# A GENERALIZATION OF THE ERDÕS-KAC THEOREM 

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#### Abstract

Given a natural number $n$, let $\omega(n)$ denote the number of distinct prime factors of $n$, let $Z$ denote a standard normal variable, and let $P_{n}$ denote the uniform distribution on $\{1, \ldots, n\}$. The Erdôs-Kac Theorem states that if $N(n)$ is a uniformly distributed variable on $\{1, \ldots, n\}$, then $\omega(N(n))$ is asymptotically normally distributed as $n \rightarrow \infty$ with both mean and variance equal to $\log \log n$. The contribution of this paper is a generalization of the Erdős-Kac Theorem to a larger class of random variables by considering perturbations of the uniform probability mass $1 / n$ in the following sense. Denote by $\mathbb{P}_{n}$ a probability distribution on $\{1, \ldots, n\}$ given by $\mathbb{P}_{n}(i)=1 / n+\varepsilon_{i, n}$. We provide sufficient conditions on $\varepsilon_{i, n}$ so that the number of distinct prime factors of a $\mathbb{P}_{n}$-distributed random variable is asymptotically normally distributed, as $n \rightarrow \infty$, with both mean and variance equal to $\log \log n$. Our main result is applied to prove that the number of distinct prime factors of a positive integer with the Harmonic $(n)$ distribution also tends to the normal distribution, as $n \rightarrow \infty$. In addition, we explore sequences of distributions on the natural numbers such that $\omega(n)$ is normally distributed in the limit. In addition, one of our theorems and its corollaries generalize a result from the literature involving the limit of Zeta (s) distributions as the parameter $s \rightarrow 1$.


## 1. Introduction

Given a natural number $n$, the number of distinct prime factors of $n$ is denoted $\omega(n)$. The function $\omega$ may be written as $\omega(n)=\sum_{p \mid n} 1$, where the sum is over all prime factors of $n$. In 1917, Hardy and Ramanujan (p. 270 of [5]) proved that the number of distinct prime factors of a natural number $n$ is about $\log \log n$. In particular, they showed that the normal order of $\omega(n)$ is $\log \log n$; i.e., for every $\varepsilon>0$, the proportion of the natural numbers for which the inequalities

$$
(1-\varepsilon) \log \log n \leq \omega(n) \leq(1+\varepsilon) \log \log n
$$

do not hold tends to 0 as $n \rightarrow \infty$-with a typical error of size $\sqrt{\log \log n}$. Informally speaking, the Erdős-Kac Theorem generalizes ${ }^{1}$ the Hardy-Ramanujan Theorem by showing that $\omega(n)$ is approximately distributed as

$$
\log \log n+Z \sqrt{\log \log n}
$$

for large $n$, where $Z$ denotes a standard normal variable. More precisely, the Erdős-Kac Theorem is the following result (p. 738 of [4]).

Theorem 1. Let $n>1$. Let $P_{n}$ denote the uniform distribution on $\{1,2, \ldots, n\}$, and let $Z$ denote $a$ standard normal variable. As $n \rightarrow \infty$,

$$
P_{n}\left(m \leq n: \omega(m)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

The contribution of this paper is to extend the Erdős-Kac Theorem to a larger class of random variables on the set $[n]:=\{1,2, \ldots, n\}$ which also have, asymptotically, $\log \log n+Z \cdot \sqrt{\log \log n}$ many distinct prime factors.

[^0]1.1. A Generalization of Erdôs-Kac Theorem for $\omega(\cdot)$. Define a probability mass function (PMF) $\mathbb{P}_{n}$ on $[n]$ given by
\[

$$
\begin{equation*}
\mathbb{P}_{n}(i)=\frac{1}{n}+\varepsilon_{i, n} \tag{1}
\end{equation*}
$$

\]

Due to the axioms of probability, the terms $\varepsilon_{i, n}, 1 \leq i \leq n$, satisfy the constraints

$$
\begin{equation*}
\sum_{i=1}^{n} \varepsilon_{i, n}=0 ; n \in \mathbb{N} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\varepsilon_{i, n} \in\left[-\frac{1}{n}, 1-\frac{1}{n}\right] ; n \in \mathbb{N}, 1 \leq i \leq n \tag{3}
\end{equation*}
$$

The motivation for defining $\mathbb{P}_{n}$ in terms of the uniform distribution is due to Durrett's proof (Theorem 3.4.16 in [3]) of the Erdős-Kac Theorem. Replacing the uniform distribution $P_{n}$ with the distribution $\mathbb{P}_{n}$ in Durrett's proof naturally yields some constraints that the terms $\varepsilon_{i, n}, 1 \leq i \leq n$, must satisfy in order to conclude that an integer-valued $\mathbb{P}_{n}$-distributed random variable has about $\log \log n+Z \sqrt{\log \log n}$ distinct prime factors. Our main result is the following theorem, where $\lfloor\cdot\rfloor$ denotes the floor function.
Theorem 2. (Generalized Erdốs-Kac Theorem for $\omega$ ) Let $Z$ denote a standard normal variable. Suppose the following statements are true.

- There exists a constant $C \in \mathbb{R}$ such that for all $n>1$ and for all primes $p$ with $p>n^{1 / \log \log n}$,

$$
\begin{equation*}
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \leq \frac{C}{p} \tag{4}
\end{equation*}
$$

- There exists a constant $D \in \mathbb{R}$ such that

$$
\begin{equation*}
\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n} \leq \frac{D}{n} \tag{5}
\end{equation*}
$$

for all $n>1$, for each $k$, and for all $k$-tuples $\left(p_{1}, \ldots, p_{k}\right)$ consisting of distinct primes of size at most $n^{1 / \log \log n}$.

- For any prime p,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}=0 \tag{6}
\end{equation*}
$$

Let $\mathbb{P}_{n}^{*}$ denote the PMF obtained by imposing the constraints $(4-6)$ on the PMF $\mathbb{P}_{n}$ given by $\mathbb{P}_{n}(i)=\frac{1}{n}+\varepsilon_{i, n}$. As $n \rightarrow \infty$,

$$
\mathbb{P}_{n}^{*}\left(m \leq n: \omega(m)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

Remark. If $\varepsilon_{i, n}=0$ for all $i \leq n$, then $\mathbb{P}_{n}^{*}=P_{n}$ and Theorem 1 is obtained.
1.2. Outline. The proof of Theorem 2 is provided in $\S 2$; the proof applies the method of moments and is motivated by Durrett's proof of the Erdős-Kac Theorem (Theorem 3.4.16 in [3]). Moreover, in §2, the constraints $(4-6)$ are applied to ensure that $\mathbb{P}_{n}^{*}$ also satisfies Durrett's method of moment bounds. In $\S \S 2.1$, Theorem 2 is applied to show that the number of distinct prime factors of a random natural number chosen according to the Harmonic ( $n$ ) distribution is asymptotically normally distributed with both mean and variance equal to $\log \log n$. In $\S 3$, Theorem 2 is used to prove statements about convex combinations of distributions satisfying constraints $(4-6)$. In $\S 4$, we define conditions that ensure $\omega\left(X_{j}(n)\right)$ is asymptotically normally distributed, with mean and variance both equal to $\log \log n$, for a sequence of random variables $\left(X_{j}(n)\right)_{j \geq 1}$ as $j \rightarrow \infty$ and $n \rightarrow \infty$. In $\S 5$, Theorem 2 is applied to show that the number of distinct prime factors of a randomly chosen integer according to any of the following distributions has the same limiting distribution as the case of a uniform variable:

- Any convex combination of the $\operatorname{Harmonic}(n)$ and uniform $(n)$ distributions,
- The Zeta ( $s$ ) distribution as $s \rightarrow 1$,
- The Logarithmic $(s)$ distribution as $s \rightarrow 1$,
- A geometric power series distribution as $s \rightarrow 1$,
- A Logarithmic-Zeta $(s, \alpha)$ distribution as $(s, \alpha) \rightarrow(1,1)$.


