# NON STABLE RATIONALITY OF PROJECTIVE APPROXIMATIONS FOR CLASSIFYING SPACES

NOBUAKI YAGITA

ABSTRACT. Let $BG$ be the classifying space of an algebraic group $G$ over a subfield $k$ of $\mathbb{C}$ of complex numbers. We compute a new stable birational invariant defined by Benoist-Ottem as the difference of two coniveau filtrations of a smooth projective (Ekedahl) approximation $X$ of $BG\times\mathbb{P}^\infty$. Then we show (by without and with the unramified cohomology) in many cases $X$ are not stable rational.

## 1. Introduction

Let $X$ be a smooth projective variety over $k\subset\mathbb{C}$. The conception of the rationality is how $X$ is near to some projective space $\mathbb{P}^n$ over $k$. Indeed, $X$ is called *rational* if $X$ is birational to a projective space $\mathbb{P}^n$. A variety $X$ is called *stable* rational if $X\times\mathbb{P}^m$ is rational for some $m\geq 0$. A variety $X$ is called *retract* rational if the rational identity map on $X$ is factorized rationally through a projective space.

Of course, the existences and properties of non these rationality for $X$ are widely studied by many authors (see explanations in [Pi]). For examples, such projective $X$ which are surface bundles of three (or four)folds are studied detailedly. These examples are computed by often using the unramified cohomology $H^*_{ur}(X;\mathbb{Z}/p)$ which is invariant of (retract) rationality.

There are another examples (exchanging $\mathbb{P}^n$ by $\mathbb{A}^n$); the quasi projective variety represented by the classifying spaces $BG$ of an affine algebraic groups $G$ [Me].

In this paper , we study the similar but different invariant $DH^*(X)$ for the projective approximation $X=X_G$ by Ekedahl for the classifying space $BG\times\mathbb{P}^\infty$. Note that stable rationality types of $BG\times\mathbb{P}^\infty$ and its projective approximation are completely different in general.

2020 *Mathematics Subject Classification.* 14C15, 14P99, 55N20.

*Key words and phrases.* birational invariant, coniveau filtrations, unramified cohomology.

For example we compare these invariants when $G = SO_{2m+1}$

$$\begin{cases} DH^*(BG) \qquad is\ not\ defined, \\ H^*_{ur}(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1\}, \\ DH^*(X_G)/2 \supset \mathbb{Z}/2\{w_3, w_5, ..., w_{2m+1}\}, \\ H^*_{ur}(X_G;\mathbb{Z}/2) \supset \mathbb{Z}/2\{1, w_2, w_4, ..., w_{2m}\}. \end{cases}$$

(The notation $A\{a, b, ...\}$ means the $A$-free module generated by $a, b, ....$)

We compute a new stable birational invariant induced from Benoist-Otten [Be-Ot] as the difference of the two coniveau filtrations. For a fixed prime $p$, define the stable birational invariant

$$DH^*(X;A)/p = N^1H^*(X;A)/(p, \tilde{N}^1H^*(X;A))$$

for the smooth projective approximation $X$ of $BG \times \mathbb{P}^\infty$. Here $H^*(X;A)$ is the Betti (or étale) cohomology and $N^1H^*(X;A)$ (resp, $\tilde{N}^1H^*(X;A)$)) is the coniveau (resp. *strong* coniveau) filtration defined by the kernel of the restriction maps to open sets of $X$ (resp. the image of of Gysin maps). For details see §2 below.

Hence $DH^*(X;A)/p$ is written as a sub-quotient module of $H^*(X;A)/p$.

Here an approximation (for *degree* $\le N$) is the projective (smooth) variety $X = X_G(N)$ such that there is a map $g : X \to BG \times \mathbb{P}^\infty$ with

$$g^* : H^*(BG \times \mathbb{P}^\infty; A) \cong H^*(X;A) \quad for\ * < N.$$

(In this paper, we say $X$ is an approximation $for\ BG$ when it is that $of\ BG \times \mathbb{P}^\infty$ strictly speaking.) Let us write $DH^*(X;\mathbb{Z})$ by $DH^*(X)$ simply as usual.

For example, let $G = G_n$ be the elementary abelian $p$-group $(\mathbb{Z}/p)^n$. Recall the $mod(p)$ cohomology (for $p$ odd)

$$H^*(BG_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ....y_n] \otimes \Lambda(x_1, ..., x_n)$$

where $|x_i| = 1$ and $Q_0(x_i) = y_i$ for the Bokstein operation $Q_0 = \beta$. (Here $\Lambda(a, b, ...)$ is the exterior algebra generated by $a, b, ...$).

**Theorem 1.1.** *For any prim $p$, take $G = G_n = (\mathbb{Z}/p)^n$, $n \ge 2$ and $\alpha_i = Q_0(x_1x_2...x_i) \in H^{n+1}(X_{G_n})$. Then we have*

$$DH^*(X_{G_n})/p \supset \mathbb{Z}/p\{\alpha_2, \alpha_3, ..., \alpha_n\} \quad * \le n+1 < N.$$

Hence $X_{G_n}$ is not stable rational. Moreover $X_{G_n}$ and $X_{G_{n'}}$ are not stable birational equivalent when $n \ne n'$.

Next we consider the (connected) case $G = SO_n$ the special orthogonal group ($p = 2$). Its cohomology is

$$H^*(BSO_{2m+1};\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, ....w_{2m+1}],$$

with $Q_0w_{2m}=w_{2m+1}$ where $w_i$ is the Stiefel-Whitney class for the embedding $SO_n\to O_n$. Hence we can identify $w_{2i+1}\in H^*(BG)$.

**Theorem 1.2.** [Ya6] *Let $X_n=X_n(N)$ be approximations for $BSO_n$ for $n\ge 3$ and $2^{m+2}<N$. Then we have*

$$DH^*(X_{2m+1})/2\supset \mathbb{Z}/2\{w_3,w_5,...,w_{2m+1}\}\quad for\ all\ 2m+1\le *<N.$$

We consider the cases $G$ is a simply connected simple group. Let $G$ contain $p$-torsion. Then we know $H^4(BG)\otimes\mathbb{Z}_p\cong\mathbb{Z}_p$, and write its generator by $w$. Then we have

**Lemma 1.3.** [Ya6] *Let $G$ be a simply connected group such that $H^*(BG)$ has $p$-torsion. Let $X=X(N)$ be an approximation for $BG$ for $N\ge 2p+3$. Then*

$$DH^4(X)/p\supset\mathbb{Z}/p\{w\}.$$

Next, we study the retract rationality of $X_G$ for the above groups $G$. We consider the Zariski cohomology $H^*_{Zar}(X,\mathcal{H}^{*'}_A)$ where $\mathcal{H}^*_A$ is the Zariski sheaf induced from the presheaf given by $U\mapsto H^*_{\acute{e}t}(U;A)$ for an open $U\subset X$. It is well known when $X$ is complete and smooth, the unramified cohomology is written

$$H^*_{ur}(X;\mathbb{Z}/p)\cong H^0_{Zar}(X;\mathcal{H}^*_{\mathbb{Z}/p}),$$

and it is an invariant of the retract rationality of $X$ (Proposition 3.1. 3.4 in [Me]).

By Totaro [Ga-Me-Se], the above cohomology is also isomorphic to the cohomological invariant (of $G$-torsors) i.e.

$$H^0_{Zar}(BG;\mathcal{H}^*_{\mathbb{Z}/p})\cong Inv^*(G;\mathbb{Z}/p).$$

Let $H^{*,*'}(X;\mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology of $X$ so that

$$H^{*,*}(X;\mathbb{Z}/p)\cong H^*_{\acute{e}t}(X;\mathbb{Z}/p)\ \ and\ \ H^{2*,*}(X;\mathbb{Z}/p)\cong CH^*(X)/p.$$

Let $0\ne\tau\in H^{0,1}(Spec(k);\mathbb{Z}/p)$. Then $\tau$ defines the map

$$\tau: H^{*,*'}(X;\mathbb{Z}/p)\to H^{*,*'+1}(X;\mathbb{Z}/p)$$

such that the cycle map is written

$$CH^*(X)/p\cong H^{2*,*}(X)/p\stackrel{\times\tau^*}{\to}H^{2*,2*}(X;\mathbb{Z}/p)\cong H^{2*}(X;\mathbb{Z}/p).$$

From Orlov-Vishik-Voevodsky [Or-Vi-Vo], ([Te-Ya] for $p:odd$,) we have

**Lemma 1.4.** *([Or-Vi-Vo]) We have the short exact sequence*

$$0\to H^{*,*}(X;\mathbb{Z}/p)/(\tau)\to H^0_{Zar}(X;\mathcal{H}^*_{\mathbb{Z}/p})\to Ker(\tau|H^{*+1,*-1}(X;\mathbb{Z}/p))\to 0.$$

Here $\quad H^{*,*}(X;\mathbb{Z}/p)/(\tau)=H^{*,*}(X;\mathbb{Z}/p)/(\tau H^{*.*-1}(X.\mathbb{Z}/p))$
$$\cong H^*(X;\mathbb{Z}/p)/N^1H^*(X;\mathbb{Z}/p).$$

This cohomology is called *stable* cohomology, and studied by Bogolomov [Bo]. [Te-Ya2].

For example, when $G=(\mathbb{Z}/p)^n$, it is known

$$Inv^*(G;\mathbb{Z}/p)\cong\Lambda(x_1,...,x_n).$$

**Theorem 1.5.** *Let $G=(\mathbb{Z}/p)^n$ and $X=X_G$. Then for $b_i=x_1...x_i$*

$$H^*_{ur}(X;\mathbb{Z}/p)\supset H^{*,*}(X;\mathbb{Z}/p)/(\tau)\supset\mathbb{Z}/2\{1,b_2,b_3,...,b_n\}.$$

*Hence each $X_n$ and $X_{n'}$ are not retract birational equivalent when $n\neq n'$.*

By Serre [Ga-Me-Se], when $G=SO_{2m+1}$, it is known

$$Inv^*(G;\mathbb{Z}/2)\cong\mathbb{Z}/2\{1,w_2,...,w_{2m}\}.$$

**Theorem 1.6.** *Let $G=SO_{2m+1}$ and $X=X_G$. Then*

$$H^*_{ur}(X;\mathbb{Z}/2)\supset H^{*,*}(X;\mathbb{Z}/2)/(\tau)\supset\mathbb{Z}/2\{1,w_2,...,w_{2m}\}.$$

*Hence each $X_m$ and $X_{m'}$ are not retract rational equivalent when $m\neq m'$.*

We also give examples of nonzero elements of $Ker(\tau)$ in Lemma 1.4.

**Theorem 1.7.** *Let $G$ be a simply connected simple group and $X=X_G$. Then there is the element $w\in H^4(X;\mathbb{Z}/p)$ such that*

$$H^3_{ur}(X;\mathbb{Z}/p)\twoheadrightarrow Ker(\tau|H^{4,2}(X;\mathbb{Z}/p))\supset\mathbb{Z}/p\{w\}.$$

*Hence $X$ is not retract rational.*

**Remark.** It is known $BSpin_n$ for $n\leq 14$ are stable rational [Ko], [Me], [Re-Sc]. Hence $BG=BSpin_n$ for $7\leq n\leq 14$ and its approximation $X=X_G$ are different stable rational type.

At the last three sections, we will do quite different arguments from the preceding sections, for quadrics $X$ over $\mathbb{R}$. Let us write

$$DH^*(X;\mathbb{Z}_p)=DH^*_{\acute{e}t}(X;\mathbb{Z}_p)$$

where $\quad H^*_{\acute{e}t}(X;\mathbb{Z}_p)=Lim_{\infty\leftarrow s}H^*_{\acute{e}t}(X;\mathbb{Z}/p^s)\cong Lim_{\infty\leftarrow s}H^{*,*}(X;\mathbb{Z}/p^s)$.

In this paper, the étale cohomology (with the integral coefficients $\mathbb{Z}_2(*)$ for even degrees) means the motivic cohomology ;

$$H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(*))\cong\begin{cases}H^{2*,2*}(X;\mathbb{Z}_2) & for\ *=even\\ H^{2*,2*+1}(X;\mathbb{Z}_2) & for\ *=odd.\end{cases}.$$

Here we see the examples that $X$ are not retract rational ($H^{4*}_{ur}(X;\mathbb{Z}_2)\neq\mathbb{Z}/2$)) while $DH^{2*}(X;\mathbb{Z}_2(*))=0$. Let $X=Q^d$ be the anisotropic quadric of dimension $d=2^n-1$ (i.e. the norm variety). Then there are elements

$$h\in H^2_{\acute{e}t}(X;\mathbb{Z}_2(1))\quad and\quad \bar{\rho}_4\in H^4_{\acute{e}t}(X;\mathbb{Z}_2(0)).$$

**Theorem 1.8.** *([Ya6]) The ring* $H^{2*}_{ét}(Q^{2^n-1};\mathbb{Z}_2(*))$ *is multiplicatively generated by* $\bar{\rho}_4$ *and* $h$ *the hyper plane section.*

**Theorem 1.9.** *Let* $X_n = Q^{2^n-1}$, $n \geq 2$ *the norm variety. Then*

$$DH^{2*}(X_n;\mathbb{Z}_2(*)) = 0,$$

$$H^{2*}_{ur}(X_n;\mathbb{Z}_2(*)) \supset \mathbb{Z}/2[\bar{\rho}_4]/(\bar{\rho}_4^{2^{n-1}}).$$

*Hence for* $n \neq n'$, *we see that* $X_n$ *and* $X_{n'}$ *are not retract birational equivalent.*

**Remark.** If $\bar{\rho}_4 \in \tilde{N}^1H^{2*}(X;\mathbb{Z}_2(*))$, the above theorem was just corollary of the Frobenius reciprocity (Lemma 2.2). But it does not hold (moreover , we see $\bar{\rho}_4 \notin N^1H^{2*}(X;\mathbb{Z}_2(*))$).

