BOUNDS ON DAO NUMBERS AND APPLICATIONS TO REGULAR LOCAL RINGS

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Abstract. The so-called Dao numbers are a sort of measure of the asymptotic behaviour of full properties of certain product ideals in a Noetherian local ring R with infinite residue field and positive depth. In this paper, we answer a question of H. Dao on how to bound such numbers. The auxiliary tools range from Castelnuovo-Mumford regularity of appropriate graded structures to reduction numbers of the maximal ideal. In particular, we substantially improve previous results (and answer questions) by the authors. As an application, we provide new characterizations of when R is regular; for instance, we show that this holds if and only if the maximal ideal of R can be generated by a d-sequence (in the sense of Huneke) if and only if the third Dao number of any (minimal) reduction of the maximal ideal vanishes.

1. Motivation: Dao's problem on the fullness of certain ideals

Throughout this note, by a *ring* we mean a commutative, Noetherian, unital ring. Let R be either a local ring with residue field K and maximal ideal $\mathfrak{m}$, or a standard graded algebra over a field K having a unique graded maximal ideal $\mathfrak{m}$. We will assume throughout that K is infinite and depth $R > 0$ (i.e., $\mathfrak{m}$ contains a non-zerodivisor), and in addition $I \subset R$ stands for an ideal which we assume to be homogeneous whenever R is graded.

In this paper we focus on the properties of $\mathfrak{m}$-fullness, fullness, and weak $\mathfrak{m}$-fullness (to be recalled in the next section) of certain ideals. More precisely, we are interested in the so-called Dao numbers of the given ideal I, i.e., three non-negative integers $\mathfrak{d}_i(I)$, $i = 1, 2, 3$, defined below, which in some sense provide a measure for the asymptotic behaviour of the full properties of certain product ideals involving I.

Definition 1.1. The *Dao numbers* of I are defined as:

$$\begin{aligned}
\mathfrak{d}_1(I) &= \min\{t \geq 0 \mid I\mathfrak{m}^k \text{ is } \mathfrak{m}\text{-full for all } k \geq t\};\\
\mathfrak{d}_2(I) &= \min\{t \geq 0 \mid I\mathfrak{m}^k \text{ is full for all } k \geq t\};\\
\mathfrak{d}_3(I) &= \min\{t \geq 0 \mid I\mathfrak{m}^k \text{ is weakly } \mathfrak{m}\text{-full for all } k \geq t\}.
\end{aligned}$$

It is worth observing that, if $(R, \mathfrak{m}, K)$ and I are as above, then as shown in [17, Proposition 2.2] the basic relations among the Dao numbers of I are

$$\mathfrak{d}_2(I) \leq \mathfrak{d}_1(I) = \mathfrak{d}_3(I).$$

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Our motivation is the following problem suggested by H. Dao, which was first addressed in [17] and then in [6].

Question 1.2. ([4, Question 4.5]) Can we find good lower and upper bounds for the $\mathfrak{d}_i(I)$'s?

In the case where I is a reduction of $\mathfrak{m}$, a lower bound for $\mathfrak{d}_3(I)$ is given by $\mathrm{r}_I(\mathfrak{m})$, the reduction number of $\mathfrak{m}$ with respect to I, which follows immediately from [17, Theorem 3.4] (the question as to whether $\mathrm{r}_I(\mathfrak{m}) \leq \mathfrak{d}_2(I)$ remains unanswered). Otherwise, for a general I (not necessarily a reduction of $\mathfrak{m}$), a more elaborated lower bound was established in [6, Proposition 1.8].

So, in the present paper, our main goal is to answer the upper bound part of Dao's question, by using two fundamental numerical invariants in commutative algebra: the Castelnuovo-Mumford regularity (of appropriate graded structures) and, again, the reduction number. The connection between these numbers and Dao's problem was first exploited in [17] and later developed even further (concerning specifically the Castelnuovo-Mumford regularity) in [6]. The present work establishes, in fact, generalizations and substantial improvements of the results given in these two papers. Additionally, we shall finally derive, as an application of some of our results, new characterizations of regular local rings.

2. Auxiliary notions and basic properties

In this section, we invoke some basic concepts and facts which we shall freely use in this note (without explicit mention).

2.1. **Full properties of ideals.** Let K be an infinite field and $(R, \mathfrak{m})$ be either a local ring with residue field K or a standard graded K-algebra having a unique homogeneous maximal ideal $\mathfrak{m}$. Assume depth $R > 0$, and let $I \subset R$ stand for an ideal (homogeneous whenever R is graded).

Definition 2.1. The following notions are central in this paper:
- (a) I is $\mathfrak{m}$-*full* if $I\mathfrak{m} : x = I$ for some element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$;
- (b) I is *full* if $I : x = I : \mathfrak{m}$ for some element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$;
- (c) I is *weakly* $\mathfrak{m}$-*full* if $I\mathfrak{m} : \mathfrak{m} = I$.

For completeness, we recall a few interesting properties. It is clear that $\mathfrak{m}$-full ideals are weakly $\mathfrak{m}$-full. If I is $\mathfrak{m}$-primary, then I is weakly $\mathfrak{m}$-full if and only if I is basically full in the sense of [8]. Moreover, $\mathfrak{m}$-full ideals satisfy the so-called *Rees property*, and if R is a normal domain then any integrally closed ideal is $\mathfrak{m}$-full; see [24] (also [7]).

