# Conformal Online Auction Design 

Jiale Han<br>JIALEHAN@UCLA.EDU<br>Department of Statistics and Data Science<br>University of California, Los Angeles, CA 90095-1554, USA

Xiaowu Dai ${ }^{*}$<br>DAI@STAT.UCLA.EDU<br>Department of Statistics and Data Science and Department of Biostatistics<br>University of California, Los Angeles, CA 90095-1554, USA


#### Abstract

This paper proposes the conformal online auction design (COAD), a novel mechanism for maximizing revenue in online auctions by quantifying the uncertainty in bidders' values without relying on assumptions about value distributions. COAD incorporates both the bidder and item features and leverages historical data to provide an incentive-compatible mechanism for online auctions. Unlike traditional methods for online auctions, COAD employs a distribution-free, prediction interval-based approach using conformal prediction techniques. This novel approach ensures that the expected revenue from our mechanism can achieve at least a constant fraction of the revenue generated by the optimal mechanism. Additionally, COAD admits the use of a broad array of modern machine-learning methods, including random forests, kernel methods, and deep neural nets, for predicting bidders' values. It ensures revenue performance under any finite sample of historical data. Moreover, COAD introduces bidder-specific reserve prices based on the lower confidence bounds of bidders' valuations, which is different from the uniform reserve prices commonly used in the literature. We validate our theoretical predictions through extensive simulations and a real-data application. All code for using COAD and reproducing results is made available on GitHub.


Keywords: Optimal auction, revenue maximization, mechanism design, conformal prediction, uncertainty quantification.

## 1 Introduction

Online auctions for advertisements have played a key role in providing individuals and businesses with the opportunity to gain from trade in e-commerce. Advertisers place advertisements on online platforms such as Google and Meta, where the ads are allocated through real-time auctions. Advertisers bid for an ad slot in these auctions, and the winner pays the platform to display their advertisement. Since these major online platforms can collect data on user behaviors, online advertising enables personalized recommendations and leads to more precise user targeting than traditional printed advertisements. Online advertisements have generated a significant fraction of the revenue for online platforms (Evans, 2008; Choi et al., 2020). The study of online auctions has become a key focus in computer science and economics. Existing works on online auctions involve the analysis of the economic properties of auction designs and their computational efficiency (e.g., Riley and Samuelson, 1981;

[^0]Milgrom and Weber, 1982; Roughgarden, 2010; Easley and Kleinberg, 2010; Mehta, 2013; Ostrovsky and Schwarz, 2023).

It is critical to design auctions in order to achieve incentive compatibility and maximize the revenue (Muthukrishnan, 2009; Milgrom, 2017). When the values of the bidders are independently drawn from a known regular distribution, Myerson (1981) developed such an auction by incorporating the Vickrey-Clarke-Groves (VCG, Vickrey, 1961; Clarke, 1971; Groves, 1973) mechanism with a reserve price. This design has laid the groundwork for most online auctions for advertisements (Easley and Kleinberg, 2010). In these auctions with a single auction item, the highest bidder is required to pay either the second-highest bid or the reserve price, whichever is higher. The platform sets the reserve price. The popularity of these auctions stems from their incentive compatibility, meaning bidders are motivated to bid exactly what they are willing to pay. However, implementing this auction mechanism in practice poses several challenges. For instance, reasonably good approximations of value distributions are often unknown in real-world auctions. Moreover, the platform's revenue is significantly influenced by how the reserve prices are set. Machine learning methods are emerging as important tools for enhancing the efficiency of online auctions. Retailers and marketplaces such as eBay, Google, and Meta are leveraging vast amounts of data to identify patterns that help them increase the efficiency of their markets.

We propose the conformal online auction design (COAD), a new mechanism aimed at maximizing revenue by quantifying the uncertainty in bidders' valuations of auctioned items. COAD admits the use of a wide range of modern machine-learning methods, including random forests, kernel methods, deep neural networks, and various hybrids and ensembles of these methods. COAD integrates three novel components. First, it leverages historical data on both item and bidder features to statistically infer each bidder's valuation for the current item, along with its uncertainty. For instance, in online advertising on search engines like Google, the bidders are advertisers, and the items are ad slots for different keywords. The features of an ad slot may include information about the keywords, while bidders' features might include the advertiser's rating level, ad brand, product information, and other relevant ad details. The use of both item and bidder features is motivated by real-world online auctions, such as those on eBay, which have heterogeneous bidders and a diverse range of items over time. Thus, it is impractical to assume a fixed distribution of values from a fixed group of bidders or to analyze value distributions for identical items.

The second novel component of COAD is a distribution-free, prediction interval-based approach that does not depend on assumptions about the distribution of bidders' values. This approach is designed to be compatible with modern machine-learning algorithms, including deep neural networks, to predict bidders' values. It ensures robust revenue performance under any finite sample of historical data. A key tool in this approach is the conformal prediction method with conditional guarantees (Gibbs et al., 2023). This method generates prediction intervals that provide exact coverage for each group within a set of finite subgroups. Given the finite types of auctioned items and a specified confidence level of $1-\alpha$, we construct prediction intervals for each bidder's value in the new auction of any specific item using conditional conformal prediction techniques. As a result, COAD does not require extensive historical data or a large number of features to offer revenue guarantees. This is a significant advantage over existing methods, which often require extensive bid data for the same item (Cole and Roughgarden, 2014, Roughgarden and Schrijvers, 2016).

The third new component is the bidder-specific reserve prices based on the lower confidence bound of the bidder's valuation for auctioned items. This approach is fundamentally different from the traditional approach of applying a uniform minimum bid value for all bidders (Riley and Samuelson, 1981, Cesa-Bianchi et al., 2014; Mohri and Medina, 2016). Bidder-specific reserve prices have already been successfully implemented in real-world auctions to generate higher revenue. For example, search engines like Google and Yahoo! have leveraged these personalized reserve prices to not only increase revenues but also to encourage the placement of high-quality advertisements (Even-Dar et al., 2008). Hence, incorporating our data-driven, bidder-specific reserve prices in practical settings is feasible. Moreover, we provide a lower bound that ensures the expected revenue is at least a constant fraction of the optimal revenue, without requiring knowing the bidders' value distributions.

Our proposed COAD mechanism differs significantly from the existing methods that use historical data to estimate empirical distributions for revenue-maximization auctions when underlying value distributions are unknown (Cole and Roughgarden, 2014; Huang et al., 2015, Roughgarden and Schrijvers, 2016). The proposed COAD mechanism offers several advantages. First, COAD incorporates features of bidder features, which allows for a more realistic scenario in online auctions where non-identical data is common. For example, new bidders might not have prior data for specific items, and bidders often participate in auctions for various items, suggesting that each bidder's distribution may differ by item. Additionally, since online auctions frequently involve different sets of random bidders, it is impractical to design mechanisms for a fixed number of bidders. Secondly, the sample size required in previous studies depends on the number of bidders, which can become impractically large. In contrast, our method does not rely on the number of new bidders. Given the unpredictable nature of bidder participation in an online setting, COAD offers a more feasible solution by not requiring prior knowledge of bidder numbers.

The COAD mechanism is also different from methods of learning optimal reserve prices for optimal auctions (e.g., Mohri and Medina, 2016; Ostrovsky and Schwarz, 2023). Among these, the approach by Mohri and Medina (2016) is most closely related to ours. It focuses on using item features to determine optimal reserve prices, assuming known upper bounds on bidders' values. In contrast, COAD expands this by incorporating both bidder and item features, allowing for bidder-specific rather than uniform reserve prices. Moreover, COAD leverages the full spectrum of bid data, unlike methods like Mohri and Medina (2016) that only use the highest and second-highest bids. Consequently, even auctions with a single bidder can provide valuable data under COAD, bypassing the limitations of methods that cannot employ data if only one bidder participates or that disregard additional data if more than two bidders are involved. In our approach, each auction can yield as many data points as there are participants. Additionally, COAD does not require the restrictive assumption of a known upper bound on bidders' valuations, making our model closely aligned with real-world auctions (Cole and Roughgarden, 2014, Yao, 2014).

### 1.1 Related Works

We briefly review related work from several fields, including optimal auction design, postedprice auction, mechanism design, and statistical uncertainty quantification.

Optimal auction design. Most theoretical work on optimal auctions for revenue maximization can be traced back to the seminal papers by Myerson (1981) and Riley and Samuelson $(1981)$. Myerson $(\sqrt{1981)}$ specifically designed auctions that integrate the Vickrey-Clarke-Groves (VCG) mechanism with a reserve price, assuming that bidders' values are independently and identically drawn from a known regular distribution. Recently, there has been a surge in literature using machine learning for related problems, such as predicting bid landscapes (Cui et al., 2011), studying pay-per-click auctions (Devanur and Kakade, 2009), and regret minimization in second-price auctions (Cesa-Bianchi et al., 2014). A line of literature highly relevant to our paper focuses on using historical data to estimate empirical distributions for revenue-maximization auctions when underlying value distributions are unknown (Cole and Roughgarden, 2014; Huang et al., 2015, Roughgarden and Schrijvers, 2016; Mohri and Medina, 2016; Ostrovsky and Schwarz, 2023). Our proposed COAD mechanism differs from these methods by incorporating bidder features and employing a distribution-free, prediction interval-based approach for revenue-maximization auctions.

Posted-price auction mechanism. We have designed a bidder-specific reserve price mechanism, which is related to the posted-price auction mechanisms in online auctions (Blum et al., 2004). In these auctions, a seller offers goods at a set price, and buyers decide whether to accept or reject this offer. Our mechanism works as a sequential posting price mechanism in online settings where bidders participate sequentially. The key difference is that our approach requires bidders to submit their bids before the seller discloses the price. There is significant research on posted-price mechanisms with known priors, including works by Blumrosen and Holenstein (2008); Chawla et al. (2010); Chakraborty et al. (2010); Feldman et al. (2014), which focus on scenarios where the price is determined based on prior knowledge of buyers' value distributions. For scenarios where the underlying distribution is unknown, Balcan et al. (2008) proposed a strategy to set prices using buyers' features. Our mechanism extends this by also considering the features of the item being auctioned. More recently, Babaioff et al. (2017) developed a mechanism that could guarantee a constant fraction of the known distribution revenue when all candidate distributions have the monotone hazard rate property and have known bounded support. Our method achieves revenue guarantees without requiring support information on the value distributions and offers a solution adaptable to practical auction settings.

Learning and robustness in mechanism design. The presence of private and unknown preferences has motivated extensive research on learning-based mechanism designs. One area of interest is designing auctions robust to errors in the distribution of bidder values, without relying on Bayesian assumptions (Bergemann and Schlag, 2011; Cai and Daskalakis, 2017; Brustle et al., 2020; Cai and Daskalakis, 2022; Anunrojwong et al., 2023). Additionally, there is a burgeoning interest in integrating machine learning into market designs (Dai and Jordan, 2021a b; Dai et al., 2022). In this paper, we employ high-confidence coverage prediction intervals developed through conformal prediction techniques on historical data. This approach ensures that our auction mechanism is not only practically implementable but also capable of achieving a high level of expected revenue.

Statistical uncertainty quantification. The conformal prediction framework, introduced by Vovk et al. (2005), is designed for effective uncertainty quantification through
the formation of prediction intervals. These intervals have a marginal coverage guarantee in finite samples without requiring assumptions about the underlying data-generating processes (Lei et al., 2013, 2018). However, achieving conditional coverage guarantee alongside marginal coverage has been shown to be challenging (Vovk, 2012; Foygel Barber et al., 2021). Recently, Gibbs et al. (2023) proposed a conformal prediction approach for groupconditional coverage and coverage under covariate shift. The conformal prediction has been widely applied in various fields, including computer vision (Angelopoulos et al., 2021) and natural language processing (Fisch et al., 2021). In our study, we extend the application of conformal prediction to auction designs in economics, demonstrating a novel use of this method.

### 1.2 Contributions and Outline

We propose a novel online auction mechanism that integrates both bidder and item features. Here are our principal methodological and theoretical contributions:

- We introduce a regression model for online auctions that considers both bidder and item features (Section 2 ). We propose a new auction mechanism design, the conformal online auction design (COAD), which employs high-confidence coverage prediction intervals of bidders' values for any given item (Section 33). This involves using dual conformal prediction intervals with conditional guarantees. We show the asymptotic equivalence of the dual prediction interval to the primal interval (Proposition 1) and establish the efficiency of the dual prediction interval (Theorem 1) when the regression model is estimated accurately.
- We provide theoretical guarantees of COAD (Section 4). It includes the incentive compatibility and individual rationality properties of COAD (Theorem 2). We establish that the expected revenue from COAD increases with the number of bidders (Theorem 3) and that it is asymptotically at least a constant fraction of the maximum expected social welfare (Theorem 4). Additionally, we discuss methods to optimize the expected revenue from COAD (Theorem 5) and compare COAD with traditional auction designs, demonstrating the advantages of COAD (Section 4.4).
- We conduct simulations to validate the properties of COAD and to compare the expected revenue it generates against that of the second-price auction and the maximum expected social welfare (Section 5). COAD consistently provides high revenue guarantees in both low-dimensional and high-dimensional scenarios. We further validate these findings using real auction data from eBay (Section 6), demonstrating the practical effectiveness of COAD.

We conclude the paper with further research directions in Section 7. All technical proofs are provided in the Appendix.

## 2 Online Auction Model with Bidder and Item Features

We begin by describing the online auction model and the learning setup. Consider a seller possessing a finite variety of indivisible items, each available in unlimited supply (e.g.,
ad slots). The seller conducts auctions at $T$ time points, with each consisting of several single-round auctions for individual items. Each time point has multiple bidders, each participating in different auctions. At each time $t=1, \ldots, T$, a bidder can participate in only one auction and each item is sold no more than once. Each item has a feature from the finite set $\mathcal{Z}=\left\{\tilde{z}_{1}, \tilde{z}_{2}, \ldots, \tilde{z}_{q}\right\}$, where each $\tilde{z}_{j} \in \mathbb{R}^{k}$ for $1 \leq j \leq q$. At time $t$, suppose there are $m^{(t)} \geq 1$ bidders. The set of bidders at time $t$ is denoted as $\left[m^{(t)}\right]=\left\{1,2, \ldots, m^{(t)}\right\}$. Each bidder $j \in\left[m^{(t)}\right]$ has a feature $x_{j}^{(t)} \in \mathcal{X} \subset \mathbb{R}^{d}$, competes in the auction for the item with feature $z_{j}^{(t)} \in \mathcal{Z}$, and her valuation for the item is $v_{j}^{(t)} \in \mathbb{R}_{\geq 0}$. In the context of online advertising, the items are ad slots associated with different keywords, where the item features could include information related to these keywords. Each time, advertisers come to the platform to bid for ad slots associated with different keywords. Each advertiser chooses one keyword at a time and bids once. The bidders' features might include the advertiser's rating level, ad brand, product information, and other relevant details about the advertisement. The online auction process is illustrated in Figure 1.


Figure 1: An illustration of the online auction process.

### 2.1 Online Auction Design

In this section, we model the decision-making process and auction design for the new auctions at time $T+1$. Our goal is to design an online auction mechanism for any specific item that not only incentivizes bidders to reveal their true values but also maximizes expected revenue by leveraging historical data up to time $T$, without knowing the bidders' value distributions. At time $T+1$, we focus on an auction for an item with feature $z^{*}$ that is randomly selected from the feature set $\mathcal{Z}$, representing a specific auction at this time point. We assume this auction attracts $m^{*}$ bidders. We let $\left[m^{*}\right]=\left\{1,2, \ldots, m^{*}\right\}$ as the set of bidders in the new auction. Consider that each bidder $i \in\left[m^{*}\right]$ with feature $x_{i}^{*} \in \mathcal{X}$ and value $v_{i}^{*}$ submits a bid $b_{i}^{*} \in \mathbb{R}_{\geq 0}$. We define $\vec{v}^{*}=\left(v_{1}^{*}, v_{2}^{*}, \ldots, v_{m^{*}}^{*}\right)$ as the vector of $m^{*}$ bidders' values, $\vec{x}^{*}=\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{m^{*}}^{*}\right)$ as the vector of their features, and $\vec{b}^{*}=\left(b_{1}^{*}, b_{2}^{*}, \ldots, b_{m^{*}}^{*}\right)$ as the vector of their bids.