## 2. Two coniveau filtrations

Let us recall the coniveau filtration of the cohomology with coefficients in $A$ for $A = \mathbb{Z}, \mathbb{Z}_p$, or $\mathbb{Z}/p$,

$$N^cH^i(X;A) = \sum_{Z\subset X} ker(j^* : H^i(X;A) \to H^i(X-Z,A))$$

where $Z \subset X$ runs through the closed subvarieties of codimension at least $c$ of $X$, and $j : X - Z \subset X$ is the complementary open immersion.

Similarly, we can define the *strong* coniveau filtration by

$$\tilde{N}^cH^i(X;A) = \sum_{f:Y\to X} im(f_* : H^{i-2r}(Y;A) \to H^i(X,A))$$

where the sum is over all proper morphism $f : Y \to X$ from a smooth complex variety $Y$ of $dim(Y) = dim(X)-r$ with $r \geq c$, and $f_*$ its transfer (Gysin map). It is immediate that $\tilde{N}^cH^*(X;A) \subset N^cH^*(X;A)$.

It is known that when $X$ is proper, $\tilde{N}^cH^i(X;\mathbb{Q}) = N^cH^i(X;\mathbb{Q})$ by Deligne. However Benoist and Ottem ([Be-Ot]) recently show that the above two coniveau filtrations are not equal for $A = \mathbb{Z}$.

Let $G$ be an algebraic group such that $H^*(BG;\mathbb{Z})$ has $p$-torsion for the classifying space $BG$ is defined by Totaro [To], and Bogomolov [Bo]. Then let us say that an (Ekedahl) *approximation* for $BG$ (for *degree* $\leq N$) is the projective (smooth) variety $X = X_G(N)$ such that there is a map $g : X \to BG \times \mathbb{P}^\infty$ with

$$g^* : H^*(BG \times \mathbb{P}^\infty;A) \cong H^*(X;A) \quad for * < N.$$

In the paper [Ya6], we try to compute the stable birational invariant of $X$ (Proposition 2.4 in [Be-Ot])

$$DH^*(X;A) = N^1H^*(X;A)/(\tilde{N}^1H^*(X;A))$$

for projective approximations $X$ for $BG$ ([Ek],[To],[Pi-Ya]).

Here we recall the Bloch-Ogus [Bl-Og] spectral sequence such that its $E_2$-term is given by

$$E(c)_2^{c,*-c} \cong H^c_{Zar}(X, \mathcal{H}_A^{*-c}) \Longrightarrow H^*_{ét}(X; A)$$

where $\mathcal{H}_A^*$ is the Zariski sheaf induced from the presheaf given by $U \mapsto H^*_{ét}(U; A)$ for an open $U \subset X$.

The filtration for this spectral sequence is defined as the coniveau filtration

$$N^c H^*_{ét}(X; A) = F(c)^{c,*-c}$$

where the infinite term $E(c)_\infty^{c,*-c} \cong F(c)^{c,*-c}/F(c)^{c+1,*-c-1}$.

Here we recall the motivic cohomology $H^{*,*'}(X; \mathbb{Z}/p)$ defined by Voevodsky and Suslin ([Vo1],[Vo3],[Vo4]) so that

$$H^{i,i}(X; \mathbb{Z}/p) \cong H^i_{ét}(X; \mathbb{Z}/p) \cong H^i(X; \mathbb{Z}/p).$$

Let us write $H^*_{ét}(X; \mathbb{Z})$ simply by $H^*_{ét}(X)$ as usual. Note that $H^*_{ét}(X) \ncong H^*(X)$ in general, while we have the natural map $H^*_{ét}(X) \to H^*(X)$.

Let $0 \neq \tau \in H^{0,1}(Spec(\mathbb{C}); \mathbb{Z}/p)$. Then by the multiplying $\tau$, we can define a map $H^{*,*'}(X; \mathbb{Z}/p) \to H^{*,*'+1}(X; \mathbb{Z}/p)$. By Deligne ( foot note (1) in Remark 6.4 in [Bl-Og]) and Paranjape (Corollary 4.4 in [Pa]), it is proven that there is an isomorphism of the coniveau spectral sequence with the $\tau$-Bockstein spectral sequence $E(\tau)_r^{*,*'}$ (see also [Te-Ya2], [Ya1]).

**Lemma 2.1.** *(Deligne) Let $A = \mathbb{Z}/p$. Then we have the isomorphism of spectral sequence $E(c)_r^{c,*-c} \cong E(\tau)_{r-1}^{*,*-c}$ for $r \geq 2$. Hence the filtrations are the same, i.e. $N^c H^*_{ét}(X; \mathbb{Z}/p) = F_\tau^{*,*-c} = Im(\times\tau^c : H^{*,*-c}(X; \mathbb{Z}/p))$. Thus we have the isomorphism*

$$H^{*,*}(X; \mathbb{Z}/p)/(\tau) \cong H^*(X; \mathbb{Z}/p)/N^1 H^*(X; \mathbb{Z}/p).$$

We recall here the Frobenius reciprocity law.

**Lemma 2.2.** *(reciprocity law) If $a \in \tilde{N}^* H^{2*}(X; A)$, then for each $g \in H^{*'}(X; A)$ we have $ag \in \tilde{N}^* H^{2*+*'}(X; A)$.*

*Proof.* Suppose we have $f : Y \to X$ with $f_*(a') = a$. Then

$$f_*(a' f^*(g)) = f_*(a')g = ag$$

by the Frobenius reciprocity law. □

Let $G$ be an algebraic group (over $\mathbb{C}$) and $r$ be a complex representation $r : G \to U_n$ the unitary group. Then we can define the Chern class in $H^*(BG)$ by $c_i = r^* c_i^U$. Here the Chern classes $c_i^U$ in $H^*(BU_n) \cong \mathbb{Z}[c_1^U, ..., c_n^U]$ ([Qu1]) are defined by using the Gysin map as $c_n^U = i_{n,*}(1)$ for

$$i_{n,*} : H^*(BU_n) \cong H^*_{U_n}(pt.) \stackrel{i_{n,*}}{\to} H^{*+2i}_{U_n}(\mathbb{C}^{\times i}) \cong H^{*+2i}(BU_n)$$

where $H_{U_n}(X) = H^*(EU_n \times_{U_n} X)$ is the $U_n$-equivaliant cohomology.

Let us write by $Ch^*(X;A)$ the Chern subring which is the subring of $H^*(X;A)$ multiplicatively generated by all Chern classes.

**Lemma 2.3.** *We have a quotient map*

$$N^1H^*(X;A)/(IdealCh^*(X;A)) \twoheadrightarrow DH^*(X;A).$$

The following lemma is proved by Colliot Thérène and Voisin [Co-Vo] by using the affirmative answer of the Bloch-Kato conjecture by Voevodsky. ([Vo3]. [Vo4])

**Lemma 2.4.** *(*[Co-Vo]*) Let $X$ be a smooth complex variety. Then any torsion element in $H^*(X)$ is in $N^1H^*(X)$.*

## 3. THE MAIN LEMMAS

The Milnor operation $Q_n$ (in $H^*(-;\mathbb{Z}/p)$) is defined by $Q_0 = \beta$ and for $n \geq 1$

$$Q_n = P^{\Delta_n}\beta - \beta P^{\Delta_n}, \quad \Delta_n = (0,..,0,\overset{n}{1},0,...).$$

(For details see [Mi], §3.1 in [Vo1]) where $\beta$ is the Bockstein operation and $P^\alpha$ for $\alpha = (\alpha_1, \alpha_2, ...)$ is the fundamental base of the module of finite sums of products of reduced powers. (For example $P^{\Delta_i}(y) = y^{p^i}$ for $|y| = 2$. and $Q_n$ is a derivative.)

**Lemma 3.1.** *Let $f_*$ be the transfer (Gysin) map (for proper smooth) $f : X \to Y$. Then $Q_nf_*(x) = f_*Q_n(x)$ for $x \in H^*(X;\mathbb{Z}/p)$.*

*Proof.* The above lemma is known (see the proof of Lemma 7.1 in [Ya4]). The transfer $f_*$ is expressed as $g^*f'_*$ such that

$$f'_*(x) = i^*(Th(1) \cdot x), \quad x \in H^*(X;\mathbb{Z}/p)$$

for some maps $g, f', i$ and the Thom class $Th(1)$. Since $Q_n(Th(1)) = 0$ and $Q_i$ is a derivation, we get the lemma. □

By Voevodsky [Vo1], [Vo2], we have the $Q_i$ operation also in the motivic cohomology $H^{*,*'}(X;\mathbb{Z}/p)$ with $deg(Q_i) = (2p^i - 1, p-1)$.

**Lemma 3.2.** *We see that $Im(cl)^+ \subset N^1H^{2*}(X;A)$.*

*Proof.* From Lemma 2.1, we see $H^{*,*'}(X;A) \subset N^{*-*'}H^*(X;A)$. We have $H^{2*,*}(X;A) \cong CH^*(X) \otimes A$. Since $2* > *$ for $* \geq 1$, we see $cl(y) \in N^1H^{2*}(X;A)$. □

Each element $y \in CH^*(X) \otimes A$ is represented by closed algebraic set supported $Y$, while $Y$ may be singular. On the other hand, by Totaro [To], we have the modified cycle map $\bar{c}l$ such that the usual cycle map is

$$cl : CH^*(X) \otimes A \xrightarrow{\bar{c}l} MU^{2*}(X) \otimes_{MU^*} A \xrightarrow{\rho} H^{2*}(X; A)$$

for the complex cobordism theory $MU^*(X)$. It is known [Qu1] that elements in $MU^{2*}(X)$ can be represented by proper maps to X from stable almost complex manifolds $Y$. (The manifold $Y$ is not necessarily a complex manifold.)

The following lemma is well known.

**Lemma 3.3.** *If $x \in Im(\rho)$ for $\rho : MU^*(X)/p \to H^*(X; \mathbb{Z}/p)$, then we have $Q_i(x) = 0$ for all $i \geq 0$.*

*Proof.* Recall the connective Morava K-theory $k(i)^*(X)$ with $k(i)^* = \mathbb{Z}/p[v_i]$, $|v_i| = -2p^i + 2$, which has natural maps

$$\rho : MU^*(X)/p \xrightarrow{\rho_1} k(i)^*(X) \xrightarrow{\rho_2} H^*(X : \mathbb{Z}/p).$$

It is known that there is an exact sequence (Sullivan exact sequence) such that

$$... \xrightarrow{\rho_2} H^*(X; \mathbb{Z}/p) \xrightarrow{\delta} k(i)^*(X) \xrightarrow{v_i} k(i)^*(X) \xrightarrow{\rho_2} H^*(X : \mathbb{Z}/p) \xrightarrow{\delta} ...$$

with $\rho_2\delta = Q_i$. Hence $Q_i\rho_2(x) = \rho_2\delta\rho_2 = 0$. which implies $Q_i\rho(x) = 0$. $\square$

The following lemma is the $Q_i$-version of one of results by Benoist and Ottem.

**Lemma 3.4.** *Let $\alpha \in N^1H^s(X)$ for $s = 3$ or $4$. If $Q_i(\alpha) \neq 0 \in H^*(X; \mathbb{Z}/p)$ for some $i \geq 1$, then*

$$DH^s(X)/p \supset \mathbb{Z}/p\{\alpha\}, \quad DH^s(X; \mathbb{Z}/p^t)/p \supset \mathbb{Z}/p\{\alpha\} \quad for\ t \geq 2.$$

*Proof.* Suppose $\alpha \in \tilde{N}^1H^s(X)$ for $s = 3$ or $4$, i.e. there is a smooth $Y$ with $f : Y \to X$ such that the transfer $f_*(\alpha') = \alpha$ for $\alpha' \in H^*(Y)$.

