beta=1$. Although the Hellinger differentiability fails for $\alpha<2$, the Hellinger distance can still be well-defined. For instance, a set of uniform distributions $\operatorname{Unif}(0, \theta)$ indexed by $\theta>0$ corresponds to $\alpha=1$. We refer to Lin et al. (2019) for more examples of irregular models. While Theorem 10 can account for this type of irregularity, Theorems 8 and 9 yield trivial lower bounds of zero; Theorem 8 fails due to the lack of the Fisher information and Theorem 9 fails as the chi-squared divergence for an irregular model is often infinite for any $n \geq 1$ and $c>0$.

## 5 Applications: Local minimax lower bounds

In this section, we provide several application of the minimax lower bounds for the estimation problems involving points of non-differentiability.

### 5.1 Nonparametric density estimation

The first application is nonparametric density estimation under the smoothness assumption.
Example 1 (Nonparametric density estimation). Let $X_{1}, \ldots, X_{n} \in \mathcal{X} \subseteq \mathbb{R}$ be $n$ IID observations from the unknown density function $f_{0}$. The functional of interest is the density value at a prespecified point $x_{0} \in \mathcal{X}$, that is, $\psi(f):=f\left(x_{0}\right)$.

We assume that the true density $f_{0}$ is $s$-times continuously differentiable at $x_{0}$. We then analyze the following class of density functions:

$$
\mathcal{F}\left(s, f_{0}, x_{0}, M\right):=\left\{f \text { is } s \text {-times differentiable at } x_{0} \text {, satisfying }\left|f^{(s)}\left(x_{0}\right)\right| \leq(1+M)\left|f_{0}^{(s)}\left(x_{0}\right)\right|\right\}
$$

for fixed $M>0$. Furthermore, we define the following localized set of density functions

$$
U(\delta ; \varepsilon):=\left\{f \in \mathcal{F}\left(s, f_{0}, x_{0}, M\right): \int_{\left|x-x_{0}\right| \leq \varepsilon}\left|f^{(k)}(x)-f_{0}^{(k)}(x)\right| d x \leq \delta \text { for all } k \in\{0,1, \ldots, s\}\right\}
$$

for fixed $\varepsilon>0$. The parameter $M$ is introduced in order to prevent the true density $f_{0}$ from lying on the boundary of the local model $U(\delta ; \varepsilon)$. As $\psi(f)$ is nonsmooth functional, we consider the following approximation via convolution:

$$
\phi(f) \equiv \phi(f ; K, h):=\int h^{-1} K\left(\frac{x-x_{0}}{h}\right) f(x) d x
$$

where $K$ is a kernel function and $h>0$ is a bandwidth parameter. The collection of approximation functionals $\Phi$ is then indexed by the choice of $K$ and the values of $h>0$. We consider any kernel function that satisfies the following conditions:
(A3) A function $K$ is assumed to satisfy the following conditions:
(a) it is uniformly bounded,
(b) it is $s$-times differentiable with the uniformly bounded $s$ th derivative,
(c) it integrates to one over its support, and
(d) for all integer $k$ where $0<k<s, \int_{-1}^{1} u^{k} K(u) d u=0$ and $\int_{-1}^{1} u^{s} K(u) d u<\infty$.

We then define the following class of approximation functionals:

$$
\Phi:=\{\phi(f ; K, h): \text { for all } h>0 \text { and } K \text {, satisfying (A3) }\} .
$$

We now present the application of Theorem 8 to the estimation of $\psi(f)$ approximated with $\phi(f)$ for any $\phi \in \Phi$ :

Lemma 14. Let $\delta_{n}:=c_{0} n^{-r}$ for $r \in\left[0,(2 s+1)^{-1}\right)$. Then Theorem 8 implies,

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{f \in U\left(\delta_{n}\right)} n^{2 s /(2 s+1)} \mathbb{E}_{f}\left|T(X)-f\left(x_{0}\right)\right|^{2} \\
& \quad \geq \sup _{K} C(s, M, K) f_{0}\left(x_{0}\right)^{2 s /(2 s+1)}\left|f_{0}^{(s)}\left(x_{0}\right)\right|^{2 /(2 s+1)}
\end{aligned}
$$

where $C(s, M, K)$ is a constant only depending on $s, M$ and $K$.
The proof of this lemma is provided in Section D of Supplementary Material. To the best of our knowledge, this is a new local asymptotic minimax constant in the context of nonparametric density estimation. A similar object to the term in the lower bound has appeared in the classical literature on kernel density estimation (Devroye and Györfi, 1985). Let $f_{0}$ be any density on $[0,1]$ with continuous sth derivative. Then Theorem 11 (Chapter 4, page 49) of Devroye and Györfi (1985) states that

$$
\liminf _{n \longrightarrow \infty} \sup _{f \in H\left(f_{0}\right)} n^{2 s /(2 s+1)} \frac{\mathbb{E}_{f} \int|T-f|^{p}}{\left(\int f^{p / 2}\right)^{2 s /(2 s+1)}\left(\int\left|f^{(s)}\right|^{p}\right)^{1 /(2 s+1)}}>0
$$

for any estimator $T$ where $H\left(f_{0}\right)$ is a set of all densities of the form $\sum_{i=1}^{\infty} \pi_{i} f_{0}\left(x+x_{i}\right), \pi_{i}$ is any probability vector and $\left\{x_{i}\right\}$ is an increasing sequence of real numbers such that $x_{i+1}-x_{i}>1$. Simple algebra also suggests that the kernel density estimator, defined as

$$
\widehat{f}:=\frac{1}{n h} \sum_{i=1}^{n} K\left(\frac{X_{i}-x_{0}}{h}\right)
$$

with the following "theoretical" choice of bandwidth

$$
h^{*} \propto\left(\frac{f_{0}\left(x_{0}\right)}{n\left|f_{0}^{(s)}\left(x_{0}\right)\right|^{2}}\right)^{1 /(2 s+1)}
$$

attains the matching upper bound including the dependency on $f_{0}$ (Abramson, 1982; Woodroofe, 1970; Hall, 1993; Brown et al., 1997). Such an estimator is often called adaptive in the nonparametric density estimation literature ${ }^{2}$. A reasonable estimate of $h^{*}$ is obtained by the "plug-in" rule where $f_{0}$ and $f_{0}^{(s)}$ in $h^{*}$ are replaced with pilot estimators $\widetilde{f}\left(x_{0}\right)$ and $\tilde{f}^{(s)}\left(x_{0}\right)$, constructed from a separate data. The final estimator is defined as

$$
\check{f}:=\frac{1}{n h_{\text {plug-in }}} \sum_{i=1}^{n} K\left(\frac{X_{i}-x_{0}}{h_{\text {plug-in }}}\right) \quad \text { where } \quad h_{\text {plug-in }} \propto\left(\frac{\widetilde{f}\left(x_{0}\right)}{n\left|\widetilde{f}(s)\left(x_{0}\right)\right|^{2}}\right)^{1 /(2 s+1)} .
$$

Assuming these pilot estimators converge in quadratic mean, Brown et al. (1997) states that the estimator $\check{f}$ asymptotically achieves the minimum pointwise risk, i.e., the risk under fixed $f_{0}$, among all choice of bandwidth parameters (See equation (2.5) of Brown et al. (1997)). Brown et al. (1997) however argues that such a pointwise assessment of the nonparametric estimator is often misleading in view of superefficiency, necessitating the evaluation under "uniform" risk. Our result does not contradict the message by Brown et al. (1997). The pointwise risk of the estimator is equivalent to the local minimiax risk over a singleton $\left\{f_{0}\right\}$, and this roughly corresponds to taking $r \longrightarrow \infty$ in our result. Lemma 14 does not allow the neighborhood $U\left(n^{-r}\right)$ to shrink too fast, slower than $n^{-1 /(1+s)}$ to be precise, which prevents the issue of superefficiency. Hence, the adaptivity to the unknown $f_{0}$ is not purely an artifact of superefficiency.

The van Trees inequality has been previously applied to analyze minimax lower bounds in nonparametric problems. See, for instance, Theorem 4 of Anevski and Soulier (2011) and Example 2.3 of Tsybakov (2008). In order to apply the classical van Trees inequality, these approaches typically reduce the original nonparametric problem to parameter estimation along a parametric submodel. This differs from our "direct" approach based on Theorem 8. Consequently, existing results do not account for risk over a shrinking neighborhood around $f_{0}$, and do not provide the dependency of the lower bound on $f_{0}$. Therefore, these results do not fully comply with the classical notion of local asymptotic minimaxity.

### 5.2 Simple directionally differentiable functionals

We now present the application of Theorem 10 to the estimation of $\psi\left(P_{\theta}\right):=\max (\theta, 0)$ for $\theta \in \mathbb{R}$. This is one of the canonical examples of directionally differentiable functionals. More complex problems such as interval regression (Fang, 2014) and testing of the shape of regression (Juditsky and Nemirovski, 2002), can be reduced to this form.

[^1]Example 2 (Estimating max $(\theta, 0)$ when $\theta=0$ ). Suppose $X_{1}, \ldots, X_{n}$ are $n$ IID observations drawn from $P_{0}$ that belongs to a local model $\left\{P_{\theta}:|\theta|<\delta, \theta \in \mathbb{R}\right\}$ for fixed $\delta>0$. We assume that this model is Hellinger differentiable with the Fisher information $\mathcal{I}(t)$ for $t \in(-\delta, \delta)$. The functional of interest is $\psi\left(P_{\theta}\right)=\max (0, \theta)$.

Then Theorem 8 implies the following non-asymptotic local minimax lower bound:
Lemma 15. Suppose that $\mathcal{Q}^{\dagger} \equiv \mathcal{Q}^{\dagger}(-1,1)$ is a collection of "nice" priors defined on $[-1,1]$ (see Definition 1) then Theorem 8 implies that

$$
\begin{align*}
\inf _{T} \sup _{|\theta|<\delta} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} & \geq \sup _{Q \in \mathcal{Q}^{+}} \frac{\left(\int_{0}^{1} q(t) d t\right)^{2}}{\delta^{-2} \mathcal{I}(Q)+n \int_{-1}^{1} \mathcal{I}(\delta t) q(t) d t}  \tag{16}\\
& \geq \sup _{a \in[0,1]} \frac{a^{2}}{4 \pi^{2} \delta^{-2} W_{a}^{-2}+n \sup _{|t|<\delta} \mathcal{I}(t)} \tag{17}
\end{align*}
$$

where $q$ is a density function of $Q$ for any $Q \in \mathcal{Q}^{\dagger}, W_{a}:=2 /\left(Y_{a}+1\right)$ and $Y_{a}$ is the inverse of the following equation:

$$
Y_{a}-\sin \left(-\pi Y_{a}\right) / \pi=|2 a-1| .
$$

The first result of Lemma 15, given by (16), is a direct consequence of the van Trees inequality since the functional $\psi(\theta)=\max (\theta, 0)$ is absolutely continuous. The second result of Lemma 15, given by (17), investigates the least-favorable prior using variational calculus and optimization under absolute moment constraints (Ernst, 2017). The corresponding derivation can be found in Section E of Supplementary Material. Our lower bound involves the inverse of the Kepler equation from Celestial Mechanics, which does not have a closed-form solution (Kepler, 1609). Hence, our lower bound relies on a computational method. This technique seems to be less known in the literature but has been mentioned in the context of non-asymptotic minimax lower bound for Gaussian mean estimation under bounded constraints (Levit, 2010).

Next, we provide a non-asymptotic lower bound for estimating $\max \left(\theta^{\alpha}, 0\right)$ for $0<\alpha \leq 1$ and $\theta \in \mathbb{R}$ based on Theorem 10. This function is also non-smooth at $\theta=0$ and the corresponding lower bound behaves differently depending on the location of the true parameter.

Example 3 (Estimating $\max \left(\theta^{\alpha}, 0\right)$ for $\left.\alpha>0\right)$. Consider the identical settings as Example 2, except the functional of interest is now $\psi\left(P_{\theta}\right)=\max \left(0, \theta^{\alpha}\right)$ for $\alpha>0$.

The following lemma provides the application of Theorem 10 to this problem:
Lemma 16. Theorem 10 implies the following non-asymptotic local minimax lower bounds:
(i) When $\theta_{0}<0$, it holds that

$$
\inf _{T} \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq \sup _{Q, \varphi_{0}} \frac{\delta^{2} \alpha^{2}\left|\mathbb{E}_{Q}\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t:\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\}\right|^{2}}{\mathcal{I}(Q)+n \delta^{2} \mathbb{E}_{Q}\left[\left\{\varphi_{0}^{\prime}(t)\right\}^{2} \mathcal{I}\left(\theta_{0}+\delta \varphi_{0}(t)\right)\right]} .
$$

(ii) When $\theta_{0}>0$, it holds that

$$
\inf _{T} \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq \sup _{Q, \varphi_{0}} \frac{\delta^{2} \alpha^{2}\left|\mathbb{E}_{Q}\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t: \varphi_{0}(t)<\theta_{0} / \delta\right\}\right|^{2}}{\mathcal{I}(Q)+n \delta^{2} \mathbb{E}_{Q}\left[\left\{\varphi_{0}^{\prime}(t)\right\}^{2} \mathcal{I}\left(\theta_{0}+\delta \varphi_{0}(t)\right)\right]} .
$$

(iii) When $\theta_{0}=0$, it holds that

$$
\inf _{T} \sup _{|\theta|<\delta} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq \sup _{Q, \varphi_{0}} \frac{\delta^{2 \alpha} \alpha^{2}\left|\mathbb{E}_{Q} \varphi_{0}(t)^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t: \varphi_{0}(t)>0\right\}\right|^{2}}{\mathcal{I}(Q)+n \delta^{2} \mathbb{E}_{Q}\left[\left\{\varphi_{0}^{\prime}(t)\right\}^{2} \mathcal{I}\left(\delta \varphi_{0}(t)\right)\right]} .
$$

The supremums are taken over any probability measure $Q$ on $\mathbb{R}$ and any increasing deffeomorphism $\varphi_{0}: \mathbb{R} \mapsto(-1,1)$ such that $\varphi_{0}(0)=0$ and $\left\|\varphi_{0}^{\prime}\right\|_{\infty}<C$ for some constant.

Here, the choice of priors no longer needs to satisfy Definition 1. Instead, Lemma 16 posits certain requirements over the choice of diffeomorphism.

Remark 6 (Minimax rates of convergence under IID observations). From the expressions above, we can deduce the local minimax rates of convergence. When $\theta_{0}>0$, the lower bound (ii) of Lemma 16 above implies that

$$
\frac{\delta^{2} O(1)}{\mathcal{I}(Q)+n \delta^{2} O(1)}=\frac{O(1)}{\delta^{-2} \mathcal{I}(Q)+n O(1)}
$$

Balancing two terms in the denominator, we choose $\delta=O\left(n^{-1 / 2}\right)$; the overall rate of convergence is $n^{-1}$, or so-called parametric rate, provided $\theta_{0}>0$. This is reasonable given that $\psi\left(\theta_{0}\right)$ is a smooth parameter when $\theta_{0}$ is bounded away from zero.

Similarly when $\theta_{0}=0$, the lower bound (iii) of Lemma 16 implies that

$$
\frac{\delta^{2 \alpha} O(1)}{\mathcal{I}(Q)+n \delta^{2} O(1)}=\frac{O(1)}{\delta^{-2 \alpha} \mathcal{I}(Q)+n \delta^{2-2 \alpha} O(1)}
$$

Balancing two terms in the denominator, we choose $\delta=O\left(n^{-1 / 2}\right)$ again; the overall rate of convergence is now $n^{-\alpha}$ when $\theta_{0}=0$, which is strictly slower than the parametric rate when $\alpha<1$. We thus conclude that the minimax rates of convergence for estimating $\max \left(\theta^{\alpha}, 0\right)$ remain the same as the rates for $\theta^{\alpha}$ discussed in Gill and Levit (1995).

Remark 7 (Local asymptotic minimax constants). Lemma 16 also recovers the precise constants for the local asymptotic minimax lower bound. In addition to the setting of Lemma 16, we further assume that $\mathcal{I}(t)$ is continuous at $\theta_{0}$. Given the observation from Remark 6, we replace $\delta$ with $c n^{-1 / 2}$ and analyze their limits. When $\theta_{0}<0$, there exists $n$ large enough that $\left|\theta_{0}\right| \geq c n^{-1 / 2}$, and hence we have

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq 0 .
$$

When $\theta_{0}>0$, there exists n large enough such that $\theta_{0} \geq c n^{-1 / 2}$, and thus $I\left\{t: \varphi_{0}(t)<\theta_{0} /\left(c n^{-1 / 2}\right)\right\}=$ 1 for all $t \in \mathbb{R}$. We then have,

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq \frac{\alpha^{2} \theta_{0}^{2 \alpha-2}}{\mathcal{I}\left(\theta_{0}\right)} \frac{\left|\mathbb{E}_{Q} \varphi_{0}^{\prime}(t)\right|^{2}}{\mathbb{E}_{Q}\left\{\varphi_{0}^{\prime}(t)\right\}^{2}}
$$

where we invoke the dominated convergence theorem in view of the continuity of $\mathcal{I}(t)$ at $t=\theta_{0}$ and the uniform boundedness of $\varphi_{0}^{\prime}$. The supremum of the last quantity involving $\varphi_{0}$ is 1 as shown in the proof of Proposition 1 (iii) (See Section C of Supplementary Material). Hence the local asymptotic
minimax constant is $\alpha^{2} \theta_{0}^{2 \alpha-2} \mathcal{I}\left(\theta_{0}\right)^{-1}$. The case with $\theta_{0}=0$ is more involved. By the analogous argument,

$$
\begin{gathered}
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1 / 2}} n^{\alpha} \mathbb{E}_{P_{\theta}^{n}}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \\
\quad \geq \liminf _{c \longrightarrow \infty} \frac{c^{2 \alpha} \alpha^{2}\left|\mathbb{E}_{Q} \varphi_{0}(t)^{\alpha-1} \varphi_{0}^{\prime}(t) I\{t>0\}\right|^{2}}{\mathcal{I}(Q)+c^{2} \mathcal{I}\left(\theta_{0}\right) \mathbb{E}_{Q}\left\{\varphi_{0}^{\prime}(t)\right\}^{2}}
\end{gathered}
$$

The liminf may not be achieved by the extreme values of $c$, and it depends on specific choices of $Q$ and $\varphi_{0}$. Taking the supremum of this object over the choice of $Q$ and $\varphi_{0}$ requires more involved analysis, which we do not pursue in this manuscript.

### 5.3 Parameter estimation under an irregular model

The final illustration provides a local asymptotic minimax result where the Hellinger differentiability fails. This is a quintessential example where the Fisher information is undefined. We demonstrate that Theorem 10 recovers a correct local minimax rate of convergence as well as local parameter dependence. We also observe that the simple choice of diffeomorphism fails to recover the correct constant and may require more delicate analyses.

Example 4 (Estimating $\theta_{0}$ from $\operatorname{Unif}\left(0, \theta_{0}\right)$ for $\theta_{0}>0$ ). Suppose $X_{1}, \ldots, X_{n}$ are $n$ IID observations from $\operatorname{Unif}\left(0, \theta_{0}\right)$, which belong to a statistical model $\{\operatorname{Unif}(0, \theta): 0<\theta\}$. The functional of interest is $\psi\left(P_{\theta}\right)=\theta$.

Then we state the following result:
Lemma 17. Theorem 10 implies the following local asymptotic minimax lower bound:

$$
\begin{aligned}
& \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} \mathbb{E}_{P_{\theta}^{n}} n^{2}|T(X)-\theta|^{2} \\
& \quad \geq \theta_{0}^{2} \sup _{\eta, Q, \varphi_{0}}\left[\frac{\left|\int \eta \varphi_{0}^{\prime}(t) d Q(t)\right|}{2\left(2-2 \int \exp \left(-\eta \varphi_{0}^{\prime}(t) / 2\right) d Q(t)\right)^{1 / 2}}-\left(\int\left\{\eta \varphi_{0}^{\prime}(t)\right\}^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2}=C \theta_{0}^{2}
\end{aligned}
$$

for $C>0$ where the supremum is over $\eta \in \mathbb{R}$, diffeomorphism $\varphi_{0}$ from $\mathbb{R}$ to $(-1,1)$ and any prior distributions $Q$ over $\mathbb{R}$.

The preceding display shows that the local minimax risk for parameter estimation under uniform distribution behaves as $O\left(\theta_{0}^{2} / n^{2}\right)$. Without taking into account the constant, this is the correct known dependency. Theorem 4.9 and Proposition 4.5 of Korostelev and Korosteleva (2011) further prove that the sharpest constant is 1 based on a similar approach using the Bayes risk with a uniform prior. This implies that Theorem 7, which is also based on the worst Bayes risk, can recover the correct constant for irregular problems. However, Theorem 10 does not readily accommodate a prior with compact support. In what follows, we present a corresponding constant based on a simple choice of diffeomorphism, which fails to recover the asymptotic constant.

Proposition 3. Consider a sequence of diffeomorphism $\varphi_{0}(t ; \eta)$, which is indexed by $\eta>0$. Assuming that $\eta \varphi_{0}^{\prime}(t ; \eta) \longrightarrow C$ as $\eta \longrightarrow \infty$ for some finite constant, we obtain

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} \mathbb{E}_{P_{\theta}^{n}} n^{2}|T(X)-\theta|^{2} \geq C^{*} \theta_{0}^{2}
$$

where $C^{*} \approx 0.0635^{2}$.

The proceeding proposition demonstrates that while lemma 11 enables simpler analysis and a sharper constant compared to Proposition 3, it still falls short of achieving the theoretically optimal constant.

Proposition 4. Under the settings as Example 4, Lemma 11 provides the following local asymptotic minimax lower bound:

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} \mathbb{E}_{P_{\theta}^{n}} n^{2}|T(X)-\theta|^{2} \geq C^{*} \theta_{0}^{2}
$$

where $C^{*} \approx 0.0558$.

## 6 Visual illustration: Upper bounds attained by plugin estimators

In this section, we investigate the precise gaps between the risk of estimators and the non-asymptotic local minimax lower bounds for fixed $\delta>0$ and $n \geq 1$. The results in this section are provided mostly for illustration purposes, demonstrating how non-asymptotic analysis of estimators beyond the efficiency bound may look like. Consider the following estimation problem:

$$
\begin{equation*}
\sup _{|\theta|<\delta} n \mathbb{E}_{\theta}|T(X)-\max (\theta, 0)|^{2} \tag{18}
\end{equation*}
$$

where we observe $n$ IID observations $X:=\left(X_{1}, \ldots, X_{n}\right)$ from $N(\theta, 1)$. We define the local model as $\{N(\theta, 1):|\theta|<\delta\}$. We construct lower bounds to (18) using two methods. The first lower bound ( vT ) is due to equation (17) of Lemma 15, which is given by

$$
\begin{equation*}
\inf _{T} \sup _{|\theta|<\delta} n \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq \sup _{a \in[0,1]} \frac{n a^{2}}{4 \pi^{2} \delta^{-2} W_{a}^{-2}+n}, \tag{vT}
\end{equation*}
$$

where $W_{a}$ is defined in Lemma 15. The second lower bound (diffeo) is based on the equation Lemma 16 (iii) when $\alpha=1$. As computing the supremum over $\varphi_{0}$ and $Q$ is challenging, we simplify the problem by focusing on the following choices:

$$
\begin{equation*}
Q \in\left\{N\left(\mu, \sigma^{2}\right):(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_{+}\right\} \text {and } \varphi_{0} \in\{t \mapsto \pi / 2 \arctan (t / \eta): \eta>0\} \tag{diffeo}
\end{equation*}
$$

and optimize the parameters $(\eta, \mu, \sigma)$. In Section G of Supplementary Material, we show that this reduces to the following simpler optimization:

$$
\inf _{T} \sup _{|\theta|<\delta} n \mathbb{E}_{\theta}|T(X)-\max (\theta, 0)|^{2} \geq \sup _{\xi_{1}, \xi_{2}} \frac{4 n \xi_{2}^{2}\left|\mathbb{E}\left[\left(1+\left(\xi_{1}+Z \xi_{2}\right)^{2}\right)^{-1} I\left(Z>-\xi_{1} / \xi_{2}\right)\right]\right|^{2}}{\pi^{2} \delta^{-2}+4 n \mathbb{E}\left[\left(1+\left(\xi_{1}+Z \xi_{2}\right)^{2}\right)^{-2}\right]}
$$

where $Z \stackrel{d}{=} N(0,1)$ and $\xi_{1}, \xi_{2} \in \mathbb{R} \times \mathbb{R}_{+}$.
The exact local minimax risk of three estimators is considered: the constant estimator, the plugin maximum likelihood estimator (MLE), and the plug-in preliminary-test (pre-test) estimator. Specifically, the plug-in pre-test estimator is an example of an irregular estimator, which may exhibit super-efficiency at a Lebesgue measure zero set. We now define the estimator. The constant estimator returns a predetermined value regardless of the observation, and we consider the case
where $S_{n}^{\text {const }}:=\delta / 2$. This is the best constant estimator that minimizes the local minimax risk, which is given by

$$
\sup _{|\theta|<\delta} n \mathbb{E}_{\theta}\left|S_{n}^{\text {const }}-\max (\theta, 0)\right|^{2}=n \delta^{2} / 4 .
$$

The remaining two estimators are defined as follows:

$$
\begin{align*}
& S_{n}^{\text {plug-in }}:=\max \left(\widehat{\theta}_{\text {MLE }}, 0\right) \quad \text { where } \quad \widehat{\theta}_{\text {MLE }}:=\bar{X}_{n}=\frac{1}{n} \sum_{i=1}^{n} X_{i}, \quad \text { and }  \tag{19}\\
& S_{n}^{\text {pre-test }}:=\max \left(\widehat{\theta}_{\text {pre-test }}, 0\right) \quad \text { where } \quad \widehat{\theta}_{\text {pre-test }}:= \begin{cases}\bar{X}_{n} & \text { If }\left|\bar{X}_{n}\right| \geq C_{n} \\
0 & \text { otherwise. }\end{cases} \tag{20}
\end{align*}
$$

This family of preliminary-test estimators (Sclove et al., 1972) is known to be super-efficient when $\theta_{0}=0$. For example, we consider the case when $C_{n}=n^{-1 / 4}$, which is equivalent to the Hodges' estimator. The following proposition provides the exact local minimax risks of two estimators.

Proposition 5. Assuming $X_{1}, \ldots, X_{n}$ are $n$ IID observation from $N(0,1)$ and the plug-in MLE is defined as (19), the local minimax risk of this estimator is given by

$$
\sup _{|\theta|<\delta} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}-\max (\theta, 0)\right|^{2}=\sup _{0 \leq \theta<\delta}\left\{\mathbb{E}\left[Z^{2} I\left(Z \geq-n^{1 / 2} \theta\right)\right]+n \theta^{2} P\left(Z<-n^{1 / 2} \theta\right)\right\} .
$$

where $Z \stackrel{d}{=} N(0,1)$. Similarly for the pre-test estimator defined as (20), the local minimax risk of this estimator is given by

$$
\begin{aligned}
\sup _{|\theta|<\delta} & n \mathbb{E}_{\theta}\left|S_{n}^{\text {pre-test }}-\max (\theta, 0)\right|^{2} \\
& =\sup _{0 \leq \theta<\delta}\left\{\mathbb{E}\left[Z^{2} I\left(Z \geq n^{1 / 4}-n^{1 / 2} \theta\right)\right]+n \theta^{2} P\left(Z<n^{1 / 4}-n^{1 / 2} \theta\right)\right\}
\end{aligned}
$$

Figure 1 shows that for any fixed sample size $n$ as $\delta$ increases, the lower bounds tend to $\sigma^{2}=1$ and the best among the estimators considered also has risk tending to $\sigma^{2}$. For any fixed sample size $n$ and "small" $\delta$, the lower bound is close to zero and the best among the estimators considered also has risk close to zero. This is expected because an estimator that is always zero will have zero risk at $\theta_{0}=0$. Finally, Figure 1 also shows an interesting comparison between different lower bounds. Neither (vT) nor (diffeo) is a clear winner. Figure 2 shows the same phenomena by fixing $\delta$ and varying the sample size $n$.

## 7 Concluding remarks

This manuscript presents new general minimax lower bound techniques for functional estimation without requiring the differentiability of functionals or the regularity of statistical models. We focus on the local minimax lower bounds based on approximation via absolutely continuous functionals (Theprem 8), the extensions of the HCR bound to the mixture model based on two divergence metrics, the chi-squared divergence (Theorem 9) and the Hellinger distance (Theorem 10). The minimax lower bound based on the Hellinger distance is more applicable to irregular estimation problems and only requires mild regularity conditions. Unlike standard minimax analysis that relies on testing reduction, this manuscript focuses on preserving a precise asymptotic. The manuscript


Figure 1: The non-asymptotic local minimax lower bounds and the risk given by different estimators.


Figure 2: The non-asymptotic local minimax lower bounds and the risk given by different estimators.
provides conditions under which the proposed lower bounds recover the well-known local asymptotic minimax theorem for both parametric (Proposition 1) and semiparametric models (Proposition 2) as well as local minimax results for irregular models (Lemma 13).

The flexibility of the proposed lower bounds offers many potential applications, especially for non-smooth functionals or estimation under irregularity. For example, one may consider the minimax lower bound for non-pathwise differentiable functionals or the irregularity that arises on the boundary of projection operations. A similar local asymptotic minimax lower bound was recently derived in the context of plug-in estimators (Fang, 2014), but there remain many open problems for general estimators. Finally, the potential application to the minimax lower bound under non-IID observations would also be of interest.

This manuscript primarily focuses on the extension of the van Trees inequality and the HCR
bound. However, other Bayes risk lower bounds exist, such as the Ziv-Zakai bound (Ziv and Zakai, 1969; Bell et al., 1997). Recently, Jeong et al. (2023) studied the extension of this bound in the context of parameter estimation where density does not exist. It would be interesting to explore the application of Ziv-Zakai bound to non-smooth functionals and its connection to the semiparametric efficiency theory.