For any $m^{*} \in \mathbb{N}_{+}$and $\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) \in \mathbb{R}_{\geq 0}^{m^{*}} \times \mathcal{X}^{m^{*}} \times \mathcal{Z}$, an auction mechanism is a mapping from the space of $\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right)$ to an allocation rule $a_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) \in[0,1]$ for each bidder $i \in\left[m^{*}\right]$. This allocation rule denotes the probability of bidder $i$ being allocated the item. Additionally, a payment rule, denoted by $p_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) \in \mathbb{R}_{\geq 0}$, specifies the price paid by bidder $i$. Since only one item is to be allocated, the allocation rule satisfies,

$$
\begin{equation*}
\sum_{i=1}^{m^{*}} a_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) \leq 1 \quad \text { and } \quad a_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) \geq 0 \tag{1}
\end{equation*}
$$

for any $i \in\left[m^{*}\right]$ and $\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) \in \mathbb{R}_{\geq 0}^{m^{*}} \times \mathcal{X}^{m^{*}} \times \mathcal{Z}$. In this paper, we focus on a deterministic mechanism so that $a_{i} \in\{0,1\}$. The utility of bidder $i$ is given by,

$$
u_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right)=v_{i}^{*} \cdot a_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right)-p_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right) .
$$

Since each bidder's valuation is private information, a bidder might not disclose their true value while bidding but may submit a different value to game the mechanism and attempt to achieve higher utility. Therefore, a primary objective in mechanism design for auctions is to incentivize bidders to bid their true values. The seller aims to design a mechanism that maximizes expected revenue, subject to the constraints of ex-post incentive compatibility (IC) given by,

$$
\begin{equation*}
u_{i}\left(v_{i}^{*}, \vec{b}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) \geq u_{i}\left(b_{i}^{*}, \vec{b}_{-i}^{*}, \vec{x}^{*}, z^{*}\right), \tag{2}
\end{equation*}
$$

for every $v_{i}^{*}, b_{i}^{*} \in \mathbb{R}_{\geq 0}$ and $\left(\vec{b}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) \in \mathbb{R}_{\geq 0}^{m^{*}-1} \times \mathcal{X}^{m^{*}} \times \mathcal{Z}$, and individual rationality (IR),

$$
\begin{equation*}
u_{i}\left(v_{i}^{*}, \vec{b}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) \geq 0 . \tag{3}
\end{equation*}
$$

Under IC and IR, the dominant strategy for the bidders to obtain the highest utility is to bid truthfully, that is, $\vec{b}^{*}=\vec{v}^{*}$.

### 2.2 Learning with Bidder and Item Features

When there is no information about the values $\vec{v}^{*}$, any deterministic auction mechanism that satisfies IC in (2) and IR in (3) can perform arbitrarily poorly in terms of revenue (e.g., Sandholm and Likhodedov, 2015). Hence, we consider a setting in which, although the valuations, prior distributions of values, and support information of values are all unknown to the seller, the seller has access to historical data from the previous auctions up to time $T$. Denote the data of historical auctions as $\mathcal{D}=\left\{\left(x_{j}^{(t)}, z_{j}^{(t)}, v_{j}^{(t)}\right) \mid j \in\left[m^{(t)}\right], t=1,2 \ldots, T\right\}$, which includes bidder features $x_{i}^{(t)}$, bidder values $v_{i}^{(t)}$, and item features $z_{i}^{(t)}$ from all auctions up to time $T$. Let $\sum_{t=1}^{T} m^{(t)}=N$. We rewrite $\mathcal{D}=\left\{\left(x_{i}, z_{i}, v_{i}\right) \mid i=1,2, \ldots, N\right\}$ for notation simplicity and make the following three assumptions.

Assumption 1 (iid historical data) The $N$ data points in $\mathcal{D}$ are independent and identically distributed (iid) copies of $(x, z, v) \sim P$, where $P$ is a distribution function on $\mathcal{X} \times \mathcal{Z} \times \mathbb{R}_{\geq 0}$. Let $\mu(x, z)=\mathbb{E}[v \mid x, z]$.

Assumption 2 (Independent noises) For $(x, z, v) \sim P$, the noise $\varepsilon=v-\mu(x, z)$ is independent of $(x, z)$, and the density of $\varepsilon$ is symmetric on 0 and has a bounded first derivative.

Assumption 3 (Independent new bidders) The pairs $\left(x_{i}^{*}, v_{i}^{*}\right), 1 \leq i \leq m^{*}$, are iid conditional on $z^{*}$. Additionally, $\left\{\left(x_{i}^{*}, z^{*}, v_{i}^{*}\right),\left(x_{1}, z_{1}, v_{1}\right), \ldots,\left(x_{N}, z_{N}, v_{N}\right)\right\} \stackrel{i i d}{\sim} P, \forall i \in\left[m^{*}\right]$.

Assumptions 1 and 2 are widely used in the auction and statistics literature (e.g., Mohri and Medina, 2016; Lei et al., 2018; Ostrovsky and Schwarz, 2023). For any $(x, z, v) \sim P$, we consider the following regression model for the remainder of this paper:

$$
\begin{equation*}
v=\mu(x, z)+\varepsilon \tag{4}
\end{equation*}
$$

For each $i \in\left[m^{*}\right]$, we denote $\varepsilon_{i}^{*}=v_{i}^{*}-\mu\left(x_{i}^{*}, v_{i}^{*}\right)$. Assumption 3 assumes that bidders participating in the new auction for the item with feature $z^{*}$ at time $T+1$ are independent. Additionally, for each bidder $i \in\left[m^{*}\right]$, the data $\left(x_{i}^{*}, z^{*}, v_{i}^{*}\right)$ is assumed to be iid relative to the historical data. We note that Assumption 3 allows various bidders to exhibit different value distributions for a specific item, depending on their bidder features. Specifically, we consider that $\left(x_{i}^{*}, v_{i}^{*}\right) \mid z^{*}$ are iid. However, when we factor in both the bidder's and item's features, $\left(x_{i}^{*}, z^{*}\right)$, the distribution of $v_{i}^{*} \mid\left(x_{i}^{*}, z^{*}\right)$ is expected to differ among bidders. This variation arises because $v_{i}^{*}=\mu\left(x_{i}^{*}, z^{*}\right)+\varepsilon_{i}^{*}$, where $\mu\left(x_{i}^{*}, z^{*}\right)$ changes due to the differences in $x_{i}^{*}$, making the mean of the value $v_{i}^{*}$ vary.

We introduce additional notations. For any $\left(x^{*}, z^{*}, v^{*}\right) \sim P$, let $F_{v^{*} \mid z^{*}}$ and $F_{v^{*}, x^{*} \mid z^{*}}$ be the distribution functions for $v^{*} \mid z^{*}$ and $\left(v^{*}, x^{*}\right) \mid z^{*}$, respectively. Let $F_{\vec{v}^{*} \mid z^{*}}$ and $F_{\vec{v}^{*}, \vec{x}^{*} \mid z^{*}}$ be the joint distributions for the vectors $\vec{v}^{*} \mid z^{*}$ and $\left(\vec{v}^{*}, \vec{x}^{*}\right) \mid z^{*}$, respectively. Under Assumption 3. $F_{\vec{v}^{*} \mid z^{*}}\left(\vec{v}^{*}\right)=\prod_{i=1}^{m^{*}} F_{v^{*} \mid z^{*}}\left(v_{i}^{*}\right)$ and $F_{\vec{v}^{*}, \vec{x}^{*} \mid z^{*}}\left(\vec{v}^{*}, \vec{x}^{*}\right)=\prod_{i=1}^{m^{*}} F_{v^{*}, x^{*} \mid z^{*}}\left(v_{i}^{*}, x_{i}^{*}\right)$.

### 2.3 Revenue Objective

For an auction mechanism $\mathcal{M}$ designed using historical data $\mathcal{D}$, let $R_{m^{*}}^{\mathcal{M} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}}\right)$ denote the expected revenue generated by $\mathcal{M}$ when there are $m^{*}$ random bidders in the auction for an item with feature $z^{*}$. That is,

$$
\begin{equation*}
R_{m^{*}}^{\mathcal{M} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}}\right)=\mathbb{E}\left[\sum_{i=1}^{m^{*}} p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}\right] . \tag{5}
\end{equation*}
$$

Here $p_{i}\left(\vec{b}^{*}, \vec{x}^{*}, z^{*}\right)$ is the price paid by bidder $i \in\left[m^{*}\right]$, and the expectation in (5) is taken over the distribution $F_{\vec{v}^{*}, \vec{x}^{*} \mid z^{*}}$. Let $W_{m^{*}}\left(F_{v^{*} \mid z^{*}}\right)$ be the maximum expected social welfare of the $m^{*}$ random bidders,

$$
\begin{equation*}
W_{m^{*}}\left(F_{v^{*} \mid z^{*}}\right)=\mathbb{E}\left[\max _{1 \leq i \leq m^{*}} v_{i}^{*} \mid z^{*}\right] \tag{6}
\end{equation*}
$$

where the expectation in (6) is taken over the distribution $F_{\vec{v}^{*} \mid z^{*}}$. Since for a given item, the payments of the bidders depend on both the features and the values of the bidders, while the maximum value among the bidders depends only on the values of the bidders, there is a difference between the underlying distributions in $R_{m^{*}}^{\mathcal{M} \mid \mathcal{D}}$ and $W_{m^{*}}$.

The maximum expected social welfare in (6) serves as a benchmark for revenue in (5) since, by employing Equations (1) and (3), it can be easily proven that for any mechanism $\mathcal{M}, \sum_{i=1}^{m^{*}} p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right) \leq \max _{1 \leq i \leq m^{*}} v_{i}^{*}$ holds for all $\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right) \in \mathbb{R}_{\geq 0}^{m^{*}} \times \mathcal{X}^{m^{*}} \times \mathcal{Z}$. Consequently, $R_{m^{*}}^{\mathcal{M} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}}\right) \leq W_{m^{*}}\left(F_{v^{*} \mid z^{*}}\right)$ holds for any mechanism $\mathcal{M}$ and distribution $P$, which means that $W_{m^{*}}\left(F_{v^{*} \mid z^{*}}\right)$ can be seen as the optimal revenue that could be obtained.

Moreover, achieving the revenue of $W_{m^{*}}\left(F_{v^{*} \mid z^{*}}\right)$ would require the seller to know the bidders' values $\vec{v}^{*}$ for the item with feature $z^{*}$ before the auction, which is impractical in real scenarios.

## 3 Conformal Online Auction Design

In this section, we present a new algorithm called conformal online auction design (COAD). The key idea is to employ prediction intervals with high-confidence coverage for bidders' values of any given item, which we derive using conformal prediction with a conditional guarantee. We also demonstrate the statistical efficiency of the conditional conformal prediction method employed in our algorithm.

### 3.1 The COAD Mechanism

Our conformal online auction design (COAD) consists of four main steps. The first step is to construct high-confidence coverage prediction intervals for the bidders' values for any given item. Specifically, for each bidder $i \in\left[m^{*}\right]$, we aim to construct a ( $1-\alpha$ )-prediction interval $\left[\hat{v}_{i}^{L}, \hat{v}_{i}^{U}\right]$ for their value $v_{i}^{*}$, given the item's feature $z^{*} \in \mathcal{Z}$. That is, $\mathbb{P}\left(v_{i}^{*} \in\left[\hat{v}_{i}^{L}, \hat{v}_{i}^{U}\right] \mid z^{*}=\right.$ $\tilde{z}) \geq 1-\alpha$, for any $\tilde{z} \in \mathcal{Z}$ and $\alpha \in(0,1)$, where $\hat{v}_{i}^{L}$ and $\hat{v}_{i}^{U}$ are functions of $x_{i}^{*}$ and $z^{*}$. Although these prediction intervals convey less information compared to the full distribution of values, they are more readily obtainable. The method for constructing these prediction intervals [ $\hat{v}_{i}^{L}, \hat{v}_{i}^{U}$ ] is presented in Section 3.2.

In the second step, we introduce a new virtual value designed to recalibrate each bid to maximize the seller's expected revenue. The key idea is to consider both the bidder's valuation and the competition among bidders. The proposed virtual values are based on the lower bounds of the prediction intervals,

$$
\begin{equation*}
c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)=v_{i}^{*} \mathbb{I}\left\{v_{i}^{*} \geq \hat{v}_{i}^{L}\right\}, \quad \forall i \in\left[m^{*}\right] . \tag{7}
\end{equation*}
$$

Here $c_{i}$ in $\sqrt{7}$ is a monotone increasing function of $v_{i}^{*}$. Note that this new virtual value in (7) differs from the classical virtual value in optimal auction theory Myerson (1981), where a virtual value is determined by the bidder's own valuation and the distribution of valuations among all bidders. Specifically, the classical virtual value formula adjusts the bidder's actual valuation based on the likelihood of higher bids occurring. However, in our setting, the distribution of valuations among all participants is unknown. Hence, we re-calibrate each bid by comparing it to the lower confidence bound.

In the third step, we determine the allocation rules $\left\{a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)\right\}_{i=1}^{m^{*}}$ as described in (1). We employ a deterministic mechanism where only one bidder can win the item, meaning that at most one value in the set $\left\{a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)\right\}_{i=1}^{m^{*}}$ is 1 , with all others set to 0 . We let the seller retain the item at the new auction if $\max _{i \in\left[m^{*}\right]} c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)=0$, or assigns it to the bidder with the highest virtual value otherwise. If the seller retains the item, then for any bidder $i \in\left[m^{*}\right], a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=0$. If there is a tie between the bidders' virtual valuations when $\max _{k \in\left[m^{*}\right]} c_{k}\left(v_{k}^{*}, x_{k}^{*}, z^{*}\right)>0$, for example,

$$
c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)=c_{j}\left(v_{j}^{*}, x_{j}^{*}, z^{*}\right)=\max _{k \in\left[m^{*}\right]} c_{k}\left(v_{k}^{*}, x_{k}^{*}, z^{*}\right),
$$

the seller may break the tie by giving the item to the bidder with the largest prediction lower bound; if there is still a tie, for example, $\hat{v}_{i}^{L}=\hat{v}_{j}^{L}$, then the seller can break the tie by

```
Algorithm 1: Conformal Online Auction Design (COAD)
    Input: Historical data \(\mathcal{D}=\left\{\left(x_{i}, z_{i}, v_{i}\right) \mid i=1,2, \ldots, N\right\}\); New auction data
    \(\left\{\left(x_{i}^{*}, z^{*}, v_{i}^{*}\right) \mid i=1,2, \ldots, m^{*}\right\}\) at time \(T+1\); Miscoverage level \(\alpha \in(0,1)\).
    for \(i=1\) to \(m^{*}\) do
        Step 1: Construct the \((1-\alpha)\)-prediction interval \(\left[\hat{v}_{i}^{L}, \hat{v}_{i}^{U}\right]\) for \(v_{i}^{*}\) by Eq. (10);
        Step 2: Obtain the virtual values \(c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)\) by Eq. (7);
    end for
    for \(i=1\) to \(m^{*}\) do
        Step 3: Determine the allocation rule \(a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)\) by the following procedure:
            if \(\max _{i \in\left[m^{*}\right]} c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)=0\) then \(a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=0\);
            else if \(c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)=\max _{i \in\left[m^{*}\right]} c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)\) then \(a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=1\);
            else \(a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=0\);
            end if
        Step 4: Calculate payment \(p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)\) by Eq. (8).
    end for
    Output: The allocations \(\left\{a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)\right\}_{i=1}^{m^{*}}\) and payments \(\left\{p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)\right\}_{i=1}^{m^{*}}\) for all
    bidders in \(\left[m^{*}\right]\).
```

giving it to the lower-numbered bidder or by using other arbitrary rules. After breaking the tie, the winner will have an allocation rule with value 1 , and all others have an allocation rule with value 0 .