## 2. Proving Theorem 2

Define $\alpha_{n}:=n^{1 / \log \log n}$.
Lemma 1. As $n \rightarrow \infty$

$$
\left(\sum_{\alpha_{n}<p \leq n}\left(\frac{1}{p}+\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}\right)\right) /(\log \log n)^{1 / 2} \rightarrow 0 .
$$

Proof. Given $n$ and any prime $p$ with $p>\alpha_{n}$, we have

$$
-\frac{\lfloor n / p\rfloor}{n} \stackrel{(3)}{\leq} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \stackrel{(4)}{\leq} \frac{C}{p}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{p}+\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \in\left[0, \frac{C+1}{p}\right] \tag{7}
\end{equation*}
$$

for all $n$. Thus,

$$
\left(\sum_{\alpha_{n}<p \leq n}\left(\frac{1}{p}+\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}\right)\right) /(\log \log n)^{1 / 2} \rightarrow 0
$$

due to (7) along with the fact that Durrett (p. 135 of [3]) shows

$$
\left(\sum_{\alpha_{n}<p \leq n} \frac{1}{p}\right) /(\log \log n)^{1 / 2} \rightarrow 0
$$

The following lemma is proved by Durrett (p. 156 of [3]).
Lemma 2. If $\varepsilon>0$, then $\alpha_{n} \leq n^{\varepsilon}$ for large $n$ and hence

$$
\begin{equation*}
\frac{\alpha_{n}^{r}}{n} \rightarrow 0 \tag{8}
\end{equation*}
$$

for all $r<\infty$.
Proof of Theorem 2. Given a natural number $m$ and a prime $p$, define $\delta_{p}(m)=1$ if $p$ divides $m$, and 0 otherwise. Let

$$
g_{n}(m)=\sum_{p \leq \alpha_{n}} \delta_{p}(m)
$$

denote the number of distinct prime factors of $m$ of size at most $\alpha_{n}$, and let $\mathbb{E}_{n}$ denote expectation with respect to $\mathbb{P}_{n}^{*}$. Then

$$
\begin{aligned}
\mathbb{E}_{n}\left(\sum_{\alpha_{n}<p \leq n} \delta_{p}\right) & =\sum_{\alpha_{n}<p \leq n} \mathbb{P}_{n}^{*}\left(m: \delta_{p}(m)=1\right) \\
& =\sum_{\alpha_{n}<p \leq n} \sum_{l=1}^{\lfloor n / p\rfloor} \mathbb{P}_{n}^{*}(m: m=l p) \\
& \stackrel{(1)}{=} \sum_{\alpha_{n}<p \leq n} \sum_{l=1}^{\lfloor n / p\rfloor}\left(\frac{1}{n}+\varepsilon_{l p, n}\right) \\
& \leq \sum_{\alpha_{n}<p \leq n}\left(\frac{1}{p}+\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}\right),
\end{aligned}
$$

so by Lemma 1 it suffices to prove Theorem 2 for $g_{n}$; i.e., replacing $\omega(m)$ with $g_{n}(m)$ in the statement of Theorem 2 does not affect the limiting distribution.

Consider a sequence $\left(X_{p}\right)_{p \geq 2}$ of independent Bernoulli random variables with prime-valued indices such that $\mathbb{P}\left(X_{p}=1\right)=1 / p$ and $\mathbb{P}\left(X_{p}=0\right)=1-1 / p$. Note that

$$
\mathbb{E}\left(\delta_{p}\right)=\frac{\lfloor n / p\rfloor}{n}+\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \xrightarrow{(6)} 1 / p
$$

as $n \rightarrow \infty$. Let

$$
\begin{aligned}
S_{n} & :=\sum_{p \leq \alpha_{n}} X_{p}, \\
b_{n} & :=\mathbb{E}\left(S_{n}\right), \\
a_{n}^{2} & :=\operatorname{Var}\left(S_{n}\right) .
\end{aligned}
$$

By Lemma $1, b_{n}$ and $a_{n}^{2}$ are both $\log \log n+o\left((\log \log n)^{1 / 2}\right)$, so it suffices to show

$$
\mathbb{P}_{n}^{*}\left(m: g_{n}(m)-b_{n} \leq x a_{n}\right) \rightarrow \mathbb{P}(Z \leq x) .
$$

An application of Theorem 3.4.10 of [3] shows

$$
\left(S_{n}-b_{n}\right) / a_{n} \rightarrow Z,
$$

and since $\left|X_{p}\right| \leq 1$, it follows from Durrett's second proof of Theorem 3.4.10 [3] that

$$
\mathbb{E}\left(\left(S_{n}-b_{n}\right) / a_{n}\right)^{r} \rightarrow \mathbb{E}\left(Z^{r}\right)
$$

for all $r$. Using the notation from that proof (and replacing $i_{j}$ by $p_{j}$ ) it follows that

$$
\mathbb{E}\left(S_{n}^{r}\right)=\sum_{k=1}^{r} \sum_{r_{i}} \frac{r!}{r_{1}!\cdots r_{k}!} \frac{1}{k!} \sum_{p_{j}} \mathbb{E}\left(X_{p_{1}}^{r_{1}} \cdots X_{p_{k}}^{r_{k}}\right),
$$

where the sum $\sum_{r_{i}}$ extends over all $k$-tuples of positive integers for which $r_{1}+\cdots+r_{k}=r$, and $\sum_{p_{j}}$ extends over all $k$-tuples of distinct primes in $[n]$. Since $X_{p} \in\{0,1\}$, the summand in $\sum_{p_{j}} \mathbb{E}\left(X_{p_{1}}^{r_{1}} \cdots X_{p_{k}}^{r_{k}}\right)$ is

$$
\mathbb{E}\left(X_{p_{1}} \cdots X_{p_{k}}\right)=\frac{1}{p_{1} \cdots p_{k}}
$$

by independence of the $X_{p}$ 's. Moreover,

$$
\begin{aligned}
\mathbb{E}_{n}\left(\delta_{p_{1}} \cdots \delta_{p_{k}}\right) & \leq \mathbb{P}_{n}\left(m: \delta_{p_{1}}(m)=\delta_{p_{2}}(m) \cdots=\delta_{p_{k}}(m)=1\right) \\
& =\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \mathbb{P}_{n}\left(m: m=l p_{1} \cdots p_{k}\right) \\
& \stackrel{(1)}{=} \sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}\left(\frac{1}{n}+\varepsilon_{l p_{1} \cdots p_{k}, n}\right) \\
& =\frac{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}{n}+\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n} \\
& \stackrel{(5)}{\leq} \frac{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}{n}+\frac{D}{n} .
\end{aligned}
$$

The two terms $\mathbb{E}\left(X_{p_{1}} \cdots X_{p_{k}}\right)$ and $\mathbb{E}_{n}\left(\delta_{p_{1}} \cdots \delta_{p_{k}}\right)$ differ by at most

$$
\max \left\{\frac{1}{p_{1} \cdots p_{k}}-\frac{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}{n}-\frac{D}{n}, \frac{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}{n}+\frac{D}{n}-\frac{1}{p_{1} \cdots p_{k}}\right\} \leq \max \left\{\frac{1-D}{n}, \frac{D}{n}\right\}
$$

Therefore, the two $r$ th moments differ by

$$
\begin{aligned}
\left|\mathbb{E}\left(S_{n}^{r}\right)-\mathbb{E}_{n}\left(g_{n}^{r}\right)\right| & \leq \sum_{k=1}^{r} \sum_{r_{i}} \frac{r!}{r_{1}!\cdots r_{k}!} \frac{1}{k!} \sum_{p_{j}} \max \left\{\frac{1-D}{n}, \frac{D}{n}\right\} . \\
& \leq \max \left\{\frac{1-D}{n}, \frac{D}{n}\right\} \cdot\left(\sum_{p \leq \alpha_{n}} 1\right)^{r} \\
& \leq \max \left\{\frac{1-D}{n}, \frac{D}{n}\right\} \cdot \alpha_{n}^{r} \\
& \xrightarrow{(8)} 0 .
\end{aligned}
$$

Using binomial expansions and the inequality above, we see that

$$
\begin{aligned}
\left|\mathbb{E}\left(\left(\left(S_{n}-b_{n}\right) / a_{n}\right)^{r}\right)-\mathbb{E}_{n}\left(\left(\left(g_{n}-b_{n}\right) / a_{n}\right)^{r}\right)\right| & =\left|1 / a_{n}^{r}\right|\left|\mathbb{E}\left(\left(S_{n}-b_{n}\right)^{r}\right)-\mathbb{E}_{n}\left(\left(g_{n}-b_{n}\right)^{r}\right)\right| \\
& \leq\left|1 / a_{n}^{r}\right| \cdot \max \left\{\frac{1-D}{n}, \frac{D}{n}\right\} \sum_{k=0}^{r}\binom{r}{k} \alpha_{n}^{k} b_{n}^{r-k} \\
& =\left|1 / a_{n}^{r}\right| \cdot \max \left\{\frac{1-D}{n}, \frac{D}{n}\right\}\left(\alpha_{n}+b_{n}\right)^{r} .
\end{aligned}
$$