Then for $s = 4$,

$$Q_i(\alpha') = (P^{\Delta_i}\beta - \beta P^{\Delta_i})(\alpha') = (-\beta P^{\Delta_i})(\alpha') = -\beta(\alpha')^{p^i}$$

$$= -p^i(\beta\alpha')(\alpha')^{p^i-1} = 0 \quad (by\ the\ Cartan\ formula)$$

since $\beta(\alpha') = 0$ and $P^{\Delta_i}(y) = y^{p^i}$ for $deg(y) = 2$. (For $s = 3$, we get also $Q_i(\alpha') = 0$ since $P^{\Delta_i}(x) = 0$ for $deg(x) = 1$.) This contradicts to the commutativity of $Q_i$ and $f_*$.

The case $A = \mathbb{Z}/p^t$, $t \geq 2$ is proved similarly, since for $\alpha' \in H^*(X; A)$ we see $\beta\alpha' = 0 \in H^*(X; \mathbb{Z}/p)$. Thus we have this lemma. $\square$

We will extend the above Lemma 3.4 to $s > 4$, by using $MU$-theory of Eilenberg-MacLane spaces. Recall that $K = K(\mathbb{Z}, n)$ is the Eilenberg-MacLane space such that the homotopy group $[X, K] \cong H^n(X;\mathbb{Z})$, i.e., each element $x \in H^n(X;\mathbb{Z})$ is represented by a homotopy map $x : X \to K$. Let $\eta_n \in H^n(K;\mathbb{Z})$ corresponding the identity map. We know the image $\rho(MU^*(K)) \subset H^*(K;\mathbb{Z})/p$ by Tamanoi.

**Lemma 3.5.** *(*[Ta], [Ra-Wi-Ya]*) Let $K = K(\mathbb{Z}, n)$ We have the isomorphism*

$$\rho : MU^*(K) \otimes_{MU^*} \mathbb{Z}/p \cong \mathbb{Z}/p[Q_{i_1}...Q_{i_{n-2}}\eta_n | 0 < i_1 < ... < i_{n-2}]$$

*where the notation $\mathbb{Z}/p[a, ...]$ exactly means $\mathbb{Z}/p[a, ...]/(a^2 | \ |a| = odd)$.*

The following lemma is an extension of Lemma 3.4 for $s > 4$. (Here we use $MU^*$-theory, and we assume $H^*(-;A)$ is the *Betti* cohomology.)

**Lemma 3.6.** *Suppose that $H^*(X;A)$ is the Betti cohomology. Let $\alpha \in N^cH^{n+2c}(X)$, $n \geq 2$, $c \geq 1$. Suppose that there is a sequence $0 < i_1 < ... < i_{n-1}$ with*

$$Q_{i_1}...Q_{i_{n-1}}\alpha \neq 0 \quad in\ H^*(X;\mathbb{Z}/p).$$

*Then $D^cH^*(X)/p = N^cH^*(X)/(p, \tilde{N}^cH^*(X)) \supset \mathbb{Z}/p\{\alpha\}$.*

*Proof.* Suppose $\alpha \in \tilde{N}^cH^{n+2c}(X)$, i.e. there is a smooth $Y$ of $dim(Y) = dim(X) - c$ with $f : Y \to X$ such that the transfer $f_*(\alpha') = \alpha$ for $\alpha' \in H^n(Y)$.

Let $r : H^*(X) \to H^*(X;\mathbb{Z}/p)$ be the reduction map. We consider the commutative diagram for $I = (i_1, ..., i_{n-2})$ and $j = i_{n-1}$

$$\begin{array}{ccc}
\alpha' \in H^n(Y) & \xrightarrow{f_*} & \alpha \in H^{n+2c}(X) \\
{\scriptstyle Q_I r}\downarrow & & {\scriptstyle Q_I r}\downarrow \\
Q_I(\alpha') \in Im(\rho|MU^*(Y)) & \xrightarrow{f_*} & H^*(X;\mathbb{Z}/p) \\
{\scriptstyle Q_j}\downarrow & & {\scriptstyle Q_j}\downarrow \\
0 = Q_{i_{n-1}}Q_I(\alpha') \in H^*(Y;\mathbb{Z}/p) & \xrightarrow{f_*} & Q_{i_{n-1}}Q_I(\alpha) \in H^*(X;\mathbb{Z}/p).
\end{array}$$

Identify the map $\alpha' : Y \to K$ with $\alpha' = (\alpha')^*\eta_n$. We still see from Lemma 3.5,

$$Q_I(\alpha') = Q_{i_1}...Q_{i_{n-2}}((\alpha')^*\eta_n) \in Im(\rho : MU^*(Y) \to H^*(Y;\mathbb{Z}/p)).$$

From Lemma 3.3, we see

$$Q_{i_{n-1}}Q_I(\alpha') = Q_{i_{n-1}}Q_{i_1}...Q_{i_{n-2}}(\alpha') = 0 \in H^*(Y;\mathbb{Z}/p).$$

Therefore $Q_{i_{n-1}}Q_I(\alpha)$ must be zero by the commutativity of $f_*$ and $Q_i$. $\square$

## 4. ABELIAN $p$-GROUPS

At first, we assume $H^*(X)$ is the *Betti* cohomology so that the main lemma (Lemma 3.6) holds. However we will see the most *irrational* results hold for each $k \subset \mathbb{C}$.

From the main lemma, we have

**Lemma 4.1.** *Let $\alpha \in N^1H^{n+2}(X)$ and $Q_I(\alpha) \neq 0 \in H^*(X;\mathbb{Z}/p)$ for some $I = (0 < i_1 < ... < i_{n-1})$. Let $X'$ be a smooth projective variety. Then*

$$DH^*(X \times X')/p \supset \mathbb{Z}/p\{\alpha \otimes 1\}.$$

*Hence $X \times X'$ is not stable rational,*

*Proof.* The (Betti) cohomology $H^*(X;\mathbb{Z}/p)$ satisfies the Kunneth formula. Hence we have

$$Q_I(\alpha \otimes 1) = Q_I(\alpha) \otimes 1 \neq 0 \ \ in \ \ \sum_{s=0} H^{*-s}(X;\mathbb{Z}/p) \otimes H^s(X';\mathbb{Z}/p).$$

From the main lemma, we have the lemma. □

Let $G_n = \mathbb{Z}/p^n$. Recall the *mod*$(p)$ cohomology (for $p$ odd)

$$H^*(BG_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ....y_n] \otimes \Lambda(x_1, ..., x_n)$$

where $|x_i| = 1$ and $Q_0(x_i) = y_i$, (for $p = 2$, $x_i^2 = y_i$).

**Corollary 4.2.** *For $n \geq 3$, let $G_n = (\mathbb{Z}/p)^n$. Then $X_{G_n}$ is not stable rational. Moreover $X_{G_n}$ and $X_{G_{n'}}$ are not stable birational equivalent for $n \neq n'$.*

*Proof.* Take $G = G_3$ and $\alpha = Q_0(x_1x_2x_3) \in H^4(X_G)$. The last statement follows from $1 \otimes ... \otimes 1 \otimes \overset{s}{\alpha} \otimes 1 \otimes ... \otimes 1 \neq 0 \in DH^*(X_{G_n})$. □

We can take also another $\alpha$ for the proof of the last statement in the above corollary.

**Lemma 4.3.** *Take $G = G_n = (\mathbb{Z}/p)^n$ and $\alpha_i = Q_0(x_1x_2...x_i) \in H^{n+1}(X_G)$. Then we have*

$$DH^*(X_{G_n}) \supset \mathbb{Z}/p\{\alpha_2, \alpha_3, ..., \alpha_n\}.$$

Since $\alpha_n = 0$ in $H^*(X_{G_{n-1}})$ we also see that $X_n$ and $X_{n-1}$ are not stable birational equivalence.

The more detailed expression of $DH^*(X)/p$ seems somewhat complicated.

**Theorem 4.4.** *Let $G = (\mathbb{Z}/p)^n$. Then we have (for fixed large $N$)*

$$DH^{s+1}(X)/p \cong \mathbb{Z}/p\{Q_0(x_{i_1}...x_{i_s}) | 1 \leq i_1 < ..., < i_s \leq n\}.$$

*Proof.* The integral cohomology (modulo $p$) is isomorphic to

$$H^*(BG)/p \cong Ker(Q_0)$$
$$\cong H(H^*(BG;\mathbb{Z}/p);Q_0) \oplus Im(Q_0)$$

where $H(-;Q_0) = Ker(Q_0)/Im(Q_0)$ is the homology with the differential $Q_0$. It is immediate that $H(H^*(B\mathbb{Z}/p;\mathbb{Z}/p);Q_0) \cong \mathbb{Z}/p$. By the Künneth formula, we have $H(H^*((BG;\mathbb{Z}/p);Q_0) \cong (\mathbb{Z}/p)^{n\otimes} \cong \mathbb{Z}/p$. Hence we have

$$H^*(BG)/p \cong \mathbb{Z}/p\{1\} \oplus Im(Q_0)$$
$$\cong \oplus_s \mathbb{Z}/p[y_1,...y_n](1, Q_0(x_{i_1}....x_{i_s})|1 \leq i_1 < ... < i_s \leq n)$$

where the notation $R(a,...,b)$ (resp. $R\{a,...,b\}$) means the $R$-submodule (resp. the free $R$-module) generated by $a,...,b$. Here we note $H^+(BG)$ is just $p$-torsion.

Also note that $y_1,...,y_n$ are represented by the Chern classes $c_1$. From Lemma 2.3, we see $Ideal(y_1,...,y_n) = 0 \in DH^*(X)$.

We know $Q_i(x_j) = y_j^{p^i}$ and $Q_j$ is a derivation. We have the theorem from Lemma 4.3 and the reciprocity law

$$Q_{i_1}...Q_{i_{s-2}}Q_0(x_{i_1}...x_{i_s}) = y_{i_1}^{p^{i_1}}...y_{i_{s-2}}^{p^{i_{s-2}}}y_{i_{s-1}}x_{i_s} + ... \neq 0.$$

(Note the $n = |\alpha'|$ in Lemma 4.3 is written by $s-1$ here.) $\square$

**Corollary 4.5.** *If $n \neq n' \geq 3$, then $X(N)_n$ and $X(N)_{n'}$ are not stable birational equivalent.*

The above corollary also holds when $ch(k) = 0$ and $k$ is an algebraic closed field by the base change theorem.

For each field $k = \bar{k}$, it is known from Voevodsky (for $p; odd$)

$$H^{*,*'}(BG_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,....y_n,\tau] \otimes \Lambda(x_1,...,x_n)$$

where $deg(x_i) = (1,1)$ and $Q_0(x_i) = y_i$. Therefor we can identify

$$Q_0(x_1...x_m) \in H^*_{\acute{e}t}(BG_n;\mathbb{Z}/p) \quad when\ \bar{k} = k.$$

Let us write $H^*_{\acute{e}t}(X;\mathbb{Z}_p)$ simply by $H^*_{\acute{e}t}(X)$. Let $G$ be an algebraic group which has an approximation $X_G$ such that

$$H^*_{\acute{e}t}(X_G;\mathbb{Z}_p) \cong H^*(BG \times \mathbb{P}^\infty) \otimes \mathbb{Z}_p \quad for * < N$$

We consider the maps

$$\psi\ :\ N^1H^*_{\acute{e}t}(X) \subset H^*_{\acute{e}t}(X) \to H^*_{\acute{e}t}(\bar{X}) \to\ H^*_{\acute{e}t}(X(\mathbb{C})) \to H^*(X(\mathbb{C})).$$

**Lemma 4.6.** *Let $k \subset \mathbb{C}$ (not assumed an algebraic closed field). Let $\alpha \in N^1H^*_{\acute{e}t}(X)$ and $Q_I(\psi(\alpha)) \neq 0 \in H^*(X(\mathbb{C});\mathbb{Z}/p)$. Then*

$$DH^*_{\acute{e}t}(X;\mathbb{Z}_p)/p \supset \mathbb{Z}/p\{\alpha\}.$$

*Hence $X$ is not stable rational.*

*Proof.* By the assumption, the main lemma implies $DH^*(X(\mathbb{C})) \supset \mathbb{Z}/p\{\psi\alpha\}$.

This implies a contradiction if $\mathbb{Z}/p\{\alpha\} = 0$ in $DH^*_{ét}(X)$. Similarly, the stable rationality for $X$ imlplies that for $X(\mathbb{C})$, which is a contradiction. (Note here, we do *not* assume of the stable birational invariance for $DH^*_{ét}(X)$.) □

For example, Lemma 4.3 holds for all $k \subset \mathbb{C}$.

## 5. CONNECTIVE GROUPS, $SO_n$

Let $SO_n$ be the special orthogonal group. Its mod(2) cohomology is

$$H^*(BSO_n; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, ..., w_n]$$

where $w_i$ is the Stiefel-Whitney class for $SO_n \subset O_n$. We know $Q_0 w_{2m} = w_{2m+1}$.

**Theorem 5.1.** *([Ya6]) Let $X_n = X_n(N)$ be approximations for $BSO_n$ for $n \geq 3$. Moreover, let $|Q_1...Q_{2m-1}(w_{2m+1})| < N$. Then we have*

$$DH^*(X_{2m+1}) \supset \mathbb{Z}/2\{w_3, w_5, ..., w_{2m+1}\} \quad for\ all\ 2m+1 \leq * < N.$$

**Remark.** When $G = SO_3$, the inclusion in the above theorem is isomorphic. However, when $G = SO_5$, we can not see whether $Q_0(w_2w_4) \in H^7(X)$ is zero or not in $DH^7(X)/2$.