Finally, the fourth step involves determining the payment. In our model, the payment is designed as follows. Let

$$
r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right)=\inf \left\{b_{i}^{*} \mid c_{i}\left(b_{i}^{*}, x_{i}^{*}, z^{*}\right) \geq 0, c_{i}\left(b_{i}^{*}, x_{i}^{*}, z^{*}\right) \geq c_{j}\left(v_{j}^{*}, x_{j}^{*}, z^{*}\right), \forall j \in\left[m^{*}\right], j \neq i\right\},
$$

which represents the lowest winning bid for bidder $i$ against values of other bidders $\vec{v}_{-i}^{*}$. Then the payment is defined by

$$
p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)= \begin{cases}r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right), & a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=1,  \tag{8}\\ 0, & a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=0,\end{cases}
$$

where $i \in\left[m^{*}\right], \vec{v}^{*} \in \mathbb{R}_{\geq 0}^{m^{*}},\left(\vec{x}^{*}, z^{*}\right) \in \mathcal{X}^{m^{*}} \times \mathcal{Z}$. We show in Section 4.1 that the payment structure in (8) ensures the COAD mechanism is IC and IR for the new auction at time $T+1$. The four-step COAD procedure is summarized in Algorithm 1 .

### 3.2 Construction of Prediction Intervals

We now construct the prediction intervals in Section 3.1 based on the historical data $\mathcal{D}$. Without loss of generality, let the number of data points $N=2 n$. We randomly split the $2 n$ data points equally into two sets: a set of training data, and a set of calibration data. To simplify the notations, we let $D_{\text {cal }}=\left\{\left(x_{i}, z_{i}, v_{i}\right) \mid i=1,2, \ldots, n\right\}$ denote the set of calibration data, and $D_{\text {train }}=\left\{\left(x_{i}, z_{i}, v_{i}\right) \mid i=n+1, n+2, \ldots, 2 n\right\}$ denote the set of training data. We can use the machine learning algorithms $\mathcal{A}_{n}$ to estimate the regression function $\mu$ in Assumption 1 based on the set of training data. That is, $\hat{\mu}_{n}=\mathcal{A}_{n}\left(D_{\text {train }}\right)$.

A recently developed conformal prediction method constructs prediction intervals for new response variables, offering conditional guarantees (Gibbs et al., 2023) that are particularly pertinent to our analysis.

We now outline the key steps of the conditional conformal prediction method, with additional details provided in Appendix A. For each bidder $i \in\left[m^{*}\right]$ in the new auction for the item with feature $z^{*}$ at time $T+1$, we define a primal prediction interval for her value $v_{i}^{*}$ as follows,

$$
\begin{equation*}
\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)=\left\{v: \mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right) \leq \hat{g}_{\mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)}^{i}\left(x_{i}^{*}, z^{*}\right)\right\} . \tag{9}
\end{equation*}
$$

Given a miscoverage level $\alpha \in(0,1)$, the function $\hat{g}_{\mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)}^{i}: \mathcal{X} \times \mathcal{Z} \rightarrow \mathbb{R}$ varies based on the calibration data $D_{\text {cal }}$ and a conformity score function $\mathcal{S}$, ensuring coverage guarantees for $v_{i}^{*}$ conditional on $z^{*}$. While the primal prediction interval accurately meets coverage guarantees, it is computationally intensive and does not provide an explicit prediction interval.

To efficiently compute the interval, we shift to employing the dual form of the prediction interval for the value $v_{i}^{*}$. Let $\mathcal{S}(\{x, z\}, v)=\left|v-\hat{\mu}_{n}(x, z)\right|$, for $(x, z, v) \in \mathcal{X} \times \mathcal{Z} \times \mathbb{R}_{\geq 0}$. The dual prediction interval can be written as,

$$
\begin{equation*}
\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)=\left[\hat{v}_{i}^{L}, \hat{v}_{i}^{U}\right]=\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}, \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right], \quad \forall i \in\left[m^{*}\right] \tag{10}
\end{equation*}
$$

Here, $S^{*}>0$ is chosen to ensure that $\mathbb{P}\left(v_{i}^{*} \in \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right) \mid z^{*}=\tilde{z}\right) \geq 1-\alpha, \forall \tilde{z} \in \mathcal{Z}$.

### 3.3 Properties of the Conformal Prediction Intervals

We first aim to demonstrate that the dual prediction interval $\hat{\mathcal{C}}_{\text {dual }}$ in 10 closely approximates the primal prediction interval $\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)$ in 9 . To establish this, we require the following mild condition on the estimator $\hat{\mu}_{n}$.

Assumption 4 (Convergence of the estimator) The machine-learning estimator $\hat{\mu}_{n}$ satisfies that $\mathbb{E}\left[\left|\hat{\mu}_{n}(x, z)-\mu(x, z)\right|^{2}\right]=O\left(n^{-2 \tau}\right)$, for some $\tau>0$.

Assumption 4 holds for a wide range of popular machine learning methods. For instance, it holds for the $l_{1}$-penalized linear regression in a variety of sparse models (e.g., Bickel et al., 2009), a class of regression trees and random forests (e.g., Wager and Walther, 2015), a class of neural networks (e.g., Chen and White, 1999), and a class of kernel methods (e.g., Dai and Li, 2023). In each of these methods, Assumption 4 is satisfied with $\tau \geq 1 / 8$.

Proposition 1 Under Assumptions 14, the primal prediction interval $\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)$ and the dual prediction interval $\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)$ satisfy that,

$$
L\left(\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right) \Delta \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)\right)=O\left(n^{-\min \{\tau, 1\}}\right), \quad \forall i \in\left[m^{*}\right]
$$

Here $L(A)$ denotes the Lebesgue measure of a set $A$, and $A \triangle B$ is the symmetric difference between two sets $A$ and $B$.

The proof is provided in Appendix B.1. Proposition 1 demonstrates that the primal and dual prediction intervals are asymptotically equivalent. Due to its computational efficiency, we use the dual prediction intervals in the proposed COAD in Algorithm 1 .

Next, we demonstrate the efficiency of the dual conformal prediction interval in (10).

Theorem 1 Let $\delta_{n}=\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)+\max _{(x, z) \in \mathcal{X} \times \mathcal{Z}}\left|\hat{\mu}_{n}(x, z)-\mu(x, z)\right|$. Then under Assumptions 1-3, the length $S^{*}$ in the dual prediction interval (10) satisfies $S^{*} \leq \delta_{n}$ almost surely. Moreover, if $\hat{\mu}_{n}=\mu$ almost surely, we have that $\hat{v}_{i}^{L} \geq 0, i \in\left[\mathrm{~m}^{*}\right]$ almost surely.

The proof is provided in Appendix B.2. Theorem 1 shows that the length of the prediction interval $S^{*}$ is controlled by $\delta_{n}$. A more accurate estimator, $\hat{\mu}_{n}$, results in a smaller $\delta_{n}$, which in turn leads to tighter conformal prediction intervals.

Finally, we point out that under Assumptions 1 and 3, the dual prediction interval exhibits the following coverage property (see, e.g., Gibbs et al., 2023),

$$
\left|\mathbb{P}\left(v_{i}^{*} \in \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right)-(1-\alpha)\right|=O_{\mathbb{P}}\left(\sqrt{\frac{q}{n}}\right) .
$$

Hence, the dual conformal prediction interval provides exact coverage as $n \rightarrow \infty$.

## 4 Analysis of Incentives and Revenues

In this section, we study the economic properties of the proposed COAD mechanism in Algorithm 1, including the incentive guarantees and revenue analysis. We also compare the COAD mechanism with alternative data-driven auction designs.

### 4.1 Incentive Guarantees

We present the following theorem indicating the incentive guarantees for the bidders under the COAD mechanism in Algorithm 1.

Theorem 2 The COAD mechanism enjoys the $I C$ and IR properties as defined in (2) and (3), respectively.

The proof, which employs the well-known envelope formula (Myerson, 1981), is provided in Appendix B.3. From Theorem 2, it is established that bidders have a dominant strategy to truthfully reveal their valuations in new auctions at the time $T+1$.

### 4.2 Revenue Guarantees

We now analyze the revenue of the COAD mechanism. First, we demonstrate that as the number of bidders $m^{*}$ increases, there is a corresponding increase in the expected revenue.

Theorem 3 For any given item with features $z^{*}=\tilde{z} \in \mathcal{Z}$, the expected revenue of the COAD mechanism will increase as the number of bidders $m^{*}$ increases. That is, for any integers $m_{1}^{*} \geq m_{2}^{*} \geq 1$,

$$
R_{m_{1}^{*}}^{C O A D \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \geq R_{m_{2}^{*}}^{C O A D \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \text {, almost surely. }
$$

The proof is provided in Appendix B.4. The result in Theorem 3 supports an intuitive property of COAD: the entry of more independent bidders into the auction increases competition, resulting in higher revenues for the seller. This finding aligns with a well-known result in second-price auctions by Bulow and Klemperer (1996), which demonstrated that adding an additional bidder and setting a zero reserve price is always preferable to setting
the optimal reserve price. Both our result in Theorem 3 and the study by Bulow and Klemperer (1996) highlight the benefits of attracting more bidders to the auction.

Next, we provide a lower bound of the expected revenue achieved by the COAD mechanism. For any distribution $F$ with density $f(x)=d F(x) / d x$, let $S(x)=1-F(x)$ denote the survival probability and $H(x)=f(x) / S(x)$ denote the hazard rate of $F$. We say a distribution $F$ has a monotone hazard rate (MHR) if $H$ is monotone non-decreasing in $x$. This MHR assumption applies to commonly used distributions such as uniform, normal, and exponential distributions and is widely used in auction design (see, Hartline, 2013; Cole and Roughgarden, 2014, Schweizer and Szech, 2019).

Theorem 4 For any given item with features $z^{*}=\tilde{z} \in \mathcal{Z}$, we assume there exists a constant $C>0$ such that

$$
\begin{equation*}
\operatorname{Var}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right) \mid \hat{\mu}_{n}, z^{*}=\tilde{z}\right) \leq C \text {, almost surely. } \tag{11}
\end{equation*}
$$

Under Assumptions 1 -4 and for $\alpha \in(0,1)$, the expected revenue of the COAD satisfies

$$
R_{m^{*}}^{C O A D \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \geq \frac{2(1-\alpha)}{(1+h) \mathcal{H}_{m^{*}}} W_{m^{*}}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right)+O_{\mathbb{P}}\left(n^{-\min \{\tau / 2,1 / 2\}}\right),
$$

for $m^{*}=1$, and for $m^{*} \geq 2$ if $F_{v^{*} \mid z^{*}=\tilde{z}}$ has a MHR. Here, $\tau$ is defined in Assumption 4. $\mathcal{H}_{m^{*}}=\sum_{i=1}^{m^{*}} i^{-1}$ is the $m^{*}$ th harmonic number, and $h=\mathbb{E}\left[\hat{v}_{1}^{U} \mid \mathcal{D}, z^{*}=\tilde{z}\right] / \mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right]$.

The proof of this theorem is given in Appendix B.5. The assumption in (11) is satisfied in many practical scenarios. For instance, if the bidders' valuations are bounded, then a good prediction $\hat{\mu}_{n}$ would also be bounded by some constant almost surely. Consequently, this would imply that the variance of $\hat{\mu}_{n}$ is also bounded because $\operatorname{Var}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right) \mid \hat{\mu}_{n}, z^{*}=\right.$ $\tilde{z}) \leq \mathbb{E}\left(\hat{\mu}_{n}^{2}\left(x_{1}^{*}, z^{*}\right) \mid \hat{\mu}_{n}, z^{*}=\tilde{z}\right)$. Theorem 4 provides a revenue guarantee by comparing the revenue to the maximum expected social welfare, $W_{m^{*}}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right)$ defined in Section 2.3. It shows that for the new auction at $T+1$, the revenue of COAD is asymptotically no less than a constant fraction of the maximum expected social welfare when conditioning on $\mathcal{D}$.

Comparing Theorems 3 and 4 , we note that the ratio between the expected revenue and the maximum expected social welfare in the worst-case scenario, $2(1-\alpha) /\left[(1+h) \mathcal{H}_{m^{*}}\right]$, decreases as the number of bidders increases. This indicates that although a larger number of bidders will yield increased revenue for the seller, it concurrently widens the discrepancy between the expected revenue and the maximum expected social welfare. This finding aligns with previous research that uses data to design mechanisms achieving a constant fraction of the optimal revenue, such as in (Cole and Roughgarden, 2014). Their work demonstrates that to maintain a fixed constant fraction of the optimal revenue, the sample size required increases polynomially with the number of bidders. This implies that as the number of bidders increases, the gap between the designed mechanism and the optimal revenue will widen if the same amount of data is used. In their study, data is used to learn the distribution, whereas in our approach, we use data to establish prediction intervals.

### 4.3 Optimizing Revenue

We can maximize the revenue of the COAD mechanism through two main strategies. The first strategy involves increasing the number of bidders $m^{*}$. By Theorem 3, this strategy is guaranteed to increase the revenue.

The second strategy focuses on developing a more accurate estimator $\hat{\mu}_{n}$. We now show that it is advantageous for maximizing revenue in the COAD mechanism. As discussed in Section 2.3, the maximum expected social welfare $W_{m^{*}}\left(F_{v^{*} \mid z^{*}}\right)$ serves as the optimal benchmark for maximizing COAD's revenue. Due to the lower bound specified in Theorem 4 , we want to maximize the ratio $2(1-\alpha) /(1+h) \mathcal{H}_{m^{*}}$. Essentially, this involves minimizing the parameter $h$ defined in Theorem 4 as $h=\mathbb{E}\left[\hat{v}_{1}^{U} \mid \mathcal{D}, z^{*}=\tilde{z}\right] / \mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right]$. The following theorem shows that the upper bound of $h$ is monotone increasing with respect to the parameter $\delta_{n}$, which is defined in Theorem 1 as $\delta_{n}=\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)+\max _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mid \hat{\mu}_{n}(x, z)-$ $\mu(x, z) \mid$.
Theorem 5 For any given item with features $z^{*}=\tilde{z} \in \mathcal{Z}$, we assume that

$$
\begin{equation*}
\mathbb{E}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right)>\delta_{n}, \text { almost surely. } \tag{12}
\end{equation*}
$$

Then under Assumptions 1.4.

$$
1 \leq h \leq 1+\frac{2 \delta_{n}}{\mathbb{E}\left(\mu\left(x_{1}^{*}, z^{*}\right) \mid z^{*}=\tilde{z}\right)+O_{\mathbb{P}}\left(n^{-\tau}\right)-\delta_{n}} .
$$

The proof is given in Appendix B.6. The assumption in 12 ) is satisfied when $\hat{\mu}_{n}$ is a good estimator. For instance, if $\hat{\mu}_{n}$ achieves uniform consistency such that $\max _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mid \hat{\mu}_{n}(x, z)-$ $\mu(x, z) \mid=o_{\mathbb{P}}(1)$, then $\mathbb{E}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right)=\left[\mathbb{E}\left(\mu\left(x_{1}^{*}, \tilde{z}\right)\right)-\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)\right]+\delta_{n}+$ $o_{\mathbb{P}}(1)$ is greater than $\delta_{n}$ in probability under mild conditions. Theorem 5 shows that a more accurate estimator $\hat{\mu}_{n}$ leads to a smaller $h$ in the worst-case scenario and, consequently, a larger lower bound for the expected revenue of COAD. Therefore, developing a more accurate estimator $\hat{\mu}_{n}$ is advantageous for maximizing revenue in the COAD mechanism.