As we discussed in Section 1.2, a precise constant associated with the minimax lower bound for non-smooth functional estimation was also discovered by Cai and Low (2011) using a slightly different approach. Their technique is based on the extension of the constrained risk inequality (Brown and Low, 1996) to two mixture distributions. Their method is also attractive as two mixture distributions are constructed based on moment-matching priors, enabling various analytic tools from approximation theory (Wu and Yang, 2020). Our current result only considers a single prior over a parametric path, and it would be interesting to explore a technique that takes advantage of such moment-matching priors.

Finally, Levit (2010) also discusses the importance of non-asymptotic constants in the minimax paradigm. In particular, Levit (2010) observes that different lower bounding methods provide sharp constants according to small, moderate, or large sample sizes. Although the results in this manuscript are valid non-asymptotically and converge to the correct constant in the limit, we did not study the sharpness of the derived constants for different finite sample sizes. While Levit (2010) focuses on bounded Gaussian mean estimations, our lower bounds can be extended to non-asymptotic analysis for more complex functional estimation problems, which is an important direction for future work.

## References

Abramson, I. S. (1982). On bandwidth variation in kernel estimates-a square root law. The annals of Statistics, pages 1217-1223.

Anevski, D. and Soulier, P. (2011). Monotone spectral density estimation. The Annals of Statistics.
Assouad, P. (1983). Deux remarques sur l'estimation. Comptes rendus des séances de l'Académie des sciences. Série 1, Mathématique, 296(23):1021-1024.

Barankin, E. W. (1949). Locally best unbiased estimates. The Annals of Mathematical Statistics, 20(4):477-501.

Bell, K. L., Steinberg, Y., Ephraim, Y., and Van Trees, H. L. (1997). Extended ziv-zakai lower bound for vector parameter estimation. IEEE Transactions on information theory, 43(2):624637.

Bhattacharyya, A. (1946). On some analogues of the amount of information and their use in statistical estimation. Sankhyā: The Indian Journal of Statistics, pages 1-14.

Bickel, P. J., Klaassen, C. A., Bickel, P. J., Ritov, Y., Klaassen, J., Wellner, J. A., and Ritov, Y. (1993). Efficient and adaptive estimation for semiparametric models, volume 4. Springer.

Birgé, L. (1987). Estimating a density under order restrictions: Nonasymptotic minimax risk. The Annals of Statistics, pages 995-1012.

Brown, L. D. and Low, M. G. (1996). A constrained risk inequality with applications to nonparametric functional estimation. The annals of Statistics, 24(6):2524-2535.

Brown, L. D., Low, M. G., and Zhao, L. H. (1997). Superefficiency in nonparametric function estimation. The Annals of Statistics, 25(6):2607-2625.

Cai, T. T. and Low, M. G. (2011). Testing composite hypotheses, hermite polynomials and optimal estimation of a nonsmooth functional. The Annals of Statistics, pages 1012-1041.

Chapman, D. G. and Robbins, H. (1951). Minimum variance estimation without regularity assumptions. The Annals of Mathematical Statistics, pages 581-586.

Chen, J. (1997). A general lower bound of minimax risk for absolute-error loss. Canadian Journal of Statistics, 25(4):545-558.

Collier, O., Comminges, L., and Tsybakov, A. B. (2020). On estimation of nonsmooth functionals of sparse normal means.

Cramér, H. (1999). Mathematical methods of statistics, volume 26. Princeton university press.
Devroye, L. and Györfi, L. (1985). Nonparametric Density Estimation: The $L_{1}$ view. Wiley, New York.

Donoho, D. L. and Liu, R. C. (1987). Geometrizing rates of convergence. Preprint.
Donoho, D. L. and Liu, R. C. (1991). Geometrizing rates of convergence, ii. The Annals of Statistics, pages 633-667.

Duchi, J. and Ruan, F. (2021). A constrained risk inequality for general losses. In International Conference on Artificial Intelligence and Statistics, pages 802-810. PMLR.

Duchi, J. C. and Ruan, F. (2018). The right complexity measure in locally private estimation: It is not the fisher information. arXiv preprint arXiv:1806.05756.

Ernst, P. A. (2017). Minimizing fisher information with absolute moment constraints. Statistics $\mathcal{B}$ Probability Letters, 129:167-170.

Fang, Z. (2014). Optimal plug-in estimators of directionally differentiable functionals. Unpublished Manuscript.

Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. Philosophical transactions of the Royal Society of London. Series A, containing papers of a mathematical or physical character, 222(594-604):309-368.

Forchini, G. and Hillier, G. H. (2005). Ill-conditioned problems, fisher information, and weak instruments. Technical report, cemmap working paper.

Gassiat, E. and Stoltz, G. (2024). The van trees inequality in the spirit of hajek and le cam. arXiv preprint arXiv:2402.06431.

Gill, R. D. and Levit, B. Y. (1995). Applications of the van trees inequality: a bayesian cramér-rao bound. Bernoulli, pages 59-79.

Hájek, J. (1970). A characterization of limiting distributions of regular estimates. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 14(4):323-330.

Hájek, J. (1972). Local asymptotic minimax and admissibility in estimation. In Proceedings of the sixth Berkeley symposium on mathematical statistics and probability, volume 1, pages 175-194.

Hall, P. (1993). On plug-in rules for local smoothing of density estimators. The Annals of Statistics, pages 694-710.

Hammersley, J. M. (1950). On estimating restricted parameters. Journal of the Royal Statistical Society. Series B (Methodological), 12(2):192-240.

Han, Y., Jiao, J., and Mukherjee, R. (2020). On estimation of l r-norms in gaussian white noise models. Probability Theory and Related Fields, 177(3-4):1243-1294.

Hencl, S. (2004). Notes on absolutely continuous functions of several variables.
Hirano, K. and Porter, J. R. (2012). Impossibility results for nondifferentiable functionals. Econometrica, 80(4):1769-1790.

Ibragimov, I. A. and Has'minskii, R. Z. (1981). Statistical estimation: asymptotic theory, volume 16. Springer Science \& Business Media.

Ibragimov, I. A., Nemirovskii, A. S., and Khas' minskii, R. (1987). Some problems on nonparametric estimation in gaussian white noise. Theory of Probability \& Its Applications, 31(3):391-406.

Jeong, M., Dytso, A., and Cardone, M. (2023). Functional properties of the ziv-zakai bound with arbitrary inputs. arXiv preprint arXiv:2305.02970.

Jiao, J., Venkat, K., Han, Y., and Weissman, T. (2015). Minimax estimation of functionals of discrete distributions. IEEE Transactions on Information Theory, 61(5):2835-2885.

Juditsky, A. and Nemirovski, A. (2002). On nonparametric tests of positivity/monotonicity/convexity. The Annals of Statistics, 30(2):498-527.

Jupp, P. E. (2010). A van trees inequality for estimators on manifolds. Journal of multivariate analysis, 101(8):1814-1825.

Kennedy, E. H., Balakrishnan, S., and Wasserman, L. (2022). Minimax rates for heterogeneous causal effect estimation. arXiv preprint arXiv:2203.0083\%.

Kepler, J. (1609). Astronomia Nova. Paris, A. Blanchard.
Korostelev, A. P. and Korosteleva, O. (2011). Mathematical statistics: asymptotic Minimax theory, volume 119. American Mathematical Soc.

Kuchibhotla, A. K., Patra, R. K., and Sen, B. (2021). Semiparametric efficiency in convexity constrained single-index model. Journal of the American Statistical Association, pages 1-15.

Le Cam, L. (1972). Limits of experiments. In Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, volume 1, pages 245-261. University of California Press Berkeley-Los Angeles.

LeCam, L. (1973). Convergence of estimates under dimensionality restrictions. The Annals of Statistics, pages 38-53.

Lehmann, E. L. and Casella, G. (2006). Theory of point estimation. Springer Science \& Business Media.

Lepski, O., Nemirovski, A., and Spokoiny, V. (1999). On estimation of the 1 r norm of a regression function. Probability theory and related fields, 113:221-253.

Levit, B. (2010). Minimax revisited. i. Mathematical Methods of Statistics, 19:283-297.
Lin, Y., Martin, R., and Yang, M. (2019). On optimal designs for nonregular models. The Annals of Statistics, 47(6):3335-3359.

Malỳ, J. (1999). Absolutely continuous functions of several variables. Journal of mathematical analysis and applications, 231(2):492-508.

Mou, W., Wainwright, M. J., and Bartlett, P. L. (2022). Off-policy estimation of linear functionals: Non-asymptotic theory for semi-parametric efficiency. arXiv preprint arXiv:2209.13075.

Nemirovski, A. (2000). Topics in non-parametric. Ecole d'Eté de Probabilités de Saint-Flour, 28:85.
Pollard, D. (2023+). Asymptotia (book in progress), Chapter 6 on "Hellinger differentiability".
Polyanskiy, Y. and Wu, Y. (2022). Information Theory: From Coding to Learning. Cambridge University Press.

Pötscher, B. M. (2002). Lower risk bounds and properties of confidence sets for ill-posed estimation problems with applications to spectral density and persistence estimation, unit roots, and estimation of long memory parameters. Econometrica, 70(3):1035-1065.

Radhakrishna Rao, C. (1945). Information and accuracy attainable in the estimation of statistical parameters. Bulletin of the Calcutta Mathematical Society, 37(3):81-91.

Sclove, S. L., Morris, C., and Radhakrishnan, R. (1972). Non-optimality of preliminary-test estimators for the mean of a multivariate normal distribution. The Annals of Mathematical Statistics, pages 1481-1490.

Shemyakin, A. (2014). Hellinger distance and non-informative priors.
Shen, Y., Gao, C., Witten, D., and Han, F. (2020). Optimal estimation of variance in nonparametric regression with random design.

Simons, G. and Woodroofe, M. (1983). The cramér-rao inequality holds almost everywhere. In Recent Advances in Statistics, pages 69-93. Elsevier.

Song, K. (2014a). Local asymptotic minimax estimation of nonregular parameters with translationscale equivariant maps. Journal of Multivariate Analysis, 125:136-158.

Song, K. (2014b). Minimax estimation of nonregular parameters and discontinuity in minimax risk. In Essays in Honor of Peter CB Phillips, volume 33, pages 557-585. Emerald Group Publishing Limited.

Šremr, J. (2010). Absolutely continuous functions of two variables in the sense of carathéodory. Electronic Journal of Differential Equations (EJDE)[electronic only], 2010:Paper-No.

Tsybakov, A. B. (2008). Introduction to Nonparametric Estimation. Springer Publishing Company, Incorporated, 1st edition.

Uhrmann-Klingen, E. (1995). Minimal fisher information distributions with compact-supports. Sankhyā: The Indian Journal of Statistics, Series A, pages 360-374.

Valiant, G. and Valiant, P. (2011). Estimating the unseen: an n/log (n)-sample estimator for entropy and support size, shown optimal via new clts. In Proceedings of the forty-third annual ACM symposium on Theory of computing, pages 685-694.

Van der Vaart, A. W. (2000). Asymptotic statistics, volume 3. Cambridge university press.
Van der Vaart, A. W. (2002). Semiparametric statistics. Lecture Notes in Math.
Van Trees, H. L. (1968). Detection, estimation, and modulation theory, part I: detection, estimation, and linear modulation theory. John Wiley \& Sons.

Weinstein, E. and Weiss, A. J. (1988). A general class of lower bounds in parameter estimation. IEEE Trans. Inf. Theory, 34:338-342.

Weiss, A. and Weinstein, E. (1985). A lower bound on the mean-square error in random parameter estimation (corresp.). IEEE Transactions on information theory, 31(5):680-682.

Woodroofe, M. (1970). On choosing a delta-sequence. The Annals of Mathematical Statistics, 41(5):1665-1671.

Wu, Y. and Yang, P. (2019). Chebyshev polynomials, moment matching, and optimal estimation of the unseen. The Annals of Statistics, 47(2):857-883.

Wu, Y. and Yang, P. (2020). Polynomial methods in statistical inference: Theory and practice. Foundations and Trends in Communications and Information Theory, 17(4):402-586.

Ziv, J. and Zakai, M. (1969). Some lower bounds on signal parameter estimation. IEEE transactions on Information Theory, 15(3):386-391.

## Supplement A Proof of the lower bound based on the approximation

We briefly review the setting and notation to which we frequently refer. Suppose we observe a random vector $X$ from an unknown distribution $P_{\theta}$, which belongs to a statistical model $\mathcal{P}:=\left\{P_{t}\right.$ : $t \in \Theta\}$ defined on a measurable space $(\mathcal{X}, \mathcal{A})$. Here, $\mathcal{A}$ denotes $\sigma$-algebra on $\mathcal{X}$. Let $\psi: \mathcal{P} \mapsto \mathbb{R}^{k}$ denote the vector-valued function of interest. In other words, the target estimand is described as the evaluation of the functional at the population parameter $\psi\left(P_{\theta}\right)$.

## A. 1 Proof of Lemma 7

Let $(\Theta, \mathcal{T})$ be a measurable space with respect to the base measure $\nu$ where $\mathcal{T}$ is a $\sigma$-algebra on $\Theta$. We denote by $\mathcal{Q} \equiv \mathcal{Q}(\Theta)$ be a collection of probability measures on $\Theta$ equipped with a density function with respect to $\mu$. We define $\Phi:=\left\{\phi: \mathcal{P} \mapsto \mathbb{R}^{k}\right\}$ to be a collection of arbitrary functionals that maps from $\mathcal{P}$ to $\mathbb{R}^{k}$. For each $Q \in \mathcal{Q}$ and $\phi \in \Phi$, it follows that

$$
\begin{aligned}
& \left(\int \mathbb{E}_{\theta}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2} \\
& \quad=\left(\int \mathbb{E}_{\theta}\left\|T(X)-\phi\left(P_{\theta}\right)+\phi\left(P_{\theta}\right)-\psi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2} \\
& \quad \geq\left(\int \mathbb{E}_{\theta}\left\|T(X)-\phi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2}-\left(\int\left\|\phi\left(P_{\theta}\right)-\psi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2}
\end{aligned}
$$

by the reverse triangle inequality. When the lower bound is negative, we replace with the trivial lower bound of zero. As the choice of $\phi$ was arbitrary, we conclude by taking the supremum over $\phi$ such that

$$
\begin{aligned}
\sup _{\theta \in \Theta} & \mathbb{E}_{\theta}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} \\
& \geq \sup _{\phi \in \Phi, Q \in \mathcal{Q}}\left[\left(\int \mathbb{E}_{\theta}\left\|T(X)-\phi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2}-\left(\int\|\psi(\theta)-\phi(\theta)\|^{2} d Q(\theta)\right)^{1 / 2}\right]_{+}^{2} .
\end{aligned}
$$

## A. 2 Proof of Theorem 8

Next, we prove Theorem 8 which follows directly by applying the van Trees inequality. We recall the necessary regularity conditions for the van Trees inequality according to Gassiat and Stoltz (2024). Finally, $q(\theta) \rightarrow 0$ as $\theta$ approaches any boundary point of $\Theta_{0}$ with finite norm along some canonical direction. A collection of probability measures $\mathcal{Q}^{\dagger}$ supported on $\Theta_{0}$ satisfies Definition 1. Note that $\mathcal{Q}^{\dagger}$ depends on $\phi$. We then state the following result:

Proof of Theorem 8. For any absolutely continuous function $\phi$, we evoke the multivariate van Trees inequality (Theorem 12 of Gassiat and Stoltz (2024)) under Definition 1 and the Hellinger differentiability of the statistical model $P_{t}$ for all $t \in \Theta_{0}$. For any vector norm $\|\cdot\|$, let $\|\cdot\|_{*}$ be its
dual. For any vector $u$ such that $\|u\|_{*} \leq 1$, we have

$$
\begin{align*}
\int \mathbb{E}_{\theta}\|T(X)-\phi(\theta)\|^{2} d Q & \geq \int_{\Theta_{0}} \mathbb{E}_{\theta}\left|u^{\top}\left(T(X)-\psi\left(P_{\theta}\right)\right)\right|^{2} d Q  \tag{21}\\
& \geq u^{\top}\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right)^{\top}\left(\mathcal{I}(Q)+\int_{\Theta_{0}} \mathcal{I}(t) d Q\right)^{-1}\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right) u \tag{22}
\end{align*}
$$

by the van Trees inequality (i.e., Theorem 12 of Gassiat and Stoltz (2024)). Combining this result with Lemma 7, we have

$$
\sup _{\theta \in \Theta_{0}} \mathbb{E}_{\theta}\left\|T(X)-\psi\left(P_{\theta}\right)\right\|^{2} \geq\left[\left(u^{\top} \Gamma_{Q, \phi} u\right)^{1 / 2}-\left(\int\left\|\phi\left(P_{\theta}\right)-\psi\left(P_{\theta}\right)\right\|^{2} d Q\right)^{1 / 2}\right]_{+}^{2}
$$

where

$$
\begin{equation*}
\Gamma_{Q, \phi}:=\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right)^{\top}\left(\mathcal{I}(Q)+\int_{\Theta_{0}} \mathcal{I}(t) d Q\right)^{-1}\left(\int_{\Theta_{0}} \nabla \phi(t) d Q\right) . \tag{23}
\end{equation*}
$$

This holds for any $\phi \in \Phi_{\mathrm{ac}}, Q \in \mathcal{Q}^{\dagger}$ and $u$ such that $\|u\|_{*} \leq 1$.

## Supplement B Proofs of the extensions of the van Trees inequality

Suppose we observe a sequence $X$ from an unknown distribution $P_{\theta}$, which belongs to a statistical model $\mathcal{P}:=\left\{P_{t}: t \in \mathbb{R}^{d}\right\}$ defined on a measurable space $(\mathcal{X}, \mathcal{A})$. Let $\psi: \mathbb{R}^{d} \mapsto \mathbb{R}^{k}$ denote the vector-valued function of interest, and let $Q$ be a prior distribution on $\mathbb{R}^{d}$ with a density function. We define two probability measures on a product space $\left(\mathcal{X} \times \mathbb{R}^{d}\right)$. Specifically, we let

$$
d \mathbb{P}_{0}(x, \theta):=d P_{\theta}(x) d Q(\theta) \quad \text { and } \quad d \mathbb{P}_{h}(x, \theta):=d P_{\theta+h}(x) d Q(\theta+h) .
$$

## B. 1 Proof of Theorem 9

For each $\lambda \in[0,1]$, we define a mixture distribution as $\mathbb{P}_{h}^{\lambda}:=(1-\lambda) \mathbb{P}_{0}+\lambda \mathbb{P}_{h}$ with the corresponding density function given by

$$
d \mathbb{P}_{h}^{\lambda}(x, \theta):=(1-\lambda) d \mathbb{P}_{0}(x, \theta)+\lambda d \mathbb{P}_{h}(x, \theta) .
$$

Throughout the proof, we denote by $\mathbb{E}_{\theta}$ the expectation under $P_{\theta}$ with a fixed parameter $\theta$, by $\mathbb{E}_{\mathbb{P}_{h}}$ the expectation under the joint probability measure $\mathbb{P}_{h}$, and by $\mathbb{E}_{\mathbb{P}_{h}^{\lambda}}$ the expectation under the joint mixture probability measure $\mathbb{P}_{h}^{\lambda}$. For any measurable functions $\psi: \mathbb{R}^{d} \mapsto \mathbb{R}^{k}, T: \mathcal{X} \mapsto \mathbb{R}^{k}$ and $h \in \mathbb{R}^{d}$, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_{h}}(T(X)-\psi(\theta)) & =\iint_{\mathcal{X} \times \mathbb{R}^{d}}(T(x)-\psi(t)) d \mathbb{P}_{h}(x, t) \\
& =\iint_{\mathcal{X} \times \mathbb{R}^{d}}(T(x)-\psi(t)) d P_{t+h}(x) d Q(t+h) \\
& =\iint_{\mathcal{X} \times\left(\mathbb{R}^{d}-h\right)}(T(x)-\psi(u-h)) d P_{u}(x) d Q(u)
\end{aligned}
$$

where $\mathbb{R}^{d}-h$ is the set $\left\{u-h: u \in \mathbb{R}^{d}\right\}$. In particular, we have $\mathbb{R}^{d}-h=\mathbb{R}^{d}$. We then obtain

$$
\mathbb{E}_{\mathbb{P}_{h}}(T(X)-\psi(\theta))=\int_{\mathbb{R}^{d}} \mathbb{E}_{t} T(X) d Q(t)-\int_{\mathbb{R}^{d}} \psi(u-h) d Q(u) .
$$

Similarly under the mixture distribution $\mathbb{P}_{h}^{\lambda}$, we have

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}_{h}^{\lambda}}( T(X)-\psi(\theta)) \\
&= \iint_{\mathcal{X} \times \mathbb{R}^{d}}(T(x)-\psi(t)) d \mathbb{P}_{h}^{\lambda}(x, t) \\
&=(1-\lambda) \iint_{\mathcal{X} \times \mathbb{R}^{d}}(T(x)-\psi(t)) d \mathbb{P}_{0}(x, t)+\lambda \iint_{\mathcal{X} \times \mathbb{R}^{d}}(T(x)-\psi(t)) d \mathbb{P}_{h}(x, t) \\
&=(1-\lambda) \iint_{\mathcal{X} \times \mathbb{R}^{d}}(T(x)-\psi(t)) d P_{t}(x) d Q(t) \\
&+\lambda \iint_{\mathcal{X} \times\left(\mathbb{R}^{d}-h\right)}(T(x)-\psi(u-h)) d P_{u}(x) d Q(u) \\
&=(1-\lambda) \int_{\mathbb{R}^{d}} \mathbb{E}_{t} T(X) d Q(t)+\lambda \int_{\mathbb{R}^{d}} \mathbb{E}_{t} T(X) d Q(t)-(1-\lambda) \int_{\mathbb{R}^{d}} \psi(u) d Q(u) \\
& \quad-\lambda \int_{\mathbb{R}^{d}} \psi(u-h) d Q(u) \\
&= \int_{\mathbb{R}^{d}} \mathbb{E}_{t} T(X) d Q(t)-(1-\lambda) \int_{\mathbb{R}^{d}} \psi(u) d Q(u)-\lambda \int_{\mathbb{R}^{d}} \psi(u-h) d Q(u) .
\end{aligned}
$$

Therefore, it follows

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}_{h}}(T(X)-\psi(\theta))-\mathbb{E}_{\mathbb{P}_{h}^{\lambda}}(T(X)-\psi(\theta))=(1-\lambda)\left(\int_{\mathbb{R}^{d}} \psi(u) d Q(u)-\int_{\mathbb{R}^{d}} \psi(u-h) d Q(u)\right) . \tag{24}
\end{equation*}
$$

Next, we consider the following ratio of the joint probability over $\mathcal{X} \times \mathbb{R}^{d}$ given by

$$
D_{h, \lambda}:(x, t) \mapsto \frac{d \mathbb{P}_{h}(x, t)-d \mathbb{P}_{h}^{\lambda}(x, t)}{d \mathbb{P}_{h}^{\lambda}(x, t)}
$$

It follows from (24) that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_{h}^{\lambda}} D_{h, \lambda}(X, \theta)(T(X)-\psi(\theta)) & =\mathbb{E}_{\mathbb{P}_{h}}(T(X)-\psi(\theta))-\mathbb{E}_{\mathbb{P}_{h}^{\lambda}}(T(X)-\psi(\theta)) \\
& =(1-\lambda) \int_{\mathbb{R}^{d}}(\psi(u)-\psi(u-h)) d Q(u) .
\end{aligned}
$$

Applying Cauchy-Schwarz inequality to the dual norm of the above display, we obtain

$$
\begin{align*}
(1-\lambda)^{2}\left\|\int_{\mathbb{R}^{d}}(\psi(t)-\psi(t-h)) d Q(t)\right\|^{2} & =\left\|\mathbb{E}_{\mathbb{P}_{h}^{\lambda}} D_{h, \lambda}(X, \theta)(T(X)-\psi(\theta))\right\|^{2} \\
& =\sup _{a:\|a\| \leq 1}\left|\mathbb{E}_{\mathbb{P}_{h}^{\lambda}} D_{h, \lambda}(X, \theta) a^{\top}(T(X)-\psi(\theta))\right|^{2} \\
& \leq\left\{\mathbb{E}_{\mathbb{P}_{h}^{\lambda}} D_{h, \lambda}^{2}(X, \theta)\right\} \sup _{a:\|a\| \leq 1}\left\{\mathbb{E}_{\mathbb{P}_{h}^{\lambda}}\left|a^{\top}(T(X)-\psi(\theta))\right|^{2}\right\} \\
& \leq\left\{\mathbb{E}_{\mathbb{P}_{h}^{\lambda}} D_{h, \lambda}^{2}(X, \theta)\right\}\left\{\mathbb{E}_{\mathbb{P}_{h}^{\lambda}}\|T(X)-\psi(\theta)\|^{2}\right\} . \tag{25}
\end{align*}
$$

We now analyze each term in the last expression. First, by the definition of the chi-squared divergence, we have

$$
\mathbb{E}_{\mathbb{P}_{h}^{\lambda}} D^{2}(X, \theta)=\iint_{\mathcal{X} \times \mathbb{R}^{d}}\left(\frac{d \mathbb{P}_{h}(x, t)-d \mathbb{P}_{h}^{\lambda}(x, t)}{d \mathbb{P}_{h}^{\lambda}(x, t)}\right)^{2} d \mathbb{P}_{h}^{\lambda}(x, t)=\chi^{2}\left(\mathbb{P}_{h} \| \lambda \mathbb{P}_{h}+(1-\lambda) \mathbb{P}_{0}\right) .
$$

Next, for the second term of the upper bound in (25), we have

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}_{h}^{\lambda}} \|\|(X)-\psi(t)\|^{2} \\
&= \iint_{\mathcal{X} \times \mathbb{R}^{d}}\|T(x)-\psi(t)\|^{2} d \mathbb{P}_{h}^{\lambda}(x, t) \\
&=(1-\lambda) \iint_{\mathcal{X} \times \mathbb{R}^{d}}\|T(x)-\psi(t)\|^{2} d P_{t}(x) d Q(t) \\
& \quad+\lambda \iint_{\mathcal{X} \times \mathbb{R}^{d}}\|T(x)-\psi(t)\|^{2} d P_{t+h}(x) d Q(t+h) \\
&=(1-\lambda) \iint_{\mathcal{X} \times \mathbb{R}^{d}}\|T(x)-\psi(t)\|^{2} d P_{t}(x) d Q(t) \\
&+\lambda \iint_{\mathcal{X} \times \mathbb{R}^{d}}\|T(x)-\psi(t)-\psi(t+h)+\psi(t+h)\|^{2} d P_{t+h}(x) d Q(t+h) \\
& \leq(1-\lambda) \int_{\mathbb{R}^{d}} \mathbb{E}_{t}\|T(X)-\psi(t)\|^{2} d Q(t) \\
& \quad+\lambda\left((1+L) \int_{\mathbb{R}^{d}-h} \mathbb{E}_{u}\|T(X)-\psi(u)\|^{2} d Q(u)\right. \\
& \quad\left.\quad+(1+1 / L) \int_{\mathbb{R}^{d}-h}\|\psi(u)-\psi(u-h)\|^{2} d Q(u)\right) \\
&=(1+L \lambda) \int_{\mathbb{R}^{d}} \mathbb{E}_{t}\|T(X)-\psi(t)\|^{2} d Q(t)+\lambda(1+1 / L) \int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)
\end{aligned}
$$

where we use $(a+b)^{2} \leq(1+L) a^{2}+(1+1 / L) b^{2}$ for any $L \geq 0$, which follows from $2 a b \leq a^{2} L+b^{2} / L$. Putting all intermediate results together, the inequality (25) implies

$$
\begin{gathered}
(25) \Longrightarrow \frac{(1-\lambda)^{2}\left\|\int_{\mathbb{R}^{d}}(\psi(t)-\psi(t-h)) d Q(t)\right\|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \lambda \mathbb{P}_{h}+(1-\lambda) \mathbb{P}_{0}\right)} \\
\leq(1+L \lambda) \int_{\mathbb{R}^{d}} \mathbb{E}_{t}\|T(X)-\psi(t)\|^{2} d Q(t)+\lambda(1+1 / L) \int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t) \\
\Longrightarrow \int_{\mathbb{R}^{d}} \mathbb{E}_{t}\|T(X)-\psi(t)\|^{2} d Q(t) \\
\geq \frac{1}{1+L \lambda}\left[\frac{(1-\lambda)^{2}\left\|\int_{\mathbb{R}^{d}}(\psi(t)-\psi(t-h)) d Q(t)\right\|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \lambda \mathbb{P}_{h}+(1-\lambda) \mathbb{P}_{0}\right)}\right. \\
\left.\quad-\lambda(1+1 / L) \int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)\right]_{+}
\end{gathered}
$$

Since the Bayes risk is bounded by minimax risk for any prior distribution $Q$, we have

$$
\begin{aligned}
\sup _{\theta \in \mathbb{R}^{d}} & \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \\
& \geq \int_{\mathbb{R}^{d}} \mathbb{E}_{t}\|T(X)-\psi(t)\|^{2} d Q(t) \\
& \geq \frac{(1-\lambda)^{2}}{1+L \lambda}\left[\frac{\left\|\int_{\mathbb{R}^{d}}(\psi(t)-\psi(t-h)) d Q(t)\right\|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \lambda \mathbb{P}_{h}+(1-\lambda) \mathbb{P}_{0}\right)}-\frac{\lambda(1+1 / L)}{(1-\lambda)^{2}} \int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)\right]_{+} .
\end{aligned}
$$

As the introduced parameter $L \geq 0$ is arbitrary, we can optimize its choice. First we observe that the lower bound is in the form

$$
\frac{(1-\lambda)^{2}}{1+L \lambda}\left[A-\frac{\lambda(1+1 / L)}{(1-\lambda)^{2}} B\right]_{+} .
$$

where

$$
A:=\frac{\left\|\int_{\mathbb{R}^{d}}(\psi(t)-\psi(t-h)) d Q(t)\right\|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \lambda \mathbb{P}_{h}+(1-\lambda) \mathbb{P}_{0}\right)} \quad \text { and } \quad B:=\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)
$$

When $\lambda=0$, the above display takes $A^{2}$ regardless of the value of $L$. The argument for $B=0$ is also similar. Hence we focus on the case with $\lambda, B>0$. First we observe that

$$
\frac{(1-\lambda)^{2}}{1+L \lambda}\left[A-\frac{\lambda(1+1 / L)}{(1-\lambda)^{2}} B\right]_{+}=\frac{\lambda^{2} B}{1+L \lambda}\left[\left(\frac{(1-\lambda)^{2} A}{\lambda^{2} B}-\frac{1}{\lambda}\right)-\frac{1}{L \lambda}\right]_{+}=\frac{\lambda^{2} B}{1+L \lambda}\left[\Gamma-\frac{1}{L \lambda}\right]_{+}
$$

where

$$
\Gamma:=\frac{(1-\lambda)^{2} A}{\lambda^{2} B}-\frac{1}{\lambda} .
$$

It is thus equivalent to optimize

$$
\max _{\ell \geq 0}\left\{\frac{1}{1+\ell}\left(\Gamma-\frac{1}{\ell}\right)\right\} .
$$

We later check if the attained maximum is positive otherwise replace the optima with zero. The optimal value of $\ell^{*}$ is given by

$$
\ell^{*}=\frac{1+\sqrt{1+\Gamma}}{\Gamma} \quad \text { when } \quad \Gamma>0
$$

The corresponding optima is

$$
\begin{aligned}
\frac{1}{1+\ell^{*}}\left(\Gamma-\frac{1}{\ell^{*}}\right) & =\frac{\Gamma}{\Gamma+1+\sqrt{1+\Gamma}}\left(\Gamma-\frac{\Gamma}{1+\sqrt{1+\Gamma}}\right) \\
& =\frac{\Gamma^{2} \sqrt{1+\Gamma}}{\sqrt{1+\Gamma}(\sqrt{\Gamma+1}+1)^{2}} \\
& =\frac{(\sqrt{\Gamma+1}-1)^{2}(\sqrt{\Gamma+1}+1)^{2}}{(\sqrt{\Gamma+1}+1)^{2}} \\
& =(\sqrt{\Gamma+1}-1)^{2} .
\end{aligned}
$$

This is non-negative when $\Gamma>0$ and thus it is a valid optimum for $(1+\ell)^{-1}[\Gamma-1 / \ell]_{+}$. Plugging this result into the original expression, we obtain

$$
\lambda^{2} B\left(\sqrt{\frac{(1-\lambda)^{2} A}{\lambda^{2} B}-\frac{1}{\lambda}+1}-1\right)^{2}=\left(\sqrt{(1-\lambda)\{A-\lambda(A+B)\}}-\sqrt{\lambda^{2} B}\right)^{2} .
$$

When $\Gamma \leq 0$, or equivalently $(1-\lambda)^{2} A \leq \lambda B$, the lower bound becomes zero. As $T$ and $h$ only appear on one side of the inequality, we conclude the claim by taking the infimum over $T$ and the supremum over $h \in \mathbb{R}^{d}$.