2.2. **Castelnuovo-Mumford regularity.** Let $S = \bigoplus_{m \geq 0} S_m$ be a finitely generated standard graded algebra over a ring S_0. As usual, by *standard* we mean that S is generated by S_1 as an S_0-algebra. We write $S_+ = \bigoplus_{m \geq 1} S_m$ for the ideal generated by all elements of S of positive degree. For a graded S-module $A = \bigoplus_{m \in \mathbb{Z}} A_m$ satisfying $A_m = 0$ for all $m \gg 0$, we let $a(A) = \max\{m \in \mathbb{Z} \mid A_m \neq 0\}$ if $A \neq 0$, and $a(A) = -\infty$ if $A = 0$.

Now, for a finitely generated graded S-module $N \neq 0$ and an integer $j \geq 0$, we take $A = H^j_{S_+}(N)$ and use the notation

$$a_j(N) := a(H^j_{S_+}(N)),$$

where $H^j_{S_+}(-)$ stands for the j-th local cohomology functor with respect to the ideal S_+. It is known that $H^j_{S_+}(N)$ is a graded module with $H^j_{S_+}(N)_n = 0$ for all $n \gg 0$ (see, e.g., [1, Proposition 15.1.5(ii)]). Thus, $a_j(N) < +\infty$.

Definition 2.2. Maintain the above setting and notations. The *Castelnuovo-Mumford regularity* of N is defined as

$$\mathrm{reg}_S N := \max\{a_j(N) + j \mid j \geq 0\}.$$

It is well-known that $\mathrm{reg}\, N$ governs the complexity of the graded structure of N and is relevant in commutative algebra and algebraic geometry, for example in the study of degrees of syzygies over polynomial rings (see, e.g., [1, Chapter 15]).

Remark 2.3. A classical instance of interest is when $S = \mathcal{R}(J)$, the Rees algebra of an ideal J in a ring R (to be recalled in the next subsection), which is known to be a finitely generated standard graded R-algebra. In particular, we can consider the case where R is local and $J = \mathfrak{m}$, the maximal ideal of R.

We recall below a few basic rules about this invariant. For details, we refer to [2, p. 277, (a), (c) and (d)], [5, Corollary 20.19] and [10, Lemma 3.1].

(i) As usual, given an integer j, we denote by $N(j)$ the module N with degrees shifted by j, that is, $N(j)_i = N_{i+j}$ for all i. Then,

$$\mathrm{reg}_S N(j) = \mathrm{reg}_S N - j.$$

(ii) Let $0 \to M \to N \to P \to 0$ be a short exact sequence of finitely generated graded S-modules. Then:
- $\mathrm{reg}_S N \leq \max\{\mathrm{reg}_S M, \mathrm{reg}_S P\}$, with equality if $\mathrm{reg}_S P \neq \mathrm{reg}_S M - 1$ or $M_k = 0$ for $k \gg 0$.
- $\mathrm{reg}_S M \leq \max\{\mathrm{reg}_S N, \mathrm{reg}_S P + 1\}$, with equality if $\mathrm{reg}_S N \neq \mathrm{reg}_S P$.
- $\mathrm{reg}_S P \leq \max\{\mathrm{reg}_S N, \mathrm{reg}_S M - 1\}$, with equality if $\mathrm{reg}_S N \neq \mathrm{reg}_S M$.

(iii) If $N_j = 0$ for all $j \gg 0$, then $\mathrm{reg}_S N = a(N)$.

2.3. **Rees algebra and Dao module.** Here, we consider some useful graded structures. Let R be a ring and J an ideal of R.

Definition 2.4. The *Rees algebra* of J is the graded ring

$$\mathcal{R}(J) = \bigoplus_{k \geq 0} J^k = R \oplus J \oplus J^2 \oplus \cdots$$

Notation 2.5. Given an R-module M, it is customary to write $\mathcal{R}(J, M) = \bigoplus_{k\geq 0} J^k M$ for the Rees module of J relative to M. In this paper, if I is another R-ideal, we will be particularly interested in

$$\mathcal{R}(J, I) = \bigoplus_{k \geq 0} I J^k.$$

Note $\mathcal{R}(J,I) = I\mathcal{R}(J)$, the extension of I to the ring $\mathcal{R}(J)$, which is therefore a finitely generated ideal of $\mathcal{R}(J)$.

Now let $(R,\mathfrak{m})$ be as in Section 1. Its associated graded ring is defined by $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{k\geq 0}\mathfrak{m}^k/\mathfrak{m}^{k+1}$. In [6], the following graded structure is introduced for a given ideal I of R.

Definition 2.6. The *Dao module* of I is given by

$$\mathfrak{D}_{\mathfrak{m}}(I) = \bigoplus_{k\geq 0}\frac{I\mathfrak{m}^{k+1} : \mathfrak{m}}{I\mathfrak{m}^k},$$

which is a graded $\mathcal{R}(\mathfrak{m})$-module.