### 4.4 Comparisons with Alternative Auction Designs

We now compare the COAD mechanism detailed in Algorithm 1 with alternative auction designs, including Myerson's optimal auction design (Myerson, 1981), the first-price auction (Harrison, 1989), the second-price auction (Vickrey, 1961), and the second-price auction with an item-specific reserve price (Riley and Samuelson, 1981, Cesa-Bianchi et al., 2014, Mohri and Medina, 2016).

### 4.4.1 Comparison with optimal auctions

Myerson's optimal auction cannot be applied in scenarios where the seller lacks prior knowledge of the bidders' value distributions. In contrast, Theorem 4 demonstrates the practical utility of COAD in these settings by guaranteeing revenues.

### 4.4.2 Comparison with first-price and second-price auctions

Unlike the first-price auction, which is not incentive-compatible, the second-price auction mechanism is known for its incentive compatibility. The COAD mechanism effectively bridges the first-price and second-price auction mechanisms. By Theorem 2, COAD ensures truthful bidding, similar to the second-price auction, but typically achieves higher revenue. This higher revenue is observed particularly when the highest bidder's reserve price lies between the highest and the second-highest bids. To illustrate this point, consider the following example.

Example 1 (Comparison with second-price auctions) If $m^{*}=1$, then the revenue from a second-price auction is 0 , while the revenue from $\operatorname{COAD}$ is that $\max \left\{0, \hat{v}_{1}^{L}\right\} \geq 0$. If $m^{*} \geq 2$, without loss of generality, let bidder $i=1$ have the highest value $v_{1}^{*}$ and bidder $i=2$ the second highest $v_{2}^{*}$. Assume that $v_{1}^{*}-v_{2}^{*} \geq 2$ and that the residuals $\varepsilon^{\prime}$ 's are standardized within the range $[-1,1]$. If the prediction rule is sufficiently accurate such that $\hat{\mu}_{n}=\mu$ and $S^{*}$ is equal to $q_{\alpha / 2}$, the $(1-\alpha / 2)$ quantile of the distribution of $\varepsilon$. Then the probability that the revenue of COAD surpasses that of a second-price auction is given by $\mathbb{P}\left(v_{1}^{*} \geq v_{1}^{L}>v_{2}^{*}\right) \geq$ $1-\alpha / 2$.

### 4.4.3 Comparison with item-Specific reserve price

The item-specific reserve price is characterized by a uniform reserve price applicable to all bidders for a given item. In contrast, the proposed COAD mechanism employs bidderspecific reserve prices, meaning the reserve price is tailored to each bidder for the same item. We present the following example to illustrate the potential for increased revenue when using bidder-specific reserve prices.

Example 2 (Comparison with item-specific reserve pricing auctions) Consider the distribution $F_{v^{*} \mid z^{*}=\tilde{z}}$ defined as follows: with probability $1 / H$, the value is $H$ where $H>1$ and is very large, and with probability $1-1 / H$, the value is 1 . The optimal auction could achieve an expected revenue of $2-1 / m^{*}$ when the seller knows the distribution of the bidders' values (Hartline, 2013), which approaches 2 as the number of bidders $m^{*}$ increases. In comparison, the best item-specific reserve price auction can only achieve an expected revenue of 1 Roughgarden and Schrijvers, 2016). By Theorems 3 and 4 , the COAD mechanism achieves an expected revenue of at least,

$$
\begin{aligned}
R_{1}^{C O A D \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) & \geq \frac{2(1-\alpha)}{(1+h) \mathcal{H}_{1}} W_{1}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right) \\
& =\frac{2(1-\alpha)}{1+h} \mathbb{E}\left(v_{1}^{*} \mid z^{*}=\tilde{z}\right)=\frac{2(1-\alpha)}{1+h}\left(2-\frac{1}{H}\right) .
\end{aligned}
$$

When $\alpha$ is small and $H$ is large, the expected revenue of COAD is at least $4 /(1+h)$, which exceeds 1 as long as $1 \leq h<3$. If $h=1$, then the expected revenue of COAD approaches the optimal revenue.

Moreover, bidder-specific reserve prices have been successfully implemented in real-world auctions to generate higher revenue. For example, search engines like Google and Yahoo! use advertiser-specific features to set unique reserve prices for each advertiser. This strategy ensures that the bids and prices per click for different advertisers have varying minimum thresholds, a strategy that not only generates higher revenues but also encourages advertisers to place high-quality ads (Even-Dar et al., 2008). This demonstrates the practical relevance of the bidder-specific reserve price strategy adopted by COAD. Additionally, COAD is the first auction mechanism that incorporates both bidder and item features in an online setting.

## 5 Simulation Studies

In this section, we provide simulation examples to support our theoretical findings. We compare the expected revenue achieved by COAD with that of the second-price auction and with the maximum expected social welfare. We set the miscoverage level at $\alpha=0.1$. In each simulation, we randomly generate $N$ iid data points based on the model specified in (4). We use half of the data as calibration data and the remaining half as training data.

### 5.1 A Low-Dimensional Example

The first example considers an auction where both the bidder's feature and the item's feature are one-dimensional. For $(x, z, v) \sim P$, let $z$ be uniformly drawn from $\mathcal{Z}=\{3,5,7\}$, $x$ be drawn from a normal distribution with mean $z / 10$ and variance 1, i.e., $x \sim \mathcal{N}(z / 10,1)$, and the residual $\varepsilon=v-\mu(x, z)$ be drawn from a truncated standard normal distribution in $[-1,1]$. Assume that

$$
\mu(x, z)=e^{x} z+1
$$

A similar example has been studied in Ostrovsky and Schwarz (2023). We employ eighthorder polynomial regression to fit the model $\mu(x, z)$. Further details on the consistency of this regression model are available in Appendix C.1.

Figure 2a presents boxplots illustrating the conditional coverage of the prediction intervals. These plots show the empirical distribution of $\mathbb{P}\left(v_{i}^{*} \in \hat{C}_{d u a l}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right)$ for $z^{*} \in \mathcal{Z}$, with $N=1000$ and $m^{*}=1000$. The results confirm that the conformal prediction method described in Section 3.2 effectively provides prediction intervals for bidders' values, achieving conditional coverage approximately equal to $1-\alpha$.

Figure 2b shows the effect of historical data size $N$ on expected revenue. Here $N \in$ $\{100,500,1000,2500,5000\}, m^{*}=50$, and the results are averaged over 1000 simulations. It is observed that as $N$ increases, both the maximum expected social welfare and the expected revenues from the second-price auction remain stable. In contrast, the expected revenue of COAD increases and surpasses that of the second-price auction. Moreover, the expected revenue from COAD approaches the maximum expected social welfare, which serves as the revenue benchmark discussed in Section 2.3. This benchmark can only be achieved when the seller knows the bidder's values $\vec{v}^{*}$ for the item $z^{*}$ before the auction, a scenario that is impractical in real-world settings. Hence, it demonstrates that COAD can asymptotically generate a large constant fraction of the optimal revenue, aligning with Theorem 4 .

Figure 2 c shows the boxplots illustrating the revenue difference between COAD and the second-price auction in each auction. That is, the COAD revenue minus the second-highest bid in each auction across different possible values of $z^{*}$, with $N=1000$ and $m^{*}=50$. It is seen that COAD frequently generates higher revenue than the second-price auction, as the majority of revenue differences are greater than 0 . These findings agree with the discussion in Section 4.4.

Figure 2 d illustrates the effect of the number of bidders $m^{*}$ on expected revenue, with $m^{*} \in\{50,100,150,200,250,300,350,400\}, N=1000$, and results averaged over 1000 simulations. As $m^{*}$ increases, the expected revenue of COAD increases, which confirms the predictions of Theorem 3. Moreover, the gap between the expected revenue of COAD and the maximum expected social welfare widens as $m^{*}$ increases, which is consistent with the implications of Theorem 4.


Figure 2: The numerical results for the example in Section 5.1 based on 1000 simulations. (a) Conditional coverage of prediction intervals for different $z^{*}$ values with $N=1000$ and $m^{*}=1000$. The red line indicates the target level of $1-\alpha=0.9$. (b) The effect of varying the number of historical data points $N \in\{100,500,1000,2500,5000\}$ on the expected revenue when $m^{*}=50$. (c) A comparison of the revenues with the second-price auction for $N=1000$ and $m^{*}=50$. The $y$-axis represents the COAD revenue minus the secondhighest bid in an auction. The red line at 0 indicates no difference. (d) The effect of varying the number of bidders $m^{*} \in\{50,100,150,200,250,300,350,400\}$ on expected revenue when $N=1000$.

### 5.2 A High-Dimensional Example Using Polynomial Regression

The second example studies an auction where both the bidder's and the item's features are 100 -dimensional, i.e., $x \in \mathbb{R}^{100}, z \in \mathcal{Z} \subset \mathbb{R}^{100}$, respectively. For any $(x, z, v) \sim P, z$ is uniformly selected from $\mathcal{Z}=\left\{\tilde{z}_{1}, \ldots, \tilde{z}_{q}\right\}$ with $q=30$. For each $i=1, \ldots, q$, we sample $\tilde{z}_{i}$ from a multivariate Gaussian distribution $\mathcal{N}\left(\boldsymbol{\mu}_{\boldsymbol{z}}, \boldsymbol{\Sigma}_{\boldsymbol{z}}\right)$, where $\boldsymbol{\mu}_{\boldsymbol{z}}=\overrightarrow{\mathbf{0}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{z}}=\boldsymbol{I}_{100}$. The bidder's feature $x$ is generated from a multivariate Gaussian distribution $\mathcal{N}\left(\boldsymbol{\mu}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}}\right)$, where $\boldsymbol{\mu}_{\boldsymbol{x}}=\left(\|z\|_{2}^{2} / 100,\|z\|_{2}^{2} / 100, \ldots,\|z\|_{2}^{2} / 100\right)$ and $\boldsymbol{\Sigma}_{\boldsymbol{x}}=\boldsymbol{I}_{100}$. The residual $\varepsilon=v-\mu(x, z)$ is drawn from a truncated standard normal distribution in $[-1,1]$. Consider the following regression model in (4),

$$
\mu(x, z)=\left(\beta_{1}^{T} x\right)^{2} \cdot\left[\sin ^{2}\left(\beta_{2}^{T} z\right)\right]+1,
$$

where each element of the vectors $\beta_{1} \in \mathbb{R}^{100}, \beta_{2} \in \mathbb{R}^{100}$ is independently drawn from a uniform distribution Unif $[-1,1]$. We employ the quadratic polynomial regression to fit the model $\mu(x, z)$.

Figure 3a shows a boxplot illustrating the conditional coverage of the prediction intervals for a randomly selected item with $N=10000$ and $m^{*}=50$. It confirms that the desired conditional coverage at the level $1-\alpha$ is achieved even in this high-dimensional setting.


Figure 3: The numerical results for the example in Section 5.2 based on 1000 simulations. (a) Conditional coverage of prediction intervals for a random item with $N=10000$ and $m^{*}=50$. The red line indicates the target level of $1-\alpha=0.9$. (b) A comparison of the revenue with the second-price auction for $N=10000$ and $m^{*}=50$. The $y$-axis represents the COAD revenue minus the second-highest bid in an auction. The red line at 0 indicates no difference. (c) The effect of varying the number of historical data points $N \in\{1000,3000,5000,7000,9000,11000,13000\}$ on expected revenue when $m^{*}=50$. (d) The effect of varying the number of bidders $m^{*} \in\{50,100,150,200,250,300\}$ on expected revenue when $N=20000$.

Figure 3 b presents a boxplot comparing the revenues between COAD and the secondprice auction in each auction. It shows the difference in revenue (COAD revenue minus the second-highest bid) for the same randomly selected item as in Figure 3a, with $N=10000$ and $m^{*}=50$. The plot demonstrates that COAD typically achieves a larger revenue than the second-price auction, corroborating the discussion in Section 4.4 in this high-dimensional setting.

Figure 3 c illustrates the effect of historical data size $N$ on expected revenue, with $N \in\{1000,3000,5000,7000,9000,11000,13000\}$ and $m^{*}=50$. For each fixed $N$, the same historical data are used for training. The results clearly show that COAD consistently outperforms the second-price auction in terms of expected revenue. Compared to the lowdimensional setting shown in Figure 2b, a high-dimensional setting requires more historical data to achieve higher revenue. This increase in data requirement is due to two factors: first, the complexity of the high-dimensional model necessitates more training data to effectively train the regression estimator; second, the extensive range of possible values for $z^{*}$ demands more calibration data to accurately produce the conditional prediction intervals.

Figure 3d shows the effect of the number of bidders $m^{*}$ on expected revenue, with $m^{*} \in\{50,100,150,200,250,300\}$ and $N=20000$. As $m^{*}$ increases, the expected revenue of COAD increases, which agrees with Theorem 3 in this high-dimensional setting.

### 5.3 A High-Dimensional Example Using Neural Networks

The third example considers an auction with a more complex regression model in a highdimensional setting. Both the bidder's and item's features are 100-dimensional, and $z$ follows the same distribution as in Section5.2. The bidder's feature $x$ is generated from a multivariate Gaussian distribution $\mathcal{N}\left(\boldsymbol{\mu}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}}\right)$, where $\boldsymbol{\mu}_{\boldsymbol{x}}=\left(\|z\|_{2}^{2} / 100,\|z\|_{2}^{2} / 100, \ldots,\|z\|_{2}^{2} / 100\right)$ and $\boldsymbol{\Sigma}_{\boldsymbol{x}}=8 \boldsymbol{I}_{100}$. The residual $\varepsilon=v-\mu(x, z)$ is drawn from Unif[-1,1]. Consider the
regression model in (4) as follows,

$$
\mu(x, z)=e^{\left|\beta_{1}^{T} x\right| / 40} \cdot\left[\cos ^{2}\left(\beta_{2}^{T} z\right)\right]+1
$$

where each element of the vectors $\beta_{1} \in \mathbb{R}^{100}, \beta_{2} \in \mathbb{R}^{100}$ is independently drawn from Unif $[-1,1]$. To fit the model $\mu(x, z)$, we employ a neural network regression with two hidden layers, each having 128 and 64 neurons respectively, and use LeakyReLU activation Maas et al., 2013) with $L_{2}$ regularization. The network also includes a $30 \%$ dropout layer to mitigate overfitting. We use the Adam optimizer with a learning rate of 0.001 and apply mean squared error as loss. The model is trained for 15 epochs with a batch size of 32 .


Figure 4: The numerical results for the example in Section 5.3 based on 1000 simulations. (a) Conditional coverage of prediction intervals for a random item with $N=10000$ and $m^{*}=50$. The red line indicates the target level of $1-\alpha=0.9$. (b) A comparison of the revenue with the second-price auction for $N=10000$ and $m^{*}=50$. The $y$-axis represents the COAD revenue minus the second-highest bid in an auction. The red line at 0 indicates no difference. (c) The effect of varying the number of historical data points $N \in\{1000,3000,5000,7000,9000,11000,13000\}$ on expected revenue when $m^{*}=50$. (d) The effect of varying the number of bidders $m^{*} \in\{50,100,150,200,250,300\}$ on expected revenue when $N=20000$.

Figure 4a shows a boxplot illustrating the conditional coverage of the prediction intervals for a randomly selected item with $N=10000$ and $m^{*}=50$. It confirms that the conformal prediction method still achieves the desired conditional coverage at the level $1-\alpha$ even in this complex high-dimensional setting.