## B. 2 Proof of Theorem 10

Following the analogous derivation leading up to equation (24) given by the proof of Theorem 9 with the special case of $\lambda=0$, we obtain

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}_{h}}(T(X)-\psi(\theta))-\mathbb{E}_{\mathbb{P}_{0}}(T(X)-\psi(\theta))=\int_{\mathbb{R}^{d}}(\psi(u)-\psi(u-h)) d Q(u) \tag{26}
\end{equation*}
$$

Next, we consider the density ratio of the joint probability over $\left(\mathcal{X} \times \mathbb{R}^{d}\right)$ given by

$$
D_{h}:=(x, t) \mapsto \frac{d \mathbb{P}_{h}(x, t)-d \mathbb{P}_{0}(x, t)}{d \mathbb{P}_{0}(x, t)} .
$$

By the application of Cauchy-Schwarz inequality to (26), we obtain

$$
\begin{align*}
\left\|\int_{\mathbb{R}^{d}}(\psi(u)-\psi(u-h)) d Q(u)\right\|^{2} & =\left\|\mathbb{E}_{\mathbb{P}_{0}} D_{h}(X, \theta)(T(X)-\psi(\theta))\right\|^{2} \\
& =\left\|\mathbb{E}_{\mathbb{P}_{0}}\left(\frac{d \mathbb{P}_{h}(X, \theta)-d \mathbb{P}_{0}(X, \theta)}{d \mathbb{P}_{0}(X, \theta)}\right)(T(X)-\psi(\theta))\right\|^{2} \\
& =\left\|\mathbb{E}_{\mathbb{P}_{0}}\left(\sqrt{\frac{d \mathbb{P}_{h}(X, \theta)}{d \mathbb{P}_{0}(X, \theta)}}-1\right)\left(\sqrt{\frac{d \mathbb{P}_{h}(X, \theta)}{d \mathbb{P}_{0}(X, \theta)}}+1\right)(T(X)-\psi(\theta))\right\|^{2} \\
& \leq\left\{\mathbb{E}_{\mathbb{P}_{0}}\left(\sqrt{\frac{d \mathbb{P}_{h}(X, \theta)}{d \mathbb{P}_{0}(X, \theta)}}+1\right)^{2}\|T(X)-\psi(\theta)\|^{2}\right\} H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right) \\
& \leq 2\left(\mathbb{E}_{\mathbb{P}_{0}}\|T(X)-\psi(\theta)\|^{2}+\mathbb{E}_{\mathbb{P}_{h}}\|T(X)-\psi(\theta)\|^{2}\right) H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right) \tag{27}
\end{align*}
$$

where we apply the inequality $(a+b)^{2} \leq 2\left(a^{2}+b^{2}\right)$ in the last step. We now analyze two terms inside the parenthesis of the above display. Since minimax risk gives an upper bound of Bayes risk, we have

$$
\mathbb{E}_{\mathbb{P}_{0}}\|T(X)-\psi(\theta)\|^{2}=\int_{\mathbb{R}^{d}} \mathbb{E}_{t}\|T(X)-\psi(t)\|^{2} d Q(t) \leq \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2}
$$

For the second term, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_{h}} \| & T(X)-\psi(\theta) \|^{2} \\
= & \int_{\mathbb{R}^{d}} \mathbb{E}_{t+h}\|T(X)-\psi(t)\|^{2} d Q(t+h) \\
= & \int_{\mathbb{R}^{d}} \mathbb{E}_{t+h}\|T(X)-\psi(t)-\psi(t+h)+\psi(t+h)\|^{2} d Q(t+h) \\
\leq & \int_{\mathbb{R}^{d}} \mathbb{E}_{t+h}\|T(X)-\psi(t+h)\|^{2} d Q(t+h) \\
& \quad+\int_{\mathbb{R}^{d}}\|\psi(t+h)-\psi(t)\|^{2} d Q(t+h) \\
& \quad+2 \sqrt{\int_{\mathbb{R}^{d}} \mathbb{E}_{t+h}\|T(X)-\psi(t+h)\|^{2} d Q(t+h)} \sqrt{\int_{\mathbb{R}^{d}}\|\psi(t+h)-\psi(t)\|^{2} d Q(t+h)} \\
= & \left(\sqrt{\int_{\mathbb{R}^{d}} \mathbb{E}_{t+h}\|T(X)-\psi(t+h)\|^{2} d Q(t+h)}+\sqrt{\int_{\mathbb{R}^{d}}\|\psi(t+h)-\psi(t)\|^{2} d Q(t+h)}\right)^{2}
\end{aligned}
$$

where the inequality follows from Cauchy-Schwarz inequality. Therefore (27) implies the following:

$$
\begin{aligned}
&(27) \Longrightarrow \frac{\left\|\int_{\mathbb{R}^{d}}(\psi(u)-\psi(u-h)) d Q(u)\right\|^{2}}{2 H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)} \\
& \leq \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \\
&+\left(\sqrt{\int_{\mathbb{R}^{d}} \mathbb{E}_{t+h}\|T(X)-\psi(t+h)\|^{2} d Q(t+h)}+\sqrt{\int_{\mathbb{R}^{d}}\|\psi(t+h)-\psi(t)\|^{2} d Q(t+h)}\right)^{2} \\
& \leq \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \\
&+\left(\sqrt{\sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2}}+\sqrt{\int_{\mathbb{R}^{d}-h}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)}\right)^{2} \\
& \leq 2\left(\sqrt{\sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2}}+\sqrt{\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)}\right)^{2}
\end{aligned}
$$

where the last step used the fact that $\sqrt{\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)} \geq 0$. We thus obtain

$$
\begin{gathered}
\frac{\left\|\int_{\mathbb{R}^{d}}(\psi(u)-\psi(u-h)) d Q(u)\right\|^{2}}{2 H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)} \\
\leq 2\left(\sqrt{\sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2}}+\sqrt{\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)}\right)^{2} \\
\Longrightarrow \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}\|T(X)-\psi(\theta)\|^{2} \\
\geq\left[\frac{\left\|\int_{\mathbb{R}^{d}}(\psi(u)-\psi(u-h)) d Q(u)\right\|}{2 H\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)}-\left(\int_{\mathbb{R}^{d}}\|\psi(t)-\psi(t-h)\|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} .
\end{gathered}
$$

As with the proof of Theorem 9 , the function $T$ and $h$ only appear on one side of the inequality, and we conclude the claim by taking the infimum over $T$ and the supremum over $h \in \Theta$.

## B. 3 Proof of Lemma 11

The main idea is provided by Theorem 6.1 of Ibragimov and Has'minskii (1981). Here, we derive the improved constant. We focus on the estimation of the real-valued functional $\psi: \Theta \mapsto \mathbb{R}$ for $\Theta \subseteq \mathbb{R}^{d}$. An analogous proof can be applied to each specific choice of the vector norm $\|\cdot\|$ (See remark 6.1 of Ibragimov and Has'minskii (1981)). For any real-valued measurable function $T:=\mathcal{X} \mapsto \mathbb{R}$, two points in the parameter space $\{\theta, \theta+h\} \in \Theta$, and an arbitrary scalar constant
$C$, we have

$$
\begin{align*}
\left|\mathbb{E}_{\theta+h} T(X)-\mathbb{E}_{\theta} T(X)\right|^{2} & =\left|\int_{\mathcal{X}}(T(x)-C)\left(d P_{\theta+h}(x)-d P_{\theta}(x)\right)\right|^{2} \\
& =\left|\int_{\mathcal{X}}(T(x)-C)\left(d P_{\theta+h}^{1 / 2}(x)-d P_{\theta}^{1 / 2}(x)\right)\left(d P_{\theta+h}^{1 / 2}(x)+d P_{\theta}^{1 / 2}(x)\right)\right|^{2} \\
& \leq 2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)\left(\int_{\mathcal{X}}|T(x)-C|^{2} d P_{\theta+h}(x)+\int_{\mathcal{X}}|T(x)-C|^{2} d P_{\theta}(x)\right) . \tag{28}
\end{align*}
$$

where the last inequalities follow by Cauchy-Schwarz inequality and $(a+b)^{2} \leq 2\left(a^{2}+b^{2}\right)$. The first integral on the right-hand side can be written out as

$$
\begin{aligned}
\int_{\mathcal{X}}|T(x)-C|^{2} d P_{\theta+h}(x) & =\int_{\mathcal{X}}\left|T(x)-\mathbb{E}_{\theta+h} T(X)+\mathbb{E}_{\theta+h} T(X)-C\right|^{2} d P_{\theta+h}(x) \\
& =\int_{\mathcal{X}}\left|T(x)-\mathbb{E}_{\theta+h} T(X)\right|^{2} d P_{\theta+h}(x)+\left|\mathbb{E}_{\theta+h} T(X)-C\right|^{2} .
\end{aligned}
$$

We also have the following standard bias-variance decomposition:

$$
\int_{\mathcal{X}}\left|T(x)-\mathbb{E}_{\theta+h} T(X)\right|^{2} d P_{\theta+h}(x)=\mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}-\left|\mathbb{E}_{\theta+h} T(X)-\psi(\theta+h)\right|^{2} .
$$

Putting together, we obtain

$$
\begin{aligned}
& \int_{\mathcal{X}}|T(x)-C|^{2} d P_{\theta+h}(x) \\
& \quad=\mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}-\left|\mathbb{E}_{\theta+h} T(X)-\psi(\theta+h)\right|+\left|\mathbb{E}_{\theta+h} T(X)-C\right|^{2}
\end{aligned}
$$

By repeating the analogous argument for $\int_{\mathcal{X}}|T(x)-C|^{2} d P_{\theta}(x)$ and plugging them into (28), we obtain

$$
\begin{aligned}
& \left|\mathbb{E}_{\theta+h} T(X)-\mathbb{E}_{\theta} T(X)\right|^{2} \\
& \quad \leq 2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)\left(\mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}-\left|\mathbb{E}_{\theta+h} T(X)-\psi(\theta+h)\right|^{2}\right. \\
& \left.\quad+\left|\mathbb{E}_{\theta+h} T(X)-C\right|^{2}+\mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2}-\left|\mathbb{E}_{\theta} T(X)-\psi(\theta)\right|^{2}+\left|\mathbb{E}_{\theta} T(X)-C\right|^{2}\right)
\end{aligned}
$$

Since the above inequality holds for an arbitrary scalar constant $C$, we choose $C$ to minimize the upper bound. As the optimal $C^{*}$ is attained by $C^{*}=1 / 2\left(\mathbb{E}_{\theta} T(X)+\mathbb{E}_{\theta+h} T(X)\right)$, we can further simplify the expression as

$$
\begin{align*}
&\left|\mathbb{E}_{\theta+h} T(X)-\mathbb{E}_{\theta} T(X)\right|^{2} \leq 2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)\left(\mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+\mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2}\right. \\
&\left.-|d(\theta+h)|^{2}-|d(\theta)|^{2}+\frac{\left|\mathbb{E}_{\theta} T(X)-\mathbb{E}_{\theta+h} T(X)\right|^{2}}{2}\right) \tag{29}
\end{align*}
$$

where $d(t):=\mathbb{E}_{t} T(X)-\psi(t)$. Hence, the above display immediately implies the following:

$$
\begin{aligned}
(29) \Longrightarrow & \mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+\mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \\
& \geq \frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)}\left|\mathbb{E}_{\theta+h} T(X)-\mathbb{E}_{\theta} T(X)\right|^{2}+|d(\theta+h)|^{2}+|d(\theta)|^{2} \\
& \Longrightarrow \mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+\mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \\
& \geq \frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)}|\psi(\theta+h)-\psi(\theta)+d(\theta+h)-d(\theta)|^{2}+|d(\theta+h)|^{2}+|d(\theta)|^{2} .
\end{aligned}
$$

The existing proofs by Ibragimov and Has'minskii (1981) and Lin et al. (2019) proceed by splitting the analysis into two cases: (i) $\max \{|d(\theta+h)|,|d(\theta)|\}<1 / 4|\psi(\theta+h)-\psi(\theta)|$ and (ii) $|d(\theta)|>$ $1 / 4|\psi(\theta+h)-\psi(\theta)|$ where the leading constant $1 / 4$ for the boundary was chosen for convenience by Ibragimov and Has'minskii (1981) as they did not focus on the optimal constant. We instead optimize this boundary to obtain a sharper constant.

We now define the following function:

$$
\eta(x, y):=A|B+x-y|^{2}+|x|^{2}+|y|^{2}
$$

with $A:=\left(1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)\right) /\left(2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)\right)$ and $B:=\psi(\theta+h)-\psi(\theta)$. The lower bound can be written as

$$
E_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+E_{\theta}|T(X)-\psi(\theta)|^{2} \geq \eta(d(\theta+h), d(\theta)) .
$$

This further implies that

$$
E_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+E_{\theta}|T(X)-\psi(\theta)|^{2} \geq \min _{x, y \in \mathbb{R}} \eta(x, y) .
$$

The minimizer of $\eta$ is given by $x^{*}=-y^{*}=-\frac{2 A B}{4 A+2}$ and hence we have,

$$
\min _{x, y} \eta(x, y)=\frac{4 A B^{2}}{(4 A+2)^{2}}+\frac{8 A^{2} B^{2}}{(4 A+2)^{2}}=\frac{4 A B^{2}(1+2 a)}{(4 A+2)^{2}}=\frac{A B^{2}}{2 A+1} .
$$

Additionally, we have

$$
\partial_{x x} \eta=2 A+2, \quad \partial_{x y} \eta=-2 A, \quad \text { and } \quad \partial_{y y} \eta=2 A+2
$$

and the discriminant is given by $(2 A+2)^{2}-4 A^{2}=4(2 A+1)$. Therefore, the optima is at a saddle point when $A<-1 / 2$. The constant $A$ is defined as $\left(1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)\right) /\left(2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)\right)$, which is monotone decreasing with respect to $H^{2}\left(P_{\theta+h}, P_{\theta}\right)$ and attains the minimum $-1 / 4$ at $H^{2}\left(P_{\theta+h}, P_{\theta}\right)=2$. Thus the minimizer of $\eta$ is well-defined for all values of $A$ and thereby all values of $H^{2}\left(P_{\theta+h}, P_{\theta}\right)$. Putting it together, the minimum of the function $\eta$ with respect to $d(\theta+h)$ and $d(\theta)$ is given by

$$
\begin{aligned}
\frac{A B^{2}}{2 A+1} & =\frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{2 H^{2}\left(P_{\theta+h}, P_{\theta}\right)}|\psi(\theta+h)-\psi(\theta)|^{2}\left(\frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{H^{2}\left(P_{\theta+h}, P_{\theta}\right)}+1\right)^{-1} \\
& =\frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{2}|\psi(\theta+h)-\psi(\theta)|^{2}
\end{aligned}
$$

When $H^{2}\left(P_{\theta+h}, P_{\theta}\right)>1$, the leading constant becomes negative and we can replace it with a trivial lower bound of zero. Therefore, we conclude

$$
\mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+\mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq\left[\frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{2}\right]_{+}|\psi(\theta+h)-\psi(\theta)|^{2} .
$$

Since minimax risk gives an upper bound of the average of risks at arbitrary two points, we have

$$
\begin{aligned}
\sup _{t \in \Theta} \mathbb{E}_{t}|T(X)-\psi(t)|^{2} & \geq \sup _{\{\theta, \theta+h\} \in \Theta} \frac{\mathbb{E}_{\theta+h}|T(X)-\psi(\theta+h)|^{2}+\mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2}}{2} \\
& \geq \sup _{\{\theta, \theta+h\} \in \Theta}\left[\frac{1-H^{2}\left(P_{\theta+h}, P_{\theta}\right)}{4}\right]_{+}|\psi(\theta+h)-\psi(\theta)|^{2} .
\end{aligned}
$$

As $T$ and $h$ only appear on one side of the inequality, we conclude the claim by taking the infimum over $T$ and the supremum over any pair of parameters $\theta, \theta+h \in \Theta$. An analogous proof can be extended to a vector-valued functional with a general vector norm $\|\cdot\|$. We only provide a result with a real-valued functional since the optimal constant depends on the choice of the norm as it requires the derivative with respect to the norm.

## Supplement C Proofs of asymptotic properties

This section provides the derivation of results from Section 4.

## C. 1 Proofs of local asymptotic minimax theorem

In this section, we use Lemma 7, Theorems 9, 10 and Lemma 11 to prove the local asymptotic minimax theorem. Throughout, we assume that $X_{1}, \ldots, X_{n}$ is an IID observation from $P_{\theta_{0}} \in\left\{P_{\theta}\right.$ : $\theta \in \Theta\}$ and each distribution in this model is Hellinger differentiable at $\theta$. Let $\psi: \Theta \mapsto \mathbb{R}$ be continuously differentiable at $\theta_{0}$ and $T: \mathcal{X}^{n} \mapsto \mathbb{R}$ be any sequence of measurable functions.

Proof of Proposition 1 (i). Since $\Phi$ is arbitrary in Lemma 7, we choose

$$
\Phi:=\left\{\phi: \Theta \mapsto \mathbb{R}: \phi \text { is continuously differentiable at } \theta_{0}\right\} .
$$

Since $\psi \in \Phi$, it follows that

$$
\begin{aligned}
\sup _{\theta \in \Theta_{0}} \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} & \geq \sup _{\phi \in \Phi, Q \in \mathcal{Q}}\left[\Gamma_{Q, \phi}^{1 / 2}-\left(\int\|\psi(\theta)-\phi(\theta)\|^{2} d Q(\theta)\right)^{1 / 2}\right]_{+}^{2} \\
& \geq\left(\int_{\Theta_{0}} \nabla \psi(t) d Q\right)^{\top}\left(\mathcal{I}(Q)+\int_{\Theta_{0}} \mathcal{I}(t) d Q\right)^{-1}\left(\int_{\Theta_{0}} \nabla \psi(t) d Q\right) .
\end{aligned}
$$

Section 4 of Gassiat and Stoltz (2024) shows that the van Trees inequality indeed recovers the optimal asymptotic constant of the LAM theorem. This concludes the claim.

Proof of Proposition 1 (iii). In this application, the parameter space considered is given by $\Theta:=B\left(\theta_{0}, c n^{-1 / 2}\right)$, which is an open $\mathbb{R}^{d}$-ball centered at $\theta_{0}$ with radius $c n^{-1 / 2}$. Following the construction of Section 3.4, a diffeomorphism $t \mapsto \varphi(t)=\theta_{0}+c n^{-1 / 2} \varphi_{0}(t)$ is defined where $\varphi_{0}$ is itself a diffeomorphism from $\mathbb{R}^{d}$ to $B([0], 1)$. It follows that $\nabla \varphi(t)=c n^{-1 / 2} \nabla \varphi_{0}(t)$. We further assume that $\left\|\nabla \varphi_{0}\right\|_{\infty}<C$ by some universal constant $C$, which is defined later. Theorem 10 is then applied to the composite function $t \mapsto \widetilde{\psi}(t):=(\psi \circ \varphi)(t)$, and the following inequality is obtained:

$$
\begin{align*}
\inf _{T} & \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\widetilde{\psi}(\theta)|^{2} \\
& \geq\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-h)) d Q(t)\right|}{2 H\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)}-\left(\int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} \tag{30}
\end{align*}
$$

for any $h \in \mathbb{R}^{d}$. Using the mean value theorem and the chain rule, it follows that

$$
\widetilde{\psi}(t)-\widetilde{\psi}(t-h)=c n^{-1 / 2} \nabla \psi\left(\lambda_{1} \varphi(t)+\left(1-\lambda_{1}\right) \varphi(t-h)\right)^{\top} \nabla \varphi_{0}\left(t-\lambda_{2} h\right)^{\top} h
$$

where $\lambda_{1}, \lambda_{2} \in[0,1]$ are constants that can possibly depend on $t$. Next, under the Hellinger differentiability and $n$ IID observations from $P_{\theta_{0}}$, it is shown by Lemma 22 in Supplementary Material that as $\|h\|_{2} \longrightarrow 0$,

$$
\begin{aligned}
H^{2}\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right) & =\frac{1}{4} h^{\top}\left(\mathcal{I}(Q)+n \int_{\mathbb{R}^{d}} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) d Q(t)\right) h+o\left(\|h\|_{2}^{2}\right) \\
& =\frac{1}{4} h^{\top}\left(\mathcal{I}(Q)+c^{2} \int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi_{0}(t) d Q(t)\right) h+o\left(\|h\|_{2}^{2}\right) .
\end{aligned}
$$

The results obtained thus far are now used to evaluate the expression (30). Since the inequality holds for any $h \in \mathbb{R}^{d}$, it also holds as $\|h\|_{2} \longrightarrow 0$. We now denote $h=u \varepsilon$ where $u \in \mathbb{S}^{d-1}$ and $\varepsilon \longrightarrow 0$. This yields the following lower bound:

$$
\begin{aligned}
\sup _{h \in \mathbb{R}^{d}} & {\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-h)) d Q(t)\right|}{2 H\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)}-\left(\int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} } \\
& \geq \sup _{u \in \mathbb{S}^{d-1}} \limsup _{\varepsilon \rightarrow 0}\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-u \varepsilon)) d Q(t)\right|}{2 H\left(\mathbb{P}_{0}, \mathbb{P}_{u \varepsilon}\right)}-\left(\int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-u \varepsilon)|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} \\
& =\sup _{u \in \mathbb{S}^{d-1}} \frac{c^{2} n^{-1}\left|\int_{\mathbb{R}^{d}} \nabla \psi(\varphi(t))^{\top} \nabla \varphi_{0}(t)^{\top} u d Q(t)\right|^{2}}{u^{\top}\left(\mathcal{I}(Q)+c^{2} \int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi_{0}(t) d Q(t)\right) u} .
\end{aligned}
$$

By multiplying both sides by $n$, we have for any $u \in \mathbb{S}^{d-1}$,

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\widetilde{\psi}(\theta)|^{2} \geq \frac{c^{2}\left|\int_{\mathbb{R}^{d}} \nabla \psi\left(\theta_{0}\right)^{\top} \nabla \varphi_{0}(t)^{\top} u d Q(t)\right|^{2}}{u^{\top}\left(\mathcal{I}(Q)+c^{2} \int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}\left(\theta_{0}\right) \nabla \varphi_{0}(t) d Q(t)\right) u} \\
\Longrightarrow & \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\widetilde{\psi}(\theta)|^{2} \geq \frac{\left.\mid \int_{\mathbb{R}^{d}} \nabla \psi\left(\theta_{0}\right)^{\top} \nabla \varphi_{0}(t)\right)\left.^{\top} u d Q(t)\right|^{2}}{u^{\top}\left(\int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}\left(\theta_{0}\right) \nabla \varphi_{0}(t) d Q(t)\right) u}
\end{aligned}
$$

where the dominated convergence theorem is invoked during the last steps to exchange the limiting and the integration operations. This follows since $\varphi(t) \longrightarrow \theta_{0}$ as $n \longrightarrow \infty$ and both $\nabla \psi$ and $\mathcal{I}$ are continuous at $\theta_{0}$ by assumption. Using $u:=\left\|M^{-1 / 2}\right\|^{-1} M^{-1 / 2}$ where $M:=$ $\int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}\left(\theta_{0}\right) \nabla \varphi_{0}(t) d Q(t)$, it implies that

$$
\begin{aligned}
& \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\widetilde{\psi}(\theta)|^{2} \\
& \quad \geq \nabla \psi\left(\theta_{0}\right)^{\top} \cdot\left\{\int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top}\left(\int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}\left(\theta_{0}\right) \nabla \varphi_{0}(t) d Q(t)\right)^{-1} \nabla \varphi_{0}(t) d Q(t)\right\} \cdot \nabla \psi\left(\theta_{0}\right)
\end{aligned}
$$

Since the choice of $\varphi_{0}$ was also arbitrary, we may consider

$$
\varphi_{0}(t)=\frac{2}{\pi} \arctan (\|t\| / \gamma) \quad \text { and } \quad \nabla \varphi_{0}(t)=\frac{2}{\pi \gamma} \frac{1}{1+(\|t\| / \gamma)^{2}} \frac{t}{\|t\|}
$$

This choice also satisfies that $\left\|\nabla \varphi_{0}\right\|_{\infty}<2(\pi \gamma)^{-1}$. Plugging them into the expression, the leading constant $2(\pi \gamma)^{-1}$ will be canceled between the numerator and the denominator. Taking $\gamma \longrightarrow \infty$, the gradient of $\varphi_{0}$ converges to a constant. Since the gradient of $\varphi_{0}$ is uniformly bounded by construction, it is concluded that

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\widetilde{\psi}(\theta)|^{2} \geq \nabla \psi\left(\theta_{0}\right)^{\top} \cdot \mathcal{I}\left(\theta_{0}\right)^{-1} \cdot \nabla \psi\left(\theta_{0}\right)
$$

in view of the dominated convergence theorem.
Proof of Proposition 1 (ii). Similar to the proof of Proposition 1 (iii), the parameter space is given by $\Theta:=B\left(\theta_{0}, c n^{-1 / 2}\right)$, an open $\mathbb{R}^{d}$-ball centered at $\theta_{0}$ with radius $c n^{-1 / 2}$. We define a diffeomorphism $t \mapsto \varphi(t)=\theta_{0}+c n^{-1 / 2} \varphi_{0}(t)$ where $\varphi_{0}$ is itself a diffeomorphism from $\mathbb{R}^{d}$ to $B([0], 1)$. We further assume that $\left\|\nabla \varphi_{0}\right\|_{\infty}<C$ by some universal constant $C$.

