A GENERALIZATION OF THE ERDŐS-KAC THEOREM

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ABSTRACT. Given a natural number n, let $\omega(n)$ denote the number of distinct prime factors of n, let Z denote a standard normal variable, and let P_n denote the uniform distribution on $\{1, \ldots, n\}$. The Erdős-Kac Theorem states that if N(n) is a uniformly distributed variable on $\{1, \ldots, n\}$, then $\omega(N(n))$ is asymptotically normally distributed as $n \to \infty$ with both mean and variance equal to log log n. The contribution of this paper is a generalization of the Erdős-Kac Theorem to a larger class of random variables by considering perturbations of the uniform probability mass 1/n in the following sense. Denote by \mathbb{P}_n a probability distribution on $\{1, \ldots, n\}$ given by $\mathbb{P}_n(i) = 1/n + \varepsilon_{i,n}$. We provide sufficient conditions on $\varepsilon_{i,n}$ so that the number of distinct prime factors of a \mathbb{P}_n -distributed random variable is asymptotically normally distributed, as $n \to \infty$, with both mean and variance equal to log log n. Our main result is applied to prove that the number of distinct prime factors of a positive integer with the Harmonic(n) distribution also tends to the normal distribution, as $n \to \infty$. In addition, we explore sequences of distributions on the natural numbers such that $\omega(n)$ is normally distributed in the limit. In addition, one of our theorems and its corollaries generalize a result from the literature involving the limit of Zeta(s) distributions as the parameter $s \to 1$.

1. INTRODUCTION

Given a natural number n, the number of distinct prime factors of n is denoted $\omega(n)$. The function ω may be written as $\omega(n) = \sum_{p|n} 1$, where the sum is over all prime factors of n. In 1917, Hardy and Ramanujan (p. 270 of [5]) proved that the number of distinct prime factors of a natural number n is about $\log \log n$. In particular, they showed that the normal order of $\omega(n)$ is $\log \log n$; i.e., for every $\varepsilon > 0$, the proportion of the natural numbers for which the inequalities

$$(1 - \varepsilon) \log \log n \le \omega(n) \le (1 + \varepsilon) \log \log n$$

do not hold tends to 0 as $n \to \infty$ -with a typical error of size $\sqrt{\log \log n}$. Informally speaking, the Erdős-Kac Theorem generalizes¹ the Hardy-Ramanujan Theorem by showing that $\omega(n)$ is approximately distributed as

$$\log \log n + Z\sqrt{\log \log n}$$

for large n, where Z denotes a standard normal variable. More precisely, the Erdős-Kac Theorem is the following result (p. 738 of [4]).

Theorem 1. Let n > 1. Let P_n denote the uniform distribution on $\{1, 2, ..., n\}$, and let Z denote a standard normal variable. As $n \to \infty$,

$$P_n\left(m \le n : \omega\left(m\right) - \log\log n \le x \left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \le x\right).$$

The contribution of this paper is to extend the Erdős-Kac Theorem to a larger class of random variables on the set $[n] := \{1, 2, ..., n\}$ which also have, asymptotically, $\log \log n + Z \cdot \sqrt{\log \log n}$ many distinct prime factors.

¹While the Hardy-Ramanujan theorem provides information about the average behavior of the number of prime factors of a natural number, the Erdős-Kac theorem offers a more detailed probabilistic description of their distribution, by taking into account not just the average number but also the variability around that average.

1.1. A Generalization of Erdős-Kac Theorem for $\omega(\cdot)$. Define a probability mass function (PMF) \mathbb{P}_n on [n] given by

(1)
$$\mathbb{P}_n(i) = \frac{1}{n} + \varepsilon_{i,n}$$

Due to the axioms of probability, the terms $\varepsilon_{i,n}$, $1 \leq i \leq n$, satisfy the constraints

(2)
$$\sum_{i=1}^{n} \varepsilon_{i,n} = 0; \ n \in \mathbb{N}$$

and

(3)
$$\varepsilon_{i,n} \in \left[-\frac{1}{n}, 1-\frac{1}{n}\right]; \ n \in \mathbb{N}, 1 \le i \le n.$$

The motivation for defining \mathbb{P}_n in terms of the uniform distribution is due to Durrett's proof (Theorem 3.4.16 in [3]) of the Erdős-Kac Theorem. Replacing the uniform distribution P_n with the distribution \mathbb{P}_n in Durrett's proof naturally yields some constraints that the terms $\varepsilon_{i,n}, 1 \leq i \leq n$, must satisfy in order to conclude that an integer-valued \mathbb{P}_n -distributed random variable has about $\log \log n + Z\sqrt{\log \log n}$ distinct prime factors. Our main result is the following theorem, where $\lfloor \cdot \rfloor$ denotes the floor function.

Theorem 2. (Generalized Erdős-Kac Theorem for ω) Let Z denote a standard normal variable. Suppose the following statements are true.

• There exists a constant $C \in \mathbb{R}$ such that for all n > 1 and for all primes p with $p > n^{1/\log \log n}$,

(4)
$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \le \frac{C}{p}.$$

• There exists a constant $D \in \mathbb{R}$ such that

(5)
$$\sum_{l=1}^{\left\lfloor \frac{n}{p_1 \cdots p_k} \right\rfloor} \varepsilon_{lp_1 \cdots p_k, n} \le \frac{D}{n}$$

for all n > 1, for each k, and for all k-tuples (p_1, \ldots, p_k) consisting of distinct primes of size at most $n^{1/\log \log n}$.

• For any prime p,

(6)
$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = 0$$

Let \mathbb{P}_n^* denote the PMF obtained by imposing the constraints (4-6) on the PMF \mathbb{P}_n given by $\mathbb{P}_n(i) = \frac{1}{n} + \varepsilon_{i,n}$. As $n \to \infty$,

$$\mathbb{P}_{n}^{*}\left(m \leq n : \omega\left(m\right) - \log\log n \leq x \left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \leq x\right).$$

Remark. If $\varepsilon_{i,n} = 0$ for all $i \leq n$, then $\mathbb{P}_n^* = P_n$ and Theorem 1 is obtained.

1.2. **Outline.** The proof of Theorem 2 is provided in §2; the proof applies the method of moments and is motivated by Durrett's proof of the Erdős-Kac Theorem (Theorem 3.4.16 in [3]). Moreover, in §2, the constraints (4-6) are applied to ensure that \mathbb{P}_n^* also satisfies Durrett's method of moment bounds. In §§2.1, Theorem 2 is applied to show that the number of distinct prime factors of a random natural number chosen according to the Harmonic(n) distribution is asymptotically normally distributed with both mean and variance equal to $\log \log n$. In §3, Theorem 2 is used to prove statements about convex combinations of distributions satisfying constraints (4-6). In §4, we define conditions that ensure $\omega(X_j(n))$ is asymptotically normally distributed, with mean and variance both equal to $\log \log n$, for a sequence of random variables $(X_j(n))_{j\geq 1}$ as $j \to \infty$ and $n \to \infty$. In §5, Theorem 2 is applied to show that the number of distinct prime factors of a randomly chosen integer according to any of the following distributions has the same limiting distribution as the case of a uniform variable:

- Any convex combination of the $\operatorname{Harmonic}(n)$ and $\operatorname{uniform}(n)$ distributions,
- The Zeta (s) distribution as $s \to 1$,

- The Logarithmic(s) distribution as $s \to 1$,
- A geometric power series distribution as $s \to 1$,
- A Logarithmic-Zeta (s, α) distribution as $(s, \alpha) \rightarrow (1, 1)$.

2. Proving Theorem 2

Define $\alpha_n \coloneqq n^{1/\log \log n}$.