Let $G = SO_5$. Indeed, we can see the homology by $Q_0$ is given

$$H(H^*(BG; \mathbb{Z}/2); Q_0) \cong \mathbb{Z}/2[c_2, c_4] \quad where\ c_i = w_i^2,$$

$$Im(Q_0) \cong \mathbb{Z}/2[c_2, c_3, c_4, c_5](Q_0(w_2).Q_0(w_4), Q_0(w_2w_4)).$$

Hence $H^*(BG)/2$ is generated by $1, w_3, w_5.Q_0(w_2w_4)$ as a $\mathbb{Z}/2[c_2, c_3, c_4, c_5]$-module. Hence we have

**Lemma 5.2.** *Let $G = SO_5$. There ie a surjection*

$$\mathbb{Z}/2\{w_3.w_5, Q_0(w_2w_4)\} \twoheadrightarrow DH^*(X_G)/2.$$

**Corollary 5.3.** *Let $X_n = X_n(N)$ be approximation for $BSO_n$ for $n \geq 3$. For $m \neq m'$, we see that $X_{2m+1}$ and $X_{2m'+1}$ are not stable rational equivalence.*

The above corollary holds for all $k \subset \mathbb{C}$, by the similar arguments done in the last places in the preceding section.

## 6. SIMPLY CONNECTED SIMPLE GROUPS

We next consider simply connected groups. Let us write by $X$ an approximation for $BG_2$ for the exceptional simple group $G_2$ of $rank = 2$. The $mod(2)$ cohomology is generated by the Stiefel-Whitney classes $w_i$ of the real representation $G_2 \to SO_7$

$$H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7], \quad P^1(w_4) = w_6,\ Q_0(w_6) = w_7,$$

$$H^*(BG_2) \cong (D' \oplus D'/2[w_7]^+) \quad where \ \ D' = \mathbb{Z}[w_4, c_6].$$

Then we have $Q_1 w_4 = w_7, Q_2(w_7) = w_7^2 = c_7$ (the Chern class).

The Chow ring of $BG_2$ is also known

$$CH^*(BG_2) \cong (D\{1, 2w_4\} \oplus D/2[c_7]^+) \quad where \ \ D = \mathbb{Z}[c_4, c_6] \ \ c_i = w_i^2.$$

In particular the cycle map $cl : CH^*(BG) \to H^*(BG)$ is injective.

It is known [Ya6] that $w_4 \in N^1H^*(X;\mathbb{Z}/2)$ and moreover we can identify $w_4 \in N^1H^*(X)$. Since $Q_1(w_4) = w_7 \neq 0$, from Lemma 4.1, we have $DH^4(X) \neq 0$. This fact is also written in [Be-Ot]. Moreover the isomorphism $H^*(BG)/(c_4, c_6, c_7) \cong \Lambda(w_4, w_7)$ implies

**Proposition 6.1.** *(*[Ya6]*) For $X$ an approximation for $BG_2$, we have the surjection*

$$\Lambda(w_4, w_7)^+ \twoheadrightarrow DH^*(X)/2 \quad for\ all\ * < N.$$

**Remark.** We can not see $w_7, w_4w_7 = 0$ or nonzero in $DH^*(X)/2$.

The cohomology of other simply connected simple groups (with 2-torsion) are written for example

$$H^*(BSpin_7;\mathbb{Z}/2) \cong H^*(BG_2;\mathbb{Z}/2) \otimes \mathbb{Z}/2[w_8],$$

$$H^*(BSpin_8;\mathbb{Z}/2) \cong H^*(BG_2;\mathbb{Z}/2) \otimes \mathbb{Z}/2[w_8, w_8'], ...$$

For the above groups $G$, there are the map $j : G_2 \to G$ and the non zero element $w \in H^*(G)$ such that $j^*w = w_4$.

**Proposition 6.2.** *(*[Ya6]*) Let $G$ be a simply connected group such that $H^*(BG)$ has $p$-torsion. Let $X = X(N)$ be an approximation for $BG$ for $N \geq 2p+3$. Then there is $w \in H^4(X)$ such that*

$$DH^4(X)/p \supset \mathbb{Z}/p\{w\}$$

*Hence these $X$ are not stable rational.*

*Proof*. It is only need to prove the theorem when $G$ is a simple group having $p$ torsion in $H^*(BG)$. Let $p = 2$. It is well known that there is an embedding $j : G_2 \subset G$ such that (see [Pi-Ya], [Ya5] for details)

$$H^4(BG) \overset{j^*}{\cong} H^4(BG_2) \cong \mathbb{Z}\{w_4\}.$$

Let $w = (j^*)^{-1}w_4 \in H^4(BG)$. From Lemma 3.1 in [Ya5], we see that $2w$ is represented by Chern classes. Hence $2w$ is the image from $CH^*(X)$, and so $2w \in N^1H^4(X)$. This means there is an open set $U \subset X$ such that $2w = 0 \in H^*(U)$ that is, $w$ is 2-torsion in $H^*(U)$. Hence from Lemma 2.4, we have $w \in N^1H^4(U)$, and so there is $U' \subset U$ such that $w = 0 \in H^4(U')$. This implies $w \in N^1H^4(X)$.

Since $j^*(Q_1x) = Q_1w_4 = w_7$, we see $Q_1w \neq 0$. From the main lemma (Lemma 4.1), we see $DH^4(X) \neq 0$ for $G$.

For the cases $p = 3, 5$, we consider the exceptional groups $F_4, E_8$ respectively. Each simply connected simple group $G$ contains $F_4$ for $p = 3$, $E_8$ for $p = 5$. There is $w \in H^4(BG)$ such that $px$ is a Chern class [Ya5], and $Q_1w) \neq 0 \in H^*(BG; \mathbb{Z}/p)$. In fact, there is embedding $j : (\mathbb{Z}/p)^3 \subset G$ with $j^*(w) = Q_0(x_1x_2x_3)$. Hence we have the theorem.

**Corollary 6.3.** *Let $X$ be an approximation for $BSpin_n$ with $n \geq 7$ or $BG$ for an exceptional group $G$. Then $X$ is not stable rational.*

## 7. RETRACT BIRATIONAL AND UNRAMIFIED COHOMOLOGY

Here we note the relations to retract rationally. Recall (in §2) that Bloch-Ogus give a spectral sequence such that its $E_2$-term is given by

$$E(c)_2^{c,*-c} \cong H^c_{Zar}(X, \mathcal{H}^{*-c}_A) \Longrightarrow H^*_{\acute{e}t}(X; A).$$

By Orlov-Vishik-Voevodsky [Or-Vi-Vo], ([Te-Ya2] for $p : odd$,) we know

**Lemma 7.1.** *([Or-Vi-Vo], [Vo5]) There is the long exact sequence*

$$H^{m-n-1}_{Zar}(X; \mathcal{H}^n_{\mathbb{Z}/p}) \to H^{m,n-1}(X; \mathbb{Z}/p) \stackrel{\times\tau}{\to} H^{m,n}(X; \mathbb{Z}/p)$$
$$\to H^{m-n}_{Zar}(X; \mathcal{H}^n_{\mathbb{Z}/p}) \to H^{m+1,n-1}(X; \mathbb{Z}/p) \stackrel{\times\tau}{\to} ....$$

In particular, when $m = n$, the first $\times\tau$ is injective.

**Corollary 7.2.** *We have the short exact sequence*

$$0 \to H^{*,*}(X; \mathbb{Z}/p)/(\tau) \to H^0_{Zar}(X; \mathcal{H}^*_{\mathbb{Z}/p})$$
$$\to Ker(\tau : H^{*+1,*-1}(X; \mathbb{Z}/p) \to H^{*+1,*}(X; \mathbb{Z}/p)) \to 0.$$

(Note $H^{*,*}(X; \mathbb{Z}/p)/(\tau) \cong H^*(X; \mathbb{Z}/p)/(N^1H^*(X; \mathbb{Z}/p))$. Hence we also write it as $H^*(X; \mathbb{Z}/p)/N^1$. This cohomology is called a stable cohomology and studied well by Bogomolov [Bo], [Te-Ya2]

**Remark.** The $\mathbb{Z}/2^s$ coffeciants version of Lemma 7.1, Corollary 7.2 also hold.

The unramified cohomology is written by this $H^0_{Zar}(X; \mathcal{H}^*_{\mathbb{Z}/p})$, when $X$ is complete,

$$H^*_{ur}(X; \mathbb{Z}/p) = H^*_{ur}(k(X); \mathbb{Z}/p) \cong H^0_{Zar}(X; \mathcal{H}^*_{\mathbb{Z}/p}),$$

and it is an invariant of the retract rationality of $X$ (Lemma 3.1, 3.4 [Me]).

By Totaro [Ga-Me-Se], the cohomological invariant of $G$ is written (while $BG$ is not complete)

$$Inv^*(G;\mathbb{Z}/p) \cong H^0_{Zar}(BG;\mathcal{H}^*_{\mathbb{Z}/p}),$$

Here we consider the following lemma which shows the relation among $DH^*(X_G)$, $Inv^*(G)$ and $H^*_{ur}(X_G)$.

**Lemma 7.3.** *Assume that $0 \neq x \in H^m(BG;\mathbb{Z}/p)/(N^1)$ and $x$ is dedicated by $A_m = (\mathbb{Z}/p)^m$ i.e. $res_{/N}(x) \neq 0$ for the restriction (of stable cohomologies)*

$$res_{/N} : H^*(BG;\mathbb{Z}/p)/N^1 \to H^*(BA_m;\mathbb{Z}/p)/N^1 \cong \Lambda(x_1, ..., x_m).$$

*Then (for projective approximation $X$ for $BG$) we have*

$$\begin{cases} Inv^*(G;\mathbb{Z}/p) \supset \mathbb{Z}/p\{x\}, \\ H^*_{un}(X;\mathbb{Z}/p) \supset \mathbb{Z}/p\{x\}, \\ DH^*(X)/p \supset \mathbb{Z}/p\{Q_0(x)\}. \end{cases}$$

*Proof.* The first formula follows from

$$Inv^*(G;\mathbb{Z}/p) \cong H^0(BG;\mathcal{H}^*_{\mathbb{Z}/p}) \supset H^*(BG;\mathbb{Z}/p)/N^1.$$

The fact $x \neq 0$ in $Inv^*(G;\mathbb{Z}/p)$ follows from that $x$ is dedicated.

The second formula comes from $H^*_{ur}(X;\mathbb{Z}/p) \cong H^0(X;\mathcal{H}^*_{\mathbb{Z}/p})$ where $X$ is smooth projective.

The last formula follows from the main lemma (Lemma 3.4). Let $Q_0(x) = \alpha \in \tilde{N}^c H^{n+2c}(X)$, $(m = n+2c-1)$, i.e. there is a smooth $Y$ of $dim(Y) = dim(X) - c$ with $f : Y \to X$ such that the transfer $f_*(\alpha') = \alpha$ for $\alpha' \in H^n(Y)$.

Identify the map $\alpha' : Y \to K$ with $\alpha' = (\alpha')^*\eta_n$. We still see from Lemma 3.5,

$$Q(\alpha') = Q_{i_1}...Q_{i_{n-2}}((\alpha')^*\eta_n) \in Im(MU^*(Y) \to H^*(Y;\mathbb{Z}/p)).$$

From Lemma 3.4, we see

$$Q_{i_{n-1}}Q(\alpha') = Q_{i_{n-1}}Q_{i_1}...Q_{i_{n-2}}(\alpha') = 0 \in H^*(Y;\mathbb{Z}/p).$$

Therefore $Q_{i_{n-1}}Q(\alpha)$ must be zero by the commutativity of $f_*$ and $Q_i$. But $Q_{i_1}...Q_{i_{n-1}}Q_0(x) \neq 0$ from the assumption that $x$ is deduced from $A_{n+1}$. In fact in $H^*(BA_{n+1};\mathbb{Z}/p)$, we see (without $mod(N^1)$)

$$Q_{i_1}...Q_{i_{n-1}}Q_0(x_1...x_{n+1}) = y_1^{p^{i_1}}...y_{n-1}^{p^{i_{n-1}}}y_nx_{n+1} + ... \neq 0.$$

$\square$

Now we consider the examples. At first, we consider the case $G = A_n = (\mathbb{Z}/p)^n$. and $X = X_G$. It is known from Garibarldy-Merkurjev-Serre [Ga-Me-Se], Theorem 6.3 in [Te-Ya2] that

$$Inv^*(G;\mathbb{Z}/2) \cong H^{*,*}(BG;\mathbb{Z}/2)/(\tau) \cong \Lambda(x_1,...,x_n).$$

Since $X$ is (proper) approximation of $BG$, we have

**Theorem 7.4.** *Let $G = G_n = (\mathbb{Z}/p)^n$ and $X = X_G$. Then*

$$H^{2*}_{ur}(X;\mathbb{Z}/p) \supset H^{2*,2*}(X;\mathbb{Z}/p)/(\tau) \cong \Lambda(x_1,x_2,...,x_n)$$

*in Corollary 7.2.*

Writing $\alpha_i = Q_0(x_1...x_i)$, we still have (Lemma 4.3)

$$DH^*(X)/p \supset \mathbb{Z}/p\{\alpha_2,\alpha_3,...,\alpha_n\}.$$

Then $X_{G_n}$ and $X_{G_{n'}}$ are not retract rational equivalent if $n \neq n'$.