Remark 2.7. The kth component of the Dao module vanishes if and only if the ideal $I\mathfrak{m}^k$ is weakly $\mathfrak{m}$-full. Since

$$\mathfrak{D}_{\mathfrak{m}}(I)_k = 0 \quad \text{for all} \quad k \geq \mathfrak{d}_3(I),$$

it follows that $\mathfrak{D}_{\mathfrak{m}}(I)$ has finite length and therefore is a finitely generated graded $\mathcal{R}(\mathfrak{m})$-module, satisfying

$$\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\,\mathfrak{D}_{\mathfrak{m}}(I) = \mathfrak{d}_3(I) - 1$$

whenever $\mathfrak{d}_3(I) \geq 1$ (e.g., if I is not weakly $\mathfrak{m}$-full).

2.4. **Ratliff-Rush operation.** Let I be an ideal of a ring R.

Definition 2.8. The *Ratliff-Rush closure* $\widetilde{I}$ of the ideal I is given by

$$\widetilde{I} = \bigcup_{m\geq 1} I^{m+1} : I^m.$$

This is an ideal of R containing I which in fact refines the integral closure of I, so that $\widetilde{I} = I$ whenever I is integrally closed. For details, see [18].

Now suppose I contains a regular element, i.e., a non-zerodivisor on R. Then it is well-known that $\widetilde{I}$ is the largest ideal that shares with I the same sufficiently high powers; hence,

$$\widetilde{I^m} = I^m \quad \text{for all} \quad m \gg 0.$$

This enables us to consider the following helpful number (inspired by [19, Proposition 4.2]).

Notation 2.9. If I contains a regular element, we set

$$s(I) = \min\{n \geq 1 \mid \widetilde{I^i} = I^i \text{ for all } i \geq n\}.$$

Remark 2.10. Let us invoke a couple of useful properties. First, according to [14, Lemma 2.2] we can write

$$\widetilde{I^{k+1}} : I = \widetilde{I^k} \quad \text{for all} \quad k \geq 0.$$

Moreover, if $\mathrm{gr}_I(R) = \bigoplus_{k\geq 0} I^k/I^{k+1}$ denotes the associated graded ring of I, then by [19, Remark 1.6] we get that $\widetilde{I^k} = I^k$ for all $k \geq 0$ (i.e., $s(I) = 1$) if and only if

$$\operatorname{depth} \mathrm{gr}_I(R) > 0.$$

2.5. **Reduction number.** One last auxiliary notion is in order.

Definition 2.11. Let J be a proper ideal of a ring R. An ideal $I \subset J$ is said to be a *reduction of* J if $IJ^r = J^{r+1}$ for some integer $r \geq 0$. Such a reduction I is *minimal* if it is minimal with respect to inclusion. If I is a reduction of J, we define the *reduction number of* J *with respect to* I as the number

$$\mathrm{r}_I(J) = \min\{m \in \mathbb{N} \mid IJ^m = J^{m+1}\},$$

and the *reduction number of* J as

$$\mathrm{r}(J) = \min\{\mathrm{r}_I(J) \mid I \text{ is a minimal reduction of } J\}.$$

Of special interest in this paper will be the case where $(R, \mathfrak{m})$ is a local ring and $J = \mathfrak{m}$.

3. Upper bounds on Dao numbers via Castelnuovo-Mumford regularity

Before establishing the main results of the section, we fix a piece of notation. For a graded R-module $M = \bigoplus_{k \geq 0} M_k$ and an integer $\ell \geq 0$, we can consider the truncation $M_{\geq \ell} = \bigoplus_{k \geq \ell} M_k$. Specifically, with notation as in (2.5) and in case $(R, \mathfrak{m}, K)$ is a local ring or a standard graded K-algebra, we will be interested in the truncation

$$\mathcal{R}(\mathfrak{m}, I)_{\geq 1} = \bigoplus_{k \geq 1} I\mathfrak{m}^k.$$

Here we are interested in tackling the upper bound part of Question 1.2 in terms of the Castelnuovo-Mumford regularity of appropriate graded structures. The first result in this direction, in case R is local and I is a reduction of $\mathfrak{m}$, was proved in [17] and can be stated as follows.

Theorem 3.1. ([17, Theorem 3.10]) *Let* $(R, \mathfrak{m})$ *be a local ring with infinite residue field and* $\operatorname{depth} R > 0$, *and let* I *be a reduction of* $\mathfrak{m}$. *Then,*

$$\mathfrak{d}_3(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}).$$

Later, in [6], the following general answer to Question 1.2 was provided.

Theorem 3.2. ([6, Theorem 1.1]) *Let* $(R, \mathfrak{m}, K)$ *be either a local ring or a standard graded* K*-algebra, with* K *infinite and* $\operatorname{depth} R > 0$. *Let* $I \subset R$ *be an ideal (homogeneous if* R *is graded). Then,*

$$\mathfrak{d}_3(I) \leq \max\{\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I),\ \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I)_{\geq 1} :_{\mathcal{R}(R)} \mathfrak{m}\}.$$

Here, we prove the following result, which establishes [6, Conjecture 0.1] in its full generality.