Lemma 1. As $n \to \infty$

$$\left(\sum_{\alpha_n$$

Proof. Given n and any prime p with $p > \alpha_n$, we have

$$-\frac{\lfloor n/p \rfloor}{n} \stackrel{(3)}{\leq} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \stackrel{(4)}{\leq} \frac{C}{p}.$$

Therefore,

(7)
$$\frac{1}{p} + \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \in \left[0, \frac{C+1}{p}\right]$$

for all n. Thus,

$$\left(\sum_{\alpha_n$$

due to (7) along with the fact that Durrett (p.135 of [3]) shows

$$\left(\sum_{\alpha_n$$

The following lemma is proved by Durrett (p. 156 of [3]).

Lemma 2. If $\varepsilon > 0$, then $\alpha_n \leq n^{\varepsilon}$ for large n and hence

(8)
$$\frac{\alpha_n^r}{n} \to 0$$

for all $r < \infty$.

Proof of Theorem 2. Given a natural number m and a prime p, define $\delta_p(m) = 1$ if p divides m, and 0 otherwise. Let

$$g_n(m) = \sum_{p \le \alpha_n} \delta_p(m)$$

denote the number of distinct prime factors of m of size at most α_n , and let \mathbb{E}_n denote expectation with respect to \mathbb{P}_n^* . Then

$$\mathbb{E}_{n}\left(\sum_{\alpha_{n}
$$= \sum_{\alpha_{n}
$$\stackrel{(1)}{=} \sum_{\alpha_{n}
$$\leq \sum_{\alpha_{n}$$$$$$$$

so by Lemma 1 it suffices to prove Theorem 2 for g_n ; i.e., replacing $\omega(m)$ with $g_n(m)$ in the statement of Theorem 2 does not affect the limiting distribution.

Consider a sequence $(X_p)_{p\geq 2}$ of independent Bernoulli random variables with prime-valued indices such that $\mathbb{P}(X_p = 1) = 1/p$ and $\mathbb{P}(X_p = 0) = 1 - 1/p$. Note that

$$\mathbb{E}\left(\delta_{p}\right) = \frac{\lfloor n/p \rfloor}{n} + \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \stackrel{(6)}{\to} 1/p$$

as $n \to \infty$. Let

$$S_n := \sum_{p \le \alpha_n} X_p,$$

$$b_n := \mathbb{E}(S_n),$$

$$a_n^2 := \operatorname{Var}(S_n)$$

By Lemma 1, b_n and a_n^2 are both $\log \log n + o\left(\left(\log \log n\right)^{1/2}\right)$, so it suffices to show

$$\mathbb{P}_{n}^{*}\left(m:g_{n}\left(m\right)-b_{n}\leq xa_{n}\right)\rightarrow\mathbb{P}\left(Z\leq x\right),$$

An application of Theorem 3.4.10 of [3] shows

$$\left(S_n - b_n\right)/a_n \to Z,$$

and since $|X_p| \leq 1$, it follows from Durrett's second proof of Theorem 3.4.10 [3] that

$$\mathbb{E}\left(\left(S_n - b_n\right)/a_n\right)^r \to \mathbb{E}\left(Z^r\right)$$

for all r. Using the notation from that proof (and replacing i_j by p_j) it follows that

$$\mathbb{E}\left(S_{n}^{r}\right) = \sum_{k=1}^{r} \sum_{r_{i}} \frac{r!}{r_{1}!\cdots r_{k}!} \frac{1}{k!} \sum_{p_{j}} \mathbb{E}\left(X_{p_{1}}^{r_{1}}\cdots X_{p_{k}}^{r_{k}}\right),$$

where the sum \sum_{r_i} extends over all k-tuples of positive integers for which $r_1 + \cdots + r_k = r$, and \sum_{p_j} extends over all k-tuples of distinct primes in [n]. Since $X_p \in \{0, 1\}$, the summand in $\sum_{p_j} \mathbb{E}\left(X_{p_1}^{r_1} \cdots X_{p_k}^{r_k}\right)$ is

$$\mathbb{E}\left(X_{p_1}\cdots X_{p_k}\right) = \frac{1}{p_1\cdots p_k}$$

by independence of the X_p 's. Moreover,

$$\mathbb{E}_{n} \left(\delta_{p_{1}} \cdots \delta_{p_{k}} \right) \leq \mathbb{P}_{n} \left(m : \delta_{p_{1}} \left(m \right) = \delta_{p_{2}} \left(m \right) \cdots = \delta_{p_{k}} \left(m \right) = 1 \right)$$

$$= \sum_{l=1}^{\left\lfloor \frac{n}{p_{1} \cdots p_{k}} \right\rfloor} \mathbb{P}_{n} \left(m : m = lp_{1} \cdots p_{k} \right)$$

$$\stackrel{(1)}{=} \sum_{l=1}^{\left\lfloor \frac{n}{p_{1} \cdots p_{k}} \right\rfloor} \left(\frac{1}{n} + \varepsilon_{lp_{1} \cdots p_{k}, n} \right)$$

$$= \frac{\left\lfloor \frac{n}{p_{1} \cdots p_{k}} \right\rfloor}{n} + \sum_{l=1}^{\left\lfloor \frac{n}{p_{1} \cdots p_{k}} \right\rfloor} \varepsilon_{lp_{1} \cdots p_{k}, n}$$

$$\stackrel{(5)}{=} \frac{\left\lfloor \frac{n}{p_{1} \cdots p_{k}} \right\rfloor}{n} + \frac{D}{n}.$$

The two terms $\mathbb{E}(X_{p_1}\cdots X_{p_k})$ and $\mathbb{E}_n(\delta_{p_1}\cdots \delta_{p_k})$ differ by at most

$$\max\left\{\frac{1}{p_1\cdots p_k} - \frac{\left\lfloor \frac{n}{p_1\cdots p_k} \right\rfloor}{n} - \frac{D}{n}, \frac{\left\lfloor \frac{n}{p_1\cdots p_k} \right\rfloor}{n} + \frac{D}{n} - \frac{1}{p_1\cdots p_k}\right\} \le \max\left\{\frac{1-D}{n}, \frac{D}{n}\right\}.$$

Therefore, the two rth moments differ by

$$\begin{aligned} |\mathbb{E}(S_n^r) - \mathbb{E}_n(g_n^r)| &\leq \sum_{k=1}^r \sum_{r_i} \frac{r!}{r_1! \cdots r_k!} \frac{1}{k!} \sum_{p_j} \max\left\{\frac{1-D}{n}, \frac{D}{n}\right\}. \\ &\leq \max\left\{\frac{1-D}{n}, \frac{D}{n}\right\} \cdot \left(\sum_{p \leq \alpha_n} 1\right)^r \\ &\leq \max\left\{\frac{1-D}{n}, \frac{D}{n}\right\} \cdot \alpha_n^r \\ &\stackrel{(8)}{\to} 0. \end{aligned}$$

Using binomial expansions and the inequality above, we see that

$$\begin{aligned} |\mathbb{E}\left(\left(\left(S_{n}-b_{n}\right)/a_{n}\right)^{r}\right)-\mathbb{E}_{n}\left(\left(\left(g_{n}-b_{n}\right)/a_{n}\right)^{r}\right)| &=\left|1/a_{n}^{r}\right|\left|\mathbb{E}\left(\left(S_{n}-b_{n}\right)^{r}\right)-\mathbb{E}_{n}\left(\left(g_{n}-b_{n}\right)^{r}\right)\right|\right) \\ &\leq\left|1/a_{n}^{r}\right|\cdot\max\left\{\frac{1-D}{n},\frac{D}{n}\right\}\sum_{k=0}^{r}\binom{r}{k}\alpha_{n}^{k}b_{n}^{r-k} \\ &=\left|1/a_{n}^{r}\right|\cdot\max\left\{\frac{1-D}{n},\frac{D}{n}\right\}\left(\alpha_{n}+b_{n}\right)^{r}.\end{aligned}$$

Therefore, since $b_n \leq \alpha_n$, we have

$$\left|\mathbb{E}\left(\left(\left(S_{n}-b_{n}\right)/a_{n}\right)^{r}\right)-\mathbb{E}_{n}\left(\left(\left(g_{n}-b_{n}\right)/a_{n}\right)^{r}\right)\right|\stackrel{(8)}{\to}0$$

for all r as well. Since $\mathbb{E}\left(\left(\left(S_n - b_n\right)/a_n\right)^r\right) \to \mathbb{E}\left(Z^r\right)$ for all r, this completes the proof of Theorem 2. \Box

The following definition is based on Theorem 2.