Theorem 9 is then applied to the composite function $t \mapsto \widetilde{\psi}(t):=(\psi \circ \varphi)(t)$, and for any $h \in \mathbb{R}^{d}$, the following inequality is obtained:

$$
\begin{align*}
& \inf _{T} \sup _{t \in \mathbb{R}^{d}} \mathbb{E}_{P_{\varphi(s)}}|T(X)-\widetilde{\psi}(t)|^{2} \\
& \quad \geq \sup _{h, L \geq 0} \frac{(1-\lambda)^{2}}{1+L \lambda}\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-h)) d Q(t)\right|^{2}}{\chi^{2}\left(\widetilde{\mathbb{P}}_{h} \| \lambda \widetilde{\mathbb{P}}_{h}+(1-\lambda) \widetilde{\mathbb{P}}_{0}\right)}-\frac{\lambda(1+1 / L)}{(1-\lambda)^{2}} \int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)|^{2} d Q(t)\right]_{+} . \tag{31}
\end{align*}
$$

Using the mean value theorem and the chain rule, it follows that

$$
\widetilde{\psi}(t)-\widetilde{\psi}(t-h)=c n^{-1 / 2} \nabla \psi\left(\lambda_{1} \varphi(t)+\left(1-\lambda_{1}\right) \varphi(t-h)\right)^{\top} \nabla \varphi_{0}\left(t-\lambda_{2} h\right)^{\top} h
$$

where $\lambda_{1}, \lambda_{2} \in[0,1]$ are constants that can possibly depend on $t$.
We further assume that the prior distribution $Q$ has a continuously differentiable Lebesgue density. Then under the $n$ IID observations from the Hellinger differentiable distribution with the Fisher defect being zero, Lemma 21 implies as $\|h\|_{2} \longrightarrow 0$,

$$
\begin{aligned}
& \chi^{2}( \left.\widetilde{\mathbb{P}}_{h} \| \lambda \widetilde{\mathbb{P}}_{h}+(1-\lambda) \widetilde{\mathbb{P}}_{0}\right) \\
&=\chi^{2}\left(Q_{h} \| Q\right)+\int_{\mathbb{R}^{d}} \chi^{2}\left(P_{\varphi(t+h)}^{n} \| \lambda P_{\varphi(t+h)}^{n}+(1-\lambda) P_{\varphi(t)}^{n}\right) \frac{d Q_{h}^{2}}{d Q} \\
& \quad=h^{\top}\left(\mathcal{I}(Q)+n(1-\lambda)^{2} \int_{\mathbb{R}^{d}} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) d Q(t)\right) h+o\left(\|h\|_{2}^{2}\right) \\
& \quad=h^{\top}\left(\mathcal{I}(Q)+c^{2}(1-\lambda)^{2} \int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi_{0}(t) d Q(t)\right) h+o\left(\|h\|_{2}^{2}\right) .
\end{aligned}
$$

The results obtained previously are now used to evaluate the expression (31). Since the inequality holds for any $h \in \mathbb{R}^{d}$, it also holds as $\|h\|_{2} \longrightarrow 0$. Here, $h$ is denoted by $u \varepsilon$ where $u \in \mathbb{S}^{d-1}$ and $\varepsilon \longrightarrow 0$. This yields the following inequality:

$$
\begin{aligned}
\frac{(1-\lambda)^{2}}{1+L \lambda} & {\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-h)) d Q(t)\right|^{2}}{\chi^{2}\left(\widetilde{\mathbb{P}}_{h} \| \lambda \widetilde{\mathbb{P}}_{h}+(1-\lambda) \widetilde{\mathbb{P}}_{0}\right)}-\frac{\lambda(1+1 / L)}{(1-\lambda)^{2}} \int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)|^{2} d Q(t)\right]_{+} } \\
\geq & \sup _{u \in \mathbb{S}^{d-1}} \limsup _{\varepsilon \rightarrow 0} \frac{(1-\lambda)^{2}}{1+L \lambda}\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-u \varepsilon)) d Q(t)\right|^{2}}{\chi^{2}\left(\widetilde{\mathbb{P}}_{u \varepsilon} \| \lambda \widetilde{\mathbb{P}}_{u \varepsilon}+(1-\lambda) \widetilde{\mathbb{P}}_{0}\right)}\right. \\
& \left.-\frac{\lambda(1+1 / L)}{(1-\lambda)^{2}} \int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-u \varepsilon)|^{2} d Q(t)\right]_{+} \\
\geq & \sup _{u \in \mathbb{S}^{d-1}} \frac{(1-\lambda)^{2}}{1+L \lambda}\left\{\frac{c^{2} n^{-1}\left|\int_{\mathbb{R}^{d}} \nabla \psi(\varphi(t))^{\top} \nabla \varphi_{0}(t)^{\top} u d Q(t)\right|^{2}}{u^{\top}\left(\mathcal{I}(Q)+c^{2}(1-\lambda)^{2} \int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi_{0}(t) d Q(t)\right) u}\right\}
\end{aligned}
$$

where the Fatou's lemma is invoked during the last step to exchange the limiting and the integration operations. Since the above display holds for any $\lambda \in[0,1]$, we let $\lambda \longrightarrow 0$, which results in

$$
\inf _{T} \sup _{t \in \mathbb{R}^{d}} \mathbb{E}_{P_{\varphi(s)}}|T(X)-\widetilde{\psi}(t)|^{2} \geq \sup _{u \in \mathbb{S}^{d-1}}\left\{\frac{c^{2} n^{-1}\left|\int_{\mathbb{R}^{d}} \nabla \psi(\varphi(t))^{\top} \nabla \varphi_{0}(t)^{\top} u d Q(t)\right|^{2}}{u^{\top}\left(\mathcal{I}(Q)+c^{2} \int_{\mathbb{R}^{d}} \nabla \varphi_{0}(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi_{0}(t) d Q(t)\right) u}\right\} .
$$

The remaining proof is identical to the proof of the second statement of Proposition 1.

Proof of Proposition 1 (iv). First, we observe that

$$
\begin{aligned}
& \quad \sup _{\left\|\theta_{0}-\theta\right\|<c n^{-1 / 2}} n \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \\
& \quad=n \sup _{\|h\|<c} \mathbb{E}_{\theta_{0}+h n^{-1 / 2}}\left|T(X)-\psi\left(\theta_{0}+h n^{-1 / 2}\right)\right|^{2} \\
& \quad \geq n \sup _{\|h\|<c} \frac{1}{2}\left(\mathbb{E}_{\theta_{0}}\left|T(X)-\psi\left(\theta_{0}\right)\right|^{2}+\mathbb{E}_{\theta_{0}+h n^{-1 / 2}}\left|T(X)-\psi\left(\theta_{0}+h n^{-1 / 2}\right)\right|^{2}\right) .
\end{aligned}
$$

We now apply Lemma 11 to two points in parameter space $\theta_{0}$ and $\theta_{0}+h n^{-1 / 2}$, which implies

$$
\begin{aligned}
& \sup _{\|h\|<c} n \mathbb{E}_{\theta_{0}+h n^{-1 / 2}}\left|T(X)-\psi\left(\theta_{0}+h n^{-1 / 2}\right)\right|^{2} \\
& \quad \geq n\left[\frac{1-H^{2}\left(P_{\theta_{0}+h n^{-1 / 2}}^{n}, P_{\theta_{0}}^{n}\right)}{4}\right]_{+}\left|\psi\left(\theta_{0}+h n^{-1 / 2}\right)-\psi\left(\theta_{0}\right)\right|^{2} \text { for all }\|h\|<c .
\end{aligned}
$$

For the remaining of the proof, we denote $h$ by $u \eta$ where $u$ is a unit vector and $\eta$ is a positive scalar such that $0 \leq \eta<c$. We treat $\eta$ and $u$ as fixed constants as $n \longrightarrow \infty$. First by the mean value theorem, we have

$$
\left|\psi\left(\theta_{0}+u \eta n^{-1 / 2}\right)-\psi\left(\theta_{0}\right)\right|=\eta n^{-1 / 2} \nabla \psi\left(\theta^{*}\right)^{\top} u
$$

where $\theta^{*}:=\lambda \theta_{0}+(1-\lambda)\left(\theta_{0}+u \eta n^{-1 / 2}\right)=\theta_{0}-\lambda u \eta n^{-1 / 2}$ for $\lambda \in[0,1]$. This implies that

$$
n\left|\psi\left(\theta_{0}+u \eta n^{-1 / 2}\right)-\psi\left(\theta_{0}\right)\right|^{2}=\eta^{2}\left(\nabla \psi\left(\theta_{0}\right)^{\top} u\right)^{2}+o(1)
$$

as $n \longrightarrow \infty$, which follows by the continuity of $\nabla \psi^{\top}$ at $\theta_{0}$. Next, by the Hellinger differentiability of $P_{\theta}$ at $\theta_{0}$, Lemma 20 implies that the Hellinger distance between $P_{\theta_{0}+u \eta n^{-1 / 2}}$ and $P_{\theta_{0}}$ associated with one observation converges to

$$
H^{2}\left(P_{\theta_{0}+u \eta n^{-1 / 2}}, P_{\theta_{0}}\right)=\frac{\eta^{2} u^{\top} \mathcal{I}\left(\theta_{0}\right) u}{4 n}+o\left(n^{-1}\right)
$$

as $n \longrightarrow \infty$. By the tensorization property of the Hellinger distance and the fact that ( $1-$ $\left.Z_{n} / n\right)^{n} \longrightarrow \exp (-Z)$ as $Z_{n} \longrightarrow Z$ for $n \longrightarrow \infty$, we have

$$
H^{2}\left(P_{\theta_{0}+u n^{-1 / 2}}^{n}, P_{\theta_{0}}^{n}\right)=2-2\left(1-\frac{H^{2}\left(P_{\theta_{0}+u \eta n^{-1 / 2}}, P_{\theta_{0}}\right)}{2}\right)^{n} \longrightarrow 2-2 \exp \left(-\frac{\eta^{2} u^{\top} \mathcal{I}\left(\theta_{0}\right) u}{8}\right)
$$

as $n \longrightarrow \infty$. Putting them together, we obtain

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \sup _{\left\|\theta_{0}-\theta\right\|<c n^{-1 / 2}} n \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \\
& \quad \geq\left[-\frac{1}{4}-\frac{1}{2} \exp \left(-\frac{\eta^{2} u^{\top} \mathcal{I}\left(\theta_{0}\right) u}{8}\right)\right]_{+} \eta^{2}\left(\nabla \psi\left(\theta_{0}\right)^{\top} u\right)^{2} \quad \text { for all } \quad 0 \leq \eta<c \quad \text { and } \quad u \in \mathbb{S}^{d-1} .
\end{aligned}
$$

It now remains to optimize the above display for $0 \leq \eta<c$ and $u \in \mathbb{S}^{d-1}$ as $c \longrightarrow \infty$. Since $u^{\top} \mathcal{I}\left(\theta_{0}\right) u$ is a scalar, we parameterize $\eta$ such that $\bar{\eta}=\eta\left(u^{\top} \mathcal{I}\left(\theta_{0}\right) u\right)^{1 / 2}$ and so we can optimize over
$\bar{\eta}$ instead. This gives us that

$$
\begin{aligned}
& \liminf _{c \longrightarrow} \liminf _{n \longrightarrow \infty} \sup _{\left\|\theta_{0}-\theta\right\|<c n^{-1 / 2}} n \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \\
& \quad \geq \sup _{u \in \mathbb{S}^{d-1}} \sup _{0 \leq \bar{\eta}<\infty}\left(-\frac{1}{4}-\frac{1}{2} \exp \left(-\frac{\bar{\eta}^{2}}{8}\right)\right)_{+} \bar{\eta}^{2}\left(\nabla \psi\left(\theta_{0}\right)^{\top} u\right)^{2}\left(u^{\top} \mathcal{I}\left(\theta_{0}\right) u\right)^{-1}
\end{aligned}
$$

For the leading constant, we obtain

$$
\sup _{0 \leq \bar{\eta}<\infty}\left\{-\frac{1}{4}+\frac{1}{2} \exp \left(-\frac{\bar{\eta}^{2}}{8}\right)\right\} \bar{\eta}^{2}=C \approx 0.28953
$$

and the optimal $u$ is given by $u^{*}:=\left\|\mathcal{I}\left(\theta_{0}\right)^{-1 / 2}\right\|^{-1} \mathcal{I}\left(\theta_{0}\right)^{-1 / 2}$. Therefore, we conclude that

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left\|\theta_{0}-\theta\right\|<c n^{-1 / 2}} n \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq C \nabla \psi\left(\theta_{0}\right)^{\top} \mathcal{I}\left(\theta_{0}\right)^{-1} \nabla \psi\left(\theta_{0}\right)
$$

where $C \approx 0.28953$.

## C. 2 Proofs of semiparametric efficiency bound

We first define the parametric path to be used for the proof of Proposition 2. Following Example 25.16 of Van der Vaart (2000), we define bounded univariate parametric paths as follows:

$$
d P_{t}(x)=\frac{1}{C_{t}} \kappa(\operatorname{tg}(x)) d P_{0}(x)
$$

where $C_{t}:=\int \kappa(\operatorname{tg}(x)) d P_{0}$. We assume that $\kappa(0)=\kappa^{\prime}(0)=1$ and $\left\|\kappa^{\prime}\right\|_{\infty} \leq K$ and $\left\|\kappa^{\prime \prime}\right\|_{\infty} \leq K$ for some constant $K$. We then use the following result from Duchi and Ruan (2021).

Lemma 18 (Lemma 1 of Duchi and Ruan (2021)). Assuming $g \in \mathcal{T}_{P_{0}}$ and $d P_{t}$ is the parametric path defined as equation (15), then as $t \longrightarrow 0$,

$$
\chi^{2}\left(P_{t, g} \| P_{0}\right)=t^{2} \int g^{2} d P_{0}+o\left(t^{2}\right)
$$

Proof of Proposition 2 (i). By an analogous argument from Proposition 1, we can choose

$$
\Phi=\left\{\phi: \mathcal{P} \mapsto \mathbb{R}: \phi \text { is pathwise differentiable relative to } \mathcal{T}_{P_{0}}\right\}
$$

for the application of Theorem 8. We then apply the van Trees inequality along each parametric path. Although Gassiat and Stoltz (2024) does not provide an explicit statement for nonparametric settings, the proof remains analogous.

Proof of Proposition 2 (ii). For each fixed $g \in L_{2}^{0}\left(P_{0}\right)$, consider a parametric path defined by $P_{t, g}$. Without loss of generality, we assume that the unknown data-generating distribution corresponds to $P_{0, g}$. We then consider the diffeomorphism $t \mapsto \varphi(t)=c n^{-1 / 2} \varphi_{0}(t)$ where $\varphi_{0}: \mathbb{R} \mapsto$ $(-1,1)$. As the score function $g$ is fixed throughout the proof, we denote the parametric path by $P_{t}$ and omit the dependency on $g$.

We apply Theorem 9 (the version with equation (8)) to the univariate functional $t \mapsto \psi\left(P_{\varphi(t)}\right)$ over the joint probability measures defined as

$$
d \widetilde{\mathbb{P}}_{0}(x, t):=d P_{\varphi(t)}^{n}(x) d Q(t) \quad \text { and } \quad d \widetilde{\mathbb{P}}_{h}(x, t):=d P_{\varphi(t+h)}^{n}(x) d Q(t+h)
$$

we then obtain that

$$
\inf _{T} \sup _{|\theta|<c n^{-1 / 2}} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \geq \sup _{h \in \mathbb{R}} \frac{\left|\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right) d Q(t)\right|^{2}}{\chi^{2}\left(\widetilde{\mathbb{P}}_{h} \| \widetilde{\mathbb{P}}_{0}\right)} .
$$

By the pathwise differentiablity of the functional, it follows that

$$
\begin{aligned}
\psi\left(P_{\varphi(t)}\right) & -\psi\left(P_{\varphi(t-h)}\right) \\
= & \psi\left(P_{c n^{-1 / 2} \varphi_{0}(t)}\right)-\psi\left(P_{c n^{-1 / 2} \varphi_{0}(t-h)}\right) \\
= & c n^{-1 / 2}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right) \int \dot{\psi}_{c n^{-1 / 2}} \varphi_{0}(t) g_{c n^{-1 / 2} \varphi_{0}(t)} d P_{c n^{-1 / 2} \varphi_{0}(t)} \\
& \quad+o\left(c n^{-1 / 2}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)\right) \\
= & c n^{-1 / 2} \varphi_{0}^{\prime}(t-\lambda h) h \int \dot{\psi}_{c n^{-1 / 2} \varphi_{0}(t)} g_{c n^{-1 / 2} \varphi_{0}(t)} d P_{c n^{-1 / 2} \varphi_{0}(t)}+o\left(c n^{-1 / 2} h\right)
\end{aligned}
$$

for some constant $\lambda \in[0,1]$ possibly depending on $t$. After multiplying by $n$ both sides and taking $n \longrightarrow \infty$, we obtain

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} n\left|\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right) d Q(t)\right| \\
& \quad=\liminf _{n \longrightarrow \infty} c^{2} h^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t-\lambda h)\left(\int \dot{\psi}_{c n^{-1 / 2} \varphi_{0}(t-h)} g_{c n^{-1 / 2} \varphi_{0}(t)} d P_{c n^{-1 / 2} \varphi_{0}(t)}\right) d Q(t)\right|^{2}+o\left(c^{2} h^{2}\right) \\
& \quad \geq c^{2} h^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t-\lambda h) d Q(t)\right|^{2}+o\left(c^{2} h^{2}\right)
\end{aligned}
$$

where we use Fatou's lemma in the last step. Specifically, the tangent space under consideration corresponds to the entire $L_{2}^{0}\left(P_{0}\right)$ and thus $g_{c n^{-1 / 2} \varphi_{0}(t-h)} \in L_{2}^{0}\left(P_{0}\right)$. Since $L_{2}^{0}\left(P_{0}\right)$ is also a complete space, it follows that

$$
g_{c n^{-1 / 2} \varphi_{0}(t)} \longrightarrow g_{0} \in L_{2}^{0}\left(P_{0}\right) .
$$

Hence, we can deduce that

$$
\int \dot{\psi}_{c n^{-1 / 2} \varphi_{0}(t-h)} g_{c n^{-1 / 2} \varphi_{0}(t)} d P_{c n^{-1 / 2} \varphi_{0}(t)} \longrightarrow \int \dot{\psi}_{0} g_{0} d P_{0}
$$

as $n \longrightarrow \infty$.
Next, we analyze the local behavior of the $\chi^{2}$-divergence on the path given by (15). By Lemma 18 provided above, we obtain

$$
\begin{aligned}
\chi^{2}\left(P_{c n^{-1 / 2} \varphi_{0}(t+h)} \| P_{c n^{-1 / 2} \varphi_{0}(t)}\right) & =c^{2} n^{-1}\left\{\varphi_{0}(t+h)-\varphi_{0}(t)\right\}^{2} \int g^{2} d P_{c n^{-1 / 2} \varphi_{0}(t)} \\
& =c^{2} n^{-1} h^{2}\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} \int g^{2} d P_{c n^{-1 / 2} \varphi_{0}(t)}
\end{aligned}
$$

for some constant $\lambda \in[0,1]$. By the tensorization property of the $\chi^{2}$-divergence, we have

$$
\begin{aligned}
\chi^{2}\left(P_{c n^{-1 / 2} \varphi_{0}(t+h)}^{n} \| P_{c n^{-1 / 2} \varphi_{0}(t)}^{n}\right) & =\left\{1+\chi^{2}\left(P_{c n^{-1 / 2} \varphi_{0}(t+h)} \| P_{c n^{-1 / 2} \varphi_{0}(t)}\right)\right\}^{n}-1 \\
& =\left[1+c^{2} n^{-1}\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} h^{2} \int g^{2} d P_{c n^{-1 / 2} \varphi_{0}(t)}\right]^{n}-1 \\
& \longrightarrow \exp \left(c^{2}\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} h^{2} \int g^{2} d P_{0}\right)-1
\end{aligned}
$$

as $n \longrightarrow \infty$. Therefore, we conclude that

$$
\begin{aligned}
\chi^{2}\left(\mathbb{P}_{h} \| \mathbb{P}_{0}\right) & =\chi^{2}\left(Q_{h} \| Q\right)+\int_{\mathbb{R}} \chi^{2}\left(P_{\varphi(t+h)}^{n} \| P_{\varphi(t)}^{n}\right) \frac{d Q_{h}^{2}}{d Q} \\
& =\chi^{2}\left(Q_{h} \| Q\right)+\int_{\mathbb{R}}\left\{\exp \left(c^{2}\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} h^{2} \int g^{2} d P_{0}\right)-1\right\} \frac{d Q_{h}^{2}}{d Q}
\end{aligned}
$$

as $h \longrightarrow 0$. Thus we obtain

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \\
& \quad \geq \liminf _{n \rightarrow \infty} \frac{\left|\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right) d Q(t)\right|^{2}}{\chi^{2}\left(\mathbb{P}_{h} \| \mathbb{P}_{0}\right)} \quad \text { for all } h \in \mathbb{R} \\
& \quad=\frac{c^{2} h^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t-\lambda h) d Q(t)\right|^{2}+o\left(c^{2} h^{2}\right)}{\chi^{2}\left(Q_{h} \| Q\right)+\int_{\mathbb{R}}\left\{\exp \left(c^{2}\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} h^{2} \int g^{2} d P_{0}\right)-1\right\} \frac{d Q_{h}^{2}}{d Q}} \quad \text { for all } h \in \mathbb{R} .
\end{aligned}
$$

Since the above inequality holds for any $h \in \mathbb{R}$, we take $h \longrightarrow 0$ to obtain

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \geq \frac{c^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t) d Q\right|^{2}}{\mathcal{I}(Q)+c^{2}\left(\int g^{2} d P_{0}\right) \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} d Q} \\
\Longrightarrow & \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \geq \frac{\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t) d Q\right|^{2}}{\left(\int g^{2} d P_{0}\right) \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} d Q} .
\end{aligned}
$$

As shown in the proof of Proposition 1, it follows that $\sup _{\varphi_{0}}\left\{\mathbb{E}_{Q} \varphi_{0}^{\prime}(t)\right\}^{2} / \mathbb{E}_{Q}\left\{\varphi_{0}^{\prime}(t)\right\}^{2}=1$. Finally, taking the supremum of the score functions $g$ over $L_{2}^{0}\left(P_{0}\right)$, we obtain
$\sup _{g \in L_{2}^{0}\left(P_{0}\right)} \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta, g}^{n}}|T(X)-\psi(\theta)|^{2} \geq \sup _{g \in L_{2}^{0}\left(P_{0}\right)} \frac{\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}}{\int g^{2} d P_{0}}=\int \dot{\psi}_{0}^{2} d P_{0}$
where the last equality follows by the fact that $L_{2}^{0}\left(P_{0}\right)$ is linear closure and by definition, the efficient influence function $\dot{\psi}_{0}$ is contained in $L_{2}^{0}\left(P_{0}\right)$ (See Lemma 2.2 of Van der Vaart (2002))

Proof of Proposition 2 (iii). Following the notation and the setting from the previous proof, we apply Theorem 10 to the univariate functional $t \mapsto \psi\left(P_{\varphi(t)}\right)$ over the joint probability measures $\widetilde{\mathbb{P}}_{0}(x, t)$ and $\widetilde{\mathbb{P}}_{h}(x, t)$, and we obtain

$$
\begin{aligned}
& \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \\
& \quad \geq\left[\frac{\left|\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right) d Q(t)\right|}{2 H\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)}-\left(\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right)^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

for all $h \in \mathbb{R}$. As shown in the previous proof, it follows that

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} n\left|\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right) d Q(t)\right| \\
& \quad \geq c^{2} h^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t-\lambda h) d Q(t)\right|^{2}+o\left(c^{2} h^{2}\right)
\end{aligned}
$$

by Fatou's lemma. Similarly, it also follows that

$$
\liminf _{n \longrightarrow \infty} n\left(\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right)^{2} d Q(t)\right)^{1 / 2}=O\left(c^{2} h^{2}\right) .
$$

Next, we study the local behavior of the Hellinger distance. Since $\left\{P_{\theta}: \theta \in \Theta\right\}$ is a QMD family, the Hellinger distance associated with one observation follows:

$$
\begin{aligned}
& H^{2}\left(P_{\varphi(t+h)}, P_{\varphi(t)}\right) \\
& \quad=\int\left(d P_{c n^{-1 / 2} \varphi_{0}(t+h)}^{1 / 2}-d P_{c n^{-1 / 2} \varphi_{0}(t)}^{1 / 2}\right)^{2} \\
& \quad=\frac{1}{4} c^{2} n^{-1} \int\left\{\varphi_{0}(t+h)-\varphi_{0}(t)\right\}^{2} g_{c n^{-1 / 2} \varphi_{0}(t)}^{2} d P_{c n^{-1 / 2} \varphi_{0}(t)}+o\left(c^{2} n^{-1}\left\{\varphi_{0}(t+h)-\varphi_{0}(t)\right\}^{2}\right) \\
& \quad=\frac{1}{4} c^{2} n^{-1} h^{2} \int\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} g_{c n^{-1 / 2} \varphi_{0}(t)}^{2} d P_{c n^{-1 / 2} \varphi_{0}(t)}+o\left(c^{2} n^{-1} h^{2}\right)
\end{aligned}
$$

and by the tensorization property,

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right) \\
& \quad=\liminf _{n \longrightarrow \infty}\left\{2-2\left(1-\frac{H^{2}\left(P_{\varphi(t+h)}, P_{\varphi(t)}\right)}{2}\right)^{n}\right\} \\
& \quad=\liminf _{n \longrightarrow \infty}\left\{2-2\left(1-\frac{c^{2} h^{2} \int\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} g_{c n^{-1 / 2} \varphi_{0}(t)}^{2} d P_{c n^{-1 / 2} \varphi_{0}(t)}+o\left(c^{2} n^{-1} h^{2}\right)}{8 n}\right)^{n}\right\} \\
& \quad=2-2 \exp \left(-\frac{c^{2} h^{2} \int\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} g^{2} d P_{0}}{8}\right) .
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \\
& \quad \geq \liminf _{n \longrightarrow \infty} \frac{n\left|\int_{\mathbb{R}}\left(\psi\left(P_{\varphi(t)}\right)-\psi\left(P_{\varphi(t-h)}\right)\right) d Q(t)\right|^{2}}{4 H^{2}\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)}-O\left(c^{2} h^{2}\right) \quad \text { for all } h \in \mathbb{R} \\
& \quad=\frac{c^{2} h^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t-\lambda h) d Q(t)\right|^{2}+o\left(c^{2} h^{2}\right)}{4\left(H^{2}\left(Q_{h}, Q_{0}\right)+\int_{\mathbb{R}}\left\{2-2 \exp \left(-\frac{c^{2} h^{2} \int\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} g^{2} d P_{0}}{8}\right)\right\} d Q_{h}^{1 / 2} d Q^{1 / 2}\right)}-O\left(c^{2} h^{2}\right)
\end{aligned}
$$

for all $h \in \mathbb{R}$. Since the above inequality holds for any $h \in \mathbb{R}$, we take $h \longrightarrow 0$ to obtain

$$
\liminf _{n \longrightarrow \infty} \inf _{T} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta}^{n}}|T(X)-\psi(\theta)|^{2} \geq \frac{c^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2}\left|\int_{\mathbb{R}} \varphi_{0}^{\prime}(t) d Q\right|^{2}}{\mathcal{I}(Q)+c^{2}\left(\int g^{2} d P_{0}\right) \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} d Q} .
$$

This follows since

$$
\begin{aligned}
\lim _{h \longrightarrow 0} & H^{2}\left(Q_{h}, Q_{0}\right)+\int_{\mathbb{R}}\left\{2-2 \exp \left(-\frac{c^{2} h^{2} \int\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} g^{2} d P_{0}}{8}\right)\right\} d Q_{h}^{1 / 2} d Q^{1 / 2} \\
& =\frac{1}{4} h^{2} \mathcal{I}(Q)+\frac{1}{4} c^{2} h^{2}\left(\int g^{2} d P_{0}\right) \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} d Q
\end{aligned}
$$

by a Taylor expansion. The rest of the proof is identical to the first statement of Proposition 2.

Proof of Proposition 2 (iv). Similar to the preceding proof, we apply Lemma 11 to the QMD parametric paths. Without loss of generality, we assume that $\theta_{0}=0$. Then for fixed $g \in \mathcal{T}_{P_{0}}$, we invoke Lemma 11 as follows:

$$
\begin{aligned}
& \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{P_{\theta, g}^{n}}\left|T(X)-\psi\left(P_{\theta, g}\right)\right|^{2} \\
& \quad \geq n \sup \frac{1}{2}\left(\mathbb{E}_{0}\left|T(X)-\psi\left(P_{0}\right)\right|^{2}+\mathbb{E}_{h n^{-1 / 2}}\left|T(X)-\psi\left(P_{h n^{-1 / 2}}\right)\right|^{2}\right) \\
& \quad \geq n\left[\frac{1-H^{2}\left(P_{h n^{-1 / 2}}^{n}, P_{0}^{n}\right)}{4}\right]_{+}\left|\psi\left(P_{h n^{-1 / 2}}\right)-\psi\left(P_{0}\right)\right|^{2} \text { for all }|h|<c .
\end{aligned}
$$

Since $\psi$ is pathwise differentiable, it follows that

$$
\liminf _{n \longrightarrow \infty} n\left|\psi\left(P_{0}\right)-\psi\left(P_{h n^{-1 / 2}}\right)\right|=h^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2} .
$$

Also by the QMD assumption of the parametric path, we have

$$
H^{2}\left(P_{h n^{-1 / 2}}, P_{0}\right)=\int\left(d P_{h n^{-1 / 2}}^{1 / 2}-d P_{0}^{1 / 2}\right)^{2}=\frac{1}{4} h^{2} n^{-1} \int g^{2} d P_{0}+o\left(h^{2} n^{-1}\right)
$$

followed by the tensorization property of the Hellinger distance,

$$
\begin{aligned}
\liminf _{n \longrightarrow \infty} H^{2}\left(P_{h n^{-1 / 2}}^{n}, P_{0}^{n}\right) & =\liminf _{n \longrightarrow \infty}\left\{2-2\left(1-\frac{H^{2}\left(P_{h n^{-1 / 2}}, P_{0}\right)}{2}\right)^{n}\right\} \\
& =\liminf _{n \longrightarrow \infty}\left\{2-2\left(1-\frac{h^{2} \int g^{2} d P_{0}+o\left(h^{2} n^{-1}\right)}{8 n}\right)^{n}\right\} \\
& =2-2 \exp \left(-\frac{h^{2} \int g^{2} d P_{0}}{8}\right) .
\end{aligned}
$$

Putting them together, we obtain

$$
\begin{aligned}
& \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \\
& \quad \geq \sup _{0 \leq h<\infty}\left[-\frac{1}{4}-\frac{1}{2} \exp \left(-\frac{h^{2} \int g^{2} d P_{0}}{8}\right)\right]_{+} h^{2}\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2} .
\end{aligned}
$$

Similar to the proof of Proposition 1, we let $\widetilde{h}=h\left(\int g^{2} d P_{0}\right)^{1 / 2}$ and optimize over $\widetilde{h}$ instead. This yields that

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq C\left(\int \dot{\psi}_{0} g d P_{0}\right)^{2} /\left(\int g^{2} d P_{0}\right)
$$

where $C \approx 0.28953$. Using Lemma 2.2 of Van der Vaart (2002) and taking the supremum over the linear closure of the tangent set, we conclude that

$$
\sup _{g \in \mathcal{T}_{P_{0}}} \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \sup _{|\theta|<c n^{-1 / 2}} n \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq C \int \dot{\psi}_{0}^{2} d P_{0}
$$

where $C \approx 0.28953$.