Theorem 3.3. *Let* R *and* I *be as in Theorem 3.2. Then,*

$$\mathfrak{d}_3(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I).$$

Proof. We may suppose $\mathfrak{d}_3(I) > 0$, otherwise there is nothing to prove. Let $\mathfrak{M} = \mathcal{R}(\mathfrak{m})_+ = \bigoplus_{k>0} \mathfrak{m}^k$ be the homogeneous maximal ideal of $\mathcal{R}(\mathfrak{m})$. It was shown in [6, proof of Theorem 1.3] that

$$\text{(1)} \qquad \operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I)) \;=\; 0 :_{\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m},I)} \mathfrak{M} \;=\; \bigoplus_{k\geq 0} \frac{(I\mathfrak{m}^{k+1} : \mathfrak{m}) \cap \mathfrak{m}^k}{I\mathfrak{m}^k}.$$

Now, since

$$\frac{(I\mathfrak{m}^{k+1} : \mathfrak{m}) \cap \mathfrak{m}^k}{I\mathfrak{m}^k} \;\subset\; \frac{I\mathfrak{m}^{k+1} : \mathfrak{m}}{I\mathfrak{m}^k} \quad \text{for all} \quad k \geq 0,$$

we have

$$\text{(2)} \qquad \operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I)) \;\subset\; \mathfrak{D}_{\mathfrak{m}}(I)$$

as a graded $\mathcal{R}(\mathfrak{m})$-submodule of $\mathfrak{D}_{\mathfrak{m}}(I)$. Next, let $t = \max\{k \mid \mathfrak{D}_{\mathfrak{m}}(I)_k \neq 0\}$. Then, $t = \operatorname{reg}_{\mathcal{R}(\mathfrak{m})}\mathfrak{D}_{\mathfrak{m}}(I) = \mathfrak{d}_3(I) - 1$. Because $\mathfrak{M} \cdot \mathfrak{D}_{\mathfrak{m}}(I)_t = 0$, it follows that

$$\mathfrak{D}_{\mathfrak{m}}(I)_t \;\subset\; [0 :_{\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m},I)} \mathfrak{M}]_t \;=\; \operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I))_t$$

and note that the opposite inclusion always holds by equation (2). Therefore,

$$\operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I))_t = \mathfrak{D}_{\mathfrak{m}}(I)_t \neq 0$$

and consequently

$$\begin{aligned} \mathfrak{d}_3(I) - 1 \;=\; t \;&\leq\; \max\{j \mid \operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I))_j \neq 0\} \\ &\leq\; \operatorname{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I) \\ &=\; \operatorname{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) - 1. \end{aligned}$$

The assertion follows. □

Naturally, we can ask when the inclusion (2) is actually an equality.

Proposition 3.4. *Let R and I be as in Theorem 3.2. If R is a standard graded K-algebra or* $\operatorname{depth} \operatorname{gr}_{\mathfrak{m}}(R) > 0$*, then* $\operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I)) = \mathfrak{D}_{\mathfrak{m}}(I)$.

Proof. In view of equation (1), it suffices to show that

$$I\mathfrak{m}^{k+1} : \mathfrak{m} \;\subset\; \mathfrak{m}^k \quad \text{for all} \quad k \geq 0.$$

First, if R is a standard graded K-algebra, we can represent it as a quotient S/J where $S = K[x_1, \ldots, x_n]$ is a standard graded polynomial ring and $J \subset S$ is a homogeneous ideal. Now let $f \in I\mathfrak{m}^{k+1} : \mathfrak{m}$ be an homogeneous element. Then

$$(x_1 + J)f \in I\mathfrak{m}^{k+1} \subset \mathfrak{m}^{k+1}$$

and so $\deg f + 1 \geq k + 1$. Thus $f \in \mathfrak{m}^k$ and consequently $I\mathfrak{m}^{k+1} : \mathfrak{m} \subset \mathfrak{m}^k$ for all $k \geq 0$. Otherwise, if $\operatorname{depth} \operatorname{gr}_{\mathfrak{m}}(R) > 0$ then, as we know, $\widetilde{\mathfrak{m}^k} = \mathfrak{m}^k$ for all $k \geq 0$. Consequently,

$$I\mathfrak{m}^{k+1} : \mathfrak{m} \subset \mathfrak{m}^{k+1} : \mathfrak{m} = \widetilde{\mathfrak{m}^{k+1}} : \mathfrak{m} = \widetilde{\mathfrak{m}^k} = \mathfrak{m}^k.$$

□

We conclude this section with some other consequences of Theorem 3.3.

Corollary 3.5. *Let R and I be as in Theorem 3.2. If $\mathfrak{d}_3(I) > 0$, then $\mathfrak{d}_3(I)$ is equal to the highest socle degree of $\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I)$ increased by one, that is,*

$$\mathfrak{d}_3(I) \ = \ \max\{j \mid \operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I))_j \neq 0\} + 1.$$

Proof. Set $h = \max\{j \mid \operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I))_j \neq 0\}$. In the proof of Theorem 3.3 we have seen that $\mathfrak{d}_3(I) \leq h + 1$. On the other hand, since $\mathfrak{d}_3(I) > 0$, we can write $\mathfrak{d}_3(I) = \max\{j \mid \mathfrak{D}_\mathfrak{m}(I)_j \neq 0\} + 1$. By equation (2) we have an inclusion $\operatorname{Soc}(\mathcal{R}(\mathfrak{m})/\mathcal{R}(\mathfrak{m}, I)) \subset \mathfrak{D}_\mathfrak{m}(I)$, so that

$$\mathfrak{d}_3(I) \ = \ \max\{j \mid \mathfrak{D}_\mathfrak{m}(I)_j \neq 0\} + 1 \ \geq \ h + 1,$$

thus completing the proof. □

Next, we estimate the Castelnuovo-Mumford regularity of some natural blowup algebras that are related to the Dao module of I.