Definition 1. We refer to distributions satisfying constraints (4-6) as **E-K distributions**.

2.1. The Harmonic Distribution. Now we will apply Theorem 2 to show that the harmonic distributions are E-K distributions. Given $n \in \mathbb{N}$, consider an integer in [n] chosen according to the Harmonic(n)distribution, whose PMF is given by

$$H_n(i) \coloneqq \frac{1}{i \sum_{j=1}^n \frac{1}{j}}, 1 \le i \le n.$$

If $i \in [n]$, then equation (1) implies

$$\varepsilon_{i,n} = \frac{1}{i\sum_{j=1}^n \frac{1}{j}} - \frac{1}{n}.$$

Therefore,

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = \sum_{l=1}^{\lfloor n/p \rfloor} \left(\frac{1}{lp \sum_{i=1}^{n} \frac{1}{i}} - \frac{1}{n} \right)$$
$$= \frac{\sum_{i=1}^{\lfloor n/p \rfloor} \frac{1}{i}}{p \sum_{l=1}^{n} \frac{1}{i}} - \frac{\lfloor n/p \rfloor}{n}$$
$$\leq \frac{1}{p} - \frac{\lfloor n/p \rfloor}{n}$$
$$\leq \frac{1}{p},$$

so (4) holds with C = 1. Moreover,

$$\sum_{l=1}^{\frac{n}{p_1\cdots p_k}} \varepsilon_{lp_1\cdots p_k,n} = \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \left(\frac{1}{lp_1\cdots p_k \sum_{i=1}^{n} \frac{1}{i}} - \frac{1}{n} \right)$$
$$= \frac{\sum_{i=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \frac{1}{i}}{p_1\cdots p_k \sum_{i=1}^{n} \frac{1}{i}} - \frac{\lfloor \frac{n}{p_1\cdots p_k} \rfloor}{n}$$
$$\leq \frac{1}{p_1\cdots p_k} - \left(\frac{\frac{n}{p_1\cdots p_k} - 1}{n}\right)$$
$$= 1/n,$$

so (5) holds with D = 1. Finally,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = \lim_{n \to \infty} \left(\frac{\sum_{i=1}^{\lfloor n/p \rfloor} \frac{1}{i}}{p \sum_{l=1}^{n} \frac{1}{i}} - \frac{\lfloor n/p \rfloor}{n} \right)$$
$$\sim \frac{1}{p} \frac{\log(n/p)}{\log n} - \frac{1}{p}$$
$$\to 1/p - 1/p$$
$$= 0,$$

so constraint (6) holds. By Theorem 2, this shows that the number of distinct prime factors of an H_n -distributed random variable is asymptotically normally distributed, as $n \to \infty$, with both mean and variance equal to $\log \log n$.

3. Convex Combinations of E-K Distributions

The following theorem shows that any convex sum of two E-K distributions is also an E-K distribution.

Theorem 3. (Convex Combinations for Erdős-Kac (CLT)) Let n > 1 and $0 \le \lambda \le 1$ be fixed. Suppose $d_{1,n}$ and $d_{2,n}$ are two PMFs on [n] satisfying the constraints (4-6) for all n > 1. Then any PMF of the form

(9)
$$\mathbb{P}_{n}^{*}(i) = \lambda d_{1,n}(i) + (1-\lambda) d_{2,n}(i), 1 \le i \le n$$

also satisfies constraints (4-6). In particular,

$$\mathbb{P}_{n}^{*}\left(m \leq n : \omega\left(m\right) - \log\log n \leq x \left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \leq x\right)$$

as $n \to \infty$.

Proof. Denote

$$\varepsilon_{i,n} = d_{1,n}(i) - 1/n,$$

 $\varepsilon'_{i,n} = d_{2,n}(i) - 1/n,$

and

$$\hat{\varepsilon}_{i,n} = \mathbb{P}_n^*(i) - 1/n$$

We have

$$\sum_{l=1}^{\lfloor n/p \rfloor} \hat{\varepsilon}_{lp,n} = \sum_{l=1}^{\lfloor n/p \rfloor} \left(\lambda d_{1,n} \left(lp \right) + (1-\lambda) d_{2,n} \left(lp \right) - \frac{1}{n} \right)$$

$$\stackrel{(1)}{=} \sum_{l=1}^{\lfloor n/p \rfloor} \left(\lambda \left(\varepsilon_{lp,n} + 1/n \right) + (1-\lambda) \left(\varepsilon'_{lp,n} + 1/n \right) - \frac{1}{n} \right)$$

$$= \sum_{l=1}^{\lfloor n/p \rfloor} \left(\lambda \varepsilon_{lp,n} + (1-\lambda) \varepsilon'_{lp,n} \right)$$

$$= \lambda \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} + (1-\lambda) \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon'_{lp,n}$$

$$\leq \frac{\lambda C_1 + (1-\lambda) C_2}{p},$$

where the latest inequality is obtained by applying constraint (4) to both $d_{1,n}$ and $d_{2,n}$; therefore, (4) holds for the PMF given by (9) with $C = \lambda C_1 + (1 - \lambda) C_2$. Similarly,

$$\begin{split} \sum_{l=1}^{\frac{n}{p_1\cdots p_k}} \hat{\varepsilon}_{lp_1\cdots p_k,n} &= \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \left(\lambda d_{1,n} \left(lp_1\cdots p_k \right) + (1-\lambda) \, d_{2,n} \left(lp_1\cdots p_k \right) - \frac{1}{n} \right) \\ & \stackrel{(1)}{=} \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \left(\lambda \left(\varepsilon_{lp_1\cdots p_k,n} + 1/n \right) + (1-\lambda) \left(\varepsilon'_{lp_1\cdots p_k,n} + 1/n \right) - \frac{1}{n} \right) \\ & = \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \left(\lambda \varepsilon_{lp_1\cdots p_k,n} + (1-\lambda) \, \varepsilon'_{lp_1\cdots p_k,n} \right) \\ & = \lambda \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \varepsilon_{lp_1\cdots p_k,n} + (1-\lambda) \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k,n} \rfloor} \varepsilon'_{lp_1\cdots p_k,n} \\ & \leq \frac{\lambda D_1 + (1-\lambda) \, D_2}{n}, \end{split}$$

where the latest inequality is obtained by applying constraint (5) to both $d_{1,n}$ and $d_{2,n}$; therefore, (5) holds for the PMF given by (9) with $D = \lambda D_1 + (1 - \lambda) D_2$. Furthermore,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \hat{\varepsilon}_{lp,n} = \lambda \lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} + (1-\lambda) \lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon'_{lp,n}$$
$$= \lambda \cdot 0 + (1-\lambda) \cdot 0$$
$$= 0,$$

where the second equation uses the fact that the distributions d_1 and d_2 satisfy constraint (6). Therefore, (6) holds for the PMF given by (9).

By Theorem 3 and mathematical induction, we obtain the following.

Corollary 1. Suppose $d_{1,n}, \ldots, d_{k,n}$ are PMFs on [n] satisfying the constraints (4-6), and suppose $(\lambda_j)_{j=1}^k$ is a sequence of nonnegative numbers which add to unity. The PMF

$$\mathbb{P}_{n}^{*}(i) = \sum_{j=1}^{k} \lambda_{j} d_{j,n}(i)$$

also satisfies constraints (4-6). Therefore,

$$\mathbb{P}_{n}^{*}\left(m \leq n : \omega\left(m\right) - \log\log n \leq x \left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \leq x\right)$$

as $n \to \infty$.