## C. 3 Proof of local minimax rate for irregular estimation

Proof of Lemma 13. To begin, we define a diffeomorphism $\varphi_{0}: \mathbb{R}^{d} \mapsto B([0], 1)$ between $\mathbb{R}^{d}$ and an open unit ball in $\mathbb{R}^{d}$. We then construct the following mapping:

$$
\begin{equation*}
\varphi(s):=\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(s) \tag{32}
\end{equation*}
$$

for all $s \in \mathbb{R}^{d}$. The resulting mapping is a valid diffeomorphism between $\mathbb{R}^{d}$ and $B\left(\theta_{0}, c n^{-1 / \alpha}\right)$. Additionally by the differentiablity of $\varphi_{0}$, we have $\nabla \varphi(s)=c n^{-1 / \alpha} \nabla \varphi_{0}(s)$. It is crucial that $\varphi_{0}$ no longer depends on $n$ or $c$. Similar to several preceding proofs such as the proof of Proposition 1, we apply Theorem 10 to the composition function $t \mapsto \psi(t):=(\psi \circ \varphi)(t)$ and the statistical models

$$
d \mathbb{P}_{0}(x, t):=d P_{\varphi(t)}^{n}(x) d Q(t) \quad \text { and } \quad d \mathbb{P}_{h}(x, t):=d P_{\varphi(t+h)}^{n}(x) d Q(t+h),
$$

which implies

$$
\begin{aligned}
& \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} \mathbb{E}_{\theta}|T(X)-\widetilde{\psi}(\theta)|^{2} \\
& \quad \geq\left[\frac{\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-h)) d Q(t)\right|}{2 H\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right)}-\left(\int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

for any $h \in \mathbb{R}^{d}$. Let $t_{*}$ be a point in $\mathbb{R}^{d}$ such that $t_{*}:=t-\lambda h$ for some $\lambda \in[0,1]$ and $\vartheta_{t}:=$ $\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t)$. Assuming that $n$ is large enough such that $\left\|\theta_{0}-\vartheta_{t}\right\|<\delta / 2$, then by the Höldersmoothness assumption of $\psi$, we have

$$
\begin{aligned}
\widetilde{\psi}(t) & -\widetilde{\psi}(t-h) \\
& =\psi\left(\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t)\right)-\psi\left(\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t-h)\right) \\
& =\psi\left(\vartheta_{t}\right)-\psi\left(\vartheta_{t}-c n^{-1 / \alpha}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)\right) \\
& =C_{2, \vartheta_{t}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right) c^{\beta} n^{-\beta / \alpha}\left\|\varphi_{0}(t)-\varphi_{0}(t-h)\right\|^{\beta}+o\left(c^{\beta} n^{-\beta / \alpha}\left\|\varphi_{0}(t)-\varphi_{0}(t-h)\right\|^{\beta}\right)} \\
& =C_{2, \vartheta_{t}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)} c^{\beta} n^{-\beta / \alpha}\left\|\nabla \varphi_{0}\left(t_{*}\right) h\right\|^{\beta}+o\left(c^{\beta} n^{-\beta / \alpha}\left\|\nabla \varphi_{0}\left(t_{*}\right) h\right\|^{\beta}\right) .
\end{aligned}
$$

By multiplying by $n^{\beta / \alpha}$ and taking the limit $n \longrightarrow \infty$, we have

$$
\begin{aligned}
& \liminf _{n \rightarrow \infty} n^{\beta / \alpha}\left|\int_{\mathbb{R}^{d}}(\widetilde{\psi}(t)-\widetilde{\psi}(t-h)) d Q(t)\right| \\
& \geq c^{\beta} \int_{\mathbb{R}^{d}} C_{2, \theta_{0}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)}\left\|\nabla \varphi_{0}(t-\lambda h) h\right\|^{\beta} d Q(t)
\end{aligned}
$$

by Fatou's lemma. We also have that

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty}\left(\int_{\mathbb{R}^{d}}|\widetilde{\psi}(t)-\widetilde{\psi}(t-h)|^{2} d Q(t)\right)^{1 / 2} \\
& \quad=c^{\beta}\left(\int_{\mathbb{R}^{d}} C_{2, \theta_{0}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)}\left\|\nabla \varphi_{0}(t-\lambda h) h\right\|^{2 \beta} d Q(t)\right)^{1 / 2} .
\end{aligned}
$$

by the dominated convergence theorem, which follows since $\nabla \varphi_{0}$ is continuous and thus bounded as $h \longrightarrow 0$. Moving onto the Hellinger distance, by the Hölder-smoothness assumption, we have

$$
\begin{aligned}
H^{2} & \left(P_{\varphi(t+h)}, P_{\varphi(t)}\right) \\
& =H^{2}\left(P_{\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t+h)}, P_{\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t)}\right) \\
& =H^{2}\left(P_{\theta_{t}+c n^{-1 / \alpha}\left(\varphi_{0}(t+h)-\varphi_{0}(t)\right)}, P_{\theta_{t}}\right) \\
& =C_{1, \vartheta_{t}, \operatorname{sign}\left(\varphi_{0}(t+h)-\varphi_{0}(t)\right)} c^{\alpha} n^{-1}\left\|\varphi_{0}(t+h)-\varphi_{0}(t)\right\|^{\alpha}+o\left(c^{\alpha} n^{-1}\left\|\varphi_{0}(t+h)-\varphi_{0}(t)\right\|^{\alpha}\right) \\
& =C_{1, \vartheta_{t}, \operatorname{sign}\left(\varphi_{0}(t+h)-\varphi_{0}(t)\right)} c^{\alpha} n^{-1}\left\|\nabla \varphi_{0}(t+\lambda h) h\right\|^{\alpha}+o\left(c^{\alpha} n^{-1}\left\|\nabla \varphi_{0}(t+\lambda h) h\right\|^{\alpha}\right),
\end{aligned}
$$

which then followed by the tensorization property of the Hellinger distance to yield

$$
\begin{aligned}
\liminf _{n \longrightarrow \infty} H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right) & =\liminf _{n \longrightarrow \infty}\left\{2-2\left(1-\frac{H^{2}\left(P_{\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t)}, P_{\theta_{0}+c n^{-1 / \alpha} \varphi_{0}(t-h)}\right)}{2}\right)^{n}\right\} \\
& =2-2 \exp \left(-\frac{\left.C_{1, \theta_{0}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)^{\alpha}\left\|\nabla \varphi_{0}(t+\lambda h) h\right\|^{\alpha}}^{2}\right)}{2}\right.
\end{aligned}
$$

where the last step follows by a first-order Taylor expansion of $(1-x / n)^{n}$ as $n \longrightarrow \infty$. We then conclude that

$$
\begin{aligned}
& H^{2}\left(\mathbb{P}_{0}, \mathbb{P}_{h}\right) \\
&=H^{2}\left(Q_{h}, Q_{0}\right)+\int_{\mathbb{R}} H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right) d Q_{h}^{1 / 2} d Q^{1 / 2} \\
&=H^{2}\left(Q_{h}, Q_{0}\right)+\int_{\mathbb{R}}\left\{2-2 \exp \left(-\frac{\left.\left.C_{1, \theta_{0}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right) c^{\alpha}\left\|\nabla \varphi_{0}(t+\lambda h) h\right\|^{\alpha}}^{2}\right)\right\} d Q_{h}^{1 / 2} d Q^{1 / 2}}{}\right.\right. \\
& \quad=2-2 \int_{\mathbb{R}} \exp \left(-\frac{C_{1, \theta_{0}, \operatorname{sign}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)} c^{\alpha}\left\|\nabla \varphi_{0}(t+\lambda h) h\right\|^{\alpha}}{2}\right) d Q_{h}^{1 / 2} d Q^{1 / 2} .
\end{aligned}
$$

For the ease of notation, we denote the constants as follows:

Then, putting these intermediate results together, we have

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \\
& \sup _{\theta \in \mathbb{R}^{d}} n^{2 \beta / \alpha} \mathbb{E}_{\theta}|T(X)-\widetilde{\psi}(\theta)|^{2} \\
& \geq {\left[\frac{c^{\beta} \int_{\mathbb{R}^{d}} \widetilde{C}_{2}(t)\left\|\nabla \varphi_{0}(t-\lambda h) h\right\|^{\beta} d Q(t)}{2\left\{2-2 \int_{\mathbb{R}^{2}} \exp \left(-\widetilde{C}_{1}(t) c^{\alpha}\left\|\nabla \varphi_{0}(t+\lambda h) h\right\|^{\alpha} / 2\right) d Q_{h}^{1 / 2} d Q^{1 / 2}\right\}^{1 / 2}}\right.} \\
&\left.\quad-c^{\beta}\left(\int_{\mathbb{R}^{d}} \widetilde{C}_{2}^{2}(t)\left\|\nabla \varphi_{0}(t-\lambda h) h\right\|^{2 \beta} d Q(t)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

where the above display holds for any $h \in \mathbb{R}^{d}$. We denote $\bar{h}=h c$ and let $c \longrightarrow \infty$. This further simplifies the expression to

$$
\begin{aligned}
& \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} n^{2 \beta / \alpha} \mathbb{E}_{\theta}|T(X)-\widetilde{\psi}(\theta)|^{2} \\
& \geq \liminf _{\bar{h} \longrightarrow \infty}\left[\frac{\int_{\mathbb{R}^{d}} \widetilde{C}_{2}(t)\left\|\nabla \varphi_{0}(t) \bar{h}\right\|^{\beta} d Q(t)}{2\left\{2-2 \int_{\mathbb{R}} \exp \left(-\widetilde{C}_{1}(t)\left\|\nabla \varphi_{0}(t) \bar{h}\right\|^{\alpha} / 2\right) d Q\right\}^{1 / 2}}\right. \\
& \left.\quad-\left(\int_{\mathbb{R}^{d}} \widetilde{C}_{2}^{2}(t)\left\|\nabla \varphi_{0}(t) \bar{h}\right\|^{2 \beta} d Q(t)\right)^{1 / 2}\right]_{+}^{2} .
\end{aligned}
$$

We claim that there exists the choice of $\varphi_{0}$ and $Q$ to make the lower bound strictly greater than zero. Since the above inequality holds for any $\varphi_{0}$ and $h \in \mathbb{R}^{d}$, we consider the class of diffeomorphism such that

$$
\left\|\nabla \varphi_{0} \bar{h}\right\| \longrightarrow \gamma \quad \text { as } \quad\|\bar{h}\| \longrightarrow \infty
$$

Denoting $\bar{h}=u \varepsilon$ for $u \in \mathbb{S}^{d-1}$, this can be obtained, for instance, by

$$
\varphi_{0}(t)=\frac{2}{\pi} \arctan \left(\frac{\pi \gamma\|t\|}{2 \varepsilon}\right) \quad \text { and } \quad \nabla \varphi_{0}(t)=\frac{\gamma}{\varepsilon}\left\{1+\left(\frac{\pi \gamma\|t\|}{2 \varepsilon}\right)^{2}\right\}^{-1} \frac{t}{\|t\|}
$$

For any choice of $\gamma>0$, the image of $\varphi_{0}$ is $B([0], 1)$ and $\left\|\nabla \varphi_{0} \bar{h}\right\| \longrightarrow \gamma$ as $\|\bar{h}\| \longrightarrow \infty$. Since $C_{2, t, \operatorname{sign}(h)} \leq \bar{C}_{1}$ and $C_{2, t, \operatorname{sign}(h)} \leq \bar{C}_{2}$ uniformly, we obtain

$$
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\theta \in \mathbb{R}^{d}} n^{2 \beta / \alpha} \mathbb{E}_{\theta}|T(X)-\widetilde{\psi}(\theta)|^{2} \geq\left[\frac{\gamma^{\beta} \int \widetilde{C}_{2}(t) d Q}{2\left\{2-2 \exp \left(-\bar{C}_{1} \gamma^{\alpha} / 2\right)\right\}^{1 / 2}}-\gamma^{\beta} \bar{C}_{2}\right]_{+}^{2}
$$

Choosing $\gamma=\left(2 \delta / \bar{C}_{1}\right)^{1 / \alpha}$ for small $\delta>0$, the lower bound becomes

$$
\left[\frac{\left(2 \delta / \bar{C}_{1}\right)^{\beta / \alpha} \int \widetilde{C}_{2}(t) d Q}{2\{2-2 \exp (-\delta)\}^{1 / 2}}-\left(2 \delta / \bar{C}_{1}\right)^{\beta / \alpha} \bar{C}_{2}\right]_{+}^{2}=\left(2 \delta / \bar{C}_{1}\right)^{2 \beta / \alpha}\left[\frac{\int \widetilde{C}_{2}(t) d Q}{2\{2-2 \exp (-\delta)\}^{1 / 2}}-\bar{C}_{2}\right]_{+}^{2}
$$

The final display is strictly greater than zero when $\delta$ is small enough as long as the prior is selected to ensure $\int \widetilde{C}_{2}(t) d Q>0$, which can be a point mass at $t$ that attains $\widetilde{C}_{2}(t)=\bar{C}_{2}$.

## Supplement D Nonparametric density estimation

Let $X_{1}, \ldots, X_{n}$ be drawn IID from the unknown density $f_{0}$. In this section, we develop a local asymptotic minimax lower bound for the density at $X=x_{0}$, which is a non-differentiable functional. We consider the approximation functional via convolution such that

$$
\psi(f):=f\left(x_{0}\right) \quad \text { and } \quad \phi(f):=\int h^{-1} K\left(\frac{x-x_{0}}{h}\right) f(x) d x
$$

where $K$ is a function, satisfying (A3), and $h>0$ is a bandwidth parameter. We assume that $f_{0}$ is $s$-times continuously differentiable at $x_{0}$. We consider the following set of density functions:

$$
\mathcal{F}\left(s, x_{0}, M\right):=\left\{f \text { is } s \text {-times differentiable at } x_{0} \text {, satisfying }\left|f^{(s)}\left(x_{0}\right)\right| \leq(1+M)\left|f_{0}^{(s)}\left(x_{0}\right)\right|\right\}
$$

Furthermore, we define the following localized set of density functions

$$
U(\delta ; \varepsilon):=\left\{f \in \mathcal{F}\left(s, x_{0}, M\right): \int_{\left|x-x_{0}\right| \leq \varepsilon}\left|f^{(k)}(x)-f_{0}^{(k)}(x)\right| d x \leq \delta \quad \text { for all } k \in\{0,1, \ldots, s\}\right\} .
$$

Proof of Lemma 14. Throughout the proof, $\varepsilon$ and $M$ are considered fixed. With the choice of a kernel function, satisfying (A3), the approximation functional $\phi$ is absolutely continuous with respect to $f$ and thus Theorem 8 applies. It then implies for any approximating functional $\phi$,

$$
\begin{equation*}
\inf _{T} \sup _{f \in U(\delta ; \varepsilon)} \mathbb{E}_{f}|T(X)-\psi(f)|^{2} \geq \sup _{Q}\left[\sqrt{\frac{\left|\int_{\Theta} \phi^{\prime}(f) d Q\right|^{2}}{\mathcal{I}(Q)+\int_{\Theta} \mathcal{I}(f) d Q}}-\|\psi-\phi\|_{L_{2}(Q)}\right]_{+}^{2} . \tag{33}
\end{equation*}
$$

where the probability measure $Q$ is placed over the collection of $f$, which we formalize by constructing a differentiable parametric submodel. Since our result holds for any choice of $\phi$, and thus for any $h>0$, the lower bound still holds by restricting ourselves to the choice of $h \in(0, \varepsilon \wedge 1)$. Similarly, the lower bound is agnostic to the choice of $K$. Thus we focus on the $K$ whose support is contained in $[-1,1]$. Throughout this section, we denote by $\mathfrak{C}$ a fixed constant that may vary line by line.

## D. 1 Local parameter space

For any function $g \in L_{0}^{2}\left(f_{0}\right)$, that is, a mean-zero and finite variance under the density $f_{0}$, we define a differentiable parametric path of densities as

$$
f_{t}(x)=(1+\operatorname{tg}(x)) f_{0}(x)
$$

for all $x \in \mathcal{X}$ and $t \geq 0$. As Theorem 8 holds for any choice of differentiable paths, we consider the choice of $g=\Phi_{0}$ such that

$$
\Phi_{0}:=x \mapsto h^{-1} K\left(\frac{x-x_{0}}{h}\right)-\int h^{-1} K\left(\frac{x-x_{0}}{h}\right) f_{0}(x) d x .
$$

It is straightforward to verify $\Phi_{0}$ is a valid score function. In fact, $\Phi_{0}$ is an efficient influence function of the functional $\mathbb{E}_{f}[\phi]$ relative to the maximal tangent space. We then consider the particular choice of paths given by $\widetilde{f}_{t}(x):=\left(1+t \Phi_{0}(x)\right) f_{0}(x)$.

First, we ensure the choice of parameter that permits $\widetilde{f}_{t}$ to belong to the class of interest for all $t$. The score function $\Phi_{0}$ is bounded uniformly as

$$
\left\|\Phi_{0}\right\|_{\infty} \leq\left\|h^{-1} K\left(\frac{x-x_{0}}{h}\right)-\int h^{-1} K\left(\frac{x-x_{0}}{h}\right) f_{0}(x) d x\right\|_{\infty} \leq 2 h^{-1}\|K\|_{\infty}
$$

Since $K$ is uniformly bounded, it follows that

$$
1+t \Phi_{0}(x) \geq 1-t\left\|\Phi_{0}\right\|_{\infty} \geq 1-2 t h^{-1}\|K\|_{\infty}
$$

Therefore the induced path generates a nonnegative function (that integrates to one) so long as $t \leq h /\left(2\|K\|_{\infty}\right)$. Next, we examine the smoothness of the induced path. Since $\Phi_{0}^{(k)}(x)=$ $h^{-(k+1)} K^{(k)}\left(\left(x-x_{0}\right) / h\right)$, it follows

$$
\begin{aligned}
\left|\widetilde{f}_{t}^{(s)}\left(x_{0}\right)\right| & \leq\left|f_{0}^{(s)}\left(x_{0}\right)\right|+\left|\sum_{k=0}^{s}\binom{s}{k} t\left(\frac{d^{s-k}}{d x^{s-k}} \Phi_{0}\left(x_{0}\right)\right) f_{0}^{(k)}\left(x_{0}\right)\right| \\
& \leq\left|f_{0}^{(s)}\left(x_{0}\right)\right|+\sum_{k=0}^{s}\binom{s}{k}\left(\frac{t}{h^{s-k+1}}\left|K^{(s-k)}(0)\right|\right)\left|f_{0}^{(k)}\left(x_{0}\right)\right| \\
& \leq\left|f_{0}^{(s)}\left(x_{0}\right)\right|+t\left\{\frac{\left|K^{(s)}(0)\right|}{h^{s+1}} f_{0}\left(x_{0}\right)+\frac{\mathfrak{C}}{h^{s}}\right\}
\end{aligned}
$$

for some constant $\mathfrak{C}>0$. We then define

$$
C_{M}:=\left(\frac{1-h \mathfrak{C}}{\left|K^{(s)}(0)\right| f_{0}\left(x_{0}\right)}\right) M\left|f_{0}^{(s)}\left(x_{0}\right)\right|
$$

and for all $t \in\left[0, C_{M} h^{s+1}\right]$, the density on the path satisfies $\left|\widetilde{f}_{t}^{(s)}\left(x_{0}\right)\right| \leq(1+M)\left|f_{0}^{(s)}\left(x_{0}\right)\right|$. Next, we observe that for any $\widetilde{f}_{t}$

$$
\begin{aligned}
\int_{\left|x-x_{0}\right| \leq \varepsilon}\left|\widetilde{f}_{t}^{(\ell)}(x)-f_{0}^{(\ell)}(x)\right| d x & =t \sum_{k=0}^{\ell}\binom{\ell}{k} \frac{1}{h^{\ell-k+1}} \int_{\left|x-x_{0}\right| \leq \varepsilon}\left|K^{(\ell-k)}\left(\frac{x-x_{0}}{h}\right) f_{0}^{(k)}(x)\right| d x \\
& \leq t h^{-\ell} \sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}(x)\right| \int_{-1}^{1}\left|K^{(\ell)}(u)\right| d u+\mathfrak{C} h^{-(\ell-1)}
\end{aligned}
$$

where the last step follows since $h<(1 \wedge \varepsilon)$. All densities on the parametric path hence belong to $U(\delta ; \varepsilon)$ so long as

$$
\begin{gathered}
t \leq \frac{\delta h^{s}}{\sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}(x)\right| \int_{-1}^{1}\left|K^{(s)}(u)\right| d u+\mathfrak{C} h}=\delta h^{s} \Delta \quad \text { where } \\
\Delta:=\left(\sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}(x)\right| \int_{-1}^{1}\left|K^{(s)}(u)\right| d u+\mathfrak{C} h\right)^{-1}
\end{gathered}
$$

It remains to analyze the lower bound given by (33) over the parametric submodel $\left\{\widetilde{f_{t}}: t \in \Theta_{0}\right\}$ where

$$
\Theta_{0}:=\left\{t: 0 \leq t \leq\left(\delta h^{s} \Delta \wedge C_{M} h^{s+1}\right)\right\}
$$

This suggests that we need to analyze two regimes, $\delta \lesssim h$ and $h \lesssim \delta$. For instance, the usual global minimax lower bound over $\mathcal{F}$ corresponds to taking $\delta$ as a fixed constant.

## D. 2 Approximation bias

Let $\mathfrak{B}:=\left(\delta h^{s} \Delta \wedge C_{M} h^{s+1}\right)$. Based on the earlier derivation, we consider a prior Q supported on $[0, \mathfrak{B}]$. Such a prior can be constructed from the following dilation:

$$
Q(t):=\frac{2}{\mathfrak{B}} Q_{0}\left(\frac{t-\mathfrak{B} / 2}{\mathfrak{B} / 2}\right) \quad \text { where } \quad Q_{0}(t):=\cos ^{2}(\pi t / 2) I(|t| \leq 1) .
$$

The choice of the cosine density is motivated by the fact that this density minimizes Fisher information among density with support over $[-1,1]$ (Uhrmann-Klingen, 1995). The corresponding Fisher information is $\mathcal{I}\left(Q_{0}\right)=\pi^{2}$. First, for any $\widetilde{f}_{t}$ over $t \in \Theta_{0}$, we have

$$
\begin{aligned}
\|\psi-\phi\|_{L_{2}(Q)} & =\left(\int\left|\psi\left(\widetilde{f}_{t}\right)-\phi\left(\widetilde{f}_{t}\right)\right|^{2} d Q(t)\right)^{1 / 2} \\
& =\left(\int\left|\widetilde{f}_{t}\left(x_{0}\right)-\int h^{-1} K\left(\frac{x-x_{0}}{h}\right) \widetilde{f}_{t}(x) d x\right|^{2} d Q(t)\right)^{1 / 2} \\
& =\left(\int\left|\int_{-1}^{1} K(u) \int_{x_{0}}^{x_{0}+u h} \frac{\widetilde{f}_{t}^{(s)}(\xi)\left(x_{0}+u h-\xi\right)^{s-1}}{(s-1)!} d \xi d u\right|^{2} d Q(t)\right)^{1 / 2} \\
& \leq\left(\int\left|\int_{-1}^{1} K(u) \frac{|u h|^{s-1}}{(s-1)!} \int_{x_{0}}^{x_{0}+u h}\right| \widetilde{f}_{t}^{(s)}(\xi)|d \xi d u|^{2} d Q(t)\right)^{1 / 2}
\end{aligned}
$$

Furthermore, it follows that

$$
\begin{aligned}
\int_{x_{0}}^{x_{0}+u h}\left|\tilde{f}_{t}^{(s)}(\xi)\right| d \xi & =\int_{x_{0}}^{x_{0}+u h}\left|f_{0}^{(s)}(\xi)+\sum_{k=0}^{s}\binom{s}{k}\left(\frac{t}{h^{s-k+1}} K^{(s-k)}\left(\left(\xi-x_{0}\right) / h\right)\right) f_{0}^{(k)}(\xi)\right| d \xi \\
& \leq \int_{x_{0}}^{x_{0}+u h}\left|f_{0}^{(s)}(\xi)\right| d \xi+\int_{x_{0}}^{x_{0}+u h}\left|\frac{t}{h^{s+1}} K^{(s)}\left(\left(\xi-x_{0}\right) / h\right) f_{0}(\xi)+\mathfrak{C} h^{-s}\right| d \xi \\
& \leq \int_{x_{0}}^{x_{0}+u h}\left|f_{0}^{(s)}(\xi)\right| d \xi+\frac{t}{h^{s}} \int_{0}^{u}\left|K^{(s)}(\zeta) f_{0}\left(x_{0}+\zeta h\right)+\mathfrak{C} h\right| d \zeta \\
& \leq|u h| \sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}^{(s)}(x)\right|+\frac{\mathfrak{B}}{h^{s}} \sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}(x)\right| \int_{0}^{u}\left|K^{(s)}(\zeta)+\mathfrak{C} h\right| d \zeta .
\end{aligned}
$$

The lower order terms cancel by the fact that $\int u^{k} K(u) d u=0$ for all $k \in\{1,2, \ldots, s-1\}$. Also by the definition of $\mathfrak{B}$, it follows $\mathfrak{B} / h^{s}=\left(\delta \Delta \wedge C_{M} h\right) \leq C_{M} h$. This leads to

$$
\begin{aligned}
& \left(\int\left|\int_{-1}^{1} K(u) \frac{|u h|^{s-1}}{(s-1)!} \int_{x_{0}}^{x_{0}+u h}\right| \widetilde{f}_{t}^{(s)}(\xi)|d \xi d u|^{2} d Q(t)\right)^{1 / 2} \\
& \quad \leq\left|\int_{-1}^{1} K(u) \frac{|u h|^{s-1}}{(s-1)!}\left(|u h| \sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}^{(s)}(x)\right|+C_{M} h \sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}(x)\right| \int_{0}^{u}\left|K^{(s)}(\zeta)+\mathfrak{C} h\right| d \zeta\right) d u\right| \\
& \quad \leq h^{s}\left|\int_{-1}^{1} \frac{K(u)|u|^{s}}{(s-1)!} d u\left(\sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}^{(s)}(x)\right|+C_{M} \int_{-1}^{1}\left|K^{(s)}(\zeta)\right| d \zeta \sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}(x)\right|+\mathfrak{C} h\right)\right|
\end{aligned}
$$

Thus, we conclude that

$$
\begin{align*}
& \|\psi-\phi\|_{L_{2}(Q)} \leq B h^{s} \\
& \quad \text { where } \quad B:=\left|\int_{-1}^{1} \frac{K(u)|u|^{s}}{(s-1)!} d u\left(\sup _{\left|x-x_{0}\right| \leq h}\left|f_{0}^{(s)}(x)\right|+\frac{C_{M}}{\Delta}+\mathfrak{C} h\right)\right| \tag{34}
\end{align*}
$$

and this term no longer depends on $Q$.