Therefore, since $b_{n} \leq \alpha_{n}$, we have

$$
\left|\mathbb{E}\left(\left(\left(S_{n}-b_{n}\right) / a_{n}\right)^{r}\right)-\mathbb{E}_{n}\left(\left(\left(g_{n}-b_{n}\right) / a_{n}\right)^{r}\right)\right| \xrightarrow{(8)} 0
$$

for all $r$ as well. Since $\mathbb{E}\left(\left(\left(S_{n}-b_{n}\right) / a_{n}\right)^{r}\right) \rightarrow \mathbb{E}\left(Z^{r}\right)$ for all $r$, this completes the proof of Theorem 2 .
The following definition is based on Theorem 2.
Definition 1. We refer to distributions satisfying constraints (4-6) as E-K distributions.
2.1. The Harmonic Distribution. Now we will apply Theorem 2 to show that the harmonic distributions are E-K distributions. Given $n \in \mathbb{N}$, consider an integer in $[n]$ chosen according to the Harmonic $(n)$ distribution, whose PMF is given by

$$
H_{n}(i):=\frac{1}{i \sum_{j=1}^{n} \frac{1}{j}}, 1 \leq i \leq n
$$

If $i \in[n]$, then equation (1) implies

$$
\varepsilon_{i, n}=\frac{1}{i \sum_{j=1}^{n} \frac{1}{j}}-\frac{1}{n}
$$

Therefore,

$$
\begin{aligned}
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} & =\sum_{l=1}^{\lfloor n / p\rfloor}\left(\frac{1}{l p \sum_{i=1}^{n} \frac{1}{i}}-\frac{1}{n}\right) \\
& =\frac{\sum_{i=1}^{\lfloor n / p\rfloor} \frac{1}{i}}{p \sum_{l=1}^{n} \frac{1}{i}}-\frac{\lfloor n / p\rfloor}{n} \\
& \leq \frac{1}{p}-\frac{\lfloor n / p\rfloor}{n} \\
& \leq \frac{1}{p}
\end{aligned}
$$

so (4) holds with $C=1$. Moreover,

$$
\begin{aligned}
\left\lfloor\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n}\right. & =\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}\left(\frac{1}{l p_{1} \cdots p_{k} \sum_{i=1}^{n} \frac{1}{i}}-\frac{1}{n}\right) \\
& =\frac{\sum_{i=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \frac{1}{i}}{p_{1} \cdots p_{k} \sum_{i=1}^{n} \frac{1}{i}}-\frac{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}{n} \\
& \leq \frac{1}{p_{1} \cdots p_{k}}-\left(\frac{{ }_{n}}{n}-1\right. \\
& =1 / n
\end{aligned}
$$

so (5) holds with $D=1$. Finally,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} & =\lim _{n \rightarrow \infty}\left(\frac{\sum_{i=1}^{\lfloor n / p\rfloor} \frac{1}{i}}{p \sum_{l=1}^{n} \frac{1}{i}}-\frac{\lfloor n / p\rfloor}{n}\right) \\
& \sim \frac{1}{p} \frac{\log (n / p)}{\log n}-\frac{1}{p} \\
& \rightarrow 1 / p-1 / p \\
& =0
\end{aligned}
$$

so constraint (6) holds. By Theorem 2, this shows that the number of distinct prime factors of an $H_{n^{-}}$ distributed random variable is asymptotically normally distributed, as $n \rightarrow \infty$, with both mean and variance equal to $\log \log n$.

## 3. Convex Combinations of E-K Distributions

The following theorem shows that any convex sum of two E-K distributions is also an E-K distribution.
Theorem 3. (Convex Combinations for Erdôs-Kac (CLT)) Let $n>1$ and $0 \leq \lambda \leq 1$ be fixed. Suppose $d_{1, n}$ and $d_{2, n}$ are two PMFs on $[n]$ satisfying the constraints $(4-6)$ for all $n>1$. Then any PMF of the form

$$
\begin{equation*}
\mathbb{P}_{n}^{*}(i)=\lambda d_{1, n}(i)+(1-\lambda) d_{2, n}(i), 1 \leq i \leq n \tag{9}
\end{equation*}
$$

also satisfies constraints $(4-6)$. In particular,

$$
\mathbb{P}_{n}^{*}\left(m \leq n: \omega(m)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $n \rightarrow \infty$.

Proof. Denote

$$
\begin{aligned}
& \varepsilon_{i, n}=d_{1, n}(i)-1 / n \\
& \varepsilon_{i, n}^{\prime}=d_{2, n}(i)-1 / n
\end{aligned}
$$

and

$$
\hat{\varepsilon}_{i, n}=\mathbb{P}_{n}^{*}(i)-1 / n .
$$

We have

$$
\begin{aligned}
\sum_{l=1}^{\lfloor n / p\rfloor} \hat{\varepsilon}_{l p, n} & =\sum_{l=1}^{\lfloor n / p\rfloor}\left(\lambda d_{1, n}(l p)+(1-\lambda) d_{2, n}(l p)-\frac{1}{n}\right) \\
& \stackrel{(1)}{=} \sum_{l=1}^{\lfloor n / p\rfloor}\left(\lambda\left(\varepsilon_{l p, n}+1 / n\right)+(1-\lambda)\left(\varepsilon_{l p, n}^{\prime}+1 / n\right)-\frac{1}{n}\right) \\
& =\sum_{l=1}^{\lfloor n / p\rfloor}\left(\lambda \varepsilon_{l p, n}+(1-\lambda) \varepsilon_{l p, n}^{\prime}\right) \\
& =\lambda \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}+(1-\lambda) \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}^{\prime} \\
& \leq \frac{\lambda C_{1}+(1-\lambda) C_{2}}{p}
\end{aligned}
$$

where the latest inequality is obtained by applying constraint (4) to both $d_{1, n}$ and $d_{2, n}$; therefore, (4) holds for the PMF given by (9) with $C=\lambda C_{1}+(1-\lambda) C_{2}$. Similarly,

$$
\begin{aligned}
& \sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \hat{\varepsilon}_{l p_{1} \cdots p_{k}, n}=\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}\left(\lambda d_{1, n}\left(l p_{1} \cdots p_{k}\right)+(1-\lambda) d_{2, n}\left(l p_{1} \cdots p_{k}\right)-\frac{1}{n}\right) \\
& \stackrel{(1)}{=} \sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}\left(\lambda\left(\varepsilon_{l p_{1} \cdots p_{k}, n}+1 / n\right)+(1-\lambda)\left(\varepsilon_{l p_{1} \cdots p_{k}, n}^{\prime}+1 / n\right)-\frac{1}{n}\right) \\
& =\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}\left(\lambda \varepsilon_{l p_{1} \cdots p_{k}, n}+(1-\lambda) \varepsilon_{l p_{1} \cdots p_{k}, n}^{\prime}\right) \\
& =\lambda \sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n}+(1-\lambda) \sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n}^{\prime} \\
& \leq \frac{\lambda D_{1}+(1-\lambda) D_{2}}{n},
\end{aligned}
$$

where the latest inequality is obtained by applying constraint (5) to both $d_{1, n}$ and $d_{2, n}$; therefore, (5) holds for the PMF given by $(9)$ with $D=\lambda D_{1}+(1-\lambda) D_{2}$. Furthermore,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \hat{\varepsilon}_{l p, n} & =\lambda \lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}+(1-\lambda) \lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}^{\prime} \\
& =\lambda \cdot 0+(1-\lambda) \cdot 0 \\
& =0
\end{aligned}
$$

where the second equation uses the fact that the distributions $d_{1}$ and $d_{2}$ satisfy constraint (6). Therefore, (6) holds for the PMF given by (9).

By Theorem 3 and mathematical induction, we obtain the following.

Corollary 1. Suppose $d_{1, n}, \ldots, d_{k, n}$ are PMFs on [n] satisfying the constraints (4-6), and suppose $\left(\lambda_{j}\right)_{j=1}^{k}$ is a sequence of nonnegative numbers which add to unity. The PMF

$$
\mathbb{P}_{n}^{*}(i)=\sum_{j=1}^{k} \lambda_{j} d_{j, n}(i)
$$

also satisfies constraints (4-6). Therefore,

$$
\mathbb{P}_{n}^{*}\left(m \leq n: \omega(m)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $n \rightarrow \infty$.
We now present an alternate proof to the remark under Corollary 3.4 in [1]; their proof uses both the Tauberian theorems for the harmonic distribution and the uniform distribution simultaneously. We present a proof based on Theorems 2 and 3 of this paper.