We now present an alternate proof to the remark under Corollary 3.4 in [1]; their proof uses both the Tauberian theorems for the harmonic distribution and the uniform distribution simultaneously. We present a proof based on Theorems 2 and 3 of this paper.

Corollary 2. (Convex Combinations of Harmonic Distribution and Uniform Distribution) Let $\lambda \in [0,1]$ and consider the PMF given by

$$\mathbb{P}_{n}^{*}(i) = \frac{\lambda}{ih_{n}} + \frac{1-\lambda}{n}; 1 \le i \le n$$

where h_n is the n^{th} harmonic number given by $h_n := \sum_{i=1}^n 1/i$, then

$$\mathbb{P}_{n}^{*}\left(m \leq n : \omega\left(m\right) - \log\log n \leq x \left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \leq x\right)$$

as $n \to \infty$.

Proof. As noted in the remark after Theorem 2, the uniform distribution satisfies constraints (4-6), and in §§2.1 we showed the harmonic distribution satisfies constraints (4-6). By Theorem 3, their convex sums also satisfy constraints (4-6).

The following theorem shows that if given the PMF $1/n + \varepsilon_{i,n}$ of a sequence of E-K distributions, under additional constraints, the PMF $1/n - \varepsilon_{i,n}$ is also the PMF of a sequence of E-K distributions.

Theorem 4. (Reflection Theorem) Let \mathbb{P}_n satisfy constraints (4) and (6) for all n > 1. In addition, assume $\varepsilon_{i,n} \in \left[-\frac{1}{n}, \frac{1}{n}\right]$ for all i and for all n, and that for all k-tuples of distinct primes p_1, \ldots, p_k , there exists a constant $D \ge 0$ such that

(10)
$$-\frac{D}{n} \le \sum_{l=1}^{\lfloor \frac{n}{p_1 \cdots p_k} \rfloor} \varepsilon_{lp_1 \cdots p_k, n} \le \frac{D}{n}$$

If

$$\mathbb{P}_{n}^{*}\left(i\right)\coloneqq\frac{1}{n}-\varepsilon_{i,n},$$

then

$$\mathbb{P}_{n}^{*}\left(m \leq n : \omega\left(m\right) - \log\log n \leq x \left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \leq x\right)$$

as $n \to \infty$.

Proof. Define $\varepsilon'_{i,n} \coloneqq -\varepsilon_{i,n}$, so

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon'_{lp,n} = -\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{i,n}$$

$$\stackrel{(3)}{\leq} \frac{\lfloor n/p \rfloor}{n}$$

$$\stackrel{\leq}{\leq} \frac{1}{p}$$

so constraint (4) holds with C = 1. Similarly,

$$\sum_{l=1}^{\left\lfloor\frac{n}{p_1\cdots p_k}\right\rfloor}\varepsilon'_{lp_1\cdots p_k,n} = -\sum_{l=1}^{\left\lfloor\frac{n}{p_1\cdots p_k}\right\rfloor}\varepsilon_{lp_1\cdots p_k,n} \stackrel{(10)}{\leq} \frac{D}{n}$$

so constraint (5) holds. Moreover,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon'_{lp,n} = -\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,r}$$
$$= 0,$$

where the latest limit is due to the assumption that \mathbb{P}_n satisfies constraint (6).

4. A Central Limit Theorem for E-K Distributions on \mathbb{N}

We would like to formalize the notion of asymptotic normality for $\omega(X_j)$ in the limit of a sequence of random variables $(X_j)_{j>1}$ on \mathbb{N} . Towards that goal we provide the following definition.

Definition 2. Given an infinite sequence of random variables X_1, X_2, \ldots defined on \mathbb{N} , let $X_j(n)$ denote the truncation of X_j on [n]. Define $\varepsilon_{l,j,n} := \mathbb{P}(X_j(n) = l) - 1/n$. The sequence X_1, X_2, \ldots is said to satisfy the **uniformity along primes** property if the following holds. There exists constants C and D such that for all n there exists a $d \ge 1$ such that for all $j \ge d$:

• For each prime $p > \alpha_n$

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,j,n} \le \frac{C}{p}.$$

• For each $k \ge 1$ and for all k-tuples (p_1, \ldots, p_k) consisting of distinct primes of size at most α_n

$$\sum_{l=1}^{\frac{n}{p_1\cdots p_k}} \varepsilon_{lp_1\cdots p_k,j,n} \le \frac{D}{n}.$$

In addition, for any prime p,

$$\lim_{j \to \infty} \lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,j,n} = 0.$$

Theorem 5. Assume $X_1, X_2, ...$ is an infinite sequence of random variables defined on \mathbb{N} satisfying the **uniformity along primes** property. Let $\mathbb{P}_{j,n}$ be the probability distribution of $X_j(n)$ on [n]. If $(n_j)_{j \in \mathbb{N}}$ is any sequence in $\mathbb{N} \setminus \{1\}$ tending to ∞ such that (4-5) hold for $X_j(n_j)$ for all $j \geq 1$, then

$$\mathbb{P}_{j,n_j}\left(m \le n_j : \omega\left(m\right) - \log\log n_j \le x \left(\log\log n_j\right)^{1/2}\right) \to \mathbb{P}\left(Z \le x\right)$$

as $j \to \infty$.

Proof. Let's proceed by contradiction. That is, suppose $(n_j)_{j \in \mathbb{N}}$ is any sequence in $\mathbb{N} \setminus \{1\}$ tending to ∞ and suppose there exist a sequence $(j_k)_{k \in \mathbb{N}}$, some $x_0 \in \mathbb{R}$, and some $\varepsilon_0 > 0$ such that

$$\left|\mathbb{P}_{j_k,n_{j_k}}\left(m \le n_{j_k} : \omega\left(m\right) - \log\log n_{j_k} \le x_0 \left(\log\log n_{j_k}\right)^{1/2}\right) - \mathbb{P}\left(Z \le x_0\right)\right| \ge \varepsilon_0$$

for all $k \ge 1$. Consider the following PMF on [n] for all n > 1, which satisfies the hypotheses of Theorem 2:

$$\mathbb{P}_{n}^{*}(i) \coloneqq \begin{cases} 1/n & \text{if } n_{j_{k}} \neq n \text{ for all } k \geq 1 \\ \mathbb{P}_{j_{k'(n)}, n}(i) & \text{if } n_{j_{k'(n)}} = n, \end{cases}$$

where k'(n) is the smallest integer satisfying $n_{j_{k'(n)}} = n$. It is clear that \mathbb{P}_n^* is defined for all n > 1 and satisfies the hypotheses of Theorem 2 due to Definition 2. Let $\varepsilon = \varepsilon_0$, then there is some $d \ge 1$ such that

$$\left|\mathbb{P}_{j_{k'(n)},n_{j_{k'(n)}}}\left(m \le n_{j_{k'(n)}} : \omega\left(m\right) - \log\log n_{j_{k'(n)}} \le x_0 \left(\log\log n_{j_{k'(n)}}\right)^{1/2}\right) - \mathbb{P}\left(Z \le x_0\right)\right| < \varepsilon$$

if $n \ge d$, by Theorem 2. The fact that at least one k'(n) exists such that $n_{j_{k'(n)}} = n$ when $n \ge d$ is due to the fact that $n_{j_k} \to \infty$ as $k \to \infty$, so there are infinitely many k'(n) such that $n_{j_{k'(n)}} = n$ when $n \ge d$. Thus, Theorem 5 holds by contradiction.

4.1. Moment Generating Functions and Characteristic Functions.

Definition 3. Let X denote a random variable. The moment generating function (MGF) of X is defined as $M_X(t) = \mathbb{E}(e^{tX})$, and the characteristic function (CF) of X is defined as $\phi_X(t) = \mathbb{E}(e^{itX})$.