## D. 3 Surrogate efficiency

We now turn to the numerator and the denominator from the first term of (33). First, by the choice of our differentiable paths, the derivative with respect to $t$ is given by $\Phi_{0}(x) f_{0}(x)$. This does not depend on $t$ as we constructed a linear path. Hence the numerator of (33) is given by

$$
\sqrt{\left|\int_{0 \leq t<\mathfrak{B}} \phi^{\prime}\left(\tilde{f}_{t}\right) d Q\right|^{2}}=\sqrt{\left|\int \frac{1}{h} K\left(\frac{x-x_{0}}{h}\right) \Phi_{0}(x) f_{0}(x) d x\right|^{2}} .
$$

This result also comes directly from the fact that the approximation functional $\phi$ is pathwise differentiable with its efficient influence function given by $\Phi_{0}$. We can further write out this term as

$$
\begin{aligned}
\int \frac{1}{h} K\left(\frac{x-x_{0}}{h}\right) \Phi_{0}(x) f_{0}(x) d x & =\int \frac{1}{h^{2}} K^{2}\left(\frac{x-x_{0}}{h}\right) f_{0}(x) d x-\left(\int h^{-1} K\left(\frac{x-x_{0}}{h}\right) f_{0}(x) d x\right)^{2} \\
& =h^{-1} \int_{-1}^{1} K^{2}(u) f_{0}\left(x_{0}+u h\right) d u-\left(\int_{-1}^{1} K(u) f_{0}\left(x_{0}+u h\right) d u\right)^{2}
\end{aligned}
$$

We introduce the notation

$$
\begin{equation*}
V_{1}:=\left|\int_{-1}^{1} K^{2}(u) f_{0}\left(x_{0}+u h\right) d u-h\left(\int_{-1}^{1} K(u) f_{0}\left(x_{0}+u h\right) d u\right)^{2}\right|, \tag{35}
\end{equation*}
$$

and the numerator can be denoted by $h^{-1} V_{1}$. We now move onto the denominator. First, the Fisher information of the dilation $Q$ is simply given by

$$
\begin{equation*}
\mathcal{I}(Q)=\frac{4 \pi^{2}}{\mathfrak{B}^{2}} \quad \text { where } \quad \mathfrak{B}:=\left(\delta h^{s} \Delta \wedge C_{M} h^{s+1}\right) . \tag{36}
\end{equation*}
$$

The Fisher information associated with $\tilde{f}_{t}$ under the $n$ IID observations is given by

$$
\begin{aligned}
& \int\left(\frac{\frac{d}{d t} \widetilde{f}_{t}^{n}\left(x_{1}, \ldots, x_{n}\right)}{\widetilde{f}_{t}^{n}\left(x_{1}, \ldots, x_{n}\right)}\right)^{2} \widetilde{f}_{t}^{n}\left(x_{1}, \ldots, x_{n}\right) d\left(x_{1}, \ldots, x_{n}\right) \\
&=\int\left(\frac{\widetilde{f}_{t}^{\prime}\left(x_{1}\right) \widetilde{f}_{t}\left(x_{2}, \ldots, x_{n}\right)+\ldots+\widetilde{f}_{t}^{\prime}\left(x_{n}\right) \widetilde{f}_{t}\left(x_{1}, \ldots, x_{n-1}\right)}{\widetilde{f}_{t}^{n}\left(x_{1}, \ldots, x_{n}\right)}\right)^{2} \widetilde{f}_{t}^{n}\left(x_{1}, \ldots, x_{n}\right) d\left(x_{1}, \ldots, x_{n}\right) \\
&=\int\left(\sum_{i=1}^{n} \frac{\widetilde{f}_{t}^{\prime}\left(x_{i}\right)}{\widetilde{f_{t}}\left(x_{i}\right)}\right)^{2} \widetilde{f}_{t}^{n}\left(x_{1}, \ldots, x_{n}\right) d\left(x_{1}, \ldots, x_{n}\right) \\
&=n \int\left(\frac{\widetilde{f}_{t}^{\prime}(x)}{\widetilde{f}_{t}(x)}\right)^{2} \widetilde{f_{t}}(x) d x+\sum_{i \neq j} \int \widetilde{f_{t}^{\prime}}\left(x_{i}\right) d x_{i} \int \widetilde{f}_{t}^{\prime}\left(x_{j}\right) d x_{j}
\end{aligned}
$$

Since the choice of our path implies $\widetilde{f_{t}^{\prime}}=\Phi_{0}(x) f_{0}(x)$, which integrates to zero, the Fisher information for our path for each $t \in \Theta_{0}$ is given by

$$
\mathcal{I}(t)=n \int \frac{\Phi_{0}^{2}(x) f_{0}^{2}(x)}{\left(1-t \Phi_{0}(x)\right) f_{0}(x)} d x=n \int \frac{\Phi_{0}^{2}(x) f_{0}(x)}{\left(1-t \Phi_{0}(x)\right)} d x
$$

The denominator of (33) is thus given by

$$
\begin{aligned}
\mathcal{I}(Q) & +n \int_{\Theta_{0}} \mathcal{I}\left(\widetilde{f}_{t}\right) d Q(t) \\
& =\frac{4 \pi^{2}}{\mathfrak{B}^{2}}+n \int_{0 \leq t \leq \mathfrak{B}} \int \frac{\Phi_{0}^{2}(x) f_{0}(x)}{(1-t \Phi(x))} d x d Q(t) \\
& =\frac{4 \pi^{2}}{\mathfrak{B}^{2}}+n \int_{-1}^{1} \int \frac{\Phi_{0}^{2}(x) f_{0}(x)}{\left(1-(\mathfrak{B} / 2+t \mathfrak{B} / 2) \Phi_{0}(x)\right)} d x d Q_{0}(t) .
\end{aligned}
$$

As it follows that

$$
\begin{aligned}
& \int \frac{\Phi_{0}^{2}(x) f_{0}(x)}{\left(1-(\mathfrak{B} / 2+t \mathfrak{B} / 2) \Phi_{0}(x)\right)} d x \\
& \quad \leq \int_{-1}^{1} \frac{h^{-1} K^{2}(u) f_{0}\left(x_{0}+u h\right)}{\left(1-(\mathfrak{B} / 2+t \mathfrak{B} / 2)\left\{K(u) f_{0}\left(x_{0}+u h\right)-\int K\left(u^{\prime}\right) f_{0}\left(x_{0}+u^{\prime} h\right) d u^{\prime}\right\}\right)} d u \\
& \quad \leq \int_{-1}^{1} \frac{h^{-1} K^{2}(u) f_{0}\left(x_{0}+u h\right)}{\left(1-(\mathfrak{B} / 2+t \mathfrak{B} / 2) K(u) f_{0}\left(x_{0}+u h\right)\right)} d u,
\end{aligned}
$$

we conclude

$$
\mathcal{I}(Q)+n \int_{\Theta_{0}} \mathcal{I}\left(\widetilde{f}_{t}\right) d Q(t) \leq \frac{4 \pi^{2}}{\mathfrak{B}^{2}}+n h^{-1} V_{2}
$$

where

$$
\begin{equation*}
V_{2}:=\int_{-1}^{1} \int_{-1}^{1} \frac{K^{2}(u) f_{0}\left(x_{0}+u h\right)}{\left(1-(\mathfrak{B} / 2+t \mathfrak{B} / 2) K(u) f_{0}\left(x_{0}+u h\right)\right)} d u d Q_{0}(t) . \tag{37}
\end{equation*}
$$

## D. 4 Optimization

Putting together all intermediate results (34), (35), (36), and (37) together and plugging them into (33), we obtain

$$
\inf _{T} \sup _{f \in U(\delta)} \mathbb{E}_{f}|T(X)-\psi(f)|^{2} \geq\left[\sqrt{\frac{h^{-2} V_{1}^{2}}{\mathfrak{B}^{-2} 4 \pi^{2}+n h^{-1} V_{2}}}-\sqrt{h^{2 s} B^{2}}\right]_{+}^{2}
$$

Now we consider two cases.
When $\mathfrak{B}=C_{M} h^{s+1}$. In this case, the lower bound becomes

$$
\begin{aligned}
& {\left[\sqrt{\frac{V_{1}^{2}}{C_{M}^{-2} h^{-2 s} 4 \pi^{2}+n h V_{2}}}-\sqrt{h^{2 s} B^{2}}\right]_{+}^{2}} \\
& \quad=\left[\sqrt{\frac{V_{1}^{2} h^{2 s}}{C_{M}^{-2} 4 \pi^{2}+n h^{2 s+1} V_{2}}}-\sqrt{h^{2 s} B^{2}}\right]_{+}^{2} \\
& \quad=\frac{h^{2 s}}{C_{M}^{-2} 4 \pi^{2}+n h^{2 s+1} V_{2}}\left[\sqrt{V_{1}^{2}}-\sqrt{B^{2}\left(C_{M}^{-2} 4 \pi^{2}+n h^{2 s+1} V_{2}\right)}\right]_{+}^{2} .
\end{aligned}
$$

Recall that our lower bound holds for any $h$, we choose an optimal choice of $h$ for the term outside of the bracket. The optimal choice here is given by

$$
h=\left(\frac{8 s C_{M}^{-2} \pi^{2}}{n V_{2}}\right)^{\frac{1}{2 s+1}}
$$

and the corresponding lower bound is

$$
\begin{aligned}
\inf _{T} \sup _{f \in U(\delta)} \mathbb{E}_{f}\left|T(X)-f\left(x_{0}\right)\right|^{2} & \geq \frac{8 s^{2 s /(2 s+1)}}{(4+8 s)}\left(C_{M}^{-2} \pi^{2}\right)^{\frac{-1}{2 s+1}}\left(\frac{1}{n V_{2}}\right)^{\frac{2 s}{2 s+1}}\left[V_{1}-B C_{M}^{-1} \pi \sqrt{(4+8 s)}\right]_{+}^{2} \\
& =\frac{8 s^{2 s /(2 s+1)}}{(4+8 s)} \pi^{-\frac{2}{2 s+1}} C_{M}^{-\frac{4 s}{2 s+1}}\left(\frac{1}{n V_{2}}\right)^{\frac{2 s}{2 s+1}}\left[V_{1} C_{M}-B \pi \sqrt{(4+8 s)}\right]_{+}^{2}
\end{aligned}
$$

The above inequality holds for any $n \geq 1$. To derive a simpler asymptotic constant, we multiply both sides on the inequality by $n^{2 s /(2 s+1)}$ and consider the limit as $n \longrightarrow \infty$. First, we recall from the definition $\mathfrak{B} \longrightarrow 0$ as $n \longrightarrow \infty$ and thus

$$
\begin{aligned}
V_{1}, V_{2} & \longrightarrow \int K^{2}(u) d u f_{0}\left(x_{0}\right) \quad, \quad C_{M} \longrightarrow\left(\frac{M\left|f_{0}^{(s)}\left(x_{0}\right)\right|}{\left|K^{(s)}(0)\right| f_{0}\left(x_{0}\right)}\right) \quad \text { and } \\
B & \longrightarrow\left|f_{0}^{(s)}\left(x_{0}\right)\right|\left|\int_{-1}^{1} \frac{K(u)|u|^{s}}{(s-1)!} d u\left(1+\frac{M \int_{-1}^{1}\left|K^{(s)}(u)\right| d u}{\left|K^{(s)}(0)\right|}\right)\right|
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \liminf _{n \longrightarrow} \inf _{T} \sup _{f \in U(\delta)} n^{2 s /(2 s+1)} \mathbb{E}_{f}\left|T(X)-f\left(x_{0}\right)\right|^{2} \\
& \geq \\
& \geq C(s) f_{0}\left(x_{0}\right)^{2 s /(2 s+1)}\left|f_{0}^{(s)}\left(x_{0}\right)\right|^{2 /(2 s+1)}\left(\frac{\left|K^{(s)}(0)\right|}{M}\right)^{\frac{4 s}{2 s+1}}\left(\int K^{2}(u) d u\right)^{2 s /(2 s+1)} \\
& \quad \quad\left[\frac{M \int K^{2}(u) d u}{\left|K^{(s)}(0)\right|}-\sqrt{\pi^{2}(4+8 s)}\left|\int_{-1}^{1} \frac{K(u)|u|^{s}}{(s-1)!} d u\left(1+\frac{M \int_{-1}^{1}\left|K^{(s)}(u)\right| d u}{\left|K^{(s)}(0)\right|}\right)\right|\right]_{+}^{2} \\
& =C(s, M, K) f_{0}\left(x_{0}\right)^{2 s /(2 s+1)}\left|f_{0}^{(s)}\left(x_{0}\right)\right|^{2 /(2 s+1)}
\end{aligned}
$$

where $C(s)$ is a constant only depending on $s$ and

$$
\begin{aligned}
C(s, M, K):= & \frac{8 s^{2 s /(2 s+1)}}{(4+8 s)} \pi^{-\frac{2}{2 s+1}}\left(\frac{\left|K^{(s)}(0)\right|}{M}\right)^{\frac{4 s}{2 s+1}}\left(\int K^{2}(u) d u\right)^{2 s /(2 s+1)} \\
& {\left[\frac{M \int K^{2}(u) d u}{\left|K^{(s)}(0)\right|}-\sqrt{\pi^{2}(4+8 s)}\left|\int_{-1}^{1} \frac{K(u)|u|^{s}}{(s-1)!} d u\left(1+\frac{M \int_{-1}^{1}\left|K^{(s)}(u)\right| d u}{\left|K^{(s)}(0)\right|}\right)\right|\right]_{+}^{2} }
\end{aligned}
$$

Crucially, the expression above does not depend on $f_{0}$. Finally, we can take $\mathfrak{B}=C_{M} h^{s+1}$ when $\Delta \delta / C_{M}>h$, meaning that

$$
\delta>\frac{C_{M}}{\Delta}\left(\frac{8 s \pi^{2}}{C_{M}^{2} V_{2}}\right)^{\frac{1}{2 s+1}} n^{-\frac{1}{2 s+1}}
$$

This is certainly the case when $\delta=c_{0} n^{-r}$ for any $r<\frac{1}{2 s+1}$. When $r=\frac{1}{2 s+1}$ and $c_{0}>$ $\frac{C_{M}}{\Delta}\left(\frac{8 s \pi^{2}}{C_{M}^{2} V_{2}}\right)^{\frac{1}{2 s+1}}$, the above bound remains valid. We consider the alternative regime of $\mathfrak{B}$ next.

When $\mathfrak{B}=\Delta h^{s} \delta$ This is the case when $\Delta \delta / C_{M}<h$. In this case, we consider the choice $h \downarrow \Delta \delta / C_{M}$. Then the corresponding lower bound is

$$
\begin{aligned}
& \inf _{T} \sup _{f \in U(\delta)} \mathbb{E}_{f}|T(X)-\psi(f)|^{2} \\
& \quad \geq\left[\sqrt{\frac{V_{1}^{2}}{\Delta^{-2} h^{-2 s}(h / \delta)^{2} 4 \pi^{2}+n h V_{2}}}-\sqrt{h^{2 s} B^{2}}\right]_{+}^{2} \\
& \quad \geq\left[\sqrt{\frac{V_{1}^{2}\left(\Delta c_{0} n^{-r} / C_{M}\right)^{2 s}}{C_{M}^{-2} 4 \pi^{2}+n\left(\Delta c_{0} n^{-r} / C_{M}\right)^{2 s+1} V_{2}}}-\sqrt{\left(\Delta c_{0} n^{-r} / C_{M}\right)^{2 s} B^{2}}\right]_{+}^{2} \\
& \quad=\frac{1}{n^{1-r}}\left[\sqrt{\frac{V_{1}^{2}\left(\Delta c_{0} n^{-r} / C_{M}\right)^{2 s}}{C_{M}^{-2} 4 \pi^{2} /\left(n^{1-r}\right)+\left(\left(\Delta c_{0} / C_{M}\right)\left(\Delta c_{0} n^{-r} / C_{M}\right)^{2 s}\right) V_{2}}}-\sqrt{n^{1-r}\left(\Delta c_{0} n^{-r} / C_{M}\right)^{2 s} B^{2}}\right]_{+}^{2} .
\end{aligned}
$$

Multiplying both sides of the expression by $n^{1-r}$ and taking the limit as $n \longrightarrow \infty$, we conclude that

$$
\liminf _{n \longrightarrow \infty} \inf _{T} \sup _{f \in U(\delta)} n^{1-r} \mathbb{E}_{f}|T(X)-\psi(f)|^{2} \geq 0
$$

since the second term in the square bracket is divergent as $n^{1-r} \longrightarrow \infty$. We can take $\mathfrak{B}=\Delta h^{s} \delta$ when $\Delta \delta / C_{M}<h$, which is the case $\delta=c_{0} n^{-r}$ for any $r>\frac{1}{2 s+1}$. When $r=\frac{1}{2 s+1}$ and $c_{0}<$ $\frac{C_{M}}{\Delta}\left(\frac{8 s \pi^{2}}{C_{M}^{2} V_{2}}\right)^{\frac{1}{2 s+1}}$, we have

$$
\liminf _{n \longrightarrow \infty} \inf _{T} \sup _{f \in U(\delta)} n^{2 s /(2 s+1)} \mathbb{E}_{f}|T(X)-\psi(f)|^{2} \geq 0
$$

and thus the lower bound does not contradict for $r=1 /(2 s+1)$ regardless of the constant $c_{0}$. Hence our derivation leads to a trivial lower bound when $r \geq 1 /(2 s+1)$.

## Supplement E Optimization under absolute moment constraints

In this section, we aim to derive a concrete constant for the lower bound given by (16) in the main text. Suppose $n$ IID observation is drawn from $P_{0}$, which belongs to the local model $\left\{P_{\theta}:|\theta|<\delta\right\}$ for any $\delta>0$. The functional of interest is $\psi(\theta)=\max (0, \theta)$, which is non-smooth at $\theta=0$. We apply Theorem 8 and obtain the lower bound

$$
\inf _{T} \sup _{|\theta|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq \inf _{Q \in \mathcal{Q}} \frac{\left(\int_{-\delta}^{\delta} I(t>0) q(t) d t\right)^{2}}{\mathcal{I}(Q)+n \int_{-\delta}^{\delta} \mathcal{I}(t) q(t) d t}
$$

where $\mathcal{Q} \equiv \mathcal{Q}(-\delta, \delta)$ is the set of "nice" probability measures on $(-\delta, \delta)$ that satisfies Definition 1 . We further define the density function $q$ as the dilation of the probability density $\nu$ such that $q(t):=\delta^{-1} \nu(t / \delta)$ for $\delta>0$. The density $\nu$ is defined on $[-1,1]$. Then by simple change of variables with $u=t / \delta$, we obtain

$$
\begin{aligned}
\int_{-\delta}^{\delta} I(t>0) q(t) d t & =\int_{-\delta}^{\delta} I(t>0) \delta^{-1} \nu(t / \delta) d t=\int_{-1}^{1} I(u>0) \nu(u) d u \\
\int_{-\delta}^{\delta} n \mathcal{I}(t) q(t) d t & =\int_{-\delta}^{\delta} n \mathcal{I}(t) \delta^{-1} \nu(t / \delta) d t=\int_{-1}^{1} n \mathcal{I}(\delta u) \nu(u) d u
\end{aligned}
$$

and $\delta^{2} \mathcal{I}(Q)=\mathcal{I}(\nu)$. Plugging them in, we obtain

$$
\inf _{T} \sup _{|\theta|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq \inf _{\nu \in \mathcal{V}(-1,1)} \frac{\left(\int_{-1}^{1} I(u>0) \nu(u) d u\right)^{2}}{\delta^{-2} \mathcal{I}(\nu)+n \int_{-1}^{1} \mathcal{I}(\delta u) \nu(u) d u}
$$

This proves the inequality in the main text.
We now provide a concrete lower bound given by (17). To make progress, we bound the denominator with the largest Fisher information $\mathcal{I}(t)$ over the local model $|t|<\delta$. When $\left\{P_{t}: t \in\right.$ $\Theta\}$ is a location model, the Fisher information is a constant for all $t \in \Theta$. We then have that

$$
\begin{align*}
\inf _{\nu \in \mathcal{V}(-1,1)} \frac{\left(\int_{-1}^{1} I(u>0) \nu(u) d u\right)^{2}}{\delta^{-2} \mathcal{I}(\nu)+n \int_{-1}^{1} \mathcal{I}(\delta u) \nu(u) d u} & \geq \inf _{\nu \in \mathcal{V}(-1,1)} \frac{\left(\int_{-1}^{1} I(u>0) \nu(u) d u\right)^{2}}{\delta^{-2} \mathcal{I}(\nu)+n \int_{-1}^{1}\left\{\sup _{t \in[-\delta, \delta]} \mathcal{I}(t)\right\} \nu(u) d u} \\
& =\sup _{a \in[0,1]} \inf _{\mathcal{I}(\nu)} \frac{a^{2}}{\delta^{-2} \mathcal{I}(\nu)+\sup _{t \in[-\delta, \delta]} n \mathcal{I}(t)} \tag{38}
\end{align*}
$$

where the infimum minimizes the Fisher information of the density $\nu$ under the constraint that $\int_{0}^{1} \nu(t) d t=a$. In other words, we need to solve the following problem:

$$
\inf _{\nu} \int_{-1}^{1} \frac{\left(\nu^{\prime}(t)\right)^{2}}{\nu(t)} d t \text { i.e., } \nu \text { is absolutely continuous, } \int_{0}^{1} \nu(t) d t=a \text {, and } \lim _{t \longrightarrow \pm \delta} \nu(t)=0 \text {. }
$$

The last constraint is introduced so $\nu$ satisfies Definition 1 required for the van Trees inequality. Theorem 2.1 of Ernst (2017) implies that the minimum Fisher information under the constraint satisfies

$$
-2 \eta^{\prime}(t)-\eta^{2}(t)=B_{1} I\{-1 \leq t \leq 0\}+B_{2} I\{0<t \leq 1\}
$$

where $\eta(t)=\nu^{\prime}(t) / \nu(t)$ and $B_{1}, B_{2}$ are constants. The general solution of this first-order nonhomogeneous ordinary differential equation is given by

$$
\eta(t)=\nu^{\prime}(t) / \nu(t)= \begin{cases}\sqrt{B_{1}} \tan \left(\frac{\sqrt{B_{1}}}{2}\left(c_{1}-t\right)\right) \\ \sqrt{B_{2}} \tan \left(\frac{\sqrt{B_{2}}}{2}\left(c_{2}-t\right)\right) & \text { where }-1 \leq x \leq 0 \\ \text { where } 0<x \leq 1\end{cases}
$$

for constants $c_{1}, c_{2}$. The general solution $\nu$ is then given by

$$
\nu(t)= \begin{cases}C_{1} \cos ^{2}\left(\frac{\sqrt{B_{1}}}{2}\left(t-c_{1}\right)\right) & \text { where }-1 \leq x \leq 0 \\ C_{2} \cos ^{2}\left(\frac{\sqrt{B_{2}}}{2}\left(t-c_{2}\right)\right) & \text { where } 0<x \leq 1\end{cases}
$$

for constants $C_{1}, C_{2}$. In other words, combining two squared cosine functions gives the minimum Fisher information prior $\nu$. We note that a squared cosine function minimizes the Fisher information over distributions supported on $[-1,1]$ (Uhrmann-Klingen, 1995). In our setting, the density $\nu$ can have support over a smaller interval.

To simplify the derivation, we focus on a case when $C \equiv C_{1}=C_{2}, B \equiv B_{1}=B_{2}$, and $c \equiv c_{1}=c_{2}$. In other words, we consider a location-scale family of squared cosine densities whose support is contained in $[-1,1]$ such that $\operatorname{supp}(\nu) \subseteq[-1,1]$. Although we investigated other cases, this simplification still resulted in the best constant. We denote the left-end of the support as $s_{-} \leq 0$ and the right-end as $s_{+} \geq 0$. The width of the support is defined as $\left(s_{+}-s_{-}\right)$. First, by the assumption that the density vanishes towards the boundary, we have

$$
\begin{gathered}
C \cos ^{2}\left(\frac{\sqrt{B}}{2}\left(s_{+}-c\right)\right)=0 \Longrightarrow \sqrt{B}\left(s_{+}-c\right)=\pi \text { and } \\
C \cos ^{2}\left(\frac{\sqrt{B}}{2}\left(s_{-}-c\right)\right)=0 \Longrightarrow \sqrt{B}\left(s_{-}-c\right)=-\pi
\end{gathered}
$$

Putting together, we have $\sqrt{B}\left(s_{+}-s_{-}\right)=2 \pi$ and $c=\left(s_{+}-s_{-}\right) / 2$. Next, by the constraint of $\int_{0}^{1} \nu(t) d t=a$, we have

$$
\begin{align*}
& C \int_{0}^{s_{+}} \cos ^{2}\left(\frac{\sqrt{B}}{2}(x-c)\right) d t=\frac{C s_{+}}{2}-\frac{C}{2 \sqrt{B}} \sin (-c \sqrt{B})=a, \text { and }  \tag{39}\\
& C \int_{s_{-}}^{0} \cos ^{2}\left(\frac{\sqrt{B}}{2}(x-c)\right) d t=-\frac{C s_{-}}{2}+\frac{C}{2 \sqrt{B}} \sin (-c \sqrt{B})=1-a \tag{40}
\end{align*}
$$

Putting together, we have $C=2 /\left(s_{+}-s_{-}\right)$. Finally, we derive the Fisher information for this parametric family. First, for each $t \in\left[s_{-}, s_{+}\right]$,

$$
\frac{\left\{\nu^{\prime}(t)\right\}^{2}}{\nu(t)}=\frac{B C^{2} \cos ^{2}\left(\frac{\sqrt{B}}{2}(t-c)\right) \sin ^{2}\left(\frac{\sqrt{B}}{2}(t-c)\right)}{C \cos ^{2}\left(\frac{\sqrt{B}}{2}(t-c)\right)}=B C \sin ^{2}\left(\frac{\sqrt{B}}{2}(t-c)\right)
$$

This implies that

$$
\begin{equation*}
\mathcal{I}(\nu)=\int_{s_{-}}^{s^{+}} B C \sin ^{2}\left(\frac{\sqrt{B}}{2}(t-c)\right) d t=\frac{B C}{2}\left(s_{+}-s_{-}\right)=\frac{4 \pi^{2}}{\left(s_{+}-s_{-}\right)^{2}} \tag{41}
\end{equation*}
$$

Hence, the Fisher information only depends on the width of the support. It thus remains to characterize the width of a squared cosine prior under the constraint $\int_{0}^{1} \nu(t) d t=a$. Plugging in the reduced expressions into (39) and (40), we obtain

$$
\begin{gathered}
\frac{C s_{+}}{2}-\frac{C}{2 \sqrt{B}} \sin (-c \sqrt{B})=a \Longrightarrow \frac{s_{+}}{s_{+}-s_{-}}-\frac{\sin \left(-\pi \frac{s_{+}+s_{-}}{s_{+}-s_{-}}\right)}{2 \pi}=a, \text { and } \\
-\frac{C s_{-}}{2}+\frac{C}{2 \sqrt{B}} \sin (-c \sqrt{B})=1-a \Longrightarrow-\frac{s_{-}}{s_{+}-s_{-}}+\frac{\sin \left(-\pi \frac{s_{+}+s_{-}}{s_{+}-s_{-}}\right)}{2 \pi}=1-a .
\end{gathered}
$$

Putting together, we obtain

$$
\begin{equation*}
Y_{a}-\frac{\sin \left(-\pi Y_{a}\right)}{\pi}=2 a-1 \text { where } Y_{a}=\frac{s_{+}+s_{-}}{s_{+}-s_{-}} . \tag{42}
\end{equation*}
$$

This expression is known as the Kepler equation and there is no closed-form solution for the inverse problem (Kepler, 1609). We thus provide a computational method to estimate the smallest Fisher information given $Y_{a}$ for each $a \in[0,1]$. Once $Y_{a}$ is estimated from $a$, it follows that

$$
\frac{s_{+}+s_{-}}{s_{+}-s_{-}}=Y_{a} \Longleftrightarrow\left(1-Y_{a}\right) s_{+}=-\left(1+Y_{a}\right) s_{-} .
$$

This concludes that

$$
s_{+}-s_{-}=\frac{2}{Y_{a}+1} s_{+}=-\frac{2}{1-Y_{a}} s_{-} .
$$

As the minimum Fisher information is achieved by maximizing the width (see Equation 41), we take $s_{+}=1$ when $a>0.5$ or $s_{-}=-1$ when $a \leq 0.5$. Denoting by $W_{a}$ the width of the support of $\nu$ that minimizes the Fisher information for each $a \in[0,1]$, we conclude that

$$
\inf \left\{\mathcal{I}(\nu): \int_{0}^{1} \nu(t) d t=a\right\}=4 \pi^{2} / W_{a}^{2} \quad \text { where } \quad W_{a}= \begin{cases}2 /\left(Y_{a}+1\right) & \text { when } a>0.5 \\ 2 /\left(1-Y_{a}\right) & \text { when } a \leq 0.5\end{cases}
$$

We also note that (42) is an odd function and the expression for $W_{a}$ is an even function. Hence, we can modify them to

$$
Y_{a}-\frac{\sin \left(-\pi Y_{a}\right)}{\pi}=|2 a-1| \text { and } W_{a}=2 /\left(Y_{a}+1\right)
$$

Finally, plugging this expression back into the van Trees inequality given by (38), we conclude

$$
\inf _{T} \sup _{|\theta|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\theta)|^{2} \geq \sup _{a \in[0,1]} \frac{a^{2}}{4 \pi^{2} \delta^{-2} W_{a}^{2}+n \sup _{t \in[-\delta, \delta]} \mathcal{I}(t)}
$$

We note that $a=0.5$ corresponds to the minimum Fisher information prior while this may not necessarily maximize the minimax lower bound due to the second term in the denominator. Figure 3 provides a visual representation of the squared cosine priors and the minimum Fisher information.


Figure 3: Visual representation of the minimum Fisher information priors in the locationscale family of squared cosine densities. The left panel displays an example of the minimum Fisher information prior when $a=0.75$. The shaded area corresponds to the constraint $\int_{0}^{1} \nu(t) d t$, which is 0.75 in this case. The center panel displays the relationship between the constraint $a$ (or the shaded area in the left panel) and the width of the support of the minimum Fisher information priors. The dot in the plot corresponds to the density with $a=0.75$ displayed in the left panel. The right panel displays the relationship between the moment constraint $a$ and the minimum Fisher information. The dot on the plot again corresponds to the density with $a=0.75$ displayed in the left panel. The minimum Fisher information is achieved by setting $a=0.5$; however, this is not necessarily the least-favorable prior for minimax lower bounds.

## Supplement F Additional derivations related to examples

In this section, we provide the remaining derivation for Examples 3 and 4 from the main text.

## F. 1 Example 3

Proof of Lemma 16. Suppose $X_{1}, \ldots, X_{n}$ is drawn from $P_{\theta_{0}}$ that belongs to a local model $\left\{P_{\theta}\right.$ : $|\theta|<\delta, \theta \in \mathbb{R}\}$ for fixed $\delta>0$. We assume that this model is Hellinger differentiable with the Fisher information $\mathcal{I}(t)$ for $t \in(-\delta, \delta)$. Let $\psi\left(P_{\theta}\right)=\max \left(0, \theta^{\alpha}\right)$ for $0<\alpha \leq 1$.