Corollary 2. (Convex Combinations of Harmonic Distribution and Uniform Distribution) Let $\lambda \in[0,1]$ and consider the PMF given by

$$
\mathbb{P}_{n}^{*}(i)=\frac{\lambda}{i h_{n}}+\frac{1-\lambda}{n} ; 1 \leq i \leq n,
$$

where $h_{n}$ is the $n^{\text {th }}$ harmonic number given by $h_{n}:=\sum_{i=1}^{n} 1 / i$, then

$$
\mathbb{P}_{n}^{*}\left(m \leq n: \omega(m)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $n \rightarrow \infty$.
Proof. As noted in the remark after Theorem 2, the uniform distribution satisfies constraints (4-6), and in $\S \S 2.1$ we showed the harmonic distribution satisfies constraints $(4-6)$. By Theorem 3, their convex sums also satisfy constraints $(4-6)$.

The following theorem shows that if given the PMF $1 / n+\varepsilon_{i, n}$ of a sequence of E-K distributions, under additional constraints, the PMF $1 / n-\varepsilon_{i, n}$ is also the PMF of a sequence of E-K distributions.

Theorem 4. (Reflection Theorem) Let $\mathbb{P}_{n}$ satisfy constraints (4) and (6) for all $n>1$. In addition, assume $\varepsilon_{i, n} \in\left[-\frac{1}{n}, \frac{1}{n}\right]$ for all $i$ and for all $n$, and that for all $k$-tuples of distinct primes $p_{1}, \ldots, p_{k}$, there exists a constant $D \geq 0$ such that

$$
\begin{equation*}
-\frac{D}{n} \leq \sum_{l=1}^{\left\lfloor\frac{n}{p_{1}, p_{k}}\right\rfloor} \varepsilon_{l p_{1} \ldots p_{k}, n} \leq \frac{D}{n} . \tag{10}
\end{equation*}
$$

If

$$
\mathbb{P}_{n}^{*}(i):=\frac{1}{n}-\varepsilon_{i, n},
$$

then

$$
\mathbb{P}_{n}^{*}\left(m \leq n: \omega(m)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $n \rightarrow \infty$.
Proof. Define $\varepsilon_{i, n}^{\prime}:=-\varepsilon_{i, n}$, so

$$
\begin{aligned}
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}^{\prime} & =-\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{i, n} \\
& \stackrel{(3)}{\leq} \frac{\lfloor n / p\rfloor}{n} \\
& \leq \frac{1}{p}
\end{aligned}
$$

so constraint (4) holds with $C=1$. Similarly,

$$
\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n}^{\prime}=-\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n} \stackrel{(10)}{\leq} \frac{D}{n}
$$

so constraint (5) holds. Moreover,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}^{\prime} & =-\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \\
& =0
\end{aligned}
$$

where the latest limit is due to the assumption that $\mathbb{P}_{n}$ satisfies constraint (6).

## 4. A Central Limit Theorem for E-K Distributions on $\mathbb{N}$

We would like to formalize the notion of asymptotic normality for $\omega\left(X_{j}\right)$ in the limit of a sequence of random variables $\left(X_{j}\right)_{j \geq 1}$ on $\mathbb{N}$. Towards that goal we provide the following definition.
Definition 2. Given an infinite sequence of random variables $X_{1}, X_{2}, \ldots$ defined on $\mathbb{N}$, let $X_{j}(n)$ denote the truncation of $X_{j}$ on $[n]$. Define $\varepsilon_{l, j, n}:=\mathbb{P}\left(X_{j}(n)=l\right)-1 / n$. The sequence $X_{1}, X_{2}, \ldots$ is said to satisfy the uniformity along primes property if the following holds. There exists constants $C$ and $D$ such that for all $n$ there exists a $d \geq 1$ such that for all $j \geq d$ :

- For each prime $p>\alpha_{n}$

$$
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, j, n} \leq \frac{C}{p}
$$

- For each $k \geq 1$ and for all $k$-tuples $\left(p_{1}, \ldots, p_{k}\right)$ consisting of distinct primes of size at most $\alpha_{n}$

$$
\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, j, n} \leq \frac{D}{n}
$$

In addition, for any prime $p$,

$$
\lim _{j \rightarrow \infty} \lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, j, n}=0
$$

Theorem 5. Assume $X_{1}, X_{2}, \ldots$ is an infinite sequence of random variables defined on $\mathbb{N}$ satisfying the uniformity along primes property. Let $\mathbb{P}_{j, n}$ be the probability distribution of $X_{j}(n)$ on $[n]$. If $\left(n_{j}\right)_{j \in \mathbb{N}}$ is any sequence in $\mathbb{N} \backslash\{1\}$ tending to $\infty$ such that $(4-5)$ hold for $X_{j}\left(n_{j}\right)$ for all $j \geq 1$, then

$$
\mathbb{P}_{j, n_{j}}\left(m \leq n_{j}: \omega(m)-\log \log n_{j} \leq x\left(\log \log n_{j}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $j \rightarrow \infty$.
Proof. Let's proceed by contradiction. That is, suppose $\left(n_{j}\right)_{j \in \mathbb{N}}$ is any sequence in $\mathbb{N} \backslash\{1\}$ tending to $\infty$ and suppose there exist a sequence $\left(j_{k}\right)_{k \in \mathbb{N}}$, some $x_{0} \in \mathbb{R}$, and some $\varepsilon_{0}>0$ such that

$$
\left|\mathbb{P}_{j_{k}, n_{j_{k}}}\left(m \leq n_{j_{k}}: \omega(m)-\log \log n_{j_{k}} \leq x_{0}\left(\log \log n_{j_{k}}\right)^{1 / 2}\right)-\mathbb{P}\left(Z \leq x_{0}\right)\right| \geq \varepsilon_{0}
$$

for all $k \geq 1$. Consider the following PMF on $[n]$ for all $n>1$, which satisfies the hypotheses of Theorem 2:

$$
\mathbb{P}_{n}^{*}(i):= \begin{cases}1 / n & \text { if } n_{j_{k}} \neq n \text { for all } k \geq 1 \\ \mathbb{P}_{j_{k^{\prime}(n)}, n}(i) & \text { if } n_{j_{k^{\prime}(n)}}=n\end{cases}
$$

where $k^{\prime}(n)$ is the smallest integer satisfying $n_{j_{k^{\prime}(n)}}=n$. It is clear that $\mathbb{P}_{n}^{*}$ is defined for all $n>1$ and satisfies the hypotheses of Theorem 2 due to Definition 2 . Let $\varepsilon=\varepsilon_{0}$, then there is some $d \geq 1$ such that

$$
\left|\mathbb{P}_{j_{k^{\prime}(n)}, n_{j_{k^{\prime}(n)}}}\left(m \leq n_{j_{k^{\prime}(n)}}: \omega(m)-\log \log n_{j_{k^{\prime}(n)}} \leq x_{0}\left(\log \log n_{j_{k^{\prime}(n)}}\right)^{1 / 2}\right)-\mathbb{P}\left(Z \leq x_{0}\right)\right|<\varepsilon
$$

if $n \geq d$, by Theorem 2. The fact that at least one $k^{\prime}(n)$ exists such that $n_{j_{k^{\prime}(n)}}=n$ when $n \geq d$ is due to the fact that $n_{j_{k}} \rightarrow \infty$ as $k \rightarrow \infty$, so there are infinitely many $k^{\prime}(n)$ such that $n_{j_{k^{\prime}(n)}}=n$ when $n \geq d$. Thus, Theorem 5 holds by contradiction.

### 4.1. Moment Generating Functions and Characteristic Functions.

Definition 3. Let $X$ denote a random variable. The moment generating function (MGF) of $X$ is defined as $M_{X}(t)=\mathbb{E}\left(e^{t X}\right)$, and the characteristic function $(\mathbf{C F})$ of $X$ is defined as $\phi_{X}(t)=\mathbb{E}\left(e^{i t X}\right)$.