We have the following two facts about the MGF and the CF that will be used in the proof of Corollary 3:

- The k^{th} derivative of $M_X(t)$ at t = 0 is the k^{th} moment of X.
- Lèvy's Continuity Theorem: Convergence in distribution, $X_n \xrightarrow{d} X$, for a sequence of random variables is equivalent to pointwise convergence, $\phi_{X_n} \to \phi_X$, of the corresponding CFs on all of \mathbb{R} .

Corollary 3 and Corollary 4 below allow us to state Theorem 7.1 of [2] in a slightly different way.

Corollary 3. Suppose X_1, X_2, \ldots is an infinite sequence of random variables defined on \mathbb{N} that satisfies the **uniformity along primes** property. Let $(n_j)_{j \in \mathbb{N}}$ be any sequence in $\mathbb{N} \setminus \{1\}$ tending to ∞ such that (4 - 5) hold for X_j (n_j) for all $j \ge 1$. Let \mathbb{P}_j be the probability distribution of X_j on \mathbb{N} , $\mathbb{P}_{j,n}$ be the probability distribution of X_j (n) on [n], and define $\mu_{j,n} \coloneqq \mathbb{E}_{j,n} (\omega(X_j(n)))$, then the following is true:

$$\mathbb{P}_{j,n_j}\left(m \le n_j : \omega\left(m\right) - \mu_{j,n_j} \le x \left(\mu_{j,n_j}\right)^{1/2}\right) \to \mathbb{P}\left(Z \le x\right)$$

as $j \to \infty$. Additionally, if $\mu_j < \infty$ for all $j \ge 1$ and

$$\lim_{j \to \infty} \left(\mu_j / \left(\log \log n_j \right)^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0,$$

then

$$\mathbb{P}_{j}\left(m:\omega\left(m\right)-\mu_{j}\leq x\left(\mu_{j}\right)^{1/2}\right)\rightarrow\mathbb{P}\left(Z\leq x\right)$$

as $j \to \infty$, where

$$\mu_j \coloneqq \mathbb{E}_j \left(\omega \left(X_j \right) \right) = \lim_{n \to \infty} \mu_{j,n}.$$

We also have the following asymptotic properties:

$$\lim_{j \to \infty} \left(\mu_{j,n_j} / \left(\log \log n_j \right)^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0,$$
$$\lim_{j \to \infty} \left(\mu_{j,n_j}^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0,$$

and

$$\lim_{j \to \infty} \frac{\mu_{j,n_j}}{\log \log n_j} = \lim_{j \to \infty} \frac{\mu_j}{\log \log n_j} = 1.$$

Proof. Let M_{j,n_i} and ϕ_{j,n_i} be the MGF and CF of

$$\frac{\omega\left(X_{j}\left(n_{j}\right)\right) - \log\log n_{j}}{\sqrt{\log\log n_{j}}}$$

and let $M_{j,\mu}$ and $\phi_{j,\mu}$ be the MGF and CF of

$$\frac{\omega\left(X_{j}\left(n_{j}\right)\right)-\mu_{j,n_{j}}}{\sqrt{\mu_{j,n_{j}}}}.$$

We have

$$M_{j,n_j}(t) = e^{-t(\log \log n_j)^{1/2}} M_{\omega(X_j(n_j))} \left((\log \log n_j)^{-1/2} t \right)$$

and

$$M_{j,\mu}(t) = e^{-t\mu_{j,n_j}^{1/2}} M_{\omega(X_j(n_j))}\left(\left(\mu_{j,n_j}\right)^{-1/2} t\right)$$

Let's compute the mean from the MGF $M_{j,n_i}(t)$:

$$\begin{aligned} M'_{j,n_j}(0) &= -\left(\log\log n_j\right)^{1/2} e^{-t(\log\log n_j)^{1/2}} M_{\omega(X_j(n_j))} \left(\left(\log\log n_j\right)^{-1/2} t\right) \\ &+ e^{-t(\log\log n_j)^{1/2}} M'_{\omega(X_j(n_j))} \left(\left(\log\log n_j\right)^{-1/2} t\right) \left(\log\log n_j\right)^{-1/2} \Big|_{t=0} \\ &= -\left(\log\log n_j\right)^{1/2} + \mu_{j,n_j} \left(\log\log n_j\right)^{-1/2} \\ &\to 0, \end{aligned}$$

where the last limit holds due to the proof of Theorem 2 (we showed the moments all approach the moments of the normal distribution) which is used in the proof of Theorem 5 above; thus,

(11)
$$\lim_{j \to \infty} \left(\mu_{j,n_j} / \left(\log \log n_j \right)^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0$$

holds. Therefore,

$$\lim_{j \to \infty} \frac{\mu_{j,n_j}}{\log \log n_j} = 1$$

From equation (11) we can conclude

$$\lim_{j \to \infty} \left(\mu_{j, n_j}^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0$$

This is because,

$$\left| \mu_{j,n_j}^{1/2} - \left(\log \log n_j \right)^{1/2} \right| = \left| \frac{\mu_{j,n_j} - \left(\log \log n_j \right)}{\mu_{j,n_j}^{1/2} + \left(\log \log n_j \right)^{1/2}} \right| \le \left| \frac{\mu_{j,n_j} - \log \log n_j}{\left(\log \log n_j \right)^{1/2}} \right| \to 0.$$

Now we want to show convergence in distribution using the characteristic functions. We use the fact that the characteristic function $\phi_{\omega(X_j(n_j))}(t)$ is continuous at t = 0. According to properties of continuous functions, we have the ratios

(12)
$$\frac{e^{-it\mu_{j,n_j}^{1/2}}}{e^{-it(\log\log n_j)^{1/2}}} = e^{-it\left(\mu_{j,n_j}^{1/2} - (\log\log n_j)^{1/2}\right)} \to e^0 = 1,$$

and also

(13)
$$\frac{\phi_{\omega(X_j(n_j))}\left(\left(\mu_{j,n_j}\right)^{-1/2}t\right)}{\phi_{\omega(X_j(n_j))}\left(\left(\log\log n_j\right)^{-1/2}t\right)} \to \frac{\phi_{\omega(X_j(n_j))}(0)}{\phi_{\omega(X_j(n_j))}(0)} = 1$$

as $j \to \infty$. This gives us the required convergence of $\phi_{j,\mu}$ at any t; i.e.,

$$\phi_{j,\mu}(t) = e^{-it\mu_{j,n_j}^{1/2}} \phi_{\omega(X_j(n_j))} \left(\left(\mu_{j,n_j}\right)^{-1/2} t \right) \stackrel{(12,13)}{\sim} e^{-it(\log\log n_j)^{1/2}} \phi_{\omega(X_j(n_j))} \left((\log\log n_j)^{-1/2} t \right) \\ \to e^{-t^2/2} = \phi_Z(t) \,.$$

By Lèvy's Continuity Theorem,

$$\mathbb{P}_{j,n_j}\left(m \le n_j : \omega\left(m\right) - \mu_{j,n_j} \le x \left(\mu_{j,n_j}\right)^{1/2}\right) \to \mathbb{P}\left(Z \le x\right).$$

Suppose additionally that $\mu_j < \infty$ for all $j \ge 1$ and

$$\lim_{j \to \infty} \left(\mu_j / \left(\log \log n_j \right)^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0.$$

Similarly, let ϕ_{j,μ_j} be the characteristic function of

$$\frac{\omega\left(X_j\right) - \mu_j}{\sqrt{\mu_j}}$$

Then

$$\phi_{j,\mu_j}(t) = e^{-it\mu_j^{1/2}} \phi_{\omega(X_j)}\left(\left(\mu_j\right)^{-1/2} t\right) \sim e^{-it(\log\log n_j)^{1/2}} \phi_{\omega(X_j(n_j))}\left(\left(\log\log n_j\right)^{-1/2} t\right) \\ \to e^{-t^2/2} = \phi_Z(t) \,,$$

for any $t \in \mathbb{R}$. Thus, again we have that

$$\mathbb{P}_{j}\left(m:\omega\left(m\right)-\mu_{j}\leq x\left(\mu_{j}\right)^{1/2}\right)\rightarrow\mathbb{P}\left(Z\leq x\right)$$

by Lèvy's Continuity Theorem. This completes the proof.