Figure 4 b presents a boxplot comparing the revenues between COAD and the secondprice auction in each auction. It shows the difference between COAD revenue and the second-highest bid for the same randomly selected item as in Figure 4a, with $N=10000$ and $m^{*}=50$. The plot demonstrates that COAD typically achieves a larger revenue than the second-price auction, corroborating the discussion in Section 4.4 in this complex highdimensional setting.

Figure 4 c illustrates the effect of historical data size $N$ on expected revenue, with $N \in\{1000,3000,5000,7000,9000,11000,13000\}$ and $m^{*}=50$. For each fixed $N$, the same historical data are used for training. The results clearly show that COAD consistently outperforms the second-price auction in terms of expected revenue. However, compared to the example in Figure 3c, this complex high-dimensional setting shows a smaller revenue
difference between COAD and the second-price auction. This small difference can be attributed to the relatively small price differences among bidders, as indicated in Figure 4 c where the maximum expected social welfare is less than 15 . Additionally, the complexity and high dimensionality of the model make it challenging to accurately set bidder-specific reserve prices between the highest and second-highest bids, especially when these differences are small. In such cases, developing an accurate estimator, such as using neural network methods, can significantly improve COAD's revenue, as shown by Theorem 5 .

Figure 4d shows the effect of the number of bidders $m^{*}$ on expected revenue, with $m^{*} \in\{50,100,150,200,250,300\}$ and $N=20000$. As $m^{*}$ increases, the expected revenue of COAD increases, which agrees with Theorem 3 in this complex high-dimensional setting. Hence, it demonstrates the substantial practical value and the effectiveness of COAD in enhancing auction revenues as bidder participation grows.

## 6 Real Data Analysis

In this section, we illustrate our method with a real data set from eBay auctions, specifically featuring 149 auctions of Palm Pilot M515 PDAs, each lasting seven days. This data set is public (https://www.modelingonlineauctions.com/datasets). Here the bidders can place multiple bids, with the requirement that each subsequent bid must exceed the previous one. Our analysis focuses on each bidder's highest bid, regarded as their final bid. However, we exclude the bidders whose final bids were placed in the last half-day of the auction, as these may not truthfully reveal their private values, which could potentially be higher. Therefore, we consider only the bidders whose final bids were made during the first six and a half days of each auction. Additionally, we treat the seller's identity as an item feature, similar to a brand. We focus on auctions by three primary sellers, labeled as $\tilde{z}_{1}$ for 'syschannel', $\tilde{z}_{2}$ for 'michael-33', and $\tilde{z}_{3}$ for 'saveking', who are responsible for the majority of the auctions in the data set. Now we have 884 historical entries, with item features categorized into the set $\mathcal{Z}=\left\{\tilde{z}_{1}, \tilde{z}_{2}, \tilde{z}_{3}\right\}$. We define the bidder's features using three key elements: bid time, bidder rating, and the average of their historical bids. The historical data represents scenarios where, at each time point, a seller offers an item (Palm Pilot M515 PDA) for sale. Multiple bidders simultaneously enter eBay to bid, choosing their preferred seller. Each bidder can select only one seller at a time, and those who choose the same seller then participate in an auction.

In our analysis, we randomly select one auction from the data set to serve as the new auction for prediction. The remaining data is then split randomly into two equal parts: a training set and a calibration set. Our analysis includes only bidders from the new auction who have previously placed bids in the remaining data. For this new auction, we perform conformal prediction conditioned on different item features using polynomial regression, and then implement our mechanism. We set the miscoverage level at $\alpha=0.1$. In each experiment, we fix an item feature and conduct 50 new auctions conditioned on that item feature. To verify the practical effectiveness of COAD, we calculate the coverage probability of the prediction interval, the revenue difference between COAD and the secondprice auction, and the ratio of COAD's revenue to the maximum social welfare (i.e., the highest bid known post-auction) for each individual auction. We then compute the average


Figure 5: The numerical results of the real data example in Section 6, based on 1000 experiments. (a) The average conditional coverage of prediction intervals for different item features $z^{*}$. (b) A comparison of the average revenues with the second-price auction, for different item features. The $y$-axis represents the average of COAD revenue minus the secondhighest bid in each experiment. (c) The average ratio of the revenue of COAD to the maximum social welfare for different item features and varying historical data sizes $N \epsilon$ $\{200,250,300,350\}$.
coverage, average revenue difference, and average revenue ratio from the 50 random new auctions in each experiment. This process is repeated 1000 times.

Figure 5 ashows boxplots illustrating the average conditional coverage of the prediction intervals for different item features. It indicates that the conformal prediction method in Section 3.2 effectively provides prediction intervals with a high coverage guarantee in practice. Figure 5b presents boxplots comparing the difference between the average COAD revenue and the second-highest bid. It demonstrates that COAD consistently generates higher revenue than the second-price auction in real-world applications. Figure 5 c shows the effect of the historical data size $N$ on the average ratio of the revenue of COAD to the maximum social welfare, across different item features. Here, maximum social welfare represents the highest bid known post-auction for each auction, which is not known in advance. The results are averaged over 1000 simulations. It shows that the expected revenue of COAD consistently exceeds a significant fraction (over 0.83 ) of the optimal expected revenue. Moreover, increasing the historical data size enhances this revenue ratio.

We make three additional remarks on the findings from this data experiment. First, COAD does not require a large data set or an extensive number of features to provide robust revenue guarantees, which demonstrates its practical utility. This shows a distinct advantage over other methods, which often depend on a large bid data for the same item
(Cole and Roughgarden, 2014, Roughgarden and Schrijvers, 2016). In real-world data sets, the number of available data is often limited, making it challenging to achieve desired outcomes with auction methods that rely on learning the distribution of bidder bids. However, COAD employs a distribution-free, prediction interval-based approach that provides coverage-guaranteed confidence intervals regardless of the number of data. Consequently, this prediction interval-based methodology is not only easier to learn but also supported by theoretical guarantees for revenues.

Second, COAD efficiently leverages historical information. In our approach, each auction can yield as many data points as there are participants. Thus, even an auction with a single bidder can provide valuable data. This is in contrast to methods such as those described by Mohri and Medina (2016), which rely on the highest and second-highest bids and cannot employ data if only one bidder participates or may disregard additional data if more than two bidders participate in an auction.

Third, practical auctions, like those illustrated in this eBay auction example, often involve heterogeneous bidders and items across various rounds. Therefore, it is impractical to assume a fixed distribution of values from a consistent group of bidders or to analyze bidders' valuation distributions for identical items. COAD addresses this variability effectively by using the features of both bidders and items. In real-world scenarios, platforms usually have access to extensive feature data on both bidders and items, which could potentially further enhance the performance of COAD.

## 7 Conclusion

We introduce a new approach to online auction mechanism design, called the conformal online auction design (COAD), which leverages data-driven methods to maximize revenue while ensuring incentive compatibility. By using historical data to predict each bidder's valuation with uncertainty quantification and introducing bidder-specific reserve prices, COAD addresses the dynamic and heterogeneous nature of online auctions. This flexibility is particularly advantageous in applications like online advertising, where bidders' features and values can vary significantly. COAD employs conformal prediction techniques that accommodate any finite sample of historical data and does not rely on assumptions about known value distributions. Our simulations and real-data applications validate that COAD outperforms traditional revenue benchmarks, offering a useful alternative to existing online auction designs. The code for reproducing the numerical results in this paper is available at https://github.com/JialeHan22/Conformal-Online-Auction-Design.git.

There are several interesting future directions. One area of interest is adapting the COAD framework to online pricing problems, which corresponds to the special case where there is only one bidder (Goldberg et al., 2006). Another possible extension is to explore the multi-item case where multiple items are auctioned simultaneously (Daskalakis, 2015). A further direction is developing a dynamic auction model where bidders sequentially enter the market, such as an online travel agency (e.g., Expedia.com or Orbitz.com) selling airline tickets to a series of bidders, each potentially offering a different price.

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## Appendix

Appendix A provides a detailed explanation of the construction of the conformal prediction interval discussed in Section 3.2. Appendix B contains all the proofs of the theories presented in the paper. Specifically, Appendix B.1 presents the proof of Proposition 1. Appendix B. 2 provides the proof of Theorem 1; Appendix B.3 includes the proof of Theorem 2; Appendix B. 4 gives the proof of Theorem 33, Appendix B.5 contains the proof of Theorem 4, Appendix B.6 includes the proof of Theorem 5. Additionally, Appendix C presents supplementary experimental results related to Section 5.1.

## Appendix A. Details of Constructing Conformal Prediction Intervals

The conformal prediction method in this paper is based on Gibbs et al. (2023) and provides a conditional guarantee for the valuation of each bidder in [ $\mathrm{m}^{*}$ ]. The construction procedure consists of two main steps. Step 1 is to generate the primal prediction interval $\hat{\mathcal{C}}_{\text {primal }}$. Step 2 is to redefine $\hat{\mathcal{C}}_{\text {primal }}$ in terms of the dual formulation and obtain the dual prediction band $\hat{\mathcal{C}}_{\text {dual }}$.

Given the conformity score function $\mathcal{S}: \mathcal{X} \times \mathcal{Z} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ defined as $\mathcal{S}(\{x, z\}, v)=$ $\left|v-\hat{\mu}_{n}(x, z)\right|$ for any $(x, z, v) \in \mathcal{X} \times \mathcal{Z} \times \mathbb{R}_{\geq 0}$, let $\tilde{S}_{i}$ denote the score $\mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v_{i}^{*}\right)$ for $i \in\left[m^{*}\right]$, and $S_{1}, \ldots, S_{n}$ denote the calibration scores $\mathcal{S}\left(\left\{x_{1}, z_{1}\right\}, v_{1}\right), \ldots, \mathcal{S}\left(\left\{x_{n}, z_{n}\right\}, v_{n}\right)$ of all the calibration data. For each $i \in\left[m^{*}\right]$, from Assumption $3,\left(x_{i}^{*}, z^{*}, v_{i}^{*}\right)$ is iid of the $D_{\text {cal }}$, so that $\tilde{S}_{i}, S_{1}, \ldots, S_{n}$ are also iid random variables.

Let $\mathcal{G}=\left\{\left\{\mathcal{X}, \tilde{z}_{1}\right\}, \ldots,\left\{\mathcal{X}, \tilde{z}_{q}\right\}\right\}$, which is a finite collection of groups in $2^{\mathcal{X} \times \mathcal{Z}}$, and it divides the domain $\mathcal{X} \times \mathcal{Z}$ into $q$ pieces. And let

$$
\mathcal{F}=\left\{(x, z) \longmapsto \sum_{G \in \mathcal{G}} \beta_{G} \mathbb{I}\{\{x, z\} \in G\} \mid \beta_{G} \in \mathbb{R}, \quad \forall G \in \mathcal{G}\right\},
$$

which is a linear function space spanned by the identification function over $\mathcal{G}$.
The augmented quantile regression estimate for the bidder $i \in\left[m^{*}\right]$ and the item with feature $z^{*}$ is defined as:

$$
\begin{equation*}
\hat{g}_{S}^{i}=\underset{g \in \mathcal{F}}{\operatorname{argmin}} \frac{1}{n+1} \sum_{j=1}^{n} \ell_{\alpha}\left(g\left(x_{j}, z_{j}\right), S_{j}\right)+\frac{1}{n+1} \ell_{\alpha}\left(g\left(x_{i}^{*}, z^{*}\right), S\right) . \tag{13}
\end{equation*}
$$

where $\ell_{\alpha}$ is the "pinball" loss and $\alpha \in(0,1)$ :

$$
\ell_{\alpha}(\theta, R)=\left\{\begin{array}{l}
(1-\alpha)(R-\theta) \quad \text { if } R \geq \theta, \\
\alpha(\theta-R) \quad \text { if } R<\theta
\end{array}\right.
$$

Then, take the prediction interval for $v_{i}^{*}$, which is the value of the bidder $i$, to be

$$
\begin{equation*}
\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)=\left\{v: \mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right) \leq \hat{g}_{\mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)}^{i}\left(x_{i}^{*}, z^{*}\right)\right\} . \tag{14}
\end{equation*}
$$

From group-conditional coverage guarantee in Gibbs et al. (2023), we have that,

$$
\begin{equation*}
\mathbb{P}\left(v_{i}^{*} \in \hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right) \mid z^{*}=\tilde{z}\right) \geq 1-\alpha \text { for all } \tilde{z} \in \mathcal{Z} . \tag{15}
\end{equation*}
$$

Before getting the dual formulation, first, we note from equation (13) that $\hat{g}_{S}^{i}$ is the optimal solution of the following unconstrained optimization problem when $x=x_{i}^{*}, z=z^{*}$ :

$$
\begin{equation*}
\underset{g \in \mathcal{F}}{\operatorname{minimize}} \frac{1}{n+1} \sum_{j=1}^{n} \ell_{\alpha}\left(g\left(x_{j}, z_{j}\right), S_{j}\right)+\frac{1}{n+1} \ell_{\alpha}(g(x, z), S) \tag{16}
\end{equation*}
$$

Let $p_{j}=\left(S_{j}-g\left(x_{j}, z_{j}\right)\right) \mathbb{I}\left(S_{j} \geq g\left(x_{j}, z_{j}\right)\right)$, and $q_{j}=\left(g\left(x_{j}, z_{j}\right)-S_{j}\right) \mathbb{I}\left(S_{j}<g\left(x_{j}, z_{j}\right)\right)$, for $j=1,2, \ldots, n$. Let $p_{n+1}=(S-g(x, z)) \mathbb{I}(S \geq g(x, z))$, and $q_{n+1}=(g(x, z)-S) \mathbb{I}(S<g(x, z))$. Denote $\boldsymbol{p}=\left(p_{1}, p_{2}, \ldots, p_{n+1}\right)$, and $\boldsymbol{q}=\left(q_{1}, q_{2}, \ldots, q_{n+1}\right)$. Then the problem (16) can be re-formulated into the following (relaxed) constrained optimization problem:

$$
\begin{align*}
\underset{p, \boldsymbol{q} \in \mathbb{R}^{n+1}, g \in \mathcal{F}}{\operatorname{minimize}} & \sum_{j=1}^{n+1}\left[(1-\alpha) p_{j}+\alpha q_{j}\right] \\
\text { sbject to } & S_{j}-g\left(x_{j}, z_{j}\right)-p_{j}+q_{j}=0, j=1,2, \ldots, n,  \tag{17}\\
& S-g(x, z)-p_{n+1}+q_{n+1}=0 \\
& p_{j}, q_{j} \geq 0, j=1,2, \ldots, n+1
\end{align*}
$$

Hence, for the constrained problem (17), we can use the Lagrange multipliers $\boldsymbol{\eta}=$ $\left(\eta_{1}, \eta_{2}, \ldots, \eta_{n+1}\right), \boldsymbol{\gamma}=\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n+1}\right), \boldsymbol{\xi}=\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n+1}\right)$ and consider its Lagrangian

$$
\begin{aligned}
\underset{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\eta}, \boldsymbol{\gamma}, \boldsymbol{\xi} \in \mathbb{R}^{n+1}, g \in \mathcal{F}}{\operatorname{minimize}} \mathcal{L}(g, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\eta}, \boldsymbol{\gamma}, \boldsymbol{\xi})= & \sum_{j=1}^{n+1}\left[(1-\alpha) p_{j}+\alpha q_{j}\right]+\sum_{j=1}^{n} \eta_{j}\left(S_{j}-g\left(x_{j}, z_{j}\right)-p_{j}+q_{j}\right) \\
& +\eta_{n+1}\left(S-g(x, z)-p_{n+1}+q_{n+1}\right)-\sum_{j=1}^{n+1}\left(\gamma_{j} p_{j}+\xi_{j} q_{j}\right) .
\end{aligned}
$$