We have the following two facts about the MGF and the CF that will be used in the proof of Corollary 3:

- The $k^{t h}$ derivative of $M_{X}(t)$ at $t=0$ is the $k^{t h}$ moment of $X$.
- Lèvy's Continuity Theorem: Convergence in distribution, $X_{n} \xrightarrow{d} X$, for a sequence of random variables is equivalent to pointwise convergence, $\phi_{X_{n}} \rightarrow \phi_{X}$, of the corresponding CFs on all of $\mathbb{R}$.
Corollary 3 and Corollary 4 below allow us to state Theorem 7.1 of [2] in a slightly different way.
Corollary 3. Suppose $X_{1}, X_{2}, \ldots$ is an infinite sequence of random variables defined on $\mathbb{N}$ that satisfies the uniformity along primes property. Let $\left(n_{j}\right)_{j \in \mathbb{N}}$ be any sequence in $\mathbb{N} \backslash\{1\}$ tending to $\infty$ such that (45) hold for $X_{j}\left(n_{j}\right)$ for all $j \geq 1$. Let $\mathbb{P}_{j}$ be the probability distribution of $X_{j}$ on $\mathbb{N}, \mathbb{P}_{j, n}$ be the probability distribution of $X_{j}(n)$ on $[n]$, and define $\mu_{j, n}:=\mathbb{E}_{j, n}\left(\omega\left(X_{j}(n)\right)\right)$, then the following is true:

$$
\mathbb{P}_{j, n_{j}}\left(m \leq n_{j}: \omega(m)-\mu_{j, n_{j}} \leq x\left(\mu_{j, n_{j}}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $j \rightarrow \infty$. Additionally, if $\mu_{j}<\infty$ for all $j \geq 1$ and

$$
\lim _{j \rightarrow \infty}\left(\mu_{j} /\left(\log \log n_{j}\right)^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0
$$

then

$$
\mathbb{P}_{j}\left(m: \omega(m)-\mu_{j} \leq x\left(\mu_{j}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $j \rightarrow \infty$, where

$$
\mu_{j}:=\mathbb{E}_{j}\left(\omega\left(X_{j}\right)\right)=\lim _{n \rightarrow \infty} \mu_{j, n}
$$

We also have the following asymptotic properties:

$$
\begin{gathered}
\lim _{j \rightarrow \infty}\left(\mu_{j, n_{j}} /\left(\log \log n_{j}\right)^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0 \\
\lim _{j \rightarrow \infty}\left(\mu_{j, n_{j}}^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0
\end{gathered}
$$

and

$$
\lim _{j \rightarrow \infty} \frac{\mu_{j, n_{j}}}{\log \log n_{j}}=\lim _{j \rightarrow \infty} \frac{\mu_{j}}{\log \log n_{j}}=1
$$

Proof. Let $M_{j, n_{j}}$ and $\phi_{j, n_{j}}$ be the MGF and CF of

$$
\frac{\omega\left(X_{j}\left(n_{j}\right)\right)-\log \log n_{j}}{\sqrt{\log \log n_{j}}}
$$

and let $M_{j, \mu}$ and $\phi_{j, \mu}$ be the MGF and CF of

$$
\frac{\omega\left(X_{j}\left(n_{j}\right)\right)-\mu_{j, n_{j}}}{\sqrt{\mu_{j, n_{j}}}}
$$

We have

$$
M_{j, n_{j}}(t)=e^{-t\left(\log \log n_{j}\right)^{1 / 2}} M_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\log \log n_{j}\right)^{-1 / 2} t\right)
$$

and

$$
M_{j, \mu}(t)=e^{-t \mu_{j, n_{j}}^{1 / 2}} M_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\mu_{j, n_{j}}\right)^{-1 / 2} t\right)
$$

Let's compute the mean from the MGF $M_{j, n_{j}}(t)$ :

$$
\begin{aligned}
M_{j, n_{j}}^{\prime}(0)= & -\left(\log \log n_{j}\right)^{1 / 2} e^{-t\left(\log \log n_{j}\right)^{1 / 2}} M_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\log \log n_{j}\right)^{-1 / 2} t\right) \\
& +\left.e^{-t\left(\log \log n_{j}\right)^{1 / 2}} M_{\omega\left(X_{j}\left(n_{j}\right)\right)}^{\prime}\left(\left(\log \log n_{j}\right)^{-1 / 2} t\right)\left(\log \log n_{j}\right)^{-1 / 2}\right|_{t=0} \\
= & -\left(\log \log n_{j}\right)^{1 / 2}+\mu_{j, n_{j}}\left(\log \log n_{j}\right)^{-1 / 2} \\
& \rightarrow 0
\end{aligned}
$$

where the last limit holds due to the proof of Theorem 2 (we showed the moments all approach the moments of the normal distribution) which is used in the proof of Theorem 5 above; thus,

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left(\mu_{j, n_{j}} /\left(\log \log n_{j}\right)^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0 \tag{11}
\end{equation*}
$$

holds. Therefore,

$$
\lim _{j \rightarrow \infty} \frac{\mu_{j, n_{j}}}{\log \log n_{j}}=1
$$

From equation (11) we can conclude

$$
\lim _{j \rightarrow \infty}\left(\mu_{j, n_{j}}^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0
$$

This is because,

$$
\left|\mu_{j, n_{j}}^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right|=\left|\frac{\mu_{j, n_{j}}-\left(\log \log n_{j}\right)}{\mu_{j, n_{j}}^{1 / 2}+\left(\log \log n_{j}\right)^{1 / 2}}\right| \leq\left|\frac{\mu_{j, n_{j}}-\log \log n_{j}}{\left(\log \log n_{j}\right)^{1 / 2}}\right| \rightarrow 0
$$

Now we want to show convergence in distribution using the characteristic functions. We use the fact that the characteristic function $\phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}(t)$ is continuous at $t=0$. According to properties of continuous functions, we have the ratios

$$
\begin{equation*}
\frac{e^{-i t \mu_{j, n_{j}}^{1 / 2}}}{e^{-i t\left(\log \log n_{j}\right)^{1 / 2}}}=e^{-i t\left(\mu_{j, n_{j}}^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)} \rightarrow e^{0}=1 \tag{12}
\end{equation*}
$$

and also

$$
\begin{equation*}
\frac{\phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\mu_{j, n_{j}}\right)^{-1 / 2} t\right)}{\phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\log \log n_{j}\right)^{-1 / 2} t\right)} \rightarrow \frac{\phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}(0)}{\phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}(0)}=1 \tag{13}
\end{equation*}
$$

as $j \rightarrow \infty$. This gives us the required convergence of $\phi_{j, \mu}$ at any $t$; i.e.,

$$
\begin{aligned}
\phi_{j, \mu}(t)=e^{-i t \mu_{j, n_{j}}^{1 / 2}} \phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\mu_{j, n_{j}}\right)^{-1 / 2} t\right) \stackrel{(12,13)}{\sim} e^{-i t\left(\log \log n_{j}\right)^{1 / 2}} \phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\log \log n_{j}\right)^{-1 / 2} t\right) \\
\rightarrow e^{-t^{2} / 2}=\phi_{Z}(t) .
\end{aligned}
$$

By Lèvy's Continuity Theorem,

$$
\mathbb{P}_{j, n_{j}}\left(m \leq n_{j}: \omega(m)-\mu_{j, n_{j}} \leq x\left(\mu_{j, n_{j}}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

Suppose additionally that $\mu_{j}<\infty$ for all $j \geq 1$ and

$$
\lim _{j \rightarrow \infty}\left(\mu_{j} /\left(\log \log n_{j}\right)^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0
$$

Similarly, let $\phi_{j, \mu_{j}}$ be the characteristic function of

$$
\frac{\omega\left(X_{j}\right)-\mu_{j}}{\sqrt{\mu_{j}}}
$$

Then

$$
\begin{aligned}
\phi_{j, \mu_{j}}(t)=e^{-i t \mu_{j}^{1 / 2}} \phi_{\omega\left(X_{j}\right)}\left(\left(\mu_{j}\right)^{-1 / 2} t\right) & \sim e^{-i t\left(\log \log n_{j}\right)^{1 / 2}} \phi_{\omega\left(X_{j}\left(n_{j}\right)\right)}\left(\left(\log \log n_{j}\right)^{-1 / 2} t\right) \\
& \rightarrow e^{-t^{2} / 2}=\phi_{Z}(t)
\end{aligned}
$$

for any $t \in \mathbb{R}$. Thus, again we have that

$$
\mathbb{P}_{j}\left(m: \omega(m)-\mu_{j} \leq x\left(\mu_{j}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

by Lèvy's Continuity Theorem. This completes the proof.
Remark. We can replace $\mu_{j}$ in Corollary 3 with any sequence $a_{j}$ that satisfies the limit

$$
\lim _{j \rightarrow \infty}\left(a_{j} /\left(\log \log n_{j}\right)^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0
$$

to get a similar result. This is a generalization of Theorem 7.1 in [2] which states this result for particular sequences of $Z$ eta $(s)$ distributions as their parameter $s \rightarrow 1$. In their result, the mean was $\sum_{p} p^{-s}$, where the sum is over all prime numbers.