Remark. We can replace μ_i in Corollary 3 with any sequence a_i that satisfies the limit

$$\lim_{j \to \infty} \left(a_j / \left(\log \log n_j \right)^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0,$$

to get a similar result. This is a generalization of Theorem 7.1 in [2] which states this result for particular sequences of Zeta(s) distributions as their parameter $s \to 1$. In their result, the mean was $\sum_{p} p^{-s}$, where the sum is over all prime numbers.

Definition 4. Given an infinite sequence of random variables X_1, X_2, \ldots defined on \mathbb{N} , let $X_j(n)$ denote the truncation of X_j on [n]. If X_1, X_2, \ldots satisfies the uniformity along primes property, and $X_j(n)$ satisfies (4 - 5) for any n and any j, then we say the sequence satisfies the **complete uniformity along primes** property.

The following lemma will be applied in Corollary 4.

Lemma 3. If X_1, X_2, \ldots has the complete uniformity along primes property and $\mu_j < \infty$ for all $j \ge 1$, then $\mu_j \to \infty$ as $j \to \infty$.

Proof. For any sequence $(n_j)_{j \in \mathbb{N}}$ in $\mathbb{N} \setminus \{1\}$ tending to ∞ we have $\log \log n_j \sim \mu_{j,n_j} \sim \mu_j$ as $j \to \infty$. \Box

Corollary 4. If X_1, X_2, \ldots has the complete uniformity along primes property and $\mu_j < \infty$ for all $j \ge 1$, then

$$\mathbb{P}_{j}\left(m:\omega\left(m\right)-\mu_{j}\leq x\left(\mu_{j}\right)^{1/2}\right)\rightarrow\mathbb{P}\left(Z\leq x\right)$$

as $j \to \infty$.

Proof. Define $n_j \coloneqq \max\left\{ \lfloor e^{e^{\mu_j}} \rfloor, 2 \right\}$ so that $n_j \to \infty$ by Lemma 3 and the limit

$$\lim_{j \to \infty} \left(\mu_j / \left(\log \log n_j \right)^{1/2} - \left(\log \log n_j \right)^{1/2} \right) = 0$$

in Corollary 3 holds.

4.2. The Zeta Distribution. We show the zeta distribution has the complete uniformity along primes property as $s \to 1$. Given s > 1, denote by Z_s the Zeta (s) distribution so that for any $j \in \mathbb{N}$,

$$Z_{s}\left(j\right) = \frac{1}{j^{s}\zeta\left(s\right)},$$

where

$$\zeta\left(s\right) = \sum_{j \ge 1} \frac{1}{j^s}$$

denotes the Riemann zeta function. Since Theorem 2 involves distributions defined on [n], restrict the Zeta(s) distribution to [n] and then normalize by dividing by $\sum_{i=1}^{n} \frac{1}{i^s \zeta(s)}$; i.e., for $j \in [n]$,

$$Z_{s,n}(j) \coloneqq \frac{\frac{1}{j^s \zeta(s)}}{\sum_{i=1}^n \frac{1}{i^s \zeta(s)}}$$
$$= \frac{1}{j^s \sum_{i=1}^n \frac{1}{i^s}};$$

and $Z_{s,n}$ is known as the Zipf distribution with parameters n and s. For a Zifp distribution, we have

$$\varepsilon_{i,n} \stackrel{(1)}{=} \frac{1/i^s}{\sum_{j=1}^n 1/j^s} - \frac{1}{n}.$$

Thus,

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = \frac{\sum_{l=1}^{\lfloor n/p \rfloor} 1/(lp)^s}{\sum_{l=1}^n 1/l^s} - \frac{\lfloor n/p \rfloor}{n}$$
$$= \frac{1}{p^s} \frac{\sum_{l=1}^{\lfloor n/p \rfloor} 1/l^s}{\sum_{l=1}^n 1/l^s} - \frac{\lfloor n/p \rfloor}{n}$$
$$\leq \frac{1}{p^s}$$
$$\leq \frac{1}{p},$$

so constraint (4) holds with C = 1 for any s. Moreover,

$$\sum_{l=1}^{\lfloor \frac{n}{p_1 \cdots p_k} \rfloor} \varepsilon_{lp_1 \cdots p_k, n} = \frac{\sum_{l=1}^{\lfloor \frac{n}{p_1 \cdots p_k} \rfloor} 1/(lp_1 \cdots p_k)^s}{\sum_{l=1}^n 1/l^s} - \frac{\lfloor \frac{n}{p_1 \cdots p_k} \rfloor}{n}$$
$$= \frac{1}{(p_1 \cdots p_k)^s} \frac{\sum_{l=1}^{\lfloor n/p \rfloor} 1/l^s}{\sum_{l=1}^n 1/l^s} - \frac{1}{p_1 \cdots p_k} + \frac{1}{n}$$
$$\leq \frac{1}{(p_1 \cdots p_k)^s} - \frac{1}{p_1 \cdots p_k} + \frac{1}{n}$$
$$\leq \frac{1}{n},$$

so constraint (5) holds with D = 1 for any s. Furthermore,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = \lim_{n \to \infty} \frac{\sum_{l=1}^{\lfloor n/p \rfloor} 1/(lp)^s}{\sum_{l=1}^n 1/l^s} - \frac{1}{p}$$
$$= \frac{1}{p^s} - \frac{1}{p}$$
$$\stackrel{s \to 1}{\to} 0;$$

therefore, (6) holds as $s \to 1$.

Corollary 5. Let $(a_j)_{j \in \mathbb{N}}$ be any sequence of real numbers such that $0 < a_j \to \infty$. If X_j is a sequence of Zeta $(1 + 1/a_j)$ -distributed random variables, then

$$\mu_j = \sum_p p^{-(1+1/a_j)} < \infty,$$
$$\lim_{j \to \infty} \mu_j = \infty,$$

and

$$\mathbb{P}_{j}\left(m:\omega\left(m\right)-\mu_{j}\leq x\left(\mu_{j}\right)^{1/2}\right)\rightarrow\mathbb{P}\left(Z\leq x\right)$$

as $j \to \infty$.

This latest limit is the statement of Theorem 7.1 in [2].

4.3. Convex Combinations of E-K Distributions on \mathbb{N} . We can also take convex combinations of sequences to form new sequences which satisfy the uniformity along primes property.

Corollary 6. Let $\lambda \in [0,1]$. Let X_1, X_2, \ldots and Y_1, Y_2, \ldots be two sequences of random variables on \mathbb{N} that both satisfy the **complete uniformity along primes** property. Then we can define a new sequence Z_1, Z_2, \ldots such that each Z_j is the convex combination of X_j and Y_j . In particular,

$$\mathbb{P}_{Z_j}(i) \coloneqq \lambda \mathbb{P}_{X_j}(i) + (1 - \lambda) \mathbb{P}_{Y_j}(i); i \ge 1, j \ge 1.$$

Then Z_j satisfies the complete uniformity along primes property and thus Theorem 5 holds for the Z_j sequence.

Proof. The proof is similar to that of Theorem 3.

Corollary 7. Apply Corollary 6 to any convex combination of a finite number of sequences

$$(X_{1,1}, X_{1,2}, \ldots), \ldots, (X_{n,1}, X_{n,2}, \ldots)$$

that all satisfy the **complete uniformity along primes** property. Then the conclusion of Corollary 6 holds for this convex combination.

Proof. Use induction and Corollary 6.