The Karush-Kuhn-Tucker (KKT) conditions for this Lagrangian also include

$$
\begin{aligned}
& \text { stationary equations } \Delta_{p} \mathcal{L}=\mathbf{0}, \Delta_{q} \mathcal{L}=\mathbf{0}, \Delta_{g} \mathcal{L}=\mathbf{0} \\
& \text { dual feasibility } \gamma_{j}, \xi_{j} \geq 0, j=1,2, \ldots, n+1 \\
& \text { complementary slackness } \gamma_{j} p_{j}=0, \xi_{j} q_{j}=0, j=1,2, \ldots, n+1
\end{aligned}
$$

Now if we focus on finding the optimal solution for $\boldsymbol{\eta}$, from the stationary equations of $\boldsymbol{p}, \boldsymbol{q}$, we have that,

$$
\begin{align*}
& \boldsymbol{\gamma}=(1-\alpha) \cdot \mathbf{1}-\boldsymbol{\eta},  \tag{18}\\
& \boldsymbol{\xi}=\alpha \cdot \mathbf{1}+\boldsymbol{\eta} .
\end{align*}
$$

By (18) and dual feasibility, it follows that the constraint on $\boldsymbol{\eta}$ is defined as,

$$
-\alpha \cdot \mathbf{1} \leq \boldsymbol{\eta} \leq(1-\alpha) \cdot \mathbf{1} .
$$

Consequently, we can derive the dual formulation with respect to $\boldsymbol{\eta}$,

$$
\begin{align*}
\underset{\eta \in \mathbb{R}^{n+1}}{\operatorname{maximize}} & \sum_{j=1}^{n} \eta_{j} S_{j}+\eta_{n+1} S+\min _{g \in \mathcal{F}}\left\{-\sum_{j=1}^{n} \eta_{j} g\left(x_{j}, z_{j}\right)-\eta_{n+1} g(x, z)\right\},  \tag{19}\\
\text { sbject to } & -\alpha \leq \eta_{j} \leq 1-\alpha, 1 \leq j \leq n+1 .
\end{align*}
$$

Gibbs et al. (2023) has shown that the primal-dual pair (17) and (19) satisfies the strong duality by Slater's condition; thus, the optimal primal-dual solution will also satisfy the KKT conditions. If we let $\left(\hat{g}_{S}^{i}, \boldsymbol{\eta}_{i}^{S}\right)$ denote the primal-dual solution for $(g, \boldsymbol{\eta})$ given $S$ when $x=x_{i}^{*}, z=z^{*}$, from (18) and the complementary slackness conditions when $j=n+1$, we obtain

$$
\eta_{i, n+1}^{S}= \begin{cases}-\alpha, & \text { if } S<\hat{g}_{S}^{i}\left(x_{i}^{*}, z^{*}\right),  \tag{20}\\ {[-\alpha, 1-\alpha],} & \text { if } S=\hat{g}_{S}^{i}\left(x_{i}^{*}, z^{*}\right), \\ 1-\alpha, & \text { if } S>\hat{g}_{S}^{i}\left(x_{i}^{*}, z^{*}\right) .\end{cases}
$$

From (20) we know that checking whether $S \leq \hat{g}_{S}^{i}\left(x_{i}^{*}, z^{*}\right)$ is nearly equivalent to check $\eta_{i, n+1}^{S}<1-\alpha$. Intuitively, we can go back to (14) by letting $S=\mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)$ in 20), so that we can replace $\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)$ by the dual prediction interval,

$$
\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)=\left\{v: \eta_{i, n+1}^{\mathcal{S}}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)<1-\alpha\right\} .
$$

We can find that $\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)$ is obtained from $\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)$ by removing a negligible portion of the points $v$ that lie on the boundary,

$$
\left\{v: \mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)=\hat{g}_{\mathcal{S}}^{i}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)\left(x_{i}^{*}, z^{*}\right)\right\} .
$$

The following theorem states that $S \mapsto \eta_{n+1}^{S}$ is non-decreasing.
Theorem 6 (Gibbs et al. (2023) Theorem 4). For all maximizes $\left\{\eta_{n+1}^{S}\right\}_{S \in \mathbb{R}}$ of (19), $S \mapsto$ $\eta_{n+1}^{S}$ in non-decreasing in $S$.

Using Theorem 6. if we compute a $S_{i}^{*}$ which is the largest value of $S$ such that $\eta_{i, n+1}^{S}<1-\alpha$, we can rewrite the dual prediction interval as

$$
\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)=\left\{v: \mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right) \leq S_{i}^{*}\right\}=\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S_{i}^{*}, \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S_{i}^{*}\right] .
$$

Gibbs et al. (2023) gives the algorithm to compute the $S_{i}^{*}$ by using the binary search, and similar to the (15), it also follows:

$$
\mathbb{P}\left(v_{i}^{*} \in \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right) \mid z^{*}=\tilde{z}\right) \geq 1-\alpha \text { for all } \tilde{z} \in \mathcal{Z} .
$$

In our setting, based on the definition of $\mathcal{F}$, for any $g \in \mathcal{F}$, the value of $g(x, z)$ is unrelated to $x$, so that $S_{1}^{*}=S_{2}^{*}=\cdots=S_{m^{*}}^{*}$, we can denote them simply by $S^{*}$.

## Appendix B. Proofs of Main Results

In this section, we provide technical proof for the main results. Appendix B. 1 presents the proof of Proposition 1; Appendix B. 2 provides the proof of Theorem 1; Appendix B.3 includes the proof of Theorem 2, Appendix B. 4 gives the proof of Theorem 3, Appendix B. 5 contains the proof of Theorem 4. Appendix B. 6 includes the proof of Theorem 5.

## B. 1 Proof of Proposition 1

Proof In the analysis presented in of Appendix A , it is established that $\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)$ is derived from $\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)$ by removing a negligible portion of the points $v$ lying on the boundary $\left\{v: \mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)=\hat{g}_{\mathcal{S}}^{i}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)\left(x_{i}^{*}, z^{*}\right)\right\}$. Thus, we can express this relationship as follows:

$$
\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right) \subset \hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right), \text { almost surely. }
$$

From equation 10, it is evident that

$$
\hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)=\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}, \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right] .
$$

Hence, we can assume that

$$
\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)=\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}-\delta_{1}, \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}+\delta_{2}\right],
$$

where $\delta_{1}, \delta_{2} \geq 0$, almost surely.
From the results in Gibbs et al. (2023), the following inequalities are established:

$$
\begin{equation*}
1-\alpha \leq \mathbb{P}\left(v_{i}^{*} \in \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)\right) \leq 1-\alpha+\frac{q}{n+1}, \tag{21}
\end{equation*}
$$

and similarly,

$$
\begin{equation*}
1-\alpha \leq \mathbb{P}\left(v_{i}^{*} \in \hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right) \leq 1-\alpha+\frac{q}{n+1} .\right. \tag{22}
\end{equation*}
$$

The inequality (21) can be reformulated as follows:

$$
\begin{align*}
& 1-\alpha \leq \mathbb{P}\left(\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*} \leq v_{i}^{*} \leq \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right) \leq 1-\alpha+\frac{q}{n+1} \\
& \Longleftrightarrow 1-\alpha \leq \mathbb{P}\left(\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*} \leq \mu\left(x_{i}^{*}, z^{*}\right)+\varepsilon_{i}^{*} \leq \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right) \leq 1-\alpha+\frac{q}{n+1} . \tag{23}
\end{align*}
$$

Let $F_{\varepsilon}$ and $f_{\varepsilon}$ denote the distribution and density functions of $\varepsilon_{i}^{*}$, respectively. It follows that $f_{\varepsilon}$ must be a bounded function. Suppose a common uniform bound for both $f_{\varepsilon}$ and $f_{\varepsilon}^{\prime}$ is $M>0$. Denote $\Delta_{n}\left(x_{i}^{*}, z^{*}\right)=\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-\mu\left(x_{i}^{*}, z^{*}\right)$, we have that,

$$
\begin{align*}
& \mathbb{P}\left(\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*} \leq \mu\left(x_{i}^{*}, z^{*}\right)+\varepsilon_{i}^{*} \leq \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right) \\
& =\mathbb{P}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*} \leq \varepsilon_{i}^{*} \leq \Delta_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right)  \tag{24}\\
& =\mathbb{E}\left[F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right)-F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}\right)\right],
\end{align*}
$$

where the last equation holds by Fubini's theorem and independence between $\varepsilon_{i}^{*}$ and $\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right), S^{*}\right)$.

Employing a first-order Taylor expansion at $S^{*}$ and $-S^{*}$, we obtain

$$
\begin{align*}
& F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right) \\
& \quad=F_{\varepsilon}\left(S^{*}\right)+\Delta_{n}\left(x_{i}^{*}, z^{*}\right) f_{\varepsilon}\left(S^{*}\right)+\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right) R\left(S^{*}, \Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right) \\
& F_{\varepsilon}\left(\Delta_{n}\left(x_{i}, z^{*}\right)-S^{*}\right)  \tag{25}\\
& \quad=F_{\varepsilon}\left(-S^{*}\right)+\Delta_{n}\left(x_{i}^{*}, z^{*}\right) f_{\varepsilon}\left(-S^{*}\right)+\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right) R\left(-S^{*}, \Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right) .
\end{align*}
$$

where $R(x, y)=\frac{1}{2} \int_{0}^{1}(1-u) f_{\varepsilon}^{\prime}(x+u y) d u$ satisfies

$$
\sup _{x, y \in \mathbb{R}}|R(x, y)| \leq M / 4
$$

Plug (25) into (24), we have that,

$$
\begin{aligned}
& \mathbb{E}\left[F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right)-F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}\right)\right] \\
& =\mathbb{E}\left[F_{\varepsilon}\left(S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)+\Delta_{n}\left(x_{i}^{*}, z^{*}\right)\left(f_{\varepsilon}\left(S^{*}\right)-f_{\varepsilon}\left(-S^{*}\right)\right)+\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right) W\right] \\
& =\mathbb{E}\left[F_{\varepsilon}\left(S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)+\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right) W\right],
\end{aligned}
$$

where the second equation holds because of the Assumption 2 and $W=R\left(S^{*}, \Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right)$ -$R\left(-S^{*}, \Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right)$ that satisfies $|W| \leq M / 2$, almost surely. Based on Assumption 4, it follows that $\mathbb{E}\left[\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right)\right]=O\left(n^{-2 \tau}\right)$. So we have that,

$$
\begin{equation*}
\mathbb{E}\left[F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right)-F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}\right)\right]=\mathbb{E}\left[F_{\varepsilon}\left(S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)\right]+O\left(n^{-2 \tau}\right) \tag{26}
\end{equation*}
$$

Combining (23), (24), and (26), we have that,

$$
\begin{aligned}
& 1-\alpha \leq \mathbb{E}\left[F_{\varepsilon}\left(S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)\right]+O\left(n^{-2 \tau}\right) \leq 1-\alpha+\frac{q}{n+1} \\
\Longleftrightarrow & 1-\alpha \leq \mathbb{P}\left(-S^{*} \leq \varepsilon_{i}^{*} \leq S^{*}\right)+O\left(n^{-2 \tau}\right) \leq 1-\alpha+\frac{q}{n+1} .
\end{aligned}
$$

Thus, it follows that,

$$
\begin{equation*}
\mathbb{P}\left(-\hat{S}^{*} \leq \varepsilon_{i}^{*} \leq \hat{S}^{*}\right)=1-\alpha+O\left(n^{-\min \{2 \tau, 1\}}\right) \tag{27}
\end{equation*}
$$

Then, doing the similar calculation to $\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right)$, from (22), we can obtain:

$$
\begin{align*}
& \mathbb{P}\left(\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}-\delta_{1} \leq \mu\left(x_{i}^{*}, z^{*}\right)+\varepsilon_{i}^{*} \leq \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}+\delta_{2}\right) \\
& =\mathbb{E}\left[F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}+\delta_{2}\right)-F_{\varepsilon}\left(\Delta_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}-\delta_{1}\right)\right] \\
& =\mathbb{E}\left[F_{\varepsilon}\left(S^{*}+\delta_{2}\right)-F_{\varepsilon}\left(-S^{*}-\delta_{1}\right)\right.  \tag{28}\\
& \left.\quad+\Delta_{n}\left(x_{i}^{*}, z^{*}\right)\left(f_{\varepsilon}\left(S^{*}+\delta_{2}\right)-f_{\varepsilon}\left(-S^{*}-\delta_{1}\right)\right)+\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right) W^{*}\right],
\end{align*}
$$

where $W^{*}=R\left(S^{*}+\delta_{2}, \Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right)-R\left(-S^{*}-\delta_{1}, \Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right)$ that satisfies $|W| \leq M / 2$, almost surely, and $\left|f_{\varepsilon}\left(S^{*}+\delta_{2}\right)-f_{\varepsilon}\left(-S^{*}-\delta_{1}\right)\right|<2 M$, almost surely. Based on Assumption 4 and Jensen's equality, we derive that,

$$
\begin{equation*}
\left|\mathbb{E}\left[\Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right]\right| \leq \mathbb{E}\left[\left|\Delta_{n}\left(x_{i}^{*}, z^{*}\right)\right|\right] \leq \sqrt{\mathbb{E}\left[\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right)\right]}=O\left(n^{-\tau}\right) . \tag{29}
\end{equation*}
$$

Hence, it follows that

$$
\mathbb{E}\left[\Delta_{n}\left(x_{i}^{*}, z^{*}\right)\left(f_{\varepsilon}\left(S^{*}+\delta_{2}\right)-f_{\varepsilon}\left(-S^{*}-\delta_{1}\right)\right)\right]=O\left(n^{-\tau}\right),
$$

and

$$
\mathbb{E}\left[\Delta_{n}^{2}\left(x_{i}^{*}, z^{*}\right) W^{*}\right]=O\left(n^{-2 \tau}\right) .
$$

Consequently, plugging into (28), we have that,

$$
\begin{align*}
& \mathbb{P}\left(\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}-\delta_{1} \leq \mu\left(x_{i}^{*}, z^{*}\right)+\varepsilon_{i}^{*} \leq \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}+\delta_{2}\right) \\
& =\mathbb{E}\left[F_{\varepsilon}\left(S^{*}+\delta_{2}\right)-F_{\varepsilon}\left(-S^{*}-\delta_{1}\right)\right]+O\left(n^{-\tau}\right) \tag{30}
\end{align*}
$$

Plugging (30) into the inequality (22), we have that,

$$
\begin{aligned}
& 1-\alpha \leq \mathbb{E}\left[F_{\varepsilon}\left(S^{*}+\delta_{2}\right)-F_{\varepsilon}\left(-S^{*}-\delta_{1}\right)\right]+O\left(n^{-\tau}\right) \leq 1-\alpha+\frac{q}{n+1} \\
\Longleftrightarrow & 1-\alpha \leq \mathbb{P}\left(-S^{*}-\delta_{1} \leq \varepsilon_{i}^{*} \leq S^{*}+\delta_{2}\right)+O\left(n^{-\tau}\right) \leq 1-\alpha+\frac{q}{n+1} .
\end{aligned}
$$

Subsequently, from the inequality above, we derive that,

$$
\begin{equation*}
\mathbb{P}\left(-S^{*}-\delta_{1} \leq \varepsilon_{i}^{*} \leq S^{*}+\delta_{2}\right)=1-\alpha+O\left(n^{-\min \{\tau, 1\}}\right) \tag{31}
\end{equation*}
$$

Let $C=\left[-S^{*}, S^{*}\right] \Delta\left[-S^{*}-\delta_{1}, S^{*}+\delta_{2}\right]$. Upon comparing (27) and (31), it follows that $\mathbb{P}\left(\varepsilon_{i}^{*} \in C\right)=O\left(n^{-\min \{\tau, 1\}}\right)$, thereby indicating that,

$$
\int_{C} f_{\varepsilon}(x) d x=O\left(n^{-\min \{\tau, 1\}}\right) .
$$

Given that $\int_{C} f_{\varepsilon}(x) d x \geq M \int_{C} d x=M \cdot L(C)$, it follows that $L(C)=O\left(n^{-\min \{\tau, 1\}}\right)$.
Recall that

$$
\begin{aligned}
& \hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right) \Delta \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right) \\
& =\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}, \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}\right] \Delta\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}-\delta_{1}, \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)+S^{*}+\delta_{2}\right] .
\end{aligned}
$$

Consequently, we have that

$$
L(C)=L\left(\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right) \Delta \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)\right), \quad \text { almost surely. }
$$

Therefore,

$$
L\left(\hat{\mathcal{C}}_{\text {primal }}\left(x_{i}^{*}, z^{*}\right) \Delta \hat{\mathcal{C}}_{\text {dual }}\left(x_{i}^{*}, z^{*}\right)\right)=O\left(n^{-\min \{\tau, 1\}}\right) .
$$

This completes the proof of Proposition 1 .