Definition 4. Given an infinite sequence of random variables $X_{1}, X_{2}, \ldots$ defined on $\mathbb{N}$, let $X_{j}(n)$ denote the truncation of $X_{j}$ on $[n]$. If $X_{1}, X_{2}, \ldots$ satisfies the uniformity along primes property, and $X_{j}(n)$ satisfies (4-5) for any $n$ and any $j$, then we say the sequence satisfies the complete uniformity along primes property.

The following lemma will be applied in Corollary 4.
Lemma 3. If $X_{1}, X_{2}, \ldots$ has the complete uniformity along primes property and $\mu_{j}<\infty$ for all $j \geq 1$, then $\mu_{j} \rightarrow \infty$ as $j \rightarrow \infty$.

Proof. For any sequence $\left(n_{j}\right)_{j \in \mathbb{N}}$ in $\mathbb{N} \backslash\{1\}$ tending to $\infty$ we have $\log \log n_{j} \sim \mu_{j, n_{j}} \sim \mu_{j}$ as $j \rightarrow \infty$.
Corollary 4. If $X_{1}, X_{2}, \ldots$ has the complete uniformity along primes property and $\mu_{j}<\infty$ for all $j \geq 1$, then

$$
\mathbb{P}_{j}\left(m: \omega(m)-\mu_{j} \leq x\left(\mu_{j}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $j \rightarrow \infty$.
Proof. Define $n_{j}:=\max \left\{\left\lfloor e^{e^{\mu_{j}}}\right\rfloor, 2\right\}$ so that $n_{j} \rightarrow \infty$ by Lemma 3 and the limit

$$
\lim _{j \rightarrow \infty}\left(\mu_{j} /\left(\log \log n_{j}\right)^{1 / 2}-\left(\log \log n_{j}\right)^{1 / 2}\right)=0
$$

in Corollary 3 holds.
4.2. The Zeta Distribution. We show the zeta distribution has the complete uniformity along primes property as $s \rightarrow 1$. Given $s>1$, denote by $Z_{s}$ the $Z e t a(s)$ distribution so that for any $j \in \mathbb{N}$,

$$
Z_{s}(j)=\frac{1}{j^{s} \zeta(s)}
$$

where

$$
\zeta(s)=\sum_{j \geq 1} \frac{1}{j^{s}}
$$

denotes the Riemann zeta function. Since Theorem 2 involves distributions defined on $[n]$, restrict the Zeta $(s)$ distribution to $[n]$ and then normalize by dividing by $\sum_{i=1}^{n} \frac{1}{i^{s} \zeta(s)}$; i.e., for $j \in[n]$,

$$
\begin{aligned}
Z_{s, n}(j) & :=\frac{\frac{1}{j^{s} \zeta(s)}}{\sum_{i=1}^{n} \frac{1}{i^{s} \zeta(s)}} \\
& =\frac{1}{j^{s} \sum_{i=1}^{n} \frac{1}{i^{s}}}
\end{aligned}
$$

and $Z_{s, n}$ is known as the Zipf distribution with parameters $n$ and $s$. For a Zifp distribution, we have

$$
\varepsilon_{i, n} \stackrel{(1)}{=} \frac{1 / i^{s}}{\sum_{j=1}^{n} 1 / j^{s}}-\frac{1}{n}
$$

Thus,

$$
\begin{aligned}
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} & =\frac{\sum_{l=1}^{\lfloor n / p\rfloor} 1 /(l p)^{s}}{\sum_{l=1}^{n} 1 / l^{s}}-\frac{\lfloor n / p\rfloor}{n} \\
& =\frac{1}{p^{s}} \frac{\sum_{l=1}^{\lfloor n / p\rfloor} 1 / l^{s}}{\sum_{l=1}^{n} 1 / l^{s}}-\frac{\lfloor n / p\rfloor}{n} \\
& \leq \frac{1}{p^{s}} \\
& \leq \frac{1}{p}
\end{aligned}
$$

so constraint (4) holds with $C=1$ for any $s$. Moreover,

$$
\begin{aligned}
\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n} & =\frac{\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} 1 /\left(l p_{1} \cdots p_{k}\right)^{s}}{\sum_{l=1}^{n} 1 / l^{s}}-\frac{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}{n} \\
& =\frac{1}{\left(p_{1} \cdots p_{k}\right)^{s}} \frac{\sum_{l=1}^{\lfloor n / p\rfloor} 1 / l^{s}}{\sum_{l=1}^{n} 1 / l^{s}}-\frac{1}{p_{1} \cdots p_{k}}+\frac{1}{n} \\
& \leq \frac{1}{\left(p_{1} \cdots p_{k}\right)^{s}}-\frac{1}{p_{1} \cdots p_{k}}+\frac{1}{n} \\
& \leq \frac{1}{n}
\end{aligned}
$$

so constraint (5) holds with $D=1$ for any $s$. Furthermore,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} & =\lim _{n \rightarrow \infty} \frac{\sum_{l=1}^{\lfloor n / p\rfloor} 1 /(l p)^{s}}{\sum_{l=1}^{n} 1 / l^{s}}-\frac{1}{p} \\
& =\frac{1}{p^{s}}-\frac{1}{p} \\
& \xrightarrow{s \rightarrow 1} 0
\end{aligned}
$$

therefore, (6) holds as $s \rightarrow 1$.
Corollary 5. Let $\left(a_{j}\right)_{j \in \mathbb{N}}$ be any sequence of real numbers such that $0<a_{j} \rightarrow \infty$. If $X_{j}$ is a sequence of Zeta $\left(1+1 / a_{j}\right)$-distributed random variables, then

$$
\begin{aligned}
\mu_{j}= & \sum_{p} p^{-\left(1+1 / a_{j}\right)}<\infty \\
& \lim _{j \rightarrow \infty} \mu_{j}=\infty
\end{aligned}
$$

and

$$
\mathbb{P}_{j}\left(m: \omega(m)-\mu_{j} \leq x\left(\mu_{j}\right)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $j \rightarrow \infty$.
This latest limit is the statement of Theorem 7.1 in [2].
4.3. Convex Combinations of E-K Distributions on $\mathbb{N}$. We can also take convex combinations of sequences to form new sequences which satisfy the uniformity along primes property.

Corollary 6. Let $\lambda \in[0,1]$. Let $X_{1}, X_{2}, \ldots$ and $Y_{1}, Y_{2}, \ldots$ be two sequences of random variables on $\mathbb{N}$ that both satisfy the complete uniformity along primes property. Then we can define a new sequence $Z_{1}, Z_{2}, \ldots$ such that each $Z_{j}$ is the convex combination of $X_{j}$ and $Y_{j}$. In particular,

$$
\mathbb{P}_{Z_{j}}(i):=\lambda \mathbb{P}_{X_{j}}(i)+(1-\lambda) \mathbb{P}_{Y_{j}}(i) ; i \geq 1, j \geq 1
$$

Then $Z_{j}$ satisfies the complete uniformity along primes property and thus Theorem 5 holds for the $Z_{j}$ sequence.

Proof. The proof is similar to that of Theorem 3.
Corollary 7. Apply Corollary 6 to any convex combination of a finite number of sequences

$$
\left(X_{1,1}, X_{1,2}, \ldots\right), \ldots,\left(X_{n, 1}, X_{n, 2}, \ldots\right)
$$

that all satisfy the complete uniformity along primes property. Then the conclusion of Corollary 6 holds for this convex combination.

Proof. Use induction and Corollary 6.