5. Illustrative Examples

In this section, we give examples from the class of distributions \mathbb{P}_n^* that satisfy the hypotheses (4-6) and examples of limits of distributions to which we can apply Theorem 5. It will be shown that the statement of Theorem 2 holds when \mathbb{P}_n^* is replaced with either the Harmonic(n) distribution or a convex combination of Harmonic and Uniform Distributions. Then we show that the Zeta (s) and the Logarithmic distribution satisfy Theorem 5 as their parameters tend towards limits. We introduce a 2-parameter family of distributions $LZ(s, \alpha)$ which includes the Logarithmic distribution and Zeta distribution as special cases (when $\alpha = 1$ and s = 1 respectfully). We also look at a geometric power series distribution that converges to the normal distribution on all truncations as $s \to 1$.

5.1. The Harmonic Distribution. This was proved in \S 2.1 above.

5.2. The Zeta Distribution. This was proved in \S 4.2 above.

5.3. The Logarithmic Distribution. Given a real number s with 0 < s < 1, a logarithmic distribution with parameter s is given by

$$L_s(i) \coloneqq \frac{-1}{\log\left(1-s\right)} \frac{s^i}{i}; \ i \in \mathbb{N}.$$

Now we will show that the number of distinct prime factors, $\omega(\cdot)$, of a truncated log-distributed variable has the same central limit theorem as the uniform distribution as $n \to \infty$ and $s \to 1$. For $i \in [n]$ we have

$$L_{s,n}(i) \coloneqq \frac{\frac{-1}{\log(1-s)} \frac{s^{i}}{i}}{\sum_{l=1}^{n} \frac{-1}{\log(1-s)} \frac{s^{l}}{l}} = \frac{\frac{s^{i}}{i}}{\sum_{l=1}^{n} \frac{s^{l}}{l}},$$

 \mathbf{SO}

$$\varepsilon_{i,n} \stackrel{(1)}{=} \frac{\frac{s^i}{i}}{\sum_{l=1}^n \frac{s^l}{l}} - \frac{1}{n}.$$

Therefore,

$$\sum_{l=1}^{\lfloor \frac{n}{p} \rfloor} \varepsilon_{lp,n} = \sum_{l=1}^{\lfloor \frac{n}{p} \rfloor} \left(\frac{\frac{(s)^{lp}}{lp}}{\sum_{l=1}^{n} \frac{s^l}{l}} - \frac{1}{n} \right)$$
$$\leq \frac{1}{p} \frac{\sum_{l=1}^{\lfloor \frac{n}{p} \rfloor} \frac{(s^p)^l}{l}}{\sum_{l=1}^{n} \frac{s^l}{l}} - \frac{\lfloor n/p \rfloor}{n}$$
$$\leq \frac{1}{p} - \frac{\lfloor n/p \rfloor}{n}$$
$$\leq \frac{1}{p},$$

so (4) holds with C = 1. Similarly,

$$\begin{split} \sum_{l=1}^{\frac{n}{p_1\cdots p_k}} \varepsilon_{lp_1\cdots p_k,n} &= \sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \left(\frac{\frac{(s)^{lp_1\cdots p_k}}{lp_1\cdots p_k}}{\sum_{l=1}^{n} \frac{s^l}{l}} - \frac{1}{n} \right) \\ &\leq \frac{1}{p_1\cdots p_k} \frac{\sum_{l=1}^{\lfloor \frac{n}{p_1\cdots p_k} \rfloor} \frac{(s^{p_1\cdots p_k})^l}{l}}{\sum_{l=1}^{n} \frac{s^l}{l}} - \frac{1}{p_1\cdots p_k} + \frac{1}{n} \\ &\leq \frac{1}{p_1\cdots p_k} - \frac{1}{p_1\cdots p_k} + \frac{1}{n} \\ &= \frac{1}{n}, \end{split}$$

so (5) holds with D = 1. Moreover, we have

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \xrightarrow{n \to \infty} \frac{1}{p} \frac{\log(1-s^p)}{\log(1-s)} - \frac{1}{p} \xrightarrow{s \to 1} 0.$$

Therefore, (6) holds, so $\omega(L_{i,s})$ is asymptotically distributed as $\mathcal{N}(\log \log n, \log \log n)$ as $n \to \infty$ and $s \to 1$.

Although Kac's heuristic for Theorem 1 is based on the asymptotic independence in the uniform case, we will show that the events {divisible by p} and {divisible by q} are not independent for any value of s. Let A_p denote the set of all positive integers divisible by p, then

$$L_{s}(A_{p}) = \sum_{l=1}^{\infty} \frac{-1}{\log(1-s)} \frac{(s^{p})^{l}}{pl}$$
$$= \frac{1}{p} \frac{\log(1-s^{p})}{\log(1-s)}$$

and

$$L_{s}(A_{p} \cap A_{q}) = \sum_{l=1}^{\infty} \frac{-1}{\log(1-s)} \frac{(s^{pq})^{l}}{pql}$$
$$= \frac{1}{pq} \frac{\log(1-s^{pq})}{\log(1-s)}.$$

It is worth noting that

$$\lim_{s \to 1} L_s (A_p \cap A_q) = \frac{1}{pq} = \lim_{s \to 1} (L_s (A_p) L_s (A_q))$$

as $s \to 1$; so independence is only approached in the limit.

5.4. Geometric Power Series Distribution. Let $s \in (0, 1)$. Define:

$$\mathbb{P}\left(i\right)\coloneqq\frac{1-s}{s}s^{i};\ i\in\mathbb{N}$$

then this distribution satisfies the hypotheses of Theorem 2 in the limit as $s \to 1$. As $s \to 1$ it seems that this distribution is converging to the uniform distribution on any truncation, so the result is to be expected.

Truncating on [n] leads to

$$\varepsilon_{i,n} \stackrel{(1)}{=} \frac{s^i}{\sum_{j=1}^n s^j} - \frac{1}{n},$$

therefore

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = \sum_{l=1}^{\lfloor n/p \rfloor} \left(\frac{s^i}{\sum_{j=1}^n s^j} - \frac{1}{n} \right)$$
$$= \frac{s - s^{\lfloor n/p \rfloor + 1}}{s - s^{n+1}} - \frac{\lfloor n/p \rfloor}{n}$$
$$\stackrel{s \to 1}{\to} \frac{\lfloor n/p \rfloor}{n} - \frac{\lfloor n/p \rfloor}{n}$$
$$= 0,$$

so constraints (4-6) hold with C = D = 1.

5.5. Convex Combination of Harmonic and Uniform. Following Cranston and Mountford in [1], let $\lambda \in [0, 1]$ and define

$$\mathbb{P}_n(i) \coloneqq \frac{\lambda}{ih_n} + (1-\lambda) \frac{1}{n}; \ i \in \mathbb{N}$$

where h_n is the n^{th} harmonic number. Then \mathbb{P}_n satisfies the hypotheses of Theorem 2. This is proved in Corollary 2 above using Theorems 2 and 3.