**Remark.** From ( Saltman [Sa]) , it is well known that there is a finite group $G$ (e.g. $|G| = p^7$, $p : odd$)) such that

$$0 \neq x_2 \in H^2_{ur}(k(W)^G;\mathbb{Z}/p) \cap H^{2,2}(BG;\mathbb{Z}/p)/(\tau) \quad .$$

Here $G$ acts freely on a $\mathbb{C}$-vector space $W$, and we have

$$H_{ur}(K(W)^G;\mathbb{Z}/p) \cong H_{Zar}(W//G;\mathcal{H}^*_{\mathbb{Z}/p}) \subset H_{Zar}(BG;\mathcal{H}^*_{\mathbb{Z}/p})$$

such that $k(W//G) \cong k(W)^G$. Hence $H^*_{un}(k(W)^G;\mathbb{Z}/p) \not\cong H^*(k;\mathbb{Z}/p)$. So $k(W)^G$ is not purely transcendent over $k$. (Hence $BG$ is not retract rational.)

**Remark.** We do *not* assume $H^0_{Zar}(X;\mathcal{H}^*_{\mathbb{Z}/p}) \cong H^0_{Zar}(X';\mathcal{H}^*_{\mathbb{Z}/p})$ for an other approximation $X'$.

Next we consider the case $G = SO_{2m+1}$ and $X = X_{G_m}$. It is known from Garibarldy-Merkurjev-Serre [Ga-Me-Se], Theorem 6.3 in [Te-Ya] that

$$Inv^*(G;\mathbb{Z}/2) \cong H^{*,*}(BG;\mathbb{Z}/2)/(\tau) \cong \mathbb{Z}/2\{1,w_2,...,w_{2m}\}.$$

Since $X$ is (proper) approximation of $BG$, we have

**Theorem 7.5.** *Let $G = SO_{2m+1}$ and $X = X_G$. Then*

$$H^{2*}_{ur}(X;\mathbb{Z}/p) \supset H^{2*,2*}(X;\mathbb{Z}/2)/(\tau) \supset \mathbb{Z}/2\{1,w_2,...,w_{2m}\}$$

*in Lemma 7.1.*

We also have (Theorem 5.2) $DH^*(X)/2 \supset \mathbb{Z}/2\{Q_0(w_2),...,Q_0(w_{2m})\}$. Hence $X_{G_m}$ and $X_{G_{m'}}$ are not retract rational if $m \neq m'$.

From Theorem 5.2 and the preceding theorem, we have

**Corollary 7.6.** *Let $G'_n = SO_n$ and $X = X_G$. Then $X_{G_n}$ and $X_{G_{n'}}$ are not retract rational if $n \neq n'$.*

*Proof.* By Serre [Ga-Me-Se], we know

$$Inv^*(BSO_{2m};\mathbb{Z}/2) \cong \mathbb{Z}/2\{1, w_2, ..., w_{2m-2}, u_{2m-1}\}$$

with $|u_{2m-1}| = 2m-1$. We see $X_{2m}$ and $X_{2m+1}$ are not retract rational since $w_{2m+1}$ is zero in the invariant for $BSO_{2m}$. We see $X_{2m-1}$ and $X_{2m}$ are not retract rational since $u_{2m}$ is zero in the invariant for $BSO_{2m-1}$. □

**Remark.** Kordonskii [Ko], Merkurjev (Corollary 5.8 in [Me]), and Reichstein-Scavia show [Re-Sc] that $BSpin_n$ itself is stably rational when $n \le 14$. These facts imply that the (Ekedahl) approximation $X$ is not stable rationally equivalent to $BG$. (The author thanks Federico Scavia who pointed out this remark.)

At last of this section, we consider the case $G = PGL_p$ projective general linear group. We have (for example Theorem 1.5,1.7 in [Ka-Ya]) additively

$$H^*(BG;\mathbb{Z}/p) \cong M \oplus N \quad with \quad M \overset{add.}{\cong} \mathbb{Z}/p[x_4, x_6, ..., x_{2p}],$$
$$N = SD \otimes \Lambda(Q_0, Q_1)\{u_2\} \quad with \quad SD = \mathbb{Z}/p[x_{2p+2}, x_{2p^2-2p}]$$

where $x_{2p+2} = Q_1Q_0u_2$ and suffix means its degree. The Chow ring is given as

$$CH^*(BG)/p \cong M \oplus SD\{Q_0Q_1(u_2)\}.$$

From Lemma 7.3, we have :

**Theorem 7.7.** *Let $p$ be odd. For an approximation $X$ for $BPGL_p$, we see*

$$DH^*(X)/p \supset \mathbb{Z}/p\{Q_0u_2\},$$
$$H^*_{un}(X;\mathbb{Z}/p) \supset \mathbb{Z}/p\{1, u_2\}, \qquad Inv^*(G;\mathbb{Z}/p) \supset \mathbb{Z}/p\{1, u_2\}.$$

In the above case, we do not see here that $DH^*(X)$ for $* < N$ is invariant of $BG$, (under taking another $X'$ as approximations for $G$).

## 8. Retract rational for simply connected $G$

We will see that simply connected groups $G$ satisfy the similar facts, but such as $Ker(\tau|H^{*+1,*-1}(X;\mathbb{Z}/p)) \ne 0$ in Lemma 7.1. In §6, we see there is $0 \ne w \in H^4(X)$ such that $DH^4(X)/p \supset \mathbb{Z}/p\{w\}$. We will see that this $w$ corresponds a nonzero element in $H^3_{un}(X;\mathbb{Z}/p)$.

**Theorem 8.1.** *(*[Ya5]*) Let $G$ be a simply connected simple group. Then there is the element (Rost invariant) such that*

$$H^3_{ur}(X;\mathbb{Z}/p) \twoheadrightarrow Ker(\tau|H^{4,2}(X;\mathbb{Z}/2)) \supset \mathbb{Z}/p\{w\}.$$

*Hence $X$ is not retract rational.*

*Proof.* We consider the following diagram

$$\begin{array}{ccc} H^*_{ur}(BG;\mathbb{Z}/p) & \xrightarrow{j^*} & H^*_{ur}(X;\mathbb{Z}/p) \\ (1)\downarrow & & (2)\ \cong\downarrow \\ Inv^*(G:\mathbb{Z}/p)\cong H^0(BG;\mathcal{H}^*_{\mathbb{Z}/p}) & \xrightarrow{j^*} & H^0(X;\mathcal{H}^*_{\mathbb{Z}/p}) \twoheadrightarrow Ker(\tau). \end{array}$$

Here $H^+_{un}(BG;\mathbb{Z}/p)=0$ when $BG$ is retract rational. (The map (1) need not isomorphism.) We see that the map (2) : $H^*_{ur}(X;\mathbb{Z}/p)\cong H^0(X;\mathcal{H}^*_{\mathbb{Z}/p})$ because $X$ is projective and smooth. Recall Lemma 7.1 that we have the surjection

$$H^0(X;\mathcal{H}^*_{\mathbb{Z}/p})\twoheadrightarrow Ker(\tau|H^{*+1,*-1}(X:\mathbb{Z}/p)).$$

Hereafter, we consider the case $*=3$. We consider the following commutative diagram.

$$\begin{array}{ccccc} pw\in H^{4,4}(BG;\mathbb{Z}/p^2) & \xrightarrow{j^*\ \cong} & pw\in H^{4,4}(X;\mathbb{Z}/p^2) & \longrightarrow & 0\in H^{4.4}(X;\mathbb{Z}/p) \\ \tau'\uparrow & & \tau'\uparrow & & \tau\ (inj.)\uparrow \\ c_2'\in H^{4,3}(BG;\mathbb{Z}/p^2) & \xrightarrow{j^*} & c_2''\in H^{4,3}(X;\mathbb{Z}/p^2) & \longrightarrow & 0\in H^{4,3}(X;\mathbb{Z}/p) \\ \tau'\uparrow & & \tau'\uparrow & & \tau\uparrow \\ c_2'\in H^{4,2}(BG;\mathbb{Z}/p^2) & \xrightarrow{j^*} & c_2''\in H^{4,2}(X;\mathbb{Z}/p^2) & \xrightarrow{r} & c'''\in H^{4.2}(X:\mathbb{Z}/p) \end{array}$$

From the proof in Proposition 6.2, we see that there is $c_2'\in H^{4,2}(BG;\mathbb{Z}/p^2)$ so that (for $\tau':H^{*,*'}(X;\mathbb{Z}/p^2)\to H^{*,*'+1}(X;\mathbb{Z}/p^2)$) we have

$$(\tau')^2c_2'=pw\in H^{4,4}(BG;\mathbb{Z}/p^2).$$

(In fact $pw$ is represented by a Chern class, but $w$ itself is not in the image of the cycle map.)

Next take $c''=j^*c_2'\in H^{4,3}(X;\mathbb{Z}/p^2)$. Since $j$ is a projective approximation, we have

$$H^{4,4}(BG;\mathbb{Z}/p^2)\cong H^{4,4}(X;\mathbb{Z}/p^2).$$

Here $(\tau')^2c''=pw$. Hence $c''\neq 0\in H^{4,2}(X;\mathbb{Z}/p^2)$.

Let us write by $c'''$ the image of $c''$ in $H^{4,2}(X;\mathbb{Z}/p)$. We note $c'''\in Ker(\tau)|H^{4,2}(X;\mathbb{Z}/p)$, because $\tau:H^{4,3}(X;\mathbb{Z}/p)\to H^{4,4}(X;\mathbb{Z}/p)$ is injective from [Or-Vi-Vo].

Moreover, $c'''$ is a module generator in $Ker(\tau)$, in fact if $c''=px$, then $\tau^2x=w$ which is not $Ker(\tau)$.

Hence there is $a\in H^3_{ur}(X;\mathbb{Z}/p)$ which corresponds $c'''\in Ker(\tau|H^{4,2}(X;\mathbb{Z}/p)$. $\square$

**Corollary 8.2.** *Let $G$ be a simply connected group having $p$-torsion in $H^*(BG)$, and $X = X_G$ be a projective approximation for $BG$. Then $H^3_{ur}(X;\mathbb{Z}/p) \neq 0$ and so $X$ is not retract rational.*

In the last of this section, we consider the case $G = F_4$, $p = 2$ the exceptional simple group of rank 4. By [Ga-Me-Se], the cohomology invariant is known

$$Inv^*(G;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1, u_3, f_5\} \quad |u_3| = 3,\ |f_5| = 5.$$

Since $H^{5,5}(BG;\mathbb{Z}/2) = 0$, we know $f_5$ corresponds

$$0 \neq x \in Ker(\tau|H^{6,4}(BG;\mathbb{Z}/2)) \twoheadleftarrow H^0(BG;\mathcal{H}^5_{\mathbb{Z}/2}).$$

But we can *not* say here that $0 \neq x \in H^{6,4}(X;\mathbb{Z}/2)$.