For fixed $\delta>0$, we consider the diffeomorphism for the local univariate parameter set $\{\theta$ : $\left.\left|\theta-\theta_{0}\right|<\delta\right\}$ defined as $\varphi(t):=\theta_{0}+\delta \varphi_{0}(t)$ where $\varphi_{0}: \mathbb{R} \mapsto(-1,1)$. As Theorem 10 holds for any choice of diffeomorphism, we further assume that $\varphi_{0}(0)=0,\left\|\varphi_{0}^{\prime}\right\|_{\infty}<C$ for some constant, and $\varphi_{0}$ is an increasing function. The derivative of this mapping is $\varphi^{\prime}(t)=\delta \varphi_{0}^{\prime}(t)$. We now apply Theorem 10 to the composite function $(\psi \circ \varphi):=t \mapsto \max \left(0, \varphi^{\alpha}(t)\right)$,

$$
\begin{aligned}
\inf _{T} & \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\varphi(\theta))|^{2} \\
& \geq \limsup _{h \longrightarrow 0}\left[\frac{\left|\int_{\mathbb{R}}(\psi(\varphi(t))-\psi(\varphi(t-h))) d Q(t)\right|}{2 H\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)}-\left(\int_{\mathbb{R}}|\psi(\varphi(t))-\psi(\varphi(t-h))|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} .
\end{aligned}
$$

By Lemma 22 in Supplementary Material, we have

$$
H^{2}\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)=\frac{h^{2}}{4}\left(\mathcal{I}(Q)+\delta^{2} \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} n \mathcal{I}\left(\theta_{0}+\delta \varphi_{0}(t)\right) d Q(t)\right)+o\left(h^{2}\right) .
$$

We now analyze the behavior of $\psi(\varphi(t))-\psi(\varphi(t-h))$ as $|h| \longrightarrow 0$ for different values of $\theta_{0}$. As the functional $\psi$ is directionally differentiable, we need to consider $h \longrightarrow 0$ from left and right.

Case $1 \theta_{0}<0$ : When $h \longrightarrow 0$ from right, or $h \downarrow 0$, we have $I\left\{t: \theta_{0}+\delta \varphi_{0}(t-h)>0\right\} \Longrightarrow I\{t:$ $\left.\theta_{0}+\delta \varphi_{0}(t)>0\right\}$ by the monotonicity of $\varphi_{0}$. We also observe that

$$
\theta_{0}+\delta \varphi_{0}(t-h)>0 \Longleftrightarrow \theta_{0}>-\delta \varphi_{0}(t-h) \Longleftrightarrow\left|\theta_{0}\right| / \delta<\varphi_{0}(t-h) .
$$

We then have

$$
\begin{aligned}
\psi(\varphi(t))-\psi(\varphi(t-h))= & \psi(\varphi(t)) I\left\{t:\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\}-\psi(\varphi(t-h)) I\left\{t:\left|\theta_{0}\right| / \delta<\varphi_{0}(t-h)\right\} \\
= & \left\{\varphi(t)^{\alpha}-\varphi(t-h)^{\alpha}\right\} I\left\{t:\left|\theta_{0}\right| / \delta<\varphi_{0}(t-h)\right\} \\
& +\varphi(t)^{\alpha} I\left\{t: \varphi_{0}(t-h) \leq\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\} .
\end{aligned}
$$

We denote by $\lambda_{1}, \lambda_{2}$ constants $0 \leq \lambda_{1}, \lambda_{2} \leq 1$, possibly depening on $t$. By the mean value theorem,

$$
\begin{aligned}
\varphi(t)^{\alpha}-\varphi(t-h)^{\alpha} & =\alpha\left\{\left(1-\lambda_{1}\right) \varphi(t)+\lambda_{1} \varphi(t-h)\right\}^{\alpha-1} \varphi^{\prime}\left(t-\lambda_{2} h\right) h \\
& =\delta \alpha\left[\theta_{0}+\delta\left\{\left(1-\lambda_{1}\right) \varphi_{0}(t)+\lambda_{1} \varphi_{0}(t-h)\right\}\right]^{\alpha-1} \varphi_{0}^{\prime}\left(t-\lambda_{2} h\right) h .
\end{aligned}
$$

Therefore, as $h \downarrow 0$, we have

$$
\varphi(t)^{\alpha}-\varphi(t-h)^{\alpha}=\delta \alpha\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) h+o(h),
$$

which follows by the continuity of $\varphi_{0}$ and $\varphi_{0}^{\prime}$. Similarly as $h \downarrow 0$, we have

$$
\begin{aligned}
\varphi(t)^{\alpha} I\left\{t: \varphi_{0}(t-h) \leq\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\} & =\varphi(t)^{\alpha} I\left\{t: \varphi_{0}(t)-h \varphi_{0}^{\prime}\left(t_{*}\right) \leq\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\} \\
& =\varphi(t)^{\alpha} I\left\{t:-h \varphi_{0}^{\prime}\left(t_{*}\right) \leq\left|\theta_{0}\right| / \delta<0\right\}
\end{aligned}
$$

for some $t_{*} \in[t-h, t]$. For any fixed $\theta_{0}, \varphi_{0}$ and $\delta$, this term is zero for $h$ small enough. Putting together, we conclude that, as $h \downarrow 0$,

$$
\psi(\varphi(t))-\psi(\varphi(t-h))=\delta \alpha\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) h I\left\{t:\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\}+o(h) .
$$

We repeat the analysis for $h \longrightarrow 0$ from left, or $h \uparrow 0$. The analysis is identical except now we have $I\left\{t: \theta_{0}+\delta \varphi_{0}(t)>0\right\} \Longrightarrow I\left\{t: \theta_{0}+\delta \varphi_{0}(t-h)>0\right\}$. All results remain the same for $h \uparrow 0$ as well.

Finally, by plugging each term into the lower bound given by Theorem 10 and taking the limit in view of the dominated convergence theorem, we conclude that

$$
\begin{aligned}
& \inf _{T} \quad \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\varphi(\theta))|^{2} \\
& \quad \geq \limsup _{h \rightarrow 0}\left[\frac{\left|\int_{\mathbb{R}}(\psi(\varphi(t))-\psi(\varphi(t-h))) d Q(t)\right|}{2 H\left(\widetilde{\mathbb{P}}_{0}, \widetilde{\mathbb{P}}_{h}\right)}-\left(\int_{\mathbb{R}}|\psi(\varphi(t))-\psi(\varphi(t-h))|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} . \\
& \quad=\frac{\delta^{2} \alpha^{2}\left|\int_{\mathbb{R}}\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t:\left|\theta_{0}\right| / \delta<\varphi_{0}(t)\right\} d Q(t)\right|^{2}}{\mathcal{I}(Q)+\delta^{2} \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} n \mathcal{I}\left(\theta_{0}+\delta \varphi_{0}(t)\right) d Q(t)} .
\end{aligned}
$$

As the upper bound does not involve $Q, \varphi_{0}$, we take the supremum over them. When $\left|\theta_{0}\right|>\delta$, the indicator in the numerator evaluates zero for all values of $t \in \mathbb{R}$. The local minimax lower bound becomes zero in such cases.

Case $2 \theta_{0}>0$ : Similar to the case with $\theta_{0}<0$, we consider $h \longrightarrow 0$ from two directions. The results are analogous thus we only provide the argument for $h \downarrow 0$. When $h \downarrow 0$, we have $I\left\{t: \theta_{0}-\delta \varphi_{0}(t)>0\right\} \Longrightarrow I\left\{t: \theta_{0}-\delta \varphi_{0}(t-h)>0\right\}$. We then have

$$
\begin{aligned}
\psi(\varphi(t))-\psi(\varphi(t-h))=\{ & \left.\varphi(t)^{\alpha}-\varphi(t-h)^{\alpha}\right\} I\left\{t: \varphi_{0}(t)<\theta_{0} / \delta\right\} \\
& -\varphi(t-h)^{\alpha} I\left\{t: \varphi_{0}(t-h)<\theta_{0} / \delta \leq \varphi_{0}(t)\right\} .
\end{aligned}
$$

By the analogous argument from $\theta_{0}<0$, we can show that as $h \downarrow 0$,

$$
\psi(\varphi(t))-\psi(\varphi(t-h))=\delta \alpha\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) h I\left\{t: \varphi_{0}(t)<\theta_{0} / \delta\right\}+o(h)
$$

In view of the dominated convergence theorem, we conclude that

$$
\inf _{T} \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\varphi(\theta))|^{2} \geq \frac{\delta^{2} \alpha^{2}\left|\int_{\mathbb{R}}\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t: \varphi_{0}(t)<\theta_{0} / \delta\right\} d Q(t)\right|^{2}}{\mathcal{I}(Q)+\delta^{2} \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} n \mathcal{I}\left(\theta_{0}+\delta \varphi_{0}(t)\right) d Q(t)}
$$

Case $3 \theta_{0}=0$ : The analysis is also similar to the previous two cases. After the analogous derivation, we arrive at

$$
\inf _{T} \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\varphi(\theta))|^{2} \geq \frac{\delta^{2} \alpha^{2}\left|\int_{\mathbb{R}}\left\{\theta_{0}+\delta \varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t: \varphi_{0}(t)<\theta_{0} / \delta\right\} d Q(t)\right|^{2}}{\mathcal{I}(Q)+\delta^{2} \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} n \mathcal{I}\left(\theta_{0}+\delta \varphi_{0}(t)\right) d Q(t)}
$$

By plugging in $\theta_{0}=0$, we obtain that

$$
\inf _{T} \sup _{\left|\theta-\theta_{0}\right|<\delta} \mathbb{E}_{\theta}|T(X)-\psi(\varphi(\theta))|^{2} \geq \frac{\delta^{2 \alpha} \alpha^{2}\left|\int_{\mathbb{R}}\left\{\varphi_{0}(t)\right\}^{\alpha-1} \varphi_{0}^{\prime}(t) I\left\{t: \varphi_{0}(t)>0\right\} d Q(t)\right|^{2}}{\mathcal{I}(Q)+\delta^{2} \int_{\mathbb{R}}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} n \mathcal{I}\left(\delta \varphi_{0}(t)\right) d Q(t)} .
$$

## F. 2 Example 4

Suppose $X_{1}, \ldots, X_{n}$ is an IID observation from $P_{\theta_{0}}$, which belongs to a model $\{\operatorname{Unif}(0, \theta): 0<$ $\theta\}$. The local parameter set we consider is given by $\Theta=\left(\theta_{0}-c n^{-1}, \theta_{0}+c n^{-1}\right)$, we define the diffeomorephism in the form of $\varphi(t):=\theta_{0}+c n^{-1} \varphi_{0}(t)$ where $\varphi_{0}: \mathbb{R} \mapsto(-1,1)$ is invertible and differentiable increasing function. Based on the analogous application of Theorem 10 as in Example 3, we obtain, for any $h \in \mathbb{R}$,

$$
\begin{align*}
\inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\theta|^{2} & =\inf _{T} \sup _{t \in \mathbb{R}} \mathbb{E}_{P_{\varphi(t)}^{n}}|T(X)-\varphi(t)|^{2} \\
& \geq\left[\frac{\left|\int_{\mathbb{R}}(\varphi(t)-\varphi(t-h)) d Q(t)\right|}{2 H\left(\widetilde{\mathbb{P}}_{0}^{n}, \widetilde{\mathbb{P}}_{h}^{n}\right)}-\left(\int_{\mathbb{R}}|\varphi(t)-\varphi(t-h)|^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} \tag{43}
\end{align*}
$$

For each $c$, we assume that $n$ is a large enough constant such that $\theta_{0}-c n^{-1}>0$. By the mean value theorom, we have

$$
\varphi(t)-\varphi(t-h)=c n^{-1}\left(\varphi_{0}(t)-\varphi_{0}(t-h)\right)=c n^{-1} h \varphi_{0}^{\prime}(t+\lambda h)
$$

for some $\lambda \in[0,1]$. The derivation thus focuses primarily on the Hellinger distance between $\operatorname{Unif}(0, \varphi(t))$ and $\operatorname{Unif}(0, \varphi(t+h))$. First, the Hellinger distance associated with one observation is given by

$$
\begin{aligned}
2-2 \int d P_{\varphi(t)}^{1 / 2} d P_{\varphi(t+h)}^{1 / 2} & =2-2 \frac{\varphi(t)}{\varphi^{1 / 2}(t) \varphi^{1 / 2}(t+h)} \\
& =2-2 \frac{\varphi^{1 / 2}(t)+\varphi^{1 / 2}(t+h)-\varphi^{1 / 2}(t+h)}{\varphi^{1 / 2}(t+h)} \\
& =2 \frac{\varphi^{1 / 2}(t+h)-\varphi^{1 / 2}(t)}{\varphi^{1 / 2}(t+h)} \\
& =\frac{\left\{\lambda_{1} \varphi(t)+\left(1-\lambda_{1}\right) \varphi(t+h)\right\}^{-1 / 2} \varphi^{\prime}\left(t+\lambda_{2} h\right) h}{\varphi^{1 / 2}(t+h)} \\
& =\frac{c \varphi_{0}^{\prime}\left(t+\lambda_{2} h\right) h}{n \theta_{0}}+o(1 / n) .
\end{aligned}
$$

Furthermore, by the tensorization property of the Hellinger distance, we have

$$
\begin{aligned}
\liminf _{n \longrightarrow \infty} H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right) & =\liminf _{n \longrightarrow \infty}\left\{2-2\left(1-\frac{H^{2}\left(P_{\varphi(t)}, P_{\varphi(t+h)}\right)}{2}\right)^{n}\right\} \\
& =2-2 \exp \left(-\frac{c \varphi_{0}^{\prime}\left(t+\lambda_{2} h\right) h}{2 \theta_{0}}\right)
\end{aligned}
$$

The last step follows by $(1-x / n)^{n} \longrightarrow \exp (-x)$ as $n \longrightarrow \infty$. Therefore we have

$$
\begin{aligned}
\liminf _{n \longrightarrow \infty} H^{2}\left(\widetilde{\mathbb{P}}_{0}^{n}, \widetilde{\mathbb{P}}_{h}^{n}\right) & =\liminf _{n \longrightarrow \infty}\left\{H^{2}\left(Q_{h}, Q\right)+\int H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right) d Q^{1 / 2}(t+h) d Q^{1 / 2}(t)\right\} \\
& =2-2 \int \exp \left(-\frac{c \varphi_{0}^{\prime}(t+\lambda h) h}{2 \theta_{0}}\right) d Q^{1 / 2}(t+h) d Q^{1 / 2}(t)
\end{aligned}
$$

Putting together, we have

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} n^{2} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\theta|^{2} \\
& \geq {\left[\frac{c h\left|\int\left(\varphi_{0}^{\prime}(t+\lambda h)\right) d Q(t)\right|}{2\left(2-2 \int \exp \left(-\frac{c \varphi_{0}^{\prime}(t+\lambda h) h}{2 \theta_{0}}\right) d Q^{1 / 2}(t+h) d Q^{1 / 2}(t)\right)^{1 / 2}}\right.} \\
&\left.\quad-c h\left(\int\left\{\varphi_{0}^{\prime}(t+\lambda h)\right\}^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

Since the above inequality holds for any $h$, we denote by $h=\left(\theta_{0} / c\right) \eta$ and it still holds for any $\eta$. Under this parameterization, we have

$$
\begin{aligned}
& \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} n^{2} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\theta|^{2} \\
& \geq\left[\frac{\theta_{0} \eta\left|\int\left(\varphi_{0}^{\prime}\left(t+\lambda\left(\theta_{0} / c\right) \eta\right)\right) d Q(t)\right|}{2\left(2-2 \int \exp \left(-\frac{\varphi_{0}^{\prime}\left(t+\lambda\left(\theta_{0} / c\right) \eta\right) \eta}{2}\right) d Q^{1 / 2}\left(t+\left(\theta_{0} / c\right) \eta\right) d Q^{1 / 2}(t)\right)^{1 / 2}}\right. \\
& \left.-\theta_{0} \eta\left(\int\left\{\varphi_{0}^{\prime}\left(t+\lambda\left(\theta_{0} / c\right) \eta\right)\right\}^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2}
\end{aligned}
$$

for any $\eta$. Now, taking $c \longrightarrow \infty$, we have

$$
\begin{aligned}
& \liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \inf _{T} \sup _{\left|\theta-\theta_{0}\right|<c n^{-1}} n^{2} \mathbb{E}_{P_{\theta}^{n}}|T(X)-\theta|^{2} \\
& \quad \geq \sup _{\eta \in \mathbb{R}, Q, \varphi_{0}} \theta_{0}^{2}\left[\frac{\eta\left|\int \varphi_{0}^{\prime}(t) d Q(t)\right|}{2\left(2-2 \int \exp \left(-\varphi_{0}^{\prime}(t) \eta / 2\right) d Q(t)\right)^{1 / 2}}-\eta\left(\int\left\{\varphi_{0}^{\prime}(t)\right\}^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} .
\end{aligned}
$$

This concludes the claim. The lower bound obtained by the preceding result can be simplified to the following analytic form.
Proof of Proposition 3. Consider the following sequence of diffeomorephisms indexed by $\eta$ such that

$$
\eta \varphi_{0}^{\prime}(t ; \eta) \longrightarrow C
$$

as $\eta \longrightarrow \infty$. This is satisfied, for instance, by $2 / \pi \arctan (t / \eta)$. Then the lower bound can be simplified as

$$
\begin{aligned}
& \sup _{\eta \in \mathbb{R}, Q, \varphi_{0}}\left[\frac{\eta\left|\int \varphi_{0}^{\prime}(t) d Q(t)\right|}{2\left(2-2 \int \exp \left(-\varphi_{0}^{\prime}(t) \eta / 2\right) d Q(t)\right)^{1 / 2}}-\eta\left(\int\left\{\varphi_{0}^{\prime}(t)\right\}^{2} d Q(t)\right)^{1 / 2}\right]_{+}^{2} \\
& \quad \geq \sup _{C}\left[\frac{C}{2(2-2 \exp (-C / 2))^{1 / 2}}-C\right]_{+}^{2} \approx 0.0635^{2}
\end{aligned}
$$

by optimizing for $C$.
Proof of Proposition 4. We now apply Lemma 11 to derive the optimal constant for the lower bound. Consider two points in parameter space $\theta_{0}$ and $\theta_{0}+h n^{-1}$, and Lemma 11 implies

$$
\begin{equation*}
\liminf _{n \longrightarrow \infty} \sup _{|h|<c} n^{2} \mathbb{E}_{\theta_{0}+h n^{-1 / 2}}\left|T(X)-\psi\left(\theta_{0}+h n^{-1 / 2}\right)\right|^{2} \geq \liminf _{n \longrightarrow \infty}\left[\frac{1-H^{2}\left(P_{\theta_{0}+h n^{-1 / 2}}^{n}, P_{\theta_{0}}^{n}\right)}{4}\right]_{+} h^{2} \tag{44}
\end{equation*}
$$

for all $|h|<c$. An analogous derivation to Example 3 will yield

$$
\liminf _{n \longrightarrow \infty} H^{2}\left(P_{\theta_{0}+h n^{-1}}^{n}, P_{\theta_{0}}^{n}\right)=2-2 \exp \left(-\frac{h}{2 \theta_{0}}\right)
$$

for $n$ IID observations from two uniform distributions. Putting together, it follows that

$$
\begin{aligned}
\liminf _{c \longrightarrow \infty} \liminf _{n \longrightarrow \infty} \sup _{|h|<\infty} n^{2} \mathbb{E}_{\theta_{0}+h n^{-1 / 2}}\left|T(X)-\psi\left(\theta_{0}+h n^{-1 / 2}\right)\right|^{2} & \geq \sup _{|h|<c}\left[-\frac{1}{4}+\frac{1}{2} \exp \left(-\frac{h}{2 \theta_{0}}\right)\right]_{+} h^{2} \\
& \geq \sup _{|\eta|<\infty} 4 \theta_{0}^{2}\left[-\frac{1}{4}+\frac{1}{2} \exp (-\eta)\right]_{+} \eta^{2} \\
& \approx 0.0558 \theta_{0}^{2}
\end{aligned}
$$

where we parameterize $h=2 \eta \theta_{0}$ and optimize for $\eta$ to obtain the result.

## Supplement G Derivation of the risk of estimators

In this section, we provide exact upper bounds and lower bounds for the truncated Gaussian mean estimation. To recall, we consider the following estimation problem:

$$
\sup _{|\theta|<\delta} n \mathbb{E}_{\theta}|T(X)-\max (\theta, 0)|^{2}
$$

where we observe $n \geq 1$ IID observations $X:=X_{1}, \ldots, X_{n}$ from $N(0,1)$. We define a local model as $\{N(\theta, 1):|\theta|<\delta\}$ for $\delta>0$.

## G. 1 Lower bound under Gaussian prior and arctan diffeomorphism

In Example 3 (iii) with $\alpha=1$, we have derived that

$$
\inf _{T} \sup _{|\theta|<\delta} n \mathbb{E}_{\theta}\left|T(X)-\psi\left(P_{\theta}\right)\right|^{2} \geq \sup _{Q, \varphi_{0}} \frac{n \delta^{2}\left|\mathbb{E}_{Q} \varphi_{0}^{\prime}(t) I\left\{t: \varphi_{0}(t)>0\right\}\right|^{2}}{\mathcal{I}(Q)+\delta^{2} \mathbb{E}_{Q}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} \mathcal{I}\left(\delta \varphi_{0}(t)\right)} .
$$

The Fisher information of distributions in our local model is given by $\mathcal{I}(\theta)=n$ for all $|\theta|<\delta$ under $n$ IID observations. To simplify the derivation, we further restrict the choice of priors and diffeomorphism to be

$$
Q \in\left\{N\left(\mu, \sigma^{2}\right):(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_{+}\right\} \text {and } \varphi_{0} \in\{t \mapsto \pi / 2 \arctan (t / \eta): \eta>0\} .
$$

We define $\xi_{1}:=\mu / \eta$ and $\xi_{2}:=\sigma / \eta$. Since $\varphi_{0}^{\prime}(t)=2(\pi \eta)^{-1}\left(1+(t / \eta)^{2}\right)^{-1}$, we have

$$
\begin{aligned}
\mathbb{E}_{Q}\left\{\varphi_{0}^{\prime}(t)\right\}^{2} & =\int \frac{4}{\pi^{2} \eta^{2}}\left(\frac{1}{1+(t / \eta)^{2}}\right)^{2} \frac{1}{\sigma \sqrt{2 \pi}} \exp \left(-(t-\mu)^{2} /\left(2 \sigma^{2}\right)\right) d t \\
& =\frac{4}{\pi^{2} \eta^{2}} \int\left(\frac{1}{1+\left(\xi_{1}+u \xi_{2}\right)^{2}}\right)^{2} \phi(u) d u
\end{aligned}
$$

where $\phi$ is a density function of a standard normal distribution. Similarly,

$$
\begin{aligned}
\left|\mathbb{E}_{Q} \varphi_{0}^{\prime}(t) I\left(\varphi_{0}(t)>0\right)\right|^{2} & =\left|\int \frac{2}{\pi \eta}\left(\frac{1}{1+(t / \eta)^{2}}\right) \frac{1}{\sigma \sqrt{2 \pi}} \exp \left(-(t-\mu)^{2} /\left(2 \sigma^{2}\right)\right) I(t>0) d t\right|^{2} \\
& =\frac{4}{\pi^{2} \eta^{2}}\left|\int\left(\frac{1}{1+(\mu / \eta+u \sigma / \eta)^{2}}\right) \frac{1}{\sqrt{2 \pi}} \exp \left(-u^{2} / 2\right) I(\mu+u \sigma>0) d u\right|^{2} \\
& =\frac{4}{\pi^{2} \eta^{2}}\left|\int\left(\frac{1}{1+\left(\xi_{1}+u \xi_{2}\right)^{2}}\right) I\left(\xi_{1}+u \xi_{2}>0\right) \phi(u) d u\right|^{2}
\end{aligned}
$$

Using the fact that $\mathcal{I}(Q)=\sigma^{-2}$, the lower bound becomes

$$
\begin{aligned}
\frac{n \delta^{2}\left|\mathbb{E}_{Q} \varphi_{0}^{\prime}(t) I(t>0)\right|^{2}}{\left(1 / \sigma^{2}\right)+\delta^{2} n \mathbb{E}_{Q}\left\{\varphi_{0}^{\prime}(t)\right\}^{2}} & =\frac{4 n \xi_{2}^{2}\left|\int\left(1+\left(\xi_{1}+u \xi_{2}\right)^{2}\right)^{-1} I\left(u>-\xi_{1} / \xi_{2}\right) \phi(u) d u\right|^{2}}{\pi^{2} \delta^{-2}+4 n \int\left(1+\left(\xi_{1}+u \xi_{2}\right)^{2}\right)^{-2} \phi(u) d u} \\
& =\frac{4 n \xi_{2}^{2}\left|\mathbb{E}\left[\left(1+\left(\xi_{1}+Z \xi_{2}\right)^{2}\right)^{-1} I\left(Z>-\xi_{1} / \xi_{2}\right)\right]\right|^{2}}{\pi^{2} \delta^{-2}+4 n \mathbb{E}\left[\left(1+\left(\xi_{1}+Z \xi_{2}\right)^{2}\right)^{-2}\right]}
\end{aligned}
$$

where $Z \stackrel{d}{=} N(0,1)$. We conclude the claim by optimizing over $\left(\xi_{1}, \xi_{2}\right) \in \mathbb{R} \times \mathbb{R}_{+}$for given $\delta>0$ and $n \geq 1$.

## G. 2 Exact upper bounds

In this section, we provide the exact local risks of the following plug-in estimators, defined as

$$
\begin{aligned}
& S_{n}^{\text {plug-in }}:=\psi\left(\widehat{\theta}_{\text {MLE }}\right) \quad \text { where } \quad \widehat{\theta}_{\text {MLE }}:=\bar{X}_{n}=\frac{1}{n} \sum_{i=1}^{n} X_{i}, \text { and } \\
& S_{n}^{\text {pre-test }}:=\psi\left(\widehat{\theta}_{\text {pre-test }}\right) \quad \text { where } \quad \widehat{\theta}_{\text {pre-test }}:= \begin{cases}\bar{X}_{n} & \text { If }\left|\bar{X}_{n}\right| \geq n^{-1 / 4} \\
0 & \text { otherwise. }\end{cases}
\end{aligned}
$$

Proof of Proposition 5. First, we consider the plug-in MLE. By splitting the risk into two cases, we have

$$
\sup _{|\theta|<\delta} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}-\max (\theta, 0)\right|^{2}=\max \left\{\sup _{0 \leq \theta<\delta} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}-\theta\right|^{2}, \sup _{-\delta<\theta<0} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}\right|^{2}\right\} .
$$

For the first case, we have

$$
\begin{aligned}
\sup _{0 \leq \theta<\delta} & n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}-\theta\right|^{2} \\
& =\sup _{0 \leq \theta<\delta} n\left\{\mathbb{E}_{\theta}\left[\left|\bar{X}_{n}-\theta\right|^{2} I\left(\bar{X}_{n} \geq 0\right)\right]+\mathbb{E}_{\theta}\left[\theta^{2} I\left(\bar{X}_{n}<0\right)\right]\right\} \\
\quad & =\sup _{0 \leq \theta<\delta}\left\{\mathbb{E}_{\theta}\left[\left|n^{1 / 2}\left(\bar{X}_{n}-\theta\right)\right|^{2} I\left(n^{1 / 2}\left(\bar{X}_{n}-\theta\right) \geq-n^{1 / 2} \theta\right)\right]+n \theta^{2} \mathbb{E}_{\theta}\left[I\left(n^{1 / 2}\left(\bar{X}_{n}-\theta\right)<-n^{1 / 2} \theta\right)\right]\right\} .
\end{aligned}
$$

Since $Z:=n^{1 / 2}\left(\bar{X}_{n}-\theta\right) \stackrel{d}{=} N(0,1)$, we conclude

$$
\sup _{0 \leq \theta<\delta} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}-\theta\right|^{2}=\sup _{0 \leq \theta<\delta}\left\{\mathbb{E}_{\theta}\left[Z^{2} I\left(Z \geq-n^{1 / 2} \theta\right)\right]+n \theta^{2} \mathbb{E}_{\theta}\left[I\left(Z<-n^{1 / 2} \theta\right)\right]\right\} .
$$

Next, we claim that

$$
\begin{equation*}
\sup _{-\delta<\theta \leq 0} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}\right|^{2} \leq \sup _{0 \leq \theta<\delta} \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}-\theta\right|^{2}, \tag{45}
\end{equation*}
$$

that is, the risk is always greater when $\theta \geq 0$ than $\theta<0$. Following an analogous argument from $\theta \geq 0$, we obtain

$$
\begin{aligned}
\sup _{-\delta<\theta \leq 0} n \mathbb{E}_{\theta}\left|S_{n}^{\text {plug-in }}\right|^{2} & =\sup _{-\delta<\theta \leq 0} n \mathbb{E}_{\theta}\left[\bar{X}_{n}^{2} I\left(\bar{X}_{n} \geq 0\right)\right] \\
& =\sup _{-\delta \leq \theta<0} \mathbb{E}\left[\left(Z+n^{1 / 2} \theta\right)^{2} I\left(Z \geq-n^{1 / 2} \theta\right)\right]
\end{aligned}
$$

where $Z \stackrel{d}{=} N(0,1)$. Denoting by $\eta=n^{1 / 2} \theta$ and by $\phi$ the density function of a standard Gaussian distribution, it follows, for any $\eta \leq 0$,

$$
\mathbb{E}(Z+\eta)^{2} I(Z \geq-\eta)=\int_{-\eta}^{\infty}(z+\eta)^{2} \phi(z) d z \leq \int_{-\eta}^{\infty}(z+\eta)^{2} \phi(z+\eta) d z=\int_{0}^{\infty} \tilde{z}^{2} \phi(\tilde{z}) d \tilde{z} .
$$

The middle inequality follows since the density $\phi(z)$ is non-increasing on $0 \leq z$ and thus $\phi(z) \leq$ $\phi(z+\eta)$ for any $\eta \leq 0$. The last quantity is equivalent to $\mathbb{E}\left[Z^{2} I(Z \geq 0)\right]$ hence the risk of the estimator for any $\eta \leq 0$ is upper bounded by the case when $\eta=0$, or equivalently when $\theta=0$. Therefore, equation (45) is implied as desired. This concludes the claim.