## B. 2 Proof of Theorem 1

Proof Based on Assumption 2, given that the value of $v=\mu(x, z)+\varepsilon$ is always non-negative and the independence between $\varepsilon$ and $x$, we have that,

$$
\begin{equation*}
\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)-\max _{\varepsilon}|\varepsilon| \geq 0, \tag{32}
\end{equation*}
$$

where the maximum is taken over the support of the density function of $\varepsilon$.
From the construction procedure of the dual prediction interval outlined in Appendix A. it is evident that $0<S^{*} \leq \hat{g}_{S^{*}}^{i}\left(x_{i}^{*}, z^{*}\right)<\infty$ for any $i \in\left[m^{*}\right]$. By referring to the definition of the augmented quantile regression estimate $\hat{g}_{S^{*}}^{l}$, as provided in (13), we can infer $\hat{g}_{S^{*}}^{i}\left(x_{i}^{*}, z^{*}\right) \leq \max \left\{S_{1}, S_{2}, \ldots, S_{n}, S^{*}\right\}$. Given that $S^{*}<\infty$, it can be deduced that
at least one of the points among $z_{1}, \ldots, z_{n}$ is equal to $z^{*}$. This is because otherwise, $\mathcal{S}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)=\hat{g}_{\mathcal{S}}^{i}\left(\left\{x_{i}^{*}, z^{*}\right\}, v\right)\left(x_{i}^{*}, z^{*}\right)$ always holds for any $v \in \mathbb{R}_{\geq 0}$, therefore resulting in $S^{*}=\infty$. Additionally, considering that $S^{*} \leq \hat{g}_{S^{*}}^{i}\left(x_{i}^{*}, z^{*}\right)$ together with the definition of the pin-loss function, we have that $S^{*} \leq \max \left\{S_{1}, S_{2}, \ldots, S_{n}\right\}$. Hence, we can derive that,

$$
\begin{align*}
S^{*} & \leq \max \left\{S_{1}, S_{2}, \ldots, S_{n}\right\} \\
& =\max \left\{\left|v_{1}-\hat{\mu}_{n}\left(x_{1}, z_{1}\right)\right|,\left|v_{2}-\hat{\mu}_{n}\left(x_{2}, z_{2}\right)\right|, \ldots,\left|v_{n}-\hat{\mu}_{n}\left(x_{n}, z_{n}\right)\right|\right\} . \tag{33}
\end{align*}
$$

If we denote $\varepsilon_{i}=v_{i}-\mu\left(x_{i}, z_{i}\right)$ for $i=1,2, \ldots, n$, then (33) implies:

$$
\begin{align*}
S^{*} & \leq \max \left\{\left|\mu\left(x_{1}, z_{1}\right)-\hat{\mu}_{n}\left(x_{1}, z_{1}\right)+\varepsilon_{1}\right|, \ldots,\left|\mu\left(x_{n}, z_{n}\right)-\hat{\mu}_{n}\left(x_{n}, z_{n}\right)+\varepsilon_{n}\right|\right\} \\
& \leq \max _{(x, z) \in \mathcal{X} \times \mathcal{Z}}\left|\hat{\mu}_{n}(x, z)-\mu(x, z)\right|+\max _{\varepsilon}|\varepsilon| \\
& \leq \max _{(x, z) \in \mathcal{X} \times \mathcal{Z}}\left|\hat{\mu}_{n}(x, z)-\mu(x, z)\right|+\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)  \tag{34}\\
& =\delta_{n} .
\end{align*}
$$

where the last inequality holds because of (32).
Recalling the definition of $\hat{v}_{i}^{L}=\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*}$. If $\hat{\mu}_{n}=\mu$ holds almost surely, then according to (34), it follows that $S^{*} \leq \min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)$. Hence, we can derive that,

$$
\hat{v}_{i}^{L} \geq \hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z)=\mu\left(x_{i}^{*}, z^{*}\right)-\min _{(x, z) \in \mathcal{X} \times \mathcal{Z}} \mu(x, z) \geq 0, \text { almost surely. }
$$

This completes the proof of Theorem 1 .

## B. 3 Proof of Theorem 2

Proof In the proposed COAD mechanism, for any bidder $i \in\left[m^{*}\right], a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=1$ implies,

$$
c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)=\max _{k \in\left[m^{*}\right]} c_{k}\left(v_{k}^{*}, x_{k}^{*}, z^{*}\right)>0 .
$$

In other words, a bidder who has a virtual value greater than all other bidders is guaranteed to win the item. Conversely, if a bidder's virtual value is less than that of any other bidder, that bidder will not win the item. That is,

$$
a_{i}\left(b_{i}^{*}, \vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right)= \begin{cases}1, & b_{i}^{*}>r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right), \\ 0, & b_{i}^{*}<r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) .\end{cases}
$$

The well-known envelope formula Myerson, 1981) indicates that if the payment is,

$$
\begin{equation*}
p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right) v_{i}^{*}-\int_{0}^{v_{i}^{*}} a_{i}\left(b_{i}^{*}, \vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) d b_{i}^{*}, \tag{35}
\end{equation*}
$$

for any $i \in\left[m^{*}\right], \vec{v}^{*} \in \mathbb{R}_{\geq 0}^{m^{*}},\left(\vec{x}^{*}, z^{*}\right) \in \mathcal{X}^{m^{*}} \times \mathcal{Z}$, then the mechanism is both IC and IR. Now we plug in the allocation rule into the envelope formula (35). Note that

$$
\int_{0}^{v_{i}^{*}} a_{i}\left(b_{i}^{*}, \vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) d b_{i}^{*}= \begin{cases}v_{i}^{*}-r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right), & v_{i}^{*} \geq r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right), \\ 0, & v_{i}^{*} \leq r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right) .\end{cases}
$$

On the one hand, if the allocation rule $a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=1$, it follows that $v_{i}^{*} \geq r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right)$, so that we can deduce $p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=v_{i}^{*}-\left(v_{i}^{*}-r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right)\right)=r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right)$. On the other hand, if $a_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=0$, it holds that $v_{i}^{*} \leq r_{i}\left(\vec{v}_{-i}^{*}, \vec{x}^{*}, z^{*}\right)$, from which we deduce that $p_{i}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=0$. The payment (8) is thus derived by the envelop formula. Following Myerson's result (Myerson, 1981), the COAD mechanism in Algorithm 1 has the IC and IR properties, which completes the proof of Theorem 2.

## B. 4 Proof of Theorem 3

Proof For any integer $m^{*} \geq 1$, to establish the desired result, it suffices to demonstrate that,

$$
R_{m^{*}+1}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \geq R_{m^{*}}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \text {, almost surely. }
$$

Subsequently, by employing recursion, the desired result can be established.
For any $m^{*}$ bidders with valuation $\left\{v_{1}^{*}, \ldots, v_{m^{*}}^{*}\right\} \in \mathbb{R}_{\geq 0}^{m^{*}}$, by using the COAD mechanism, we can calculate the virtual value $c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)$ of each bidder. If $\max _{i \in\left[m^{*}\right]}\left(c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)\right)=$ 0 , the seller keeps the item and gets the 0 revenue.

If $\max _{i \in\left[m^{*}\right]}\left(c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)\right)>0$, without generation, suppose that the first $k \in \mathbb{N}_{+}$bidders among these $m^{*}$ bidders satisfy $v_{i}^{*} \geq \hat{v}_{i}^{L}$. We can rewrite the set $\left\{v_{1}^{*}, \ldots, v_{k}^{*}, \hat{v}_{1}^{L}, \ldots, \hat{v}_{k}^{L}\right\}$ as $\left\{a_{1}, \ldots, a_{2 k}\right\}$ and let $a_{(2 k-1)}$ be the second largest value among $\left\{a_{1}, \ldots, a_{2 k}\right\}$. Then the COAD mechanism will sell the item with price $\max \left\{0, a_{(2 k-1)}\right\}$.

Now, for each new bidder joining the auction with a valuation of $v_{m^{*}+1}^{*} \in \mathbb{R}_{\geq 0}$, we can implement the COAD mechanism for the $m^{*}+1$ bidders.

- If $v_{m^{*}+1}^{*}<\hat{v}_{m^{*}+1}^{L}$, then implementing the COAD mechanism for these $m^{*}+1$ bidders is equivalent to implementing the COAD mechanism for the first $m^{*}$ bidders, which means that the COAD mechanism will also sell the item at price $\max \left\{0, a_{(2 k-1)}\right\}$ if $\max _{i \in\left[m^{*}\right]}\left(c_{i}\left(v_{i}^{*}, x_{i}^{*}, z^{*}\right)\right)>0$, otherwise keep the item and get 0 revenue.
- If $v_{m^{*}+1}^{*} \geq \hat{v}_{m^{*}+1}^{L}$, then we define $a_{2 k+1}$ as $v_{m^{*}+1}^{*}$ and $a_{2 k+2}$ as $\hat{v}_{m^{*}+1}^{L}$, and let $a_{(2 k+1)}$ denote the second highest value among $\left\{a_{1}, \ldots, a_{2 k}, a_{2 k+1}, a_{2 k+2}\right\}$. In such a scenario, the COAD mechanism will sell the item with price $\max \left\{0, a_{(2 k+1)}\right\}$. It follows that $a_{(2 k+1)} \geq a_{(2 k-1)}$ for any $k \geq 1$, given that the second highest value in the set $\left\{a_{1}, \ldots, a_{2 k}\right\}$ will not exceed the second highest value in the set $\left\{a_{1}, \ldots, a_{2 k}, a_{2 k+1}, a_{2 k+2}\right\}$.
Based on the two different cases, conditioning on any first $m^{*}$ bidders, the revenue generated by the COAD mechanism on $m^{*}+1$ bidders will be no less than that achieved with $m^{*}$ bidders for any additional bidder $m^{*}+1$, thus we have that,

$$
R_{m^{*}+1}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \geq R_{m^{*}}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \text {, almost surely. }
$$

Subsequently, employing recursion, we finalize the proof of Theorem 3 .

## B. 5 Proof of Theorem 4

Proof of the case when $m^{*}=1$ : By equation (5), it is defined that,

$$
\begin{equation*}
R_{1}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right)=\mathbb{E}\left[p_{1}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right] \tag{36}
\end{equation*}
$$

Under the COAD mechanism and $m^{*}=1$, we have $p_{1}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right)=\hat{v}_{1}^{L} \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\}$. Thus,

$$
\mathbb{E}\left[p_{1}\left(\vec{v}^{*}, \vec{x}^{*}, z^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]=\mathbb{E}\left[\hat{v}_{1}^{L} \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]
$$

By the definition of $\hat{v}_{1}^{L}$, we have that,

$$
\begin{align*}
& \mathbb{E}\left[\hat{v}_{1}^{L} \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right] \\
& =\mathbb{E}\left[\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right) \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right] \\
& =\mathbb{E}\left[\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right] \mathbb{E}\left[\mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]  \tag{37}\\
& \quad+\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, Z=\tilde{z}\right),
\end{align*}
$$

where the last equality uses the definition of covariance.
Given that the value of $\mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\}$ is equal to or greater than $\mathbb{I}\left\{\hat{v}_{1}^{U} \geq v_{1}^{*} \geq \hat{v}_{1}^{L}\right\}$ almost surely, we have,

$$
\begin{align*}
& \mathbb{E}\left[\mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right] \\
& \geq \mathbb{E}\left[\mathbb{I}\left\{\hat{v}_{1}^{U} \geq v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]  \tag{38}\\
& =\mathbb{P}\left(\hat{v}_{1}^{U} \geq v_{1}^{*} \geq \hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right) .
\end{align*}
$$

It is known that (see, Gibbs et al., 2023),

$$
\left|\mathbb{P}\left(v_{1}^{*} \in \hat{\mathcal{C}}_{\text {dual }}\left(x_{1}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right)-(1-\alpha)\right|=O_{\mathbb{P}}\left(\sqrt{\frac{q}{n}}\right) .
$$

Hence,

$$
\begin{equation*}
\mathbb{P}\left(\hat{v}_{1}^{U} \geq v_{1}^{*} \geq \hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right)=1-\alpha+O_{\mathbb{P}}\left(n^{-1 / 2}\right) \tag{39}
\end{equation*}
$$

By combining equations (36)-(39), we obtain that,

$$
\begin{align*}
& R_{1}^{\operatorname{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \\
& \geq \mathbb{E}\left[\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]\left(1-\alpha+O_{\mathbb{P}}\left(n^{-1 / 2}\right)\right)  \tag{40}\\
& \quad+\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right) .
\end{align*}
$$

From the model of $v_{1}^{*}$ (i.e., equation (4), it follows that $v_{1}^{*}$ can be defined as:

$$
\begin{equation*}
v_{1}^{*}=\mu\left(x_{1}^{*}, z^{*}\right)+\varepsilon_{1}^{*}, \tag{41}
\end{equation*}
$$

where the residual $\varepsilon_{1}^{*}=v_{1}^{*}-\mu\left(x_{1}^{*}, z^{*}\right)$ is independent of $\left(x_{1}^{*}, z^{*}\right)$. Together with (41) and the definition of $\hat{v}_{1}^{L}$, we have that,

$$
\begin{align*}
& \operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right) \\
& =\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\mu\left(x_{1}^{*}, z^{*}\right)+\varepsilon_{1}^{*} \geq \hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right)  \tag{42}\\
& =\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\varepsilon_{1}^{*} \geq \Delta_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right),
\end{align*}
$$

where $\Delta_{n}\left(x_{1}^{*}, z^{*}\right)=\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-\mu\left(x_{1}^{*}, z^{*}\right)$.
Let $E=\left[\Delta_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \infty\right) \Delta\left[-S^{*}, \infty\right)$, we have that,

$$
\begin{equation*}
\mathbb{I}\left\{\varepsilon_{1}^{*} \geq \Delta_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right\} \leq \mathbb{I}\left\{\varepsilon_{1}^{*} \geq-S^{*}\right\}+\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\}, \text { almost surely. } \tag{43}
\end{equation*}
$$

By the results in Appendix A. $S^{*}$ is the largest value such that $\eta_{1, n+1}^{S}<1-\alpha$. Consequently, given $D_{\text {train }}, D_{\text {cal }}$, and $z^{*}=\tilde{z}$, then $S^{*}$ is determined accordingly. Additionally, when $D_{\text {train }}$ is given, $\hat{\mu}_{n}(\cdot)$ is also determined, so that $\Delta_{n}(\cdot)$ is determined.