**Proposition 8.3.** *If there is an approximation such that $H^{6,4}(BG;\mathbb{Z}/2) \cong H^{6,4}(X;\mathbb{Z}/2)$, then*

$$H^*_{ur}(X;\mathbb{Z}/2) \supset \mathbb{Z}/2\{u_3, f_5\}.$$

*Hence if the assumption is correct. then $X_{G_2}$ and $X_{F_4}$ are not retract rational equivalent.*

## 9. EXTRASPECIAL $p$-GROUPS

We assume at first that $p$ is an odd prime. The extraspecial $p$-group $E(n) = p_+^{1+2n}$ is the group such that exponent is $p$, its center is $C \cong \mathbb{Z}/p$ and there is the extension

$$0 \to C \to E(n) \stackrel{\pi}{\to} V_n \to 0$$

with $V = \oplus^{2n}\mathbb{Z}/p$. (For details of the cohomology of $E(n)$ see [Te-Ya1].) We can take generators $a_1, ..., a_{2n}, c \in E(n)$ such that $\pi(a_1), .., \pi(a_{2n})$ (resp. $c$ ) make a base of $V_n$ (resp. $C$) such that commutators are

$$[a_{2i-1}, a_{2i}] = c \quad and \quad [a_{2i-1}, a_j] = 1 \ \ if\ j \neq 2i.$$

We note that $E(n)$ is also the central product of the $n$-copies of $E(1)$

$$E_n \cong E(1) \cdots E(1) = E(1) \times_{\langle c\rangle} E(1) ... \times_{\langle c\rangle} E(1).$$

Take cohomologies

$$H^*(BC;\mathbb{Z}/p) \cong \mathbb{Z}/p[u] \otimes \Lambda(z), \quad \beta z = u,$$

$$H^*(BV_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ..., y_{2n}] \otimes \Lambda(x_1, ...x_{2n}), \quad \beta x_i = y_i,$$

identifying the dual of $a_i$ (resp.$c$) with $x_i$ (resp. $z$). That means

$$H^1(E(n);\mathbb{Z}/p) \cong Hom(E(n);\mathbb{Z}/p) \ni x_i : a_j \mapsto \delta_{ij}.$$

The central extension is expressed by

$$f = \sum_{i=1}^{n} x_{2i-1}x_{2i} \in H^2(BV_n;\mathbb{Z}/p).$$

Hence $\pi^* f = 0$ in $H^2(BE(n);\mathbb{Z}/p)$. We consider the Hochshild-Serre spectral sequence

$$E_2^{*,*'} \cong H^*(BV_n;\mathbb{Z}/p)\otimes H^*(BC;\mathbb{Z}/p) \Longrightarrow H^*(BE(n);\mathbb{Z}/p).$$

Hence the first nonzero differential is $d_2(z) = f$ and the next differential is

$$d_3(u) = d_3(Q_0(z)) = Q_0(f) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1}.$$

In particular

$$E_4^{0,*} \cong \mathbb{Z}/p[y_1,...,y_{2n}]\otimes \Lambda(x_1,...x_{2n})/(f, Q_0(f)).$$

**Lemma 9.1.** *We have the inclusion*

$$\Lambda(x_1,...,x_{2n})/(f) \subset H^*(BE(n);\mathbb{Z}/p).$$

*Proof.* We consider similar group $E(n)'$ such that its center is $C \cong \mathbb{Z}/p$ and there is the extension

$$0 \longrightarrow C \xrightarrow{i} E(n)' \xrightarrow{\pi} V_n' \longrightarrow 0$$

but $V_n' = \oplus^{2n}\mathbb{Z}_p$ such that there is the quotient map $q : E(n)' \to E(n)$. We also consider the spectral sequence

$$E_2^{*,*'} \cong H^*(BV';\mathbb{Z}/p)\otimes H^*(BC;\mathbb{Z}/p) \Longrightarrow H^*(BE(n)';\mathbb{Z}/p).$$

Here $H^*(BV_n';\mathbb{Z}/p) \cong \Lambda(x_1,...x_{2n})$. The first nonzero differential is $d_2(z) = f$ but the second differential is

$$d_3(u) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1} = 0 \in \Lambda(x_1,...,x_{2n}).$$

Hence $E_4^{*,*'}$ is (multiplicatively) generated by $u$ and $x_i$ (permanent cycles). So $E_4^{*,*'} \cong E_\infty^{*,*'}$. Therefore we have

$$H^*(BE(n)';\mathbb{Z}/p) \cong \mathbb{Z}/p[u]\otimes \Lambda(x_1,...,x_{2n})/(f).$$

From the map $q^* : H^*(BE(n);\mathbb{Z}/p) \to H^*(BE(n)';\mathbb{Z}/p)$, we get the result. $\square$

However $H^*(BE(n);\mathbb{Z}/p)/(N^1) \ncong \Lambda(x_1,...,x_{2n})/(f)$, in fact, when $n = 1$, from Theorem 3.3 in [Ya6] we see

**Proposition 9.2.** *We have*

$$H^*(BE(1);\mathbb{Z}/p)/(N^1) \cong \mathbb{Z}/p\{1, x_1, x_2, a_1', a_2'\} \quad deg(a_i') = 2.$$

$$DH^*(X)/2 \cong \mathbb{Z}/2\{Q_0(a_1'), Q_0(a_2')\}.$$

**Lemma 9.3.** *Let $n \geq 2$. Then*

$$y_i^p y_j - y_i y_j^p \neq 0 \quad \in \ H^*(BE(n); \mathbb{Z}/p).$$

*Proof.* By the inclusion $E(2) \subset E(n)$ and induced quotient map

$$H^*(BE(n); \mathbb{Z}/p) \to H^*(BE(2); \mathbb{Z}/p)$$

we only need to see

$$y_1^p y_2 - y_1 y_2^p \neq 0 \quad \in H^*(BE(2); \mathbb{Z}/p) \otimes \bar{\mathbb{F}}_p$$

for the algebraic closure $\bar{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$.

Let $n = 2$. Note here

$$Q_i Q_0(f) = y_1^{p^i} y_2 - y_1 y_2^{p^i} + y_3^{p^i} y_4 - y_3 y_4^{p^i}$$
$$= y_2 \Pi_{\lambda \in \mathbb{F}_{p^i}} (y_1 - \lambda y_2) + y_4 \Pi_{\lambda \in \mathbb{F}_{p^i}} (y_3 - \lambda y_4).$$

Hence this formula $Q_i Q_0(f)$ is a sum of multiplies of

$$y_1^p y_2 - y_1 y_2^p = y_2 \Pi_{\lambda \in \mathbb{F}_p} (y_1 - \lambda y_2) \quad and \quad y_3^p y_4 - y_3 y_4^p,$$

Suppose that $y_1^p y_2 - y_1 y_2^p = 0$. Then by the symmetry of the group. we see $y_3^p y_4 - y_3 y_4^p = 0$. But it is known [Te-Ya1] $(Q_1 Q_0(f), Q_2 Q_0(f))$ is regular in $\mathbb{Z}/p[y_1, y_2, y_3.y_4]$. This is a contradiction. □

The more concrete expression of $DH^*(X)/p$ seems somewhat complicated. So we only give it for $* = 3$.

**Proposition 9.4.** *Let $G = E(n),\ n > 1$. Then we have*

$$DH^3(X)/p \cong \mathbb{Z}/p\{Q_0(x_i x_j) | (i,j) \neq (1,2),\ 1 \leq i < j \leq n\}.$$

$$H^2_{ur}(X; \mathbb{Z}/p),\ Inv^2(G : \mathbb{Z}/p) \supset \mathbb{Z}/p\{x_i x_j | (i,j) \neq (1,2),\ 1 \leq i < j \leq n\}.$$

*Proof.* The degree 3 integral cohomology $mod(p)$ $H^3(X)/p$ is generated as a $\mathbb{Z}/p[y_1 ...., y_n]$-module by $Q_0(x_i x_j)$. The proposition follows from the main lemma and

$$Q_1 Q_0(x_i x_j) = y_i^p y_j - y_i y_j^p \neq 0 \quad in \ H^*(X; \mathbb{Z}/p).$$

□

Bogomolov-Bohning-Pirutka study the kernel of the map

$$K = Ker(q^*_{/N^1} : H^*(BV_n; \mathbb{Z}/p)/N^1 \to H^*(BG; \mathbb{Z}/p)/N^1).$$

where $H^*(BV_n; \mathbb{Z}/p)/N^1 \cong \Lambda(x_1, ...., x_{2n})$. Their theorem in [Bo-Bo-Pi] induces

**Theorem 9.5.** *(Theorem 1.3 in [Bo-Bo-Pi]) If $p \geq n$, $G$ is extraspecial group of order $p^{1+2n}$ then $Ker(q^*_{/N^1}) \cong (f)$. Hence*

$$H^*_{ur}(X; \mathbb{Z}/p) \supset \Lambda(x_1, ..., x_{2n})/(f).$$

**Remark.** There is the another group $p_-^{1+2n}$ with the degree $2^n+1$.

When $p=2$, the situation becomes changed. The extraspecial 2-group $D(n)=2_+^{1+2n}$ in the $n$-th central extension of the dihedral group $D_8$ of order 8. It has the central extension

$$0 \to \mathbb{Z}/2 \to D(n) \to V_n \to 0$$

with $V_n=\oplus^{2n}\mathbb{Z}/2$. Hence $H^*(BV_n;\mathbb{Z}/2)\cong \mathbb{Z}/2[x_1,...,x_{2n}]$. Then using the Hochschild-Serre spectral sequence, Quillen proved [Qu]

$$H^*(BD(n);\mathbb{Z}/2)\cong \mathbb{Z}/2[x_1,...,x_{2n}]/(f,Q_0(f),...,Q_{n-2}(f))\otimes \mathbb{Z}[w_{2^n}(\Delta)].$$

Here $w_{2^n}(\Delta))$ is the Stiefel-Whitney class of $2^n$-dimensional (spin) representation $\Delta$ which restricts nonzero on the center. Moreover Quillen proves the following two theorems (Theorem 5.10-11 in [Qu])

**Theorem 9.6.** *([Qu]) $H^*(BD(n);\mathbb{Z}/2)$ is detected by the product of cohomology of maximal elementary abeian groups.*

**Theorem 9.7.** *([Qu]) The nonzero Stiefel-Whitney $w_i(\Delta)$ are those of degrees $2^n$ and $2^n-2^i$ for $0\le i<n$.*

In fact $w_i(\Delta)$ generates the Dickson algebra in the cohomology of the maximal elementary abelian 2-groups.

**Proposition 9.8.** *When $n>2$, there is the surjection*

$$\Lambda(x_1,...,x_{2n})/(f)\to H^*(BD(n);\mathbb{Z}/2)/(N^1).$$

*Proof.* By the same arguments with $p=odd$, we see

$$\Lambda=\Lambda(x_1,...,x_{2n})/(f)\subset H^*(BD(n);\mathbb{Z}/2).$$

The fact $w_2(\Delta)=0$ follows from the above third Quillen's theorem. Hence we have $w_{2^n}(\Delta)\in N^1$ from Becher's theorem (Theorem 6.2 in [Te-Ya2]). i.e., $w_i$ is multiplicative generated by $w_1$ and $w_2$. Thus we get the proposition. □

However this map (in Proposition 9.8) is not need injective. In fact, in [Bo-Bo-Pi], it is proven that the above map is not injective when $G=D(3)=2_+^{1+6}$. They also see that the map in the proposition is injective when we restrict the degree $*=2$

**Theorem 9.9.** *Let $G=D(3)$ and $X=X_G$. Then we have*

$$DH^3(X)/2\cong \mathbb{Z}/p\{Q_0(x_ix_j)|(i,j)\neq(1,2),\ 1\le i<j\le 3\}.$$

$$H^2_{ur}(X;\mathbb{Z}/p),\ Inv^2(G:\mathbb{Z}/p)\supset \mathbb{Z}/p\{x_ix_j|(i,j)\neq(1,2),\ 1\le i<j\le 3\}.$$

*However, the map in Proposition 9.8 is not injective for some $*>2$.*

## 10. The motivic cohomology of quadrics over $\mathbb{R}$ with coefficients $\mathbb{Z}/2$

Let $X$ be a smooth variety over the field $\mathbb{R}$ of real numbers, and we consider the cohomologies of $\mathbb{Z}/2$ coefficients. In this paper the $mod(2)$ étale cohomology means the motivic cohomology of the same first and the second degrees $H^*_{\acute{e}t}(X;\mathbb{Z}/2) \cong H^{*,*}(X.\mathbb{Z}/2)$.

It is well known ([Vo1], [Vo2])

$$H^*_{\acute{e}t}(Spec(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2, \quad H^{*,*'}(Spec(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau],$$

$$H^*_{\acute{e}t}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho], \quad H^{*,*'}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau,\rho]$$

where $0 \neq \tau \in H^{0,1}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2$ and where

$$\rho = -1 \in \mathbb{R}^*/(\mathbb{R}^*)^2 \cong K_1^M(\mathbb{R})/2 \cong H^1_{\acute{e}t}(Spec(\mathbb{R});\mathbb{Z}/2).$$

We recall the cycle map from the Chow ring to the étale cohomology

$$cl/2 : CH^*(X)/2 \to H^{2*}_{\acute{e}t}(X;\mathbb{Z}/2).$$

This map is also written as $H^{2*,*}(X;\mathbb{Z}/2) \stackrel{\times\tau^*}{\to} H^{2*,2*}(X;\mathbb{Z}/2)$.

Let $X = Q^d$ be an anisotropic quadric of dimension $2^n - 1$ (i.e. the norm variety for $(\rho^{n+1} \in K^M_{n+1}(\mathbb{R})/2)$). Then we have the Rost motive $M \subset Q^d$ [Ro]. It is known ( the remark page 575 in [Ya2])

$$H^*_{\acute{e}t}(M;\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]/(\rho^{2^{n+1}-1}) \cong \mathbb{Z}/2\{1,\rho,\rho^2,...,\rho^{2^{n+1}-2}\}.$$

The Chow ring is also known [Ro]

$$CH^*(M)/2 \cong \mathbb{Z}/2\{1,c_0,c_1....,c_{n-1}\}, \quad cl(c_i) = \rho^{2^{n+1}-2^{i+1}}.$$

The cycle map $cl/2$ is injective. The elements $c_i$ is also written as

$$c_i = \rho^{2^{n+1}-2^{i+1}}\tau^{-2^n+2^i} \quad in\ CH^*(M)/2 \subset H^{2*}_{\acute{e}t}(M:\mathbb{Z}/2)[\tau^{-1}]$$

The mod(2) motivic cohomology is known (Theorem 5.3 in [Ya2]).