For the pre-test estimator, the derivation is almost analogous except that we have

$$
\begin{aligned}
\sup _{0 \leq \theta<\delta} n \mathbb{E}_{\theta}\left|S_{n}^{\text {pre-test }}-\theta\right|^{2} & =\sup _{0 \leq \theta<\delta} n\left\{\mathbb{E}_{\theta}\left[\left|\bar{X}_{n}-\theta\right|^{2} I\left(\bar{X}_{n} \geq n^{-1 / 4}\right)\right]+\mathbb{E}_{\theta}\left[\theta^{2} I\left(\bar{X}_{n}<n^{-1 / 4}\right)\right]\right\} \\
& =\sup _{0 \leq \theta<\delta}\left\{\mathbb{E}_{\theta}\left[Z^{2} I\left(Z \geq n^{1 / 4}-n^{1 / 2} \theta\right)\right]+n \theta^{2} \mathbb{E}_{\theta}\left[I\left(Z<n^{1 / 4}-n^{1 / 2} \theta\right)\right]\right\}
\end{aligned}
$$

The remaining derivation is omitted.

## Supplement H Supporting lemmas on divergence metrics

In this section, we provide the supporting lemmas related to the local behaviors of the chi-squared divergence and the Hellinger distance. We provide the derivation for completeness and to unify terminology between Pollard (2023) and Polyanskiy and Wu (2022). Throughout this section, we use the following notation consistently; Let $\varphi: \mathbb{R}^{d} \mapsto \Theta$ be a continuously differentiable mapping and $\|\nabla \varphi\|_{\infty}<C$ for some universal constant. Unless specified otherwise, we denote the mixture probability measures by

$$
d \widetilde{\mathbb{P}}_{0}^{n}(x, t):=d P_{\varphi(t)}^{n}(x) d Q(t) \quad \text { and } \quad d \widetilde{\mathbb{P}}_{h}^{n}(x, t):=d P_{\varphi(t+h)}^{n}(x) d Q(t+h),
$$

where $P_{\varphi(t)^{n}}$ is an $n$-fold product measure of $P_{\varphi(t)}, Q(\cdot)$ is a probability measure on $\mathbb{R}^{d}$ with a bounded and absolutely continuous density function $q$ with respect to the base measure on $\Theta$.

We recall the regularity condition defined in Section 2:
Definition 3 (Multivariate absolute continuity). A function $\omega: \mathbb{R}^{d} \mapsto \mathbb{R}$ is absolutely continuous over an open $\mathbb{R}^{d}$-ball $B(\theta, \delta)$ if for all directions $u \in \mathbb{S}^{d-1}$, the induced univariate function $t \mapsto$ $\omega(\theta+t u)$ is absolutely continuous over $|t|<\delta$.

The following regularity conditions are placed:
(A1) There exits $\delta>0$ such that
(a) for $\nu$-almost everywhere, the mapping $t \mapsto d P_{t}$ is absolutely continuous over an open $\mathbb{R}^{d}$-ball $B([0], \delta)$ with the gradient $t \mapsto u^{\top} \dot{\rho}_{t, u}$ for each $u$,
(b) for all $u \in \mathbb{S}^{d-1}$, the gradient is continuous such that $\lim _{t \rightarrow 0} u^{\top} \dot{\rho}_{t, u}=u^{\top} \dot{\rho}_{0}$ for $\nu$-almost everywhere, and
(c) for all $0 \leq|t|<\delta$ and $u \in \mathbb{S}^{d-1}, d P_{0}(x)=0$ implies $u^{\top} \dot{\rho}_{t, u}(x)=0$, and

$$
\int \sup _{0 \leq t_{1}, t_{2}<\delta} \frac{\dot{\rho}_{t_{1}, u} \dot{\rho}_{t_{2}, u}^{\top}}{d P_{0}}<\infty .
$$

(A2) There exits $\delta>0$ such that
(a) for $\nu$-almost everywhere, the mapping $t \mapsto d P_{t}^{1 / 2}$ is absolutely continuous over an open $\mathbb{R}^{d}$-ball $B([0], \delta)$ with the gradient $t \mapsto u^{\top} \dot{\gamma}_{t, u}$ for each $u$,
(b) for all $u \in \mathbb{S}^{d-1}$, the gradient is continuous such that $\lim _{t \rightarrow 0} u^{\top} \dot{\gamma}_{t, u}=u^{\top} \dot{\gamma}_{0}$ for $\nu$-almost everywhere, and

$$
\int \sup _{0 \leq t_{1}, t_{2}<\delta} \dot{\gamma}_{t_{1}, u} \dot{\gamma}_{t_{2}, u}^{\top}<\infty
$$

Proof of $(\boldsymbol{A} 1) \Longrightarrow(\boldsymbol{A} 2)$. Given a small positive scalar $\eta>0$, we define $d P_{t, \eta}:=d P_{t}+\eta$. The resulting object is no longer a probability density since it does not integrate to one. Since $d P_{t, \eta}$ is bounded away from zero, the gradient of $d P_{t, \eta}^{1 / 2}$ exists and is given by $\frac{1}{2}\left\{\dot{\rho}_{t, u} /\left(d P_{t}+\eta\right)^{1 / 2}\right\}$. It now follows for any $0<b<\delta$ that

$$
d P_{t, \eta}^{1 / 2}(u b)-d P_{t, \eta}^{1 / 2}(0)=\int_{0}^{b} \frac{u^{\top} \dot{\rho}_{t, u}}{2\left(d P_{t}+\eta\right)^{1 / 2}} d t=\int_{0}^{b} \frac{u^{\top} \dot{\rho}_{t, u}}{2\left(d P_{t}+\eta\right)^{1 / 2}} I\left(d P_{t}>0\right) d t .
$$

The second equality follows by (A1)(c) as $d P_{0}(x)=0$ implies $u^{\top} \dot{\rho}_{t, u}(x)=0$, which justifies to insert the indicator $I\left(d P_{t}>0\right)$ inside the integral. Assuming for a moment that we can invoke the dominated convergence theorem, it follows that

$$
\lim _{\eta \longrightarrow 0}\left\{d P_{t, \eta}^{1 / 2}(u b)-d P_{t, \eta}^{1 / 2}(0)\right\}=\int_{0}^{b} \lim _{\eta \longrightarrow 0} \frac{u^{\top} \dot{\rho}_{t, u}}{2\left(d P_{t}+\eta\right)^{1 / 2}} I\left(d P_{t}>0\right) d t=\int_{0}^{b} \frac{u^{\top} \dot{\rho}_{t, u}}{2 d P_{t}^{1 / 2}} I\left(d P_{t}>0\right) d t
$$

Hence, we conclude (A2)(a) with

$$
\begin{equation*}
\dot{\gamma}_{t, u}=\frac{\dot{\rho}_{t, u}}{2 d P_{t}^{1 / 2}} I\left(d P_{t}>0\right) \tag{46}
\end{equation*}
$$

It thus remains to check the condition for the dominated convergence theorem. This follows since we have that

$$
\begin{aligned}
\int_{0}^{b}\left|\frac{u^{\top} \dot{\rho}_{t, u}}{2\left(d P_{t}+\eta\right)^{1 / 2}} I\left(d P_{t}>0\right)\right| d t & \leq \int_{0}^{b}\left|\frac{u^{\top} \dot{\rho}_{t, u}}{2\left(d P_{t}\right)^{1 / 2}} I\left(d P_{t}>0\right)\right| d t \\
& \leq\left(\frac{1}{2} \int_{0}^{b} \frac{u^{\top} \dot{\rho}_{t, u} \dot{\rho}_{t, u}^{\top} u}{d P_{t}} I\left(d P_{t}>0\right) d t\right)^{1 / 2}
\end{aligned}
$$

The last term is finite by (A1)(c) and hence the dominated convergence theorem holds. The second statement (A2)(b) follows directly from (46), (A1)(b) and (A1)(c).

Lemma 19. Assuming that the collection of paths $h \mapsto P_{\varphi(t+h)}$ satisfies (A1),

$$
\chi^{2}\left(P_{\varphi(t+h)}, P_{\varphi(t)}\right)=h^{\top} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) h^{\top}+o\left(\|h\|_{2}^{2}\right)
$$

as $\|h\|_{2} \longrightarrow 0$ with any mapping $\varphi$ with continuous derivative at $\varepsilon(t)$.
Proof of Lemma 19. We define $\operatorname{sign}(x):=x /\|x\|$ and $u:=\operatorname{sign}(\varphi(t+h)-\varphi(t))$. Since it is assumed that the path $h \mapsto P_{\varphi(t+h)}$ is absolutely continuous, there exists a gradient such that

$$
d P_{\varphi(t+h)}-d P_{\varphi(t)}=\int_{\varphi(t)}^{\varphi(t+h)} \dot{\rho}_{s, u} d s
$$

for any $u \in \mathbb{S}^{d-1}$. Using this result, the chi-squared divergence can be written out as follows:

$$
\begin{aligned}
\chi^{2}\left(P_{\varphi(t+h)} \| P_{\varphi(t)}\right) & =\int \frac{\left(d P_{\varphi(t+h)}-d P_{\varphi(t)}\right)^{2}}{d P_{\varphi(t)}} d \nu \\
& =\int \frac{\left(\int_{\varphi(t)}^{\varphi(t+h)} \dot{\rho}_{s, u} d s\right)^{2}}{d P_{\varphi(t)}} d \nu \\
& =\int \frac{\left\{\int_{0}^{1}(\nabla \varphi(t+h \widetilde{s}) h)^{\top} \dot{\rho}_{\varphi(t+h \widetilde{s}), u} d \widetilde{s}\right\}^{2}}{d P_{\varphi(t)}(x)} d \nu \\
& =h^{\top}\left(\int \frac{\int_{0}^{1} \int_{0}^{1} \nabla \varphi^{\top}\left(t+h \widetilde{s}_{1}\right) \dot{\rho}_{\varphi\left(t+h \widetilde{s}_{1}\right), u} \dot{\rho}_{\varphi\left(t+h \widetilde{s}_{2}\right), u}^{\top} \nabla \varphi\left(t+h \widetilde{s}_{2}\right) d \widetilde{s}_{1} d \widetilde{s}_{2}}{\left.d P_{\varphi(t)} d \nu\right) h}\right.
\end{aligned}
$$

The last steps use the change of variables. By the uniform integrability assumption and the continuity of $\dot{\rho}_{t}$, both asserted by (A1), it follows that

$$
\chi^{2}\left(P_{\varphi(t+h)} \| P_{\varphi(t)}\right)=h^{\top} \nabla \varphi(t)^{\top}\left(\int \frac{\nabla \dot{\rho}_{\varphi(t)} \nabla \dot{\rho}_{\varphi(t)}^{\top}}{d P_{\varphi(t)}} d \nu\right) \nabla \varphi(t) h+o\left(\|h\|_{2}^{2}\right)
$$

by the dominated convergence. As the middle term in the parenthesis is finite under (A1), Lemma 6 of Gassiat and Stoltz (2024) implies that the path $d P_{\varphi(t)}^{1 / 2}$ is Hellinger differentiable with the gradient

$$
\dot{\xi}_{\varphi(t)}:=\frac{1}{2} \frac{\nabla \dot{\rho}_{\varphi(t)}}{d P_{\varphi(t)}^{1 / 2}} I\left(d P_{\varphi(t)}>0\right)
$$

We can use this to show that:

$$
\mathcal{I}(\varphi(t))=4 \int \dot{\xi}_{\varphi(t)} \dot{\xi}_{\varphi(t)}^{\top} d \nu=\left(\int \frac{\nabla \dot{\rho}_{\varphi(t)} \nabla \dot{\rho}_{\varphi(t)}^{\top}}{d P_{\varphi(t)}} d \nu\right)
$$

and this leads us to conclude

$$
\chi^{2}\left(P_{\varphi(t+h)} \| P_{\varphi(t)}\right)=h^{\top} \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) h+o\left(\|h\|_{2}^{2}\right) .
$$

as desired.
Lemma 20. Assuming that the collection of paths $h \mapsto d P_{\varphi(t+h)}$ is Hellinger differentiable at $h=0$ and $\nabla \varphi$ is continuous at $t$,

$$
H^{2}\left(P_{\varphi(t+h)}, P_{\varphi(t)}\right)=\frac{1}{4} h^{\top} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) h^{\top}+o\left(\|h\|_{2}^{2}\right)
$$

as $\|h\|_{2} \longrightarrow 0$.
Proof of Lemma 20. By the definition of the Hellinger differentiability, we have

$$
\begin{aligned}
H^{2}\left(P_{\varphi(t+h)}, P_{\varphi(t)}\right) & =\int\left(d P_{\varphi(t+h)}^{1 / 2}-d P_{\varphi(t)}^{1 / 2}\right)^{2} \\
& =\int\left(d P_{\varphi(t)+\varphi(t+h)-\varphi(t)}^{1 / 2}-d P_{\varphi(t)}^{1 / 2}\right)^{2} \\
& =\int\left((\varphi(t+h)-\varphi(t))^{\top} \dot{\xi}_{\varphi(t)}\right)^{2}+o\left(\|\varphi(t+h)-\varphi(t)\|_{2}^{2}\right) \\
& \left.=\int\left((\nabla \varphi(t+\lambda h) h)^{\top} \dot{\xi}_{\varphi(t)}\right)^{2}+o(\| \nabla \varphi(t+\lambda h) h) \|_{2}^{2}\right) \\
& \left.=h^{\top} \nabla \varphi^{\top}(t+\lambda h)\left(\int \dot{\xi}_{\varphi(t)} \dot{\xi}_{\varphi(t)}^{\top} d \nu\right) \nabla \varphi(t+\lambda h) h^{\top}+o(\| \nabla \varphi(t+\lambda h) h) \|_{2}^{2}\right)
\end{aligned}
$$

for some constant $\lambda \in[0,1]$ possibley depending on $t$. Since the Fisher information is defined as $4 \int \dot{\xi}_{\varphi(t)} \dot{\xi}_{\varphi(t)}^{\top} d \nu$ and $\nabla \varphi(t+\lambda h) \longrightarrow \nabla \varphi(t)$ as $\|h\|_{2} \longrightarrow 0$ by the continuity, we conclude that

$$
H^{2}\left(P_{\varphi(t+h)}, P_{\varphi(t)}\right)=\frac{1}{4} h^{\top} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) h^{\top}+o\left(\|h\|_{2}^{2}\right)
$$

as $\|h\|_{2} \longrightarrow 0$.

Lemma 21. Assuming that the collection of paths $h \mapsto P_{\varphi(t+h)}$ satisfies (A2),

$$
\begin{aligned}
& \chi^{2}\left(\widetilde{\mathbb{P}}_{h}^{n} \| \lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right) \\
& \quad=(1-\lambda)^{2} h^{\top}\left\{\mathcal{I}(Q)+n \int \nabla \varphi(t)^{\top}\left(\mathcal{I}(\varphi(t))+\frac{1-4 \lambda}{4 \lambda} \mathcal{I}^{\circ}(\varphi(t))\right) \nabla \varphi(t) d Q(t)\right\} h+o\left(\|h\|_{2}^{2}\right) .
\end{aligned}
$$

Proof of Lemma 21. The analogous proof can be found in Section 7.14 of Polyanskiy and Wu (2022). To begin, we observe that the $\chi^{2}$-divergence can be written out as

$$
\begin{aligned}
\chi^{2}\left(\widetilde{\mathbb{P}}_{h}^{n} \| \lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right) & =\iint_{\mathcal{X} \times \Theta} \frac{\left\{d \widetilde{\mathbb{P}}_{h}^{n}-\left(\lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right)\right\}^{2}}{\lambda d \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) d \widetilde{\mathbb{P}}_{0}^{n}} \\
& =(1-\lambda)^{2} \iint_{\mathcal{X} \times \Theta} \frac{\left(d \widetilde{\mathbb{P}}_{h}^{n}-d \widetilde{\mathbb{P}}_{0}^{n}\right)^{2}}{\lambda d \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) d \widetilde{\mathbb{P}}_{0}^{n}} .
\end{aligned}
$$

We will consider the function

$$
\phi:=f \mapsto \iint_{\mathcal{X} \times \Theta} \frac{f^{2}-d \widetilde{\mathbb{P}}_{0}^{n}}{\left\{\lambda f^{2}+(1-\lambda) d \widetilde{\mathbb{P}}_{0}^{n}\right\}^{1 / 2}} .
$$

It can then be seen that $\chi^{2}\left(\widetilde{\mathbb{P}}_{h}^{n} \| \lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right)=(1-\lambda)^{2} \iint_{\mathcal{X} \times \Theta} \phi\left(d \widetilde{\mathbb{P}}_{h}^{n / 2}\right)^{2}$. When $\phi$ is Lipshitz continuous and $d \widetilde{\mathbb{P}}_{h}^{n / 2}$ is absolutely continuous with the gradient $t \mapsto \dot{\gamma}_{t, u}$, then the composition function $\phi\left(d \widetilde{\mathbb{P}}_{h}^{n / 2}\right)$ is also absolutely continuous with the gradient $t \mapsto \phi^{\prime}\left(d \widetilde{\mathbb{P}}_{t}^{n / 2}\right) \dot{\gamma}_{t, u}$. In Section 7.14 of Polyanskiy and Wu (2022), it is shown that $\phi$ is indeed Lipshitz continuous with the constant $(2-\lambda) /\left\{(1-\lambda) \lambda^{1 / 2}\right\}$. Additionally, $\phi$ has a continuous derivative and we have

$$
\phi^{\prime}\left(d \widetilde{\mathbb{P}}_{0}^{n}\right):=\left\{\begin{array}{lll}
2, & \text { if } & (x, t): d \widetilde{\mathbb{P}}_{0}^{n}(x, t)>0 \\
\lambda^{-1 / 2} & \text { if } & (x, t): d \mathbb{P}_{0}^{n}(x, t)=0 .
\end{array}\right.
$$

Putting together, we have

$$
\lim _{t \longrightarrow 0} \phi^{\prime}\left(d \widetilde{\mathbb{P}}_{t}^{n / 2}\right) \dot{\gamma}_{t, u}=\dot{\gamma}_{0, u}\left(2 I\left(d \widetilde{\mathbb{P}}_{0}^{n}>0\right)+\frac{1}{\lambda^{1 / 2}} I\left(d \widetilde{\mathbb{P}}_{0}^{n}=0\right)\right)
$$

Noting that $\phi\left(d \widetilde{\mathbb{P}}_{0}^{n / 2}\right)=0$, it follows that

$$
\begin{aligned}
\chi^{2}\left(\widetilde{\mathbb{P}}_{h}^{n} \| \lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right) & =(1-\lambda)^{2} \iint_{\mathcal{X} \times \Theta} \phi\left(d \widetilde{\mathbb{P}}_{h}^{n / 2}\right)^{2} \\
& =(1-\lambda)^{2} \iint_{\mathcal{X} \times \Theta}\left(\phi\left(d \widetilde{\mathbb{P}}_{h}^{n / 2}\right)-\phi\left(d \widetilde{\mathbb{P}}_{0}^{n / 2}\right)\right)^{2} \\
& =(1-\lambda)^{2} \iint_{\mathcal{X} \times \Theta}\left(\int_{0}^{h} \phi^{\prime}\left(d \widetilde{\mathbb{P}}_{s}^{n / 2}\right) \dot{\gamma}_{s, u} d s\right)^{2} \\
& =(1-\lambda)^{2} h^{\top} \iint_{\mathcal{X} \times \Theta}\left(\int_{0}^{1} \int_{0}^{1} \phi^{\prime}\left(d \widetilde{\mathbb{P}}_{h s_{1}}^{n / 2}\right) \phi^{\prime}\left(d \widetilde{\mathbb{P}}_{h s_{2}}^{n / 2}\right) \dot{\gamma}_{h s_{1}, u} \dot{\gamma}_{h s_{2}, u}^{\top} d s_{1} d s_{1}\right) h .
\end{aligned}
$$

Since $\phi^{\prime}$ is continuous and $\dot{\gamma}$ is the uniform integrability, asserted by (A2), it follows that

$$
\begin{aligned}
& \chi^{2}\left(\widetilde{\mathbb{P}}_{h}^{n} \| \lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right) \\
&=(1-\lambda)^{2} h^{\top}\left\{\iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top}\left(4 I\left(d \widetilde{\mathbb{P}}_{0}^{n}>0\right)+\frac{1}{\lambda} I\left(d \widetilde{\mathbb{P}}_{0}^{n}=0\right)\right)\right\} h+o\left(\|h\|_{2}^{2}\right) \\
& \quad=(1-\lambda)^{2} h^{\top} 4\left\{\iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top}+\iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top}\left(\frac{1-4 \lambda}{4 \lambda} I\left(\widetilde{\mathbb{P}}_{0}^{n}=0\right)\right)\right\} h+o\left(\|h\|_{2}^{2}\right)
\end{aligned}
$$

as $\|h\|_{2} \longrightarrow 0$ by the dominated convergence theorem. Finally, Lemma 23 states that the Hellinger differentiablity is implied by (A2), and that the corresponding Fisher information can be computed using $4 \iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top}$. By the definition of the Hellinger differentiability, it follows that

$$
H^{2}\left(\widetilde{\mathbb{P}}_{h}^{n}, \widetilde{\mathbb{P}}_{0}^{n}\right)=\int\left(d \widetilde{\mathbb{P}}_{h}^{n / 2}-d \widetilde{\mathbb{P}}_{0}^{n / 2}\right)^{2}=h^{\top}\left(\iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top}\right) h+o\left(\|h\|_{2}^{2}\right)
$$

as $\|h\|_{2} \longrightarrow 0$. Thus in view of Lemma 22, we have

$$
\iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top}=\frac{1}{4}\left(\mathcal{I}(Q)+n \int_{\mathbb{R}^{d}} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) d Q(t)\right)
$$

and similarly

$$
\iint_{\mathcal{X} \times \Theta} \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top} I\left(d \widetilde{\mathbb{P}}_{0}^{n}=0\right)=\frac{1}{4}\left(\mathcal{I}^{\circ}(Q)+n \int_{\mathbb{R}^{d}} \nabla \varphi(t)^{\top} \mathcal{I}^{\circ}(\varphi(t)) \nabla \varphi(t) d Q(t)\right)
$$

where $\mathcal{I}^{\circ}(0):=4 \int \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top} I\left(d \widetilde{\mathbb{P}}_{0}^{n}=0\right)$ is the Fisher defect. When the prior $Q$ has a continuously differentiable density $q$, it must follow that $\nabla q(t)=0$ whenever $q(t)=0$. This implies that the Fisher defect $\mathcal{I}^{\circ}(Q)$ is zero. Thereby, we conclude that

$$
\begin{aligned}
& \chi^{2}\left(\widetilde{\mathbb{P}}_{h}^{n} \| \lambda \widetilde{\mathbb{P}}_{h}^{n}+(1-\lambda) \widetilde{\mathbb{P}}_{0}^{n}\right) \\
& \quad=(1-\lambda)^{2} h^{\top}\left\{\mathcal{I}(Q)+n \int \nabla \varphi(t)^{\top}\left(\mathcal{I}(\varphi(t))+\frac{1-4 \lambda}{4 \lambda} \mathcal{I}^{\circ}(\varphi(t))\right) \nabla \varphi(t) d Q(t)\right\} h+o\left(\|h\|_{2}^{2}\right) .
\end{aligned}
$$

as desired.
Lemma 22. Assuming $Q$ has a density function that is bounded and continuously differentiable, and $P_{t}$ is Hellinger differentiable at $t=\varphi(t)$ such that $\|\nabla \varphi\|_{\infty}<C$,

$$
H^{2}\left(\widetilde{\mathbb{P}}_{h}^{n}, \widetilde{\mathbb{P}}_{0}^{n}\right)=\frac{1}{4} h^{\top}\left(\mathcal{I}(Q)+n \int_{\mathbb{R}^{d}} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) d Q(t)\right) h+o\left(\|h\|_{2}^{2}\right)
$$

Proof of Lemma 22. First, the Hellinger distance between $\widetilde{\mathbb{P}}_{h}^{n}$ and $\widetilde{\mathbb{P}}_{0}^{n}$ can be written out as

$$
\begin{aligned}
H^{2}\left(\widetilde{\mathbb{P}}_{h}, \widetilde{\mathbb{P}}_{0}\right) & =2-2 \iint_{\varphi(t+h)} d P^{n / 2} d Q^{1 / 2}(t+h) d P_{\varphi(t)}^{n / 2} d Q^{1 / 2}(t) \\
& =2-2 \int_{\mathbb{R}^{d}} d Q^{1 / 2}(t+h) d Q^{1 / 2}(t) \int_{\mathcal{X}} d P_{\varphi(t+h)}^{n / 2} d P_{\varphi(t)}^{n / 2} \\
& =2-2 \int_{\mathbb{R}^{d}} d Q^{1 / 2}(t+h) d Q^{1 / 2}(t)\left(1-\frac{1}{2} H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right)\right) \\
& =H^{2}\left(Q_{h}, Q\right)+\int_{\mathbb{R}^{d}} d Q^{1 / 2}(t+h) d Q^{1 / 2}(t) H^{2}\left(P_{\varphi(t+h)}^{n}, P_{\varphi(t)}^{n}\right) .
\end{aligned}
$$

When the density function $Q$ is continuously differentiable, the induced location family is differentiable in quadratic mean (Example 7.8 of Van der Vaart (2000)), which implies the Hellinger differentiability. By the analogous result to Lemma 20 for $\varphi(t)=t$, we have

$$
H^{2}\left(Q_{h}, Q\right)=\frac{1}{4} h^{\top} \mathcal{I}(Q) h+o\left(\|h\|_{2}^{2}\right)
$$

Next, using the tensorization property of the Hellinger distance as well as Lemma 20, it implies

$$
H^{2}\left(P_{\varphi(t)}^{n}, P_{\varphi(t+h)}^{n}\right)=2-2\left(1-\frac{H^{2}\left(P_{\varphi(t)}, P_{\varphi(t+h)}\right)}{2}\right)^{n}=2-2\left(1-h^{\top} C h+o\left(\|h\|_{2}^{2}\right)\right)^{n}
$$

as $\|h\|_{2} \longrightarrow 0$ where $C:=\frac{1}{8} u^{\top} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) u$. By treating $n \geq 1$ as a fixed constant as $\|h\|_{2} \longrightarrow 0$, we have

$$
2-2\left(1-h^{\top} C h+o\left(\|h\|_{2}^{2}\right)\right)^{n}=2-2\left(1-n h^{\top} C h+o\left(n\|h\|_{2}^{2}\right)\right)=2 n h^{\top} C h+o\left(n\|h\|_{2}^{2}\right),
$$

by a first-order Taylor expansion. Putting them together, we have

$$
H^{2}\left(\widetilde{P}_{h}^{n}, \widetilde{\mathbb{P}}_{0}^{n}\right)=\frac{1}{4} h^{\top}\left(\mathcal{I}(Q)+n \int_{\mathbb{R}^{d}} \nabla \varphi(t)^{\top} \mathcal{I}(\varphi(t)) \nabla \varphi(t) d Q(t)\right) h+o\left(\|h\|_{2}^{2}\right)
$$

where we invoke the dominated convergence theorem to interchange the limit and integration operations, which follows by the boundedness of $\nabla \varphi$ and $\mathcal{I}(\cdot)$.

Lemma 23. If $\left\{P_{t}: t \in \Theta\right\}$ satisfies (A2) for some $\delta>0$, it is Hellinger differentiable at $t=0$ with the corresponding Fisher information is given by

$$
\mathcal{I}(0)=4 \int \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top} d \nu
$$

Proof of Lemma 23. First, we claim that the regularity (A2) implies the Hellinger differentiability. By the absolute continuity of the square root density, it follows

$$
d P_{t}^{1 / 2}-d P_{0}^{1 / 2}=\int_{0}^{t} \dot{\gamma}_{s, \operatorname{sign}(t)} d s=\int_{0}^{1} \dot{\gamma}_{t \widetilde{s}, \operatorname{sign}(t)} t d \widetilde{s}
$$

and hence we have

$$
\begin{aligned}
\int\left(d P_{t}^{1 / 2}-d P_{0}^{1 / 2}\right)^{2} d \nu & =\int\left(\int_{0}^{1} \dot{\gamma}_{t \widetilde{s}, \operatorname{sign}(t)} t d \widetilde{s}\right)^{2} d \nu \\
& =t^{\top}\left\{\int\left(\int_{0}^{1} \int_{0}^{1} \dot{\gamma}_{t \widetilde{s}_{1}, \operatorname{sign}(t)} \dot{\gamma}_{t_{s_{2}}, \operatorname{sign}(t)}^{\top} d \widetilde{s}_{1} d \widetilde{s}_{2}\right) d \nu\right\} t
\end{aligned}
$$

By the assumption that $\lim _{s \longrightarrow 0} \dot{\gamma}_{s, u}=\dot{\gamma}_{0}$ for $\nu$-almost where and uniform integrability $s \mapsto \dot{\gamma}_{s, u}$, we conclude

$$
\int\left(d P_{t}^{1 / 2}-d P_{0}^{1 / 2}\right)^{2} d \nu=t^{\top}\left(\int \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top} d \nu\right) t+o\left(\|t\|_{2}^{2}\right)
$$

as $\|t\|_{2} \longrightarrow 0$. This concludes the claim that the local path $t \mapsto P_{t}$ is Hellinger differentiable at $t=0$ with $\dot{\xi}_{0}=\dot{\gamma}_{0}, \nu$-almost everywhere. As a result, we have

$$
\mathcal{I}(0)=4 \int \dot{\xi}_{0} \dot{\xi}_{0}^{\top} d \nu=4 \int \dot{\gamma}_{0} \dot{\gamma}_{0}^{\top} d \nu
$$

This figure "bounds.png" is available in "png" format from: http://arxiv.org/ps/2405.06437v1


[^0]:    ${ }^{1} \mathrm{~A}$ diffeomorphism is an isomorphism of smooth manifolds. It is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable.

[^1]:    ${ }^{2}$ This notion of adaptivity is completely separate from the adaptation to unknown smoothness $s$.