Definition 3.6. The *associated graded module of* $\mathfrak{m}$ *relative to* I is the $\mathcal{R}(\mathfrak{m})$-module

$$\operatorname{gr}_\mathfrak{m}(I) = \mathcal{R}(\mathfrak{m}, I)/\mathfrak{m}\mathcal{R}(\mathfrak{m}, I) = \bigoplus_{k \geq 0} I\mathfrak{m}^k / I\mathfrak{m}^{k+1}.$$

Note that, for each $k \geq 0$, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & I\mathfrak{m}^{k+1} \arrow[r, equal] \arrow[d] & I\mathfrak{m}^{k+1} \arrow[d] & & \\
0 \arrow[r] & I\mathfrak{m}^k \arrow[r] \arrow[d] & I\mathfrak{m}^{k+1} : \mathfrak{m} \arrow[r] \arrow[d] & \dfrac{I\mathfrak{m}^{k+1} : \mathfrak{m}}{I\mathfrak{m}^k} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{I\mathfrak{m}^k}{I\mathfrak{m}^{k+1}} \arrow[r] \arrow[d] & \dfrac{I\mathfrak{m}^{k+1} : \mathfrak{m}}{I\mathfrak{m}^{k+1}} \arrow[r] \arrow[d] & \dfrac{I\mathfrak{m}^{k+1} : \mathfrak{m}}{I\mathfrak{m}^k} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Notation 3.7. For simplicity, we set

$$\mathcal{P}_\mathfrak{m}(I) = \mathcal{R}(\mathfrak{m}, I)_{\geq 1} :_{\mathcal{R}(R)} \mathfrak{m} \quad \text{and} \quad \mathcal{Q}_\mathfrak{m}(I) = (\mathcal{R}(\mathfrak{m}, I)_{\geq 1} :_{\mathcal{R}(R)} \mathfrak{m})/\mathcal{R}(\mathfrak{m}, I)_{\geq 1}.$$

Taking the direct sum in the above diagram, we obtain the following commutative diagram of finitely generated graded $\mathcal{R}(\mathfrak{m})$-modules with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{R}(\mathfrak{m}, I)_{\geq 1}(1) \arrow[r, equal] \arrow[d] & \mathcal{R}(\mathfrak{m}, I)_{\geq 1}(1) \arrow[d] & & \\
0 \arrow[r] & \mathcal{R}(\mathfrak{m}, I) \arrow[r] \arrow[d] & \mathcal{P}_{\mathfrak{m}}(I)(1) \arrow[r] \arrow[d] & \mathfrak{D}_{\mathfrak{m}}(I) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(I) \arrow[r] \arrow[d] & \mathcal{Q}_{\mathfrak{m}}(I)(1) \arrow[r] \arrow[d] & \mathfrak{D}_{\mathfrak{m}}(I) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The lemma below is a special case of [25, Corollary 3].

Lemma 3.8. *There is an equality* $\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathrm{gr}_{\mathfrak{m}}(I)$.

Now we prove:

Corollary 3.9. *Let R and I be as in Theorem 3.2. Then,*

$$\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{P}_{\mathfrak{m}}(I),\ \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{Q}_{\mathfrak{m}}(I) \ \leq\ \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) + 1.$$

In particular,

$$\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{P}_{\mathfrak{m}}(I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{Q}_{\mathfrak{m}}(I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) + 1$$

if either $\mathfrak{d}_3(I) = 0$ or $\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) > \mathfrak{d}_3(I) > 0$.

Proof. There are short exact sequences of finitely generated graded $\mathcal{R}(\mathfrak{m})$-modules

$$0 \to \mathcal{R}(\mathfrak{m}, I) \to \mathcal{P}_{\mathfrak{m}}(I)(1) \to \mathfrak{D}_{\mathfrak{m}}(I) \to 0,$$
$$0 \to \mathrm{gr}_{\mathfrak{m}}(I) \to \mathcal{Q}_{\mathfrak{m}}(I)(1) \to \mathfrak{D}_{\mathfrak{m}}(I) \to 0.$$

Recall that Lemma 3.8 gives $\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathrm{gr}_{\mathfrak{m}}(I)$.

First, assume that $\mathfrak{d}_3(I) = 0$. Then, $\mathfrak{D}_{\mathfrak{m}}(I) = 0$. In this case, $\mathcal{P}_{\mathfrak{m}}(I)(1) \cong \mathcal{R}(\mathfrak{m}, I)$ and $\mathcal{Q}_{\mathfrak{m}}(I)(1) \cong \mathrm{gr}_{\mathfrak{m}}(I)$, so that

$$\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{P}_{\mathfrak{m}}(I) - 1 = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathrm{gr}_{\mathfrak{m}}(I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{Q}_{\mathfrak{m}}(I) - 1.$$

Now, suppose $\mathfrak{d}_3(I) > 0$. In particular $\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathfrak{D}_{\mathfrak{m}}(I) = \mathfrak{d}_3(I) - 1$. On the other hand, Theorem 3.3 yields

$$\mathfrak{d}_3(I) \leq \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I) = \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathrm{gr}_{\mathfrak{m}}(I)$$

and thus the above exact sequence implies

$$\begin{aligned}
\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{P}_{\mathfrak{m}}(I)(1) &\leq \max\{\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I), \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathfrak{D}_{\mathfrak{m}}(I)\} \\
&= \max\{\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I), \mathfrak{d}_3(I) - 1\} \\
&= \mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}, I).
\end{aligned}$$