**Theorem 10.1.** *(Theorem 5.3 in* [Ya2]*) The cohomology $H^{*,*'}(M_n;\mathbb{Z}/2)$ is isomorphic to the $\mathbb{Z}/2[\rho,\tau]$-subalgebra of*

$$\mathbb{Z}/2[\rho,\tau,\tau^{-1}]/(\rho^{2^{n+1}-1})$$

*generated by $a = \rho^{n+1}$, $a' = a\tau^{-1}$, and elements in $\Lambda(Q_0,...,Q_{n-1})\{a'\}$.*

The following lemma is used in the next sections.

**Lemma 10.2.** *We have $Q_0(\tau^{-1}) = \rho\tau^{-2}$. Hence $Q_0(a') = \rho a\tau^{-2}$, while $Q_0(a) = 0$.*

*Proof.* We see the first equation from

$$0 = Q_0(1) = Q_0(\tau\tau^{-1}) = \rho\tau^{-1} + \tau Q_0(\tau^{-1}).$$

□

**Lemma 10.3.** *(Lemma 5.13 in* [Ya2] *) Let $X_d$ be anisotropic quadric of the degree $d$. Let $h \in H^{2,1}(X_d)$ be the hyper plain section. If $2^n - 2 < d$, then we have a graded ring isomorphism*

$$H^{*,*'}(X_d;\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau,h] \quad when \ \ * \le n.$$

*In particular, $H^{*,*-1}(X_d:\mathbb{Z}/2) = 0 \ mod(Ideal(h))$ for $* \le n$.*

## 11. The cohomology of quadrics with coefficients in $\mathbb{Z}_2$

In this section we consider integral coefficients case. In this paper, the 2-adic integral $\mathbb{Z}_2$ cohomology means the inverse limit

$$H^*_{\acute{e}t}(M;\mathbb{Z}_2) = Lim_{\infty\leftarrow s} H^{*,*}(M;\mathbb{Z}/2^s)$$

of motivic cohomologies.

We recall here the Lichtenberg cohomology [Vo1,2] such that

$$H_L^{*,*'}(X;\mathbb{Z}) \cong H^{*,*'}(X;\mathbb{Z}) \quad for \ * \le *' + 1.$$

(The right side is the motivic cohomology.) By the five lemma, we see (for $1/s \in k$)

$$H_L^{*,*'}(X;\mathbb{Z}/s) \cong H^{*,*'}(X;\mathbb{Z}/s) \quad for \ \ * \le *'.$$

Moreover we have $H_L^{2*,*'}(X;\mathbb{Z}/s) \cong H^{2*}_{\acute{e}t}(X;\mu_s^{*'\otimes})$.

In this paper we consider the cycle maps to this Lichitenberg (or motivic) cohomology in stead of the étale cohomology itself. The cycle map is written

$$cl : CH^*(X)\otimes\mathbb{Z}_2 \cong H^{2*,*}(X;\mathbb{Z}_2) \to H_L^{2*,*}(X;\mathbb{Z}_2) \cong H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(*))$$

where $\mathbb{Z}(*)$ is the Galois module, when $k=\mathbb{R}$, it acts as $(-1)^*$. Here we can write

$$H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(*)) = \oplus_{m\ge 0}(H^{4m}_{\acute{e}t}X;\mathbb{Z}_2) \oplus H^{4m+2}_{\acute{e}t}(X;\mathbb{Z}_2(1)).$$

Note that it is the (graded) ring.

Let $k=\mathbb{R}$. Moreover let $* = even$. Then the right hand side cohomology is written

$$H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(*)) \cong H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(even)) \cong H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(2*))$$
$$\cong H_L^{2*,2*}(X;\mathbb{Z}_2) \cong H^{2*,2*}(X;\mathbb{Z}_2).$$

Similarly, when $* = odd$, we see $H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(*)) \cong H^{2*,2*+1}(X;\mathbb{Z}_2)$.

Thus in this paper, the cycle map means ;

$$cl : CH^*(X)\otimes\mathbb{Z}_2 \to H^{2*}_{\acute{e}t}(X;\mathbb{Z}_2(*)) \cong \begin{cases} H^{2*,2*}(X;\mathbb{Z}_2) & for \ * = even \\ H^{2*,2*+1}(X;\mathbb{Z}_2) & for \ * = odd. \end{cases}.$$

We say that $x \in H^{2*}_{\acute{e}t}(X;\mathbb{Z}(*)$ is non-algebraic if $x \ne 0 \ mod(Im(cl))$.

The short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ induces the long exact sequence of motivic cohomology

$$... \to H^{*-1,*}(M;\mathbb{Z}/2) \xrightarrow{\delta} H^{*,*}(M;\mathbb{Z}) \xrightarrow{2} H^{*,*}(M;\mathbb{Z}) \xrightarrow{r} H^{*,*}(M;\mathbb{Z}/2) \to ...$$

By Voevodsky [Vo1], [Vo2]), it is known $\beta(\tau) = \rho$ for the Bockstein operation $\beta$. Let us write $\delta(\tau\rho^{i-1}) = \bar{\rho}_i \in H^{*,*}(M;\mathbb{Z})$ so that $r(\bar{\rho}_i) = \beta(\tau\rho^{i-1}) = \rho^i$ since $r\delta = \beta$. Moteover $\bar{\rho}_i$ is 2-torsion from the above exact sequence.

Hence for all $1 \le c \le 2^{n+1} - 2$, we see $H^{c,c}(M;\mathbb{Z}) \neq 0$. The same fact holds each $H^c_{\acute{e}t}(M;\mathbb{Z}/2^s)$ and so $H^c_{\acute{e}t}(M;\mathbb{Z}_2)$.

**Lemma 11.1.** *Let $N = 2^{n+1} - 2$. Then*

$$\mathbb{Z}_2\{1, cl(c_0)\} \oplus \mathbb{Z}/2\{\bar{\rho}_1, ..., \bar{\rho}_N\} \subset H^*_{\acute{e}t}(M;\mathbb{Z}_2) \oplus H^*_{\acute{e}t}(M;\mathbb{Z}_2(1)).$$

*The element $\bar{\rho}_c$ with $c = 0 \ mod(4)$ and $c \neq 2^{n+1} - 2^{i+1}$ is a non-algebraic element (i.e., not in the image of the cycle map).*

**Remark.** When $c = 2 \ mod(4)$, the element $\bar{\rho}_c \in H^c(M;\mathbb{Z}_2)$ but not in $H^c(M;\mathbb{Z}_2(1))$. So we identify here $\bar{\rho}_c$ is not in $H^{2*}_{\acute{e}t}(M;\mathbb{Z}_2(*))$.

Writing $\pi = cl(c_0)$, we have the following theorem.

**Theorem 11.2.** *([Ya7]) Let $M_n \subset Q^{2^n-1}$ be the Rost motive of the norm variety. Then there are element $\pi \in H^{2^{n+1}-2}_{\acute{e}t}(M_n;\mathbb{Z}_2(1))$ and $\bar{\rho}_{4m} \in H^{4m}_{\acute{e}t}(M_n;\mathbb{Z}_2(0))$ such that*

$$H^{2*}_{\acute{e}t}(M_n;\mathbb{Z}_2(*)) \cong \mathbb{Z}_2\{1,\pi\} \oplus \mathbb{Z}/2\{\bar{\rho}_4, \bar{\rho}_8, ..., \bar{\rho}_{2^{n+1}-4}\}$$
$$\cong \mathbb{Z}_2\{1,\pi\} \oplus \mathbb{Z}/2[\bar{\rho}_4]^+/(\bar{\rho}_4^{2^{n-1}}).$$

*The image of the cycle map is given*

$$CH^*(M_n) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2\{1,\pi\} \oplus \mathbb{Z}/2\{\bar{\rho}_{2^{n+1}-2^n}, \bar{\rho}_{2^{n+1}-2^{n-1}}, ..., \bar{\rho}_{2^{n+1}-4}\}.$$

## 12. NORM VARIETIES

Let $X = Q^{2^n-1}$ be the norm variety, and $M_n$ be its Rost motive. We have the decomposition of motives ([Ro], §6 in[Ya1])

$$M(Q^{2^n-1}) \cong M_n \oplus M_{n-1} \otimes M(\tilde{\mathbb{P}}^{2^{n-1}-1})$$

where $M(\tilde{\mathbb{P}}^s) \cong \mathbb{T} \oplus ... \oplus \mathbb{T}^{s\otimes}$.

Hence we have the additive structure from Theorem 11.2 in the preceding section. More strongly, we can prove

**Theorem 12.1.** [Ya6] *We have a ring isomorphism*

$$H^{2*}_{\acute{e}t}(Q^{2^n-1};\mathbb{Z}_2(*)) \cong \mathbb{Z}_2[h, \bar{\rho}_4]/(h^{2^n}, 2\bar{\rho}_4, h\bar{\rho}_4^{2^{n-2}}, \bar{\rho}_4 h^{2^{n-1}}, \bar{\rho}_4^{2^{n-1}}).$$

*Here $h \in H^2(Q^{2^n-1};\mathbb{Z}_2(1))$ is the hyper plain section, and we can take $\pi = h^{2^n-1}$. (The ring is generated by only two elements.)*

We give only an outline of the proof for $Q^7$ here, for ease of arguments.

**Lemma 12.2.** *We have a ring isomorphism*

$$H_{\acute{e}t}^{2*}(Q^7;\mathbb{Z}_2(*)) \cong \mathbb{Z}_2[h]/(h^8) \oplus \mathbb{Z}/2[h]/(h^4)\{\bar{\rho}_4\} \otimes \mathbb{Z}/2\{\bar{\rho}_4^2,\bar{\rho}_4^3\}$$
$$\cong \mathbb{Z}_2[h,\bar{\rho}_4]/(h^8, 2\bar{\rho}_4, h^4\bar{\rho}_4, h\bar{\rho}_4^2, \bar{\rho}_4^4)$$

*where* $h^7 = c_0 = \pi,\ c_1 = \bar{\rho}_4^3,\ c_2 = \bar{\rho}_4^2$ *and* $c_1'h = h\bar{\rho}_4$. *Hence we have*

$$H_{\acute{e}t}^{2*}(Q^7;\mathbb{Z}_2(*))/(Im(cl)) \cong \mathbb{Z}/2\{\bar{\rho}_4\}.$$

*Proof.* From the decomposition of the motive, we see (additively)

$$H_{\acute{e}t}^{2*}(Q^7;\mathbb{Z}_2(*)) \cong H^{2*}(M_3;\mathbb{Z}_2(*)) \oplus H^{2*}(M_2;\mathbb{Z}_2(*)) \otimes \mathbb{Z}_2\{h,h^2,h^3\}.$$

Hence it can be written additively (with $|c_0| = 14$, $|c_1| = 12$, $|c_2| = 8$, $|c_0'| = 6$, $|c_1'| = 4$)

$$(\mathbb{Z}_2\{1,c_0\} \oplus \mathbb{Z}/2\{\bar{\rho}_4,c_1,c_2\}) \oplus (\mathbb{Z}_2\{1,c_0'\} \oplus \mathbb{Z}/2\{c_1'\}) \otimes \mathbb{Z}_2\{h,h^2,h^3\}.$$

It is well known (for $\bar{X} = X(\mathbb{C})$)

$$H^*(\bar{X};\mathbb{Z}_2) \cong \mathbb{Z}_2[h,y]/(h^8, 2y = h^4, y^2).$$

Hence, from the restriction map, the ring $H^*(X;\mathbb{Z}_2) \supset Z_2[h]/(h^8)$.

First note

$$\mathbb{Z}_2\{c_0'h, c_0'h^2, c_0'h^3\} \cong \mathbb{Z}_2\{h^4,h^5,h^6\}.$$

Thus we have

$$\mathbb{Z}_2\{1,h,...,h^7\} \cong \mathbb{Z}_2\{1,h,h^2,h^3,hc_0',h^2c_0',h^3c_0',c_0\}.$$

So we have the above $H^{2*}(Q^7;\mathbb{Z}_2(*))$ is isomorphic to

$$\mathbb{Z}_2[h]/(h^8) \oplus \mathbb{Z}/2\{\bar{\rho}_4,c_2,c_1\} \oplus \mathbb{Z}_2\{c_1'h, c_1'h^2, c_1'h^3\}.$$

Taking $c_2 = \bar{\rho}_4^2,\ c_1 = \bar{\rho}_4^3,\ hc_1' = h\bar{\rho}_4$, we have the result. □

We want to see the following theorem.