## 5. Illustrative Examples

In this section, we give examples from the class of distributions $\mathbb{P}_{n}^{*}$ that satisfy the hypotheses $(4-6)$ and examples of limits of distributions to which we can apply Theorem 5. It will be shown that the statement of Theorem 2 holds when $\mathbb{P}_{n}^{*}$ is replaced with either the Harmonic $(n)$ distribution or a convex combination of Harmonic and Uniform Distributions. Then we show that the Zeta $(s)$ and the Logarithmic distribution satisfy Theorem 5 as their parameters tend towards limits. We introduce a 2-parameter family of distributions $L Z(s, \alpha)$ which includes the Logarithmic distribution and Zeta distribution as special cases (when $\alpha=1$ and $s=1$ respectfully). We also look at a geometric power series distribution that converges to the normal distribution on all truncations as $s \rightarrow 1$.
5.1. The Harmonic Distribution. This was proved in $\S \S 2.1$ above.
5.2. The Zeta Distribution. This was proved in $\S \S 4.2$ above.
5.3. The Logarithmic Distribution. Given a real number $s$ with $0<s<1$, a logarithmic distribution with parameter $s$ is given by

$$
L_{s}(i):=\frac{-1}{\log (1-s)} \frac{s^{i}}{i} ; i \in \mathbb{N} .
$$

Now we will show that the number of distinct prime factors, $\omega(\cdot)$, of a truncated log-distributed variable has the same central limit theorem as the uniform distribution as $n \rightarrow \infty$ and $s \rightarrow 1$. For $i \in[n]$ we have

$$
L_{s, n}(i):=\frac{\frac{-1}{\log (1-s)} \frac{s^{i}}{i}}{\sum_{l=1}^{n} \frac{-1}{\log (1-s)} \frac{s^{l}}{l}}=\frac{\frac{s^{i}}{i}}{\sum_{l=1}^{n} \frac{s^{l}}{l}},
$$

so

$$
\varepsilon_{i, n} \stackrel{(1)}{=} \frac{\frac{s^{i}}{i}}{\sum_{l=1}^{n} \frac{s^{l}}{l}}-\frac{1}{n} .
$$

Therefore,

$$
\begin{aligned}
\sum_{l=1}^{\left\lfloor\frac{n}{p}\right\rfloor} \varepsilon_{l p, n} & =\sum_{l=1}^{\left\lfloor\frac{n}{p}\right\rfloor}\left(\frac{\frac{(s)^{l_{p}}}{l p}}{\sum_{l=1}^{n} \frac{s^{l}}{l}}-\frac{1}{n}\right) \\
& \leq \frac{1}{p} \frac{\sum_{l=1}^{\left\lfloor\frac{n}{p}\right\rfloor} \frac{\left(s^{p}\right)^{l}}{l}}{\sum_{l=1}^{n} \frac{s^{l}}{l}}-\frac{\lfloor n / p\rfloor}{n} \\
& \leq \frac{1}{p}-\frac{\lfloor n / p\rfloor}{n} \\
& \leq \frac{1}{p}
\end{aligned}
$$

so (4) holds with $C=1$. Similarly,

$$
\begin{aligned}
\left\lfloor\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n}\right. & =\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor}\left(\frac{\frac{(s)^{l_{1} \cdots p_{k}}}{l p_{1} \cdots p_{k}}}{\sum_{l=1}^{n} \frac{s^{l}}{l}}-\frac{1}{n}\right) \\
& \leq \frac{1}{p_{1} \cdots p_{k}} \frac{\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor \frac{\left(s^{p_{1} \cdots p_{k}}\right)^{l}}{l}}}{\sum_{l=1}^{n} \frac{s^{l}}{l}}-\frac{1}{p_{1} \cdots p_{k}}+\frac{1}{n} \\
& \leq \frac{1}{p_{1} \cdots p_{k}}-\frac{1}{p_{1} \cdots p_{k}}+\frac{1}{n} \\
& =\frac{1}{n}
\end{aligned}
$$

so (5) holds with $D=1$. Moreover, we have

$$
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \xrightarrow{n \rightarrow \infty} \frac{1}{p} \frac{\log \left(1-s^{p}\right)}{\log (1-s)}-\frac{1}{p} \xrightarrow{s \rightarrow 1} 0 .
$$

Therefore, (6) holds, so $\omega\left(L_{i, s}\right)$ is asymptotically distributed as $\mathcal{N}(\log \log n, \log \log n)$ as $n \rightarrow \infty$ and $s \rightarrow 1$.
Although Kac's heuristic for Theorem 1 is based on the asymptotic independence in the uniform case, we will show that the events $\{$ divisible by $p\}$ and $\{$ divisible by $q\}$ are not independent for any value of $s$. Let $A_{p}$ denote the set of all positive integers divisible by $p$, then

$$
\begin{aligned}
L_{s}\left(A_{p}\right) & =\sum_{l=1}^{\infty} \frac{-1}{\log (1-s)} \frac{\left(s^{p}\right)^{l}}{p l} \\
& =\frac{1}{p} \frac{\log \left(1-s^{p}\right)}{\log (1-s)}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{s}\left(A_{p} \cap A_{q}\right) & =\sum_{l=1}^{\infty} \frac{-1}{\log (1-s)} \frac{\left(s^{p q}\right)^{l}}{p q l} \\
& =\frac{1}{p q} \frac{\log \left(1-s^{p q}\right)}{\log (1-s)}
\end{aligned}
$$

It is worth noting that

$$
\lim _{s \rightarrow 1} L_{s}\left(A_{p} \cap A_{q}\right)=\frac{1}{p q}=\lim _{s \rightarrow 1}\left(L_{s}\left(A_{p}\right) L_{s}\left(A_{q}\right)\right)
$$

as $s \rightarrow 1$; so independence is only approached in the limit.
5.4. Geometric Power Series Distribution. Let $s \in(0,1)$. Define:

$$
\mathbb{P}(i):=\frac{1-s}{s} s^{i} ; i \in \mathbb{N}
$$

then this distribution satisfies the hypotheses of Theorem 2 in the limit as $s \rightarrow 1$. As $s \rightarrow 1$ it seems that this distribution is converging to the uniform distribution on any truncation, so the result is to be expected.

Truncating on $[n]$ leads to

$$
\varepsilon_{i, n} \stackrel{(1)}{=} \frac{s^{i}}{\sum_{j=1}^{n} s^{j}}-\frac{1}{n},
$$

therefore

$$
\begin{aligned}
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} & =\sum_{l=1}^{\lfloor n / p\rfloor}\left(\frac{s^{i}}{\sum_{j=1}^{n} s^{j}}-\frac{1}{n}\right) \\
& =\frac{s-s^{\lfloor n / p\rfloor+1}}{s-s^{n+1}}-\frac{\lfloor n / p\rfloor}{n} \\
& \stackrel{s \rightarrow 1}{\rightarrow} \frac{\lfloor n / p\rfloor}{n}-\frac{\lfloor n / p\rfloor}{n} \\
& =0,
\end{aligned}
$$

so constraints $(4-6)$ hold with $C=D=1$.
5.5. Convex Combination of Harmonic and Uniform. Following Cranston and Mountford in [1], let $\lambda \in[0,1]$ and define

$$
\mathbb{P}_{n}(i):=\frac{\lambda}{i h_{n}}+(1-\lambda) \frac{1}{n} ; i \in \mathbb{N}
$$

where $h_{n}$ is the $n^{\text {th }}$ harmonic number. Then $\mathbb{P}_{n}$ satisfies the hypotheses of Theorem 2 . This is proved in Corollary 2 above using Theorems 2 and 3 .
5.6. A Logarithmic-Zeta Distribution. There is a 2-parameter family of power series distributions on $\mathbb{N}$ for $(s, \alpha) \in \mathbb{R}^{2}$ such that $0<s \leq 1, \alpha \geq 1$, and $s=\alpha=1$ is not allowed; the PMF is given by

$$
L Z_{s, \alpha}(i):=\frac{1}{\sum_{j=1}^{\infty} \frac{s^{j}}{j^{\alpha}}} \frac{s^{i}}{i^{\alpha}} ; i \in \mathbb{N} .
$$

Similarly, there is a truncation of $L Z$ given by

$$
L Z_{s, \alpha, n}(i):=\frac{1}{\sum_{j=1}^{n} \frac{s^{j}}{j^{\alpha}}} \frac{s^{i}}{i^{\alpha}} ; i \leq n
$$

and a description of the $\varepsilon_{i, n}$ given by

$$
\varepsilon_{i, n} \stackrel{(1)}{=} \frac{1}{\sum_{j=1}^{n} \frac{s^{j}}{j^{\alpha}}} \frac{s^{i}}{i^{\alpha}}-\frac{1}{n}
$$

It is clear that a similar type of argument shows that Theorem 2 holds for the truncated logarithmic-zeta distribution as $(s, \alpha) \rightarrow(1,1)$; in particular, we obtain

$$
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \leq \frac{1}{p^{\alpha}} \leq \frac{1}{p}
$$

and

$$
\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n} \leq \frac{1}{p_{1}^{\alpha} \cdots p_{k}^{\alpha}}-\frac{1}{p_{1} \cdots p_{k}}+\frac{1}{n} \leq \frac{1}{n}
$$

Furthermore,

$$
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \xrightarrow{s \rightarrow 1} \frac{1}{p^{\alpha}}-\frac{1}{p} \xrightarrow{\alpha \rightarrow 1} 0 .
$$

5.7. Passage to the Limit: $\lim _{\substack{s \rightarrow 1 \\ \alpha \rightarrow 1}} L Z_{s, \alpha}$. We would like to look at the behavior of $\omega(X)$ for $X$ distributed as $L Z_{s, \alpha}$ as $(s, \alpha) \rightarrow(1,1)$. The motivation for this comes from the fact that as $(s, \alpha) \rightarrow(1,1)$ the truncated distributions resemble the harmonic distribution which we know behaves similarly to the uniform distribution on $[n]$ when $n \rightarrow \infty$ according to Theorem 2 .