5.6. A Logarithmic-Zeta Distribution. There is a 2-parameter family of power series distributions on \mathbb{N} for $(s, \alpha) \in \mathbb{R}^2$ such that $0 < s \leq 1$, $\alpha \geq 1$, and $s = \alpha = 1$ is not allowed; the PMF is given by

$$LZ_{s,\alpha}\left(i\right) \coloneqq \frac{1}{\sum_{j=1}^{\infty} \frac{s^{j}}{j^{\alpha}}} \frac{s^{i}}{i^{\alpha}}; \ i \in \mathbb{N}.$$

Similarly, there is a truncation of LZ given by

$$LZ_{s,\alpha,n}(i) \coloneqq \frac{1}{\sum_{j=1}^{n} \frac{s^{j}}{j^{\alpha}}} \frac{s^{i}}{i^{\alpha}}; \ i \le n,$$

and a description of the $\varepsilon_{i,n}$ given by

$$\varepsilon_{i,n} \stackrel{(1)}{=} \frac{1}{\sum_{j=1}^{n} \frac{s^{j}}{j^{\alpha}}} \frac{s^{i}}{i^{\alpha}} - \frac{1}{n}.$$

It is clear that a similar type of argument shows that Theorem 2 holds for the truncated logarithmic-zeta distribution as $(s, \alpha) \rightarrow (1, 1)$; in particular, we obtain

$$\sum_{l=1}^{n/p} \varepsilon_{lp,n} \le \frac{1}{p^{\alpha}} \le \frac{1}{p}$$

and

$$\sum_{l=1}^{\overline{p_1\cdots p_k}} \varepsilon_{lp_1\cdots p_k,n} \le \frac{1}{p_1^{\alpha}\cdots p_k^{\alpha}} - \frac{1}{p_1\cdots p_k} + \frac{1}{n} \le \frac{1}{n}$$

Furthermore,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \stackrel{s \to 1}{\to} \frac{1}{p^{\alpha}} - \frac{1}{p} \stackrel{\alpha \to 1}{\to} 0.$$

5.7. Passage to the Limit: $\lim_{\substack{s \to 1 \\ \alpha \to 1}} LZ_{s,\alpha}$. We would like to look at the behavior of $\omega(X)$ for X distributed as $LZ_{s,\alpha}$ as $(s,\alpha) \to (1,1)$. The motivation for this comes from the fact that as $(s,\alpha) \to (1,1)$ the truncated distributions resemble the harmonic distribution which we know behaves similarly to the uniform distribution

on [n] when $n \to \infty$ according to Theorem 2.

In Theorem 5.5 of [2], they provide the moment generating function for $\omega(X_{\alpha})$ when X_{α} is a Zeta (α) distributed random integer with parameter $\alpha > 1$. They go on to prove that

$$\hat{\omega}(X_{\alpha}) = \frac{\omega(X_{\alpha}) - \sum_{p} p^{-\alpha}}{\sqrt{\sum_{p} p^{-\alpha}}} \stackrel{d}{\to} Z$$

as $\alpha \to 1$ in Theorem 7.1 of [2] by using this moment generating function. Later, Cranston and Mountford [1] give a new proof of Theorem 1. The proof uses Theorem 7.1 about zeta distributions to prove Erdős-Kac in a way that translates over to settings where zeta functions still make sense.

We generalize Peltzer and Cranston's Theorem 7.1 in [2] in the following way. As $(s, \alpha) \rightarrow (1, 1)$, we show that

$$\hat{\omega}\left(X_{s,\alpha}\right) = \frac{\omega\left(X_{s,\alpha}\right) - \mu_{s,\alpha}}{\sqrt{\mu_{s,\alpha}}} \stackrel{d}{\to} Z,$$

where $\mu_{s,\alpha}$ is the mean of $\omega(X_{s,\alpha})$ when $X_{s,\alpha}$ is distributed as $LZ_{s,\alpha}$. In fact, we do not need independence, nor do we need to compute the mean for any particular s, α in order to conclude this (neither do we need to compute a MGF).

In §4, we proved a stronger statement than the above statement about $LZ_{s,\alpha}$; we proved that in general

$$\hat{\omega}\left(X_{j}\right) = \frac{\omega\left(X_{j}\right) - \mu_{j}}{\sqrt{\mu_{j}}} \stackrel{d}{\to} Z,$$

as long as the truncated variables $X_j(n)$ have the **complete uniformity along primes** property. Here we assume that $\mu_j = \mathbb{E}(\omega(X_j)) < \infty$. When $(s, \alpha) \to (1, 1)$, we recover the above statement about $LZ_{s,\alpha}$.

5.8. A Non-Example: Zeroing at a Set of Primes. Fix $n \in \mathbb{N}$ and let $p \leq n$ denote a prime. Consider the PMF defined by

$$\mathbb{P}_{n,p}\left(i\right) = \begin{cases} \frac{1}{\#\left([n] \setminus p\mathbb{N}\right)} & i \notin p\mathbb{N}, \\ 0 & i \in p\mathbb{N}. \end{cases}$$

We have

$$\varepsilon_{i,n} = \begin{cases} \frac{1}{\#([n] \setminus p\mathbb{N})} - 1/n & i \notin p\mathbb{N}, \\ -1/n & i \in p\mathbb{N}. \end{cases}$$

Therefore,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = \lim_{n \to \infty} -\frac{\lfloor n/p \rfloor}{n}$$
$$= -1/p$$
$$\neq 0.$$

Thus, this PMF does not satisfy constraint (6). We conjecture that the conclusion of Theorem 2 does not hold for this distribution.

5.9. An Erdős-Kac Theorem for Continuous Variables. Consider a continuous uniform random variable N_n on the interval (0, n]. Then $\lceil N_n \rceil$ is a uniform variable on [n]. Therefore, we have

Corollary 8. Let Z denote a standard normal variable, and let X_n be a continuous random variable on (0,n]. Define the $\varepsilon_{i,n}$ according to the following relation: $\mathbb{P}_n(\lceil X_n \rceil = i) = \frac{1}{n} + \varepsilon_{i,n}$. If the constraints

• There exists a constant C such that for all n > 1 and for all primes p with $p > \alpha_n$,

$$\sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} \le \frac{C}{p}.$$

• There exists a constant D such that

$$\sum_{l=1}^{\frac{n}{p_1\cdots p_k}} \varepsilon_{lp_1\cdots p_k,n} \le \frac{D}{n}$$

for all n > 1 and, for each k, all k-tuples (p_1, \ldots, p_k) consisting of distinct primes of size at most α_n , and

• For any prime p,

$$\lim_{n \to \infty} \sum_{l=1}^{\lfloor n/p \rfloor} \varepsilon_{lp,n} = 0$$

all hold, then

$$\mathbb{P}_n\left(t \le n : \omega\left(\lceil t \rceil\right) - \log\log n \le x\left(\log\log n\right)^{1/2}\right) \to \mathbb{P}\left(Z \le x\right)$$

as $n \to \infty$.

6. Conclusion

Theorem 2 generalizes the Erdős-Kac Theorem for $\omega(\cdot)$ to distributions other than the uniform distribution, and this theorem was proved by imposing constraints (4-6) on a PMF of the form $\mathbb{P}(i) = 1/n + \varepsilon_{i,n}$. We showed that the uniform and harmonic distributions satisfy these constraints; then, we showed that any convex sum of these PMFs also satisfies the constraints (4-6).

The uniformity property provides a natural way to examine asymptotic properties of truncations of variables with support \mathbb{N} . Given an infinite sequence X_1, X_2, \ldots of random variables on \mathbb{N} satisfying uniformity along primes, Theorem 5 showed that for any sequence with $n_j \to \infty$ as $j \to \infty$, the distribution of $\omega(X_j(n_j))$ is asymptotically normally distributed with mean and variance both equal to $\log \log n_j$ as long as $X_j(n_j)$ satisfies (4 - 6).

The definition of complete uniformity along primes allows us to obtain central limit theorems regardless of how $n \to \infty$, and allows us to make asymptotic statements involving the mean of $\omega(X_j)$. This generalized a statement from [2] involving the mean of $\omega(X_s)$ as $s \to 1$ when X_s is a random Zeta (s)-distributed variable.

Another way to generalize Theorem 2 would be to incorporate it with other generalizations, e.g., [1, 2, 6, 7]. By incorporating Theorem 2 with these, further generalizations can be made in which the original setting is not [n], the underlying distribution of the random-integer is not uniform, and $\omega(n)$ can be replaced with a more general strongly additive function f(n).

We also showed the complete uniformity property holds, and thus normality in the limit, for Zeta(s) and a number of similar distributions. It is suspected by the authors, but not known, whether or not the hypotheses (4-6) are necessary and sufficient for the conclusion of Theorem 2; we conjecture that is the case.

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