Similarly, $\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{Q}_{\mathfrak{m}}(I)(1) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I)$. Hence,

$$\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{P}_{\mathfrak{m}}(I), \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{Q}_{\mathfrak{m}}(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I) + 1.$$

Finally, if we suppose that $\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I) > \mathfrak{d}_3(I) > 0$, we derive

$$\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I) - 1 \neq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathfrak{D}_{\mathfrak{m}}(I),$$

and the equalities $\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{P}_{\mathfrak{m}}(I) = \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{Q}_{\mathfrak{m}}(I) = \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}, I) + 1$ follow. □

Next, we strengthen [6, Proposition 1.6] by finding the relation between the regularity of an ideal as in (2.5) and that of the ambient Rees algebra $\mathcal{R}(J)$.

Lemma 3.10. *Let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded K-algebra. Let I, J be ideals of R. Then,*

$$\operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J) \leq \operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I),$$

with equality if I is a reduction of J.

Proof. Since $\mathcal{R}(J, I)$ is a homogeneous ideal of $\mathcal{R}(J)$, the short exact sequence

$$0 \to \mathcal{R}(J, I) \to \mathcal{R}(J) \to \frac{\mathcal{R}(J)}{\mathcal{R}(J, I)} \to 0$$

yields

$$\begin{aligned} \operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J) &\leq \max\left\{\operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I), \operatorname{reg}_{\mathcal{R}(J)} \frac{\mathcal{R}(J)}{\mathcal{R}(J, I)}\right\} \\ &= \max\left\{\operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I), \operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I) - 1\right\} \\ &= \operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I). \end{aligned}$$

For the equality part, the proof of $\operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J) \geq \operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I)$ is exactly the same as the one given in [6, Proposition 1.6] for the case $J = \mathfrak{m}$. □

Remark 3.11. Notice that, as a direct consequence of Theorem 3.3 and Lemma 3.10, we rediscover Theorem 3.1.

In our view, and in connection to Lemma 3.10, it is worth asking the following.

Question 3.12. Let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded K-algebra. Let $I \subset J$ be ideals of R. When does the condition $\operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J) = \operatorname{reg}_{\mathcal{R}(J)} \mathcal{R}(J, I)$ force I to be a reduction of J?

Inspired by Theorem 3.1 and Theorem 3.3, we might wonder whether the comparison

$$\mathfrak{d}_3(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m})$$

holds in general. However, in the case where I is not a reduction of $\mathfrak{m}$, the relationship with the regularity of $\mathcal{R}(\mathfrak{m})$ may become rather wild, as we can see in the examples below.

Example 3.13. Let $(R, \mathfrak{m})$ be a local ring with infinite residue field and depth $R > 0$. By [20, Proposition 1.5], we have $\mathfrak{m}^{n+1} : \mathfrak{m} = \mathfrak{m}^n$ for all $n \geq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m})$. Now set $I = \mathfrak{m}^k$ for any given $k \geq 2$ (note I cannot be a reduction of $\mathfrak{m}$). We can write

$$I\mathfrak{m}^n = \mathfrak{m}^{n+k} = \mathfrak{m}^{n+k+1} : \mathfrak{m} = I\mathfrak{m}^{n+1} : \mathfrak{m} \tag{3}$$

for all $n \geq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m})$ and therefore $\mathfrak{d}_3(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m})$. Furthermore notice that, by using (3) whenever $n \geq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}) - k \geq 0$, we obtain

$$\mathfrak{d}_3(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}) - k < \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}).$$

Finally, if $k \geq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m})$ then it is easy to see that $\mathfrak{d}_3(I) = 0$. In particular, if $\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}) \leq 2$ then $\mathfrak{d}_3(I) = 0$.

Example 3.14. Let R be as in [17, Example 4.3]. As observed there, $\operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}) = 8$. Consider the ideal $I = \mathfrak{m}^2$. By Example 3.13 above, we can write

$$\mathfrak{d}_3(I) \leq \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}) - 2 = 6,$$

whereas, on the other hand, a computation shows $I\mathfrak{m}^5 \neq I\mathfrak{m}^6 : \mathfrak{m}$. Therefore, we must have $\mathfrak{d}_3(I) = 6$.

We are also able to illustrate the opposite situation, as follows.

Example 3.15. Let $I = (x^a, y^a) \subset R = k[\![x, y]\!]$, where k is a field and $a \geq 2$. Clearly, I is not a reduction of $\mathfrak{m} = (x, y)$. From [4, Example 4.1] we can write

$$\mathfrak{d}_3(I) = a - 1 > 0 = \operatorname{reg}_{\mathcal{R}(\mathfrak{m})} \mathcal{R}(\mathfrak{m}),$$

where the last equality holds because R is a regular local ring (see Theorem 5.3 for a more general statement).

4. Approach (and a conjecture) via reduction numbers

We begin recalling the following result in dimension one.

Proposition 4.1. ([17, Corollary 3.7]) *If $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring with infinite residue field, then*

$$\mathfrak{d}_3(I) = \mathrm{r}(\mathfrak{m})$$

for any minimal reduction I of $\mathfrak{m}$.