Plugging the inequality (43) into (42), we have that,

$$
\begin{align*}
& \operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right) \\
& \leq \operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\varepsilon_{1}^{*} \geq-S^{*}\right\}+\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right) \\
& =\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\varepsilon_{1}^{*} \geq-S^{*}\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)  \tag{44}\\
& \quad+\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right) \\
& =\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right),
\end{align*}
$$

where the last equation holds because $\varepsilon_{1}^{*}$ is independent of $x_{1}^{*}$. Subsequently, by CauchySchwarz inequality, it follows that

$$
\begin{align*}
& \operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right) \\
& \leq \sqrt{\operatorname{Var}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right) \operatorname{Var}\left(\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)}  \tag{45}\\
& =\sqrt{\operatorname{Var}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right) \mid \hat{\mu}_{n}, z^{*}=\tilde{z}\right) \operatorname{Var}\left(\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)} .
\end{align*}
$$

By assumption (11), we have that,

$$
\sqrt{\operatorname{Var}\left(\hat{\mu}_{n}\left(x_{1}, z^{*}\right) \mid \hat{\mu}_{n}, z^{*}=\tilde{z}\right)} \leq \sqrt{C}, \text { almost surely. }
$$

Observe that,

$$
\begin{align*}
& \operatorname{Var}\left(\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right) \\
& =\mathbb{E}\left[\mathbb{I}^{2}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right]-\left(\mathbb{E}\left[\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right]\right)^{2} \\
& =\mathbb{E}\left[\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right]\left(1-\mathbb{E}\left[\mathbb{I}\left\{\varepsilon_{1}^{*} \in E\right\} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right]\right)  \tag{46}\\
& =\mathbb{P}\left(\varepsilon_{1}^{*} \in E \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)\left(1-\mathbb{P}\left(\varepsilon_{1}^{*} \in E \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)\right) \\
& \leq \mathbb{P}\left(\varepsilon_{1}^{*} \in E \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right), \text { almost surely. }
\end{align*}
$$

If we denote the cumulative distribution function of $\varepsilon_{1}^{*}$ as $F_{\varepsilon}$, then we can obtain:

$$
\begin{equation*}
\mathbb{P}\left(\varepsilon_{1}^{*} \in E \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)=\mathbb{E}\left[\left|F_{\varepsilon}\left(\Delta_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)\right| \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right] . \tag{47}
\end{equation*}
$$

Recalling (25), we have,

$$
F_{\varepsilon}\left(\Delta_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)=\Delta_{n}\left(x_{1}^{*}, z^{*}\right) f_{\varepsilon^{*}}\left(-S^{*}\right)+\Delta_{n}^{2}\left(x_{1}^{*}, z^{*}\right) R\left(-S^{*}, \Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right) .
$$

Hence, we have that,

$$
\begin{align*}
& \mathbb{E}\left[\left|F_{\varepsilon}\left(\Delta_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right)-F_{\varepsilon}\left(-S^{*}\right)\right| \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right] \\
& =\mathbb{E}\left[\left|\Delta_{n}\left(x_{1}^{*}, z^{*}\right) f_{\varepsilon}\left(-S^{*}\right)+\Delta_{n}^{2}\left(x_{1}^{*}, z^{*}\right) R\left(-S^{*}, \Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right)\right| \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right]  \tag{48}\\
& \leq M \mathbb{E}\left[\left|\Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right]+(M / 4) \mathbb{E}\left[\left|\Delta_{n}^{2}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right], \text { almost surely. }
\end{align*}
$$

Based on Assumption 4 , it follows that $\mathbb{E}\left[\Delta_{n}^{2}\left(x_{1}^{*}, z^{*}\right)\right]=O\left(n^{-2 \tau}\right)$, and from (29), we have that $\mathbb{E}\left[\left|\Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right|\right]=O\left(n^{-\tau}\right)$. Consequently, by employing Markov's inequality, we obtain that

$$
\mathbb{E}\left[\left|\Delta_{n}^{2}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right]=O_{\mathbb{P}}\left(n^{-2 \tau}\right),
$$

and

$$
\mathbb{E}\left[\left|\Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right]=O_{\mathbb{P}}\left(n^{-\tau}\right) .
$$

Therefore, we can derive that,

$$
\begin{equation*}
M \mathbb{E}\left[\left|\Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right]+(M / 4) \mathbb{E}\left[\left|\Delta_{n}^{2}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right]=O_{\mathbb{P}}\left(n^{-\tau}\right) . \tag{49}
\end{equation*}
$$

Combining equations (44)-(49), we deduce that

$$
\begin{equation*}
\operatorname{Cov}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}, \mathbb{I}\left\{v_{1}^{*} \geq \hat{v}_{1}^{L}\right\} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right)=O_{\mathbb{P}}\left(n^{-\tau / 2}\right) \tag{50}
\end{equation*}
$$

By (40) and (50), we can derive that,

$$
\begin{align*}
& R_{1}^{\mathrm{COAD} \mid D}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \\
& \geq \mathbb{E}\left[\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]\left(1-\alpha+O_{\mathbb{P}}\left(n^{-1 / 2}\right)\right)+O_{\mathbb{P}}\left(n^{-\tau / 2}\right)  \tag{51}\\
& =\mathbb{E}\left[\hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]\left(1-\alpha+O_{\mathbb{P}}\left(n^{-1 / 2}\right)\right)+O_{\mathbb{P}}\left(n^{-\tau / 2}\right)
\end{align*}
$$

According to the definitions of $\hat{v}_{1}^{L}, \hat{v}_{1}^{U}$, and $v_{1}^{*}$, we have that,

$$
\begin{aligned}
& \mathbb{E}\left(v_{1}^{*} \mid z^{*}=z\right) \\
& =\mathbb{E}\left(\mu\left(x_{1}^{*}, z^{*}\right)+\varepsilon_{1}^{*} \mid z^{*}=\tilde{z}\right) \\
& =\mathbb{E}\left(\mu\left(x_{1}^{*}, z^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right)+\mathbb{E}\left(\varepsilon_{1}^{*}\right) \\
& =\mathbb{E}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right) \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right)+O_{\mathbb{P}}\left(n^{-\tau}\right)+0 \\
& =\frac{\mathbb{E}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)+\mathbb{E}\left(\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)+S^{*} \mid \hat{\mu}_{n}, S^{*}, z^{*}=\tilde{z}\right)}{2}+O_{\mathbb{P}}\left(n^{-\tau}\right) \\
& =\frac{\mathbb{E}\left[\hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]+\mathbb{E}\left[\hat{v}_{1}^{U} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]}{2}+O_{\mathbb{P}}\left(n^{-\tau}\right),
\end{aligned}
$$

where the third equation uses $\mathbb{E}\left[\left|\Delta_{n}\left(x_{1}^{*}, z^{*}\right)\right| \mid \Delta_{n}, z^{*}=\tilde{z}\right]=O_{\mathbb{P}}\left(n^{-\tau}\right)$ and Assumption 2 .
Hence, we have that

$$
\begin{align*}
& \mathbb{E}\left[\hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right] \\
& =\frac{2\left(\mathbb{E}\left(v_{1}^{*} \mid z^{*}=\tilde{z}\right)+O_{\mathbb{P}}\left(n^{-\tau}\right)\right)}{\mathbb{E}\left[\hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]+\mathbb{E}\left[\hat{v}_{1}^{U} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]} \mathbb{E}\left[\hat{v}_{1}^{L} \mid D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]  \tag{52}\\
& =\frac{2}{1+h} \mathbb{E}\left(v_{1}^{*} \mid z^{*}=\tilde{z}\right)+O_{\mathbb{P}}\left(n^{-\tau}\right),
\end{align*}
$$

where

$$
\left.\left.h=\frac{\mathbb{E}\left[\hat{v}_{1}^{U}\right.}{} \right\rvert\, D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right], \begin{aligned}
& \mathbb{E}\left[\hat{v}_{1}^{L}\right. \\
& \left.D_{\text {train }}, D_{\text {cal }}, z^{*}=\tilde{z}\right]
\end{aligned}=1+\frac{2 S^{*}}{\mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right]} .
$$

Combining equations (51) and (52), we have that,

$$
\begin{align*}
& R_{1}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \\
& \geq \frac{2(1-\alpha)}{1+h} \mathbb{E}\left(v_{1}^{*} \mid z^{*}=\tilde{z}\right)+O_{\mathbb{P}}\left(n^{-\min \{\tau / 2,1 / 2\}}\right)  \tag{53}\\
& =\frac{2(1-\alpha)}{1+h} W_{1}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right)+O_{\mathbb{P}}\left(n^{-\min \{\tau / 2,1 / 2\}}\right) .
\end{align*}
$$

This completes the proof of the situation $m^{*}=1$.
Proof of the case when $m^{*} \geq 2$ and $F_{v \mid z=\tilde{z}}$ has a Monotone hazard rate: In order to prove the result, we will use the following lemma.

Lemma 1 Babaioff et al. (2017) Lemma 5.3). If $\left\{v_{1}^{*}, \ldots, v_{m^{*}}^{*}\right\}$ are drawn iid from a monotone hazard rate distribution $F_{v^{*}}$, we have

$$
\mathbb{E}_{\vec{v}^{*} \sim F_{v^{*}}^{m^{*}}}\left[\max _{1 \leq i \leq m^{*}} v_{i}^{*}\right] \leq \mathcal{H}_{m^{*}} \mathbb{E}_{v_{1}^{*} \sim F_{v^{*}}}\left(v_{1}^{*}\right),
$$

where $\mathcal{H}_{m^{*}}=\sum_{i=1}^{m^{*}} i^{-1}$ denotes the $m^{*}$ th harmonic number.
In our setting, for any given item with feature $z^{*}=\tilde{z}$, since $F_{v^{*} \mid z^{*}=\tilde{z}}$ has a monotone hazard rate, it follows from Lemma 1 that,

$$
\begin{align*}
W_{m^{*}}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right) & =\mathbb{E}_{\vec{v}^{*} \sim F_{v^{*} \mid z^{*}=\tilde{z}}^{m^{*}}}\left[\max _{1 \leq i \leq m^{*}} v_{i}^{*}\right]  \tag{54}\\
& \leq \mathcal{H}_{m^{*}} \mathbb{E}_{v_{1}^{*} \sim F_{v^{*}} \mid z^{*}=\tilde{z}}\left(v_{1}^{*}\right)=\mathcal{H}_{m^{*}} W_{1}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right) .
\end{align*}
$$

Furthermore, in accordance with Theorem 3, where $m^{*} \geq 2$, it follows that,

$$
\begin{equation*}
R_{m^{*}}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \geq R_{1}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) . \tag{55}
\end{equation*}
$$

By leveraging inequality (53) and inequality (55), we derive that,

$$
\begin{equation*}
R_{m^{*}}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \geq \frac{2(1-\alpha)}{1+h} W_{1}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right)+O_{\mathbb{P}}\left(n^{-\min \{\tau / 2,1 / 2\}}\right) . \tag{56}
\end{equation*}
$$

Finally, by the direct derivation from inequalities (54) and (56), we obtain:

$$
\begin{aligned}
& R_{m^{*}}^{\mathrm{COAD} \mid \mathcal{D}}\left(F_{v^{*}, x^{*} \mid z^{*}=\tilde{z}}\right) \\
& \geq \frac{2(1-\alpha)}{1+h} \frac{W_{m^{*}}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right)}{\mathcal{H}_{m^{*}}}+O_{\mathbb{P}}\left(n^{-\min \{\tau / 2,1 / 2\}}\right) \\
& =\frac{2(1-\alpha)}{(1+h) \mathcal{H}_{m^{*}}} W_{m^{*}}\left(F_{v^{*} \mid z^{*}=\tilde{z}}\right)+O_{\mathbb{P}}\left(n^{-\min \{\tau / 2,1 / 2\}}\right) .
\end{aligned}
$$

This completes the proof of Theorem 4 .

## B. 6 Proof of Theorem 5

Proof From the prescribed definition of $h=\mathbb{E}\left[\hat{v}_{1}^{U} \mid \mathcal{D}, z^{*}=\tilde{z}\right] / \mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right]$, coupled with the definitions of $\hat{v}_{1}^{U}=\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)+S^{*}$ and $\hat{v}_{1}^{L}=\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-S^{*}$, we derive the following:

$$
\begin{equation*}
h=\frac{\mathbb{E}\left[\hat{v}_{1}^{U} \mid \mathcal{D}, z^{*}=\tilde{z}\right]}{\mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right]}=1+\frac{2 S^{*}}{\mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right]}=1+\frac{2 S^{*}}{\mathbb{E}\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right]-S^{*}} . \tag{57}
\end{equation*}
$$

By Theorem 1, it follows that $S^{*} \leq \delta_{n}$ almost surely. Consequently,

$$
\begin{aligned}
\mathbb{E}\left[\hat{v}_{1}^{L} \mid \mathcal{D}, z^{*}=\tilde{z}\right] & =\mathbb{E}\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right)-S^{*} \mid \mathcal{D}, z^{*}=\tilde{z}\right] \\
& =\mathbb{E}\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right]-S^{*} \\
& \geq \mathbb{E}\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right]-\delta_{n} \\
& >0, \text { almost surely. }
\end{aligned}
$$

Hence, $h$ is an increasing function of $S^{*}$. From the construction procedure of the dual prediction interval, it follows that $S^{*} \geq 0$, thereby ensuring that $0 \leq S^{*} \leq \delta_{n}$. Substituting the range of $S^{*}$ into equation (57), we can deduce that,

$$
\begin{equation*}
1 \leq h \leq 1+\frac{2 \delta_{n}}{\mathbb{E}\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right]-\delta_{n}} . \tag{58}
\end{equation*}
$$

By Assumption 4 and using Markov's inequality, we have,

$$
\mathbb{E}\left[\left|\hat{\mu}_{n}\left(x_{1}^{*}, z^{*}\right)-\mu\left(x_{1}^{*}, z^{*}\right)\right| \mid \mathcal{D}, z^{*}=\tilde{z}\right]=O_{\mathbb{P}}\left(n^{-\tau}\right),
$$

from which we have that $\mathbb{E}\left[\hat{\mu}_{n}\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right]=\mathbb{E}\left[\mu\left(x_{i}^{*}, z^{*}\right) \mid \mathcal{D}, z^{*}=\tilde{z}\right]+O_{\mathbb{P}}\left(n^{-\tau}\right)$. Therefore, combined with inequality (58), we obtain:

$$
1 \leq h \leq 1+\frac{2 \delta_{n}}{\mathbb{E}\left(\mu\left(x_{1}^{*}, z^{*}\right) \mid z^{*}=\tilde{z}\right)+O_{\mathbb{P}}\left(n^{-\tau}\right)-\delta_{n}} .
$$

This completes the proof of Theorem 5 .


Figure 6: Conformal prediction conditioning on item features with $N=1000, m^{*}=1000$.

## Appendix C. Additional Numerical Results

## C. 1 Consistency of the Polynomial Regression

Table 1 shows the Mean Squared Error (MSE) and Standard Deviation (SD) of the errors between the predicted regression value and the true regression value of the calibration data, where the model is trained on the training data. Here $N \in\{100,500,1000,5000,10000,50000\}$, and the results are averaged over 500 simulations. It is observed that as $N$ goes large, both the average MSE and SD approach to 0 . Hence, Assumption 4 is asymptotically satisfied.

| $N$ | 100 | 500 | 1000 | 5000 | 10000 | 50000 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| Average MSE | 1764137 | 44.81 | 3.86 | 0.056 | 0.018 | 0.005 |
| Standard Deviation | 333.60 | 2.72 | 0.92 | 0.143 | 0.087 | 0.045 |

Table 1: Average MSE and SD.
Figure 6 illustrates the accuracy of the fitting model and part of the conformal prediction bands when conditioned on different values of $z$, with $N=1000, m^{*}=1000$. The black solid line is the true regression function, the red dashed line is the fitted line of the polynomial regression, and the light blue area is the conditional prediction band. It is seen that most of the points are covered by the conformal prediction bands.

## References

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[^0]:    *. Address for correspondence: Xiaowu Dai, Department of Statistics, University of California, Los Angeles, 8125 Math Sciences Bldg \#951554, Los Angeles, CA 90095, USA. Email: dai@stat.ucla.edu.