In Theorem 5.5 of [2], they provide the moment generating function for $\omega\left(X_{\alpha}\right)$ when $X_{\alpha}$ is a Zeta ( $\alpha$ ) distributed random integer with parameter $\alpha>1$. They go on to prove that

$$
\hat{\omega}\left(X_{\alpha}\right)=\frac{\omega\left(X_{\alpha}\right)-\sum_{p} p^{-\alpha}}{\sqrt{\sum_{p} p^{-\alpha}}} \stackrel{d}{\rightarrow} Z
$$

as $\alpha \rightarrow 1$ in Theorem 7.1 of [2] by using this moment generating function. Later, Cranston and Mountford [1] give a new proof of Theorem 1. The proof uses Theorem 7.1 about zeta distributions to prove Erdôs-Kac in a way that translates over to settings where zeta functions still make sense.

We generalize Peltzer and Cranston's Theorem 7.1 in [2] in the following way. As $(s, \alpha) \rightarrow(1,1)$, we show that

$$
\hat{\omega}\left(X_{s, \alpha}\right)=\frac{\omega\left(X_{s, \alpha}\right)-\mu_{s, \alpha}}{\sqrt{\mu_{s, \alpha}}} \xrightarrow{d} Z,
$$

where $\mu_{s, \alpha}$ is the mean of $\omega\left(X_{s, \alpha}\right)$ when $X_{s, \alpha}$ is distributed as $L Z_{s, \alpha}$. In fact, we do not need independence, nor do we need to compute the mean for any particular $s, \alpha$ in order to conclude this (neither do we need to compute a MGF).

In $\S 4$, we proved a stronger statement than the above statement about $L Z_{s, \alpha}$; we proved that in general

$$
\hat{\omega}\left(X_{j}\right)=\frac{\omega\left(X_{j}\right)-\mu_{j}}{\sqrt{\mu_{j}}} \xrightarrow{d} Z,
$$

as long as the truncated variables $X_{j}(n)$ have the complete uniformity along primes property. Here we assume that $\mu_{j}=\mathbb{E}\left(\omega\left(X_{j}\right)\right)<\infty$. When $(s, \alpha) \rightarrow(1,1)$, we recover the above statement about $L Z_{s, \alpha}$.
5.8. A Non-Example: Zeroing at a Set of Primes. Fix $n \in \mathbb{N}$ and let $p \leq n$ denote a prime. Consider the PMF defined by

$$
\mathbb{P}_{n, p}(i)= \begin{cases}\frac{1}{\#([n] \backslash p \mathbb{N})} & i \notin p \mathbb{N} \\ 0 & i \in p \mathbb{N}\end{cases}
$$

We have

$$
\varepsilon_{i, n}= \begin{cases}\frac{1}{\#([n] \backslash p \mathbb{N})}-1 / n & i \notin p \mathbb{N} \\ -1 / n & i \in p \mathbb{N}\end{cases}
$$

Therefore,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} & =\lim _{n \rightarrow \infty}-\frac{\lfloor n / p\rfloor}{n} \\
& =-1 / p \\
& \neq 0 .
\end{aligned}
$$

Thus, this PMF does not satisfy constraint (6). We conjecture that the conclusion of Theorem 2 does not hold for this distribution.
5.9. An Erdôs-Kac Theorem for Continuous Variables. Consider a continuous uniform random variable $N_{n}$ on the interval $(0, n]$. Then $\left\lceil N_{n}\right\rceil$ is a uniform variable on $[n]$. Therefore, we have

Corollary 8. Let $Z$ denote a standard normal variable, and let $X_{n}$ be a continuous random variable on $(0, n]$. Define the $\varepsilon_{i, n}$ according to the following relation: $\mathbb{P}_{n}\left(\left\lceil X_{n}\right\rceil=i\right)=\frac{1}{n}+\varepsilon_{i, n}$. If the constraints

- There exists a constant $C$ such that for all $n>1$ and for all primes $p$ with $p>\alpha_{n}$,

$$
\sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n} \leq \frac{C}{p}
$$

- There exists a constant $D$ such that

$$
\sum_{l=1}^{\left\lfloor\frac{n}{p_{1} \cdots p_{k}}\right\rfloor} \varepsilon_{l p_{1} \cdots p_{k}, n} \leq \frac{D}{n}
$$

for all $n>1$ and, for each $k$, all $k$-tuples $\left(p_{1}, \ldots, p_{k}\right)$ consisting of distinct primes of size at most $\alpha_{n}$, and

- For any prime p,

$$
\lim _{n \rightarrow \infty} \sum_{l=1}^{\lfloor n / p\rfloor} \varepsilon_{l p, n}=0
$$

all hold, then

$$
\mathbb{P}_{n}\left(t \leq n: \omega(\lceil t\rceil)-\log \log n \leq x(\log \log n)^{1 / 2}\right) \rightarrow \mathbb{P}(Z \leq x)
$$

as $n \rightarrow \infty$.

## 6. Conclusion

Theorem 2 generalizes the Erdős-Kac Theorem for $\omega(\cdot)$ to distributions other than the uniform distribution, and this theorem was proved by imposing constraints $(4-6)$ on a PMF of the form $\mathbb{P}(i)=1 / n+\varepsilon_{i, n}$. We showed that the uniform and harmonic distributions satisfy these constraints; then, we showed that any convex sum of these PMFs also satisfies the constraints $(4-6)$.

The uniformity property provides a natural way to examine asymptotic properties of truncations of variables with support $\mathbb{N}$. Given an infinite sequence $X_{1}, X_{2}, \ldots$ of random variables on $\mathbb{N}$ satisfying uniformity along primes, Theorem 5 showed that for any sequence with $n_{j} \rightarrow \infty$ as $j \rightarrow \infty$, the distribution of $\omega\left(X_{j}\left(n_{j}\right)\right)$ is asymptotically normally distributed with mean and variance both equal to $\log \log n_{j}$ as long as $X_{j}\left(n_{j}\right)$ satisfies (4-6).

The definition of complete uniformity along primes allows us to obtain central limit theorems regardless of how $n \rightarrow \infty$, and allows us to make asymptotic statements involving the mean of $\omega\left(X_{j}\right)$. This generalized a statement from [2] involving the mean of $\omega\left(X_{s}\right)$ as $s \rightarrow 1$ when $X_{s}$ is a random Zeta $(s)$-distributed variable.

Another way to generalize Theorem 2 would be to incorporate it with other generalizations, e.g., [1, 2, 6, 7]. By incorporating Theorem 2 with these, further generalizations can be made in which the original setting is not [ $n$ ], the underlying distribution of the random-integer is not uniform, and $\omega(n)$ can be replaced with a more general strongly additive function $f(n)$.

We also showed the complete uniformity property holds, and thus normality in the limit, for Zeta (s) and a number of similar distributions. It is suspected by the authors, but not known, whether or not the hypotheses $(4-6)$ are necessary and sufficient for the conclusion of Theorem 2 ; we conjecture that is the case.

## References

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[^0]:    ${ }^{1}$ While the Hardy-Ramanujan theorem provides information about the average behavior of the number of prime factors of a natural number, the Erdős-Kac theorem offers a more detailed probabilistic description of their distribution, by taking into account not just the average number but also the variability around that average.