It is worth pointing out that this proposition is no longer true if the reduction I is not required to be minimal, as exemplified in [17, Example 4.2].

In higher dimension, there is the conjecture below.

Conjecture 4.2. ([17, Conjecture 3.8]) *If $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with infinite residue field and* $\dim R \geq 2$, *then*

$$\mathfrak{d}_3(I) = \mathrm{r}_I(\mathfrak{m})$$

for any minimal reduction I of $\mathfrak{m}$.

Our main purpose in this section is to provide partial answers to this conjecture. A crucial tool in this investigation is given by the following result.

Theorem 4.3. ([17, Theorem 3.4]) *Let* $(R, \mathfrak{m})$ *be a local ring with infinite residue field and* depth $R > 0$*, and let* I *be a reduction of* $\mathfrak{m}$*. Then*

$$\mathfrak{d}_3(I) = \max\{\mathrm{r}_I(\mathfrak{m}), s(\mathfrak{m}) - 1\}.$$

Now we can illustrate Conjecture 4.2 in the case $\dim R = 2$.

Example 4.4. Let $(R, \mathfrak{m})$ be the local ring of a rational triple point as in [17, Example 4.1], where we highlighted that $s(\mathfrak{m}) = 1$. Then, by Theorem 4.3, we obtain $\mathfrak{d}_3(I) = \mathrm{r}_I(\mathfrak{m})$ for any reduction (in particular, minimal reduction) I of $\mathfrak{m}$.

The above example actually shows that Conjecture 4.2 is true in the case $s(\mathfrak{m}) = 1$, or equivalently, depth $\mathrm{gr}_{\mathfrak{m}}(R) > 0$ (see also Proposition 4.9 below). Furthermore, it is clear that $\widetilde{\mathfrak{m}} = \mathfrak{m}$, which forces $s(\mathfrak{m}) \neq 2$. Therefore, in tackling the conjecture we can suppose $s(\mathfrak{m}) \geq 3$.

Theorem 4.5. *Conjecture 4.2 holds true in case* $s := s(\mathfrak{m}) \geq 3$ *and*

$$\widetilde{I\mathfrak{m}^{s-2}} = I\mathfrak{m}^{s-2}.$$

Proof. By virtue of Theorem 4.3, it suffices to show that $\mathrm{r}_I(\mathfrak{m}) \geq s - 1$. Suppose, by way of contradiction, that

$$\mathrm{r}_I(\mathfrak{m}) \leq s - 2.$$

Since I is a minimal reduction of $\mathfrak{m}$, we can apply [14, Proposition 2.4 and Theorem 2.10] to obtain $\widetilde{I\mathfrak{m}^k} = \widetilde{\mathfrak{m}^{k+1}}$ for all $k \geq \mathrm{r}_I(\mathfrak{m})$. Consequently,

$$\widetilde{\mathfrak{m}^{s-1}} = \widetilde{I\mathfrak{m}^{s-2}} = I\mathfrak{m}^{s-2} = \mathfrak{m}^{s-1},$$

which contradicts the definition of s. □

Since $\mathfrak{m} = \widetilde{\mathfrak{m}}$ is always true, the case $s(\mathfrak{m}) = 3$ is a straightforward consequence of Theorem 4.5.

Corollary 4.6. *Conjecture 4.2 is true if* $s(\mathfrak{m}) = 3$.

Example 4.7. Let K be an infinite field and

$$R = \frac{K[\![x_1, x_2, x_3, x_4, x_5, x_6, x_7]\!]}{(x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^3 - x_1x_5, x_3^3 - x_1x_6, x_4^3 - x_1x_7)},$$

which is a three-dimensional Cohen-Macaulay local ring. Notice that

$$x_1 \in \widetilde{\mathfrak{m}^2} \setminus \mathfrak{m}^2$$

and $\widetilde{\mathfrak{m}^n} = \mathfrak{m}^n$ for all $n \geq 3$. Thus, $s(\mathfrak{m}) = 3$. By Corollary 4.6, we obtain $\mathfrak{d}_3(I) = \mathrm{r}_I(\mathfrak{m})$ for any minimal reduction I of $\mathfrak{m}$.

When $s := s(\mathfrak{m}) \geq 4$, it may be difficult to determine whether $\widetilde{I\mathfrak{m}^{s-2}} = I\mathfrak{m}^{s-2}$. However, by employing a different argument, we are able to establish Conjecture 4.2 in the case $s(\mathfrak{m}) = 4$.

Proposition 4.8. *Conjecture 4.2 is true if* $s(\mathfrak{m}) = 4$.

Proof. Suppose $\mathrm{r}_I(\mathfrak{m}) \leq s(\mathfrak{m}) - 2 = 2$. Then, by [3, Theorem 3.6], the ring $\mathrm{gr}_{\mathfrak{m}}(R)$ must be Cohen-Macaulay. Hence $\operatorname{depth} \mathrm{gr}_{\mathfrak{m}}(R) = \dim \mathrm{gr}_{\mathfrak{m}}(R) = \dim R > 0$. But this implies $s(\mathfrak{m}) = 1$, which is a contradiction. Therefore,

$$\mathrm{r}_I(\mathfrak{m}) \geq s(\mathfrak{m}) - 1$$

and the result follows from Theorem 4.3. □

We close the section with yet another affirmative case of the conjecture.