**Theorem 12.3.** *Let* $X_n = Q^{2^n-1}$, $n \geq 2$ *the norm variety. Then*

$$DH^{2*}(X_n;\mathbb{Z}_2(*)) = 0.$$

$$H_{ur}^{2*}(X_n;\mathbb{Z}_2(*)) \supset \mathbb{Z}/2[\bar{\rho}_4]/(\bar{\rho}_4^{2^{n-1}}).$$

*Hence for* $n \neq n'$, *we see that* $X_n$ *and* $X_{n'}$ *are not retract rationally equivaliant.*

**Remark.** When $n = 1$, we see $X_1 \cong \mathbb{P}^1$ that is, $X_1$ stable birational.

**Corollary 12.4.** *The second and the last formulas in the above theorem, hold when* $k$ *is a real number field.*

*Proof.* Recall that the norm variety $X_n = X_n(k)$ is defined naturaly.

Let $r_1 \geq 1$ be real embedding number. Then we have the restriction

$$r : H^*_{et}(Spec(k); \mathbb{Z}/2) \to \oplus^{r_1} H^*_{et}(Spec(\mathbb{R}); \mathbb{Z}/2)$$

such that $r$ is surjective for $* \geq 1$ and isomorphic for $* \geq 3$, Hence we can define $\rho(k) \in H^1_{et}(Spec(k); \mathbb{Z}/2)$ so that $r(\rho(k)) = \rho$. Similarly we can define $\bar{\rho}_4(k)$ in $H^*_{un}(X; \mathbb{Z}_2)$. It is nonzero since so for $k = \mathbb{R}$. □

Since the hyper plain section $h$ is represented by a first Chern class and $\pi = c_0 = h^{2^n-1}$. By Frobenius reciprocity, we only check elements

$$\bar{\rho}_4^i \notin N^1 H^{2*}(X; \mathbb{Z}_2(*))$$

for the first equality in theorem. Then $Ideal(h) \in N^1$ and

$$H^{4*}_{ur}(X; \mathbb{Z}/2) \supset H^{4*}(X : \mathbb{Z}/2)/N^1 \supset \mathbb{Z}/2[\bar{\rho}_4]/(\bar{\rho}_4^{2^{n-1}})$$

implies the second formula.

Since $H^{2*}(M_n; \mathbb{Z}_2(*))$ is a direct summand of $H^{2*}(X; \mathbb{Z}_2(*))$ and $\bar{\rho}_4$ is defined in $H^{2*}(M_n; \mathbb{Z}_2(*))$, we only need to see the following Lemma 12.5, (by using Lemma 10.1-10.3) for the proof of the above theorem.

**Lemma 12.5.** *We have $\bar{p}_4^i \notin N^1 H^*(M_n; \mathbb{Z}_2(*))$ for $i \geq 1$.*

*Proof.* Consider the following diagram

$$\begin{array}{ccccc}
\bar{\rho}_s \in H^{*,*}(X; \mathbb{Z}_2) & \xrightarrow{r} & \rho^s \in H^{*,*}(X : \mathbb{Z}/2) & & \\
\tau' \uparrow & & \tau \uparrow & & \\
x \in H^{*,*-1}(X : \mathbb{Z}_2) & \xrightarrow{r} & H^{*,*-1}(X; \mathbb{Z}/2) & \xrightarrow{Q_0} &
\end{array}$$

Suppose $\bar{\rho}_s \in N^1 H^{*,*}(X; \mathbb{Z}_2)$, which means that there is $x \in H^{*,*-1}(X : \mathbb{Z}_2)$ such that $\tau' x = \bar{\rho}_s$. We consider the reduction maps $r$ to the cohomology of $\mathbb{Z}/2$ cefficients. Then $\tau r(x) = \rho^s$. and $Q_0(r(x))$ must be zero (since $x$ is in the integral coefficients $\mathbb{Z}_2$). We will prove this does not happen.

Recall $a = \rho^{n+1}$ and $a' = a\tau^{-1}$ in $H^{*.*-1}(M_n; \mathbb{Z}/2)$.

The case $* \leq n$ ; The cohomology $H^{*,*'}(M_n; \mathbb{Z}/2) = 0 \ mod(Ideal(h))$ for $* > *'$ from Lemma 10.3. Hence there is no non zero element $\tau^{-1}\rho^* \in H^{*.*-1}(M_n; \mathbb{Z}/2) \ mod(Ideal(h)$ (where $h \in \tilde{N}^1$).

The case $* = n+1$; Then there is $a'$ such that $\tau a' = a$. But this element $a'$ is not in the integral $H^{*.*-1}(M_a; \mathbb{Z}_2)$, because

$$Q_0(a') = Q_0(\rho^{n+1}\tau^{-1}) = \rho^{n+2}\tau^{-2}.$$

which is nonzero in $H^{*,*'}(M_n; \mathbb{Z}/2)$, and so $a \notin N^1 H^{*,*}(M_n; \mathbb{Z}_2)$.

The case $* > n+1$ . Let us write $b' = Q_0(a') = \rho^{n+2}\tau^{-2}$. Next consider the element $b = \tau b'$. Then we note

$$\tau b = \tau^2 b' = \tau^2\rho^{n+2}\tau^{-2} = a\rho = \rho^{n+2}.$$

That is $b = \rho^{n+2}\tau^{-1}$and $b = \tau Q_0(a')$. Hence from Theorem 10.1, we see $b \in H^{*.*'}(M_n;\mathbb{Z}/2)$.

Since $Q_0(b') = Q_0Q_0(a') = 0$, we can compute

$$Q_0(b) = Q_0(\tau b') = \rho b' + \tau Q_0(b') = \rho b'$$

is nonzero in $H^{*,*'}(X;\mathbb{Z}/2)$ and hence $b$ is not in the integral $H^{*,*'}(X;\mathbb{Z}_2)$. Therefore $\rho^{n+2} \notin N^1H^*(X;\mathbb{Z}_2)$.

Similarly we can show for $j > n+2$, the element $\rho^j$ is not in $N^1H^*(X;\mathbb{Z}_2)$. $\square$

The elements $a, ..., b'$ are written in $\mathbb{Z}/2[\rho,\tau,\tau^{-1}]/(\rho^{2^n-1})$ as follows. (Recall Theorem 10.1 and Lemma 10.2.)

$$\begin{tikzcd}
 & & \rho^{n+2} \in H^{n+2,n+2} & \\
 & a = \rho^{n+1} \in H^{n+1,n+1} & b = \rho^{n+2}\tau^{-1} \arrow[u, "\tau"] \arrow[r, "Q_0"] & \rho^{n+3}\tau^{-2} \\
\rho^n \in H^{n.n} & a' = \rho^{n+1}\tau^{-1} \arrow[u, "\tau"] \arrow[r, "Q_0"] & b' = \rho^{n+2}\tau^{-2} \arrow[u, "\tau"] \arrow[r, "Q_0"] & 0 \\
0 = H^{n,n-1} \arrow[u, "\tau"] & & &
\end{tikzcd}$$

## References


[Be] O. Benoist, Steenrod operations and algebraic classes. *arXiv:220 9.03685. [math.AG]*

[Be-Ot] O. Benoist and J. Ottem. Two coniveau filtrations. *Duke Math J.* **170** (2021) 2719-2715.

[Bl-Og] S. Bloch and A. Ogus. Gersten's conjecture and the homology of schemes. *Ann.Scient.Éc.Norm.Sup.* **7** (1974) 181-202.

[Bo] F.Bogomolov. The Brauer group of quotient spaces by linear group actions. *Math. USSR Izvestiya* **30** (1988) 455-485.

[Bo-Bo-Pi] F. Bogomolov, C. Bohning and A. Pirutka. On stable cohomology of central extensions of elementary abelian groups, In: Ahmadinezhad, H. and Kasprzyk, A. and Mori, S., (eds.) London Mathematical Society. Lecture Note Series. Cambridge University Press (2021).

[Co-Vo] J.L. Colliot Thérène and C. Voisin. Cohomologie non ramifieet conjecture de Hodge entíere. *Duke Math. J.* **161** (2012), 735-801.

[Ek] T.Ekedahl. Approximating classifying spaces by smooth projective varieties *preprint* **44** (2009) arXiv:0905.1538.

[Ga-Me-Se] S.Garibaldi, A.Merkurjev and J.P.Serre. Cohomological invariants in Galois cohomology. *University lect. series vol(28) AMS* (2003).

[Ka-Ya] M.Kameko and N.Yagita. The Brown-Peterson cohomology of the classifying spaces of the projective unitary groups $PU(p)$ and exceptional Lie group *Trans. of A.M.S.* **360** (2008), 2265-2284.

[Ko] E. Kordonskii. Stable rationality of the group $Spin_{10}$. *Russian Math. Soc.* **360** (2000) 178-179.

[Me] A. Merkurjev. Rationaly problem for classifying spaces of Spinor groups. *Proc. Skeklov Inst. Math.* **307** (2019) 115-124.

[Mi] J. Milnor. The Steenrod algebra and its dual. *Ann. Math.* **67** (1958) 150-171.

[Mi-To] M. Mimura and H. Toda, Topology of Lie groups I and II, Translations of Math. Monographs, Amer. Math. Soc, **91**, 1991

[Or-Vi-Vo] D.Orlov, A.Vishik and V.Voevodsky. An exact sequence for Milnor's K-theory with applications to quadric forms. *Ann. of Math.* **165** (2007) 1-13.

[Sc-Su] F. Scavia, F. Suzuki, Non-injectivity of the cycle class map in continuous $\ell$-adic cohomology. arXiv:2208.06713. [Math.AG]

[Pa] K. Paranjape. Some spectral sequences for filtered complexes and applications. *J. Algebra* **186** (1996) 793-806.

[Pi] A. Pirutka. Varieties that are not stable retract, zero cycles and unramified cohomology. in Algebraic Geometry (Salt Lake City, UT, 2015), Proc. Sympos. Pure Math., 97, 459-484. Amer. Math. Soc., Providence, RI, 2018.

[Pi-Ya] A. Pirutka and N. Yagita. Note on the counterexamples for the integral Tate conjecture over finite fields. *Doc. Math.* (2015) Extra vol. : Alexander S. Merkurjev's sixties birthday. 501-511.

[Qu1] D.Quillen. Elementary proofs of some results of the cobordism theory using Steenrod operations. *Advances in Math.* **7** (1971), 29-56.

[Qu2] D.Quillen. The mod 2 cohomology rings of extra-special 2-groups and the spinor groups. *Math. Ann.* **194** (1971), 197-212.

[Ra-Wi-Ya] D. Ravenel, S. Wilson and N.Yagita. Brown-Peterson cohomology from Morava $K$-theory. *$K$-theory.* **15** (1998), 147-199.

[Re-Sc] Z. Reichstein and F. Scavia. The Noether problem for spinor groups of small rank. *J. Algebra* **48** (2020), 134-152.

[Ro] M.Rost, Some new results on Chowgroups of quadrics, preprint (1990).

[Sc-Su] F. Scavia, F. Suzuki, Non-injectivity of the cycle class map in continuous $\ell$-adic cohomology. arXiv:2208.06713. [Math.AG]

[Ta] M. Tamanoi. The image of the BP-Thom map for Eilenberg-MacLane spaces. *Trans. of AMS.* **249** (1997), 1209-1237.

[Te-Ya1] M. Tezuka and N.Yagita. The varieties of the mod $p$ cohomology rings of extraspecial $p$-groups for an odd prime $p$. *Math. Proc. Cambridge Phil. Soc.* **94** (1983) 449-459.

[Te-Ya2] M.Tezuka and N.Yagita. The image of the map from group cohomology to Galois cohomology. *Trans. of AMS.* **363** (2011), 4475-4503.

[To] B. Totaro. The Chow ring of classifying spaces. *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington,Seattle)* **67** (1999), 248-281.

[Vo1] V. Voevodsky. The Milnor conjecture. *www.math.uiuc.edu/K-theory/0170* (1996).

[Vo2] V.Voevodsky. Reduced power operations in motivic cohomology. *Publ.Math. IHES* **98** (2003),1-57.

[Vo3] V. Voevodsky. Motivic cohomology with $\mathbb{Z}/2$ coefficient. *Publ. Math. IHES* **98** (2003), 59-104.

[Vo4] V.Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. *Ann. Math.* **174** (2011) 401-438.

[Ya1] N. Yagita. Chow rings of excellent quadrics. *J. Pure and Appl. Algebra* **212** (2008), 2440-2449.

[Ya2] N. Yagita, Motivic cohomology of quadrics and the coniveau spectral sequence. J. K-theory **6** (2010) 547-589.

[Ya4] N. Yagita. Algebraic $BP$-theory and norm varieties. *Hokkaido Math. J.* **41** (2012), 275-316.

[Ya5] N. Yagita. Chern classes and the Rost cohomological invariant. *Kodai Math. J.* **36** (2013), 174-178.

[Ya6] N. Yagita. Coniveau filtrations and Milnor operations $Q_n$. *Math. Proc. Cambridge Phil. Soc.* **175** (2023) 521-538.

[Ya7] N. Yagita. Étal cohomologies of quadrics over $\mathbb{R}$. *Arxiv.2106.0761* (math AT).



[N. Yagita] Department of Mathematics, Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan

*Email address*: `nobuaki.yagita.math@vc.ibaraki.ac.jp`,