Proposition 4.9. *Conjecture 4.2 is true if R has minimal multiplicity.*

Proof. For such R, the ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay (see [22, Theorem 2]), hence its depth is equal to $\dim R \geq 2 > 0$. In this case, as we already know, the conjecture holds true. □

5. Application: Regular local rings

In this section, as applications of some of our results, we provide new characterizations of regular local rings, and describe a potential approach to the long-standing Zariski-Lipman conjecture about derivations.

5.1. **Characterizations of regular local rings.** Our result in this part is Theorem 5.3 below. We observe that the implication (a) $\Rightarrow$ (d) recovers [17, Corollary 3.11]. Moreover, a crucial fact here (which, as far as we know, is new) is given by (b) $\Rightarrow$ (a), i.e., $(R, \mathfrak{m})$ must be regular if $\mathfrak{m}$ is generated by a d-sequence (see [11], [12] for the definition of this type of sequence and its properties), which in particular solves the problem suggested in [17, Remark 3.12]. Finally, the equivalence between assertions (a) and (e) reveals the curious role played by $\mathfrak{d}_3(I)$ in regard to the theory of regular local rings, which can be re-expressed by means of equivalence to the structural assertion (f).

For the proof, the following two interesting facts will be useful.

Lemma 5.1. ([23, Corollary 5.2]) *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. Then, $\mathfrak{m}$ is generated by a d-sequence if and only if $\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}) = 0$.*

Lemma 5.2. ([15, Theorem 1.1]) *Let $(R, \mathfrak{m})$ be a local ring with $\operatorname{depth} R > 0$. If Q is a parameter ideal of R such that Q^n is $\mathfrak{m}$-full for some $n \geq 1$, then R is regular.*

Theorem 5.3. *Let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded K-algebra, with K infinite and $\operatorname{depth} R > 0$. The following assertions are equivalent:*

(a) *R is regular;*
(b) *$\mathfrak{m}$ is generated by a d-sequence;*
(c) *$s(\mathfrak{m}) = 1$ and $\mathrm{r}_I(\mathfrak{m}) = 0$, for any (minimal) reduction I of $\mathfrak{m}$;*
(d) *$\mathfrak{d}_1(I) = \mathfrak{d}_2(I) = \mathfrak{d}_3(I) = 0$, for any (minimal) reduction I of $\mathfrak{m}$;*
(e) *$\mathfrak{d}_3(I) = 0$, for any (minimal) reduction I of $\mathfrak{m}$;*
(f) *$\mathcal{R}(\mathfrak{m}, I) = (\mathcal{R}(\mathfrak{m}, I)_{\geq 1} :_{\mathcal{R}(R)} \mathfrak{m})(1)$, for any (minimal) reduction I of $\mathfrak{m}$.*

Proof. (a) $\Rightarrow$ (b) If R is regular, then $\mathfrak{m}$ is generated by a regular sequence, which is therefore a d-sequence.

(b) $\Rightarrow$ (c) According to [20, Theorem 2.1(ii)], we have $\max\{\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}), 1\} \geq s(\mathfrak{m})$. Hence, using Lemma 5.1, we obtain $s(\mathfrak{m}) = 1$. In order to deal with $\mathrm{r}_I(\mathfrak{m})$, consider first the case where the reduction I is minimal. Applying [20, Theorem 1.3] and [21, p. 12], we derive

$$\mathrm{reg}_{\mathcal{R}(\mathfrak{m})}\mathcal{R}(\mathfrak{m}) \geq \mathrm{r}_I(\mathfrak{m}),$$

which gives $\mathrm{r}_I(\mathfrak{m}) = 0$. Now, if I is not minimal, then it necessarily contains a minimal reduction J of $\mathfrak{m}$, so that $\mathrm{r}_I(\mathfrak{m}) \leq \mathrm{r}_J(\mathfrak{m}) = 0$.

(c) $\Rightarrow$ (d) This follows readily from the fact (proved in [17, Theorem 3.4]) that

$$\mathfrak{d}_3(I) \leq \max\{\mathrm{r}_I(\mathfrak{m}), s(\mathfrak{m}) - 1\}.$$

Now, by [17, Proposition 2.2], the vanishing of $\mathfrak{d}_3(I)$ forces that of $\mathfrak{d}_1(I)$ and $\mathfrak{d}_2(I)$.

(d) $\Rightarrow$ (e) This is obvious.

(e) $\Rightarrow$ (a) In the present setting, $\mathfrak{m}$ admits a reduction Q which is a parameter ideal (see, e.g., [13, Exercise 8.11(ii)]). As $\mathfrak{d}_3(Q) = \mathfrak{d}_1(Q)$, we obtain

$$\mathfrak{d}_1(Q) = 0$$

and consequently Q is $\mathfrak{m}$-full. We are now in a position to apply Lemma 5.2 with $n = 1$ to conclude that R is regular.

It remains to prove that assertions (a) and (f) are equivalent. To this end, simply note that (f) holds if and only if $(\mathcal{R}(\mathfrak{m}, I))_k = (\mathcal{R}(\mathfrak{m}, I)_{k\geq 1} :_{\mathcal{R}(R)} \mathfrak{m})(1)_k$ for all $k \geq 0$ and any (minimal) reduction I of $\mathfrak{m}$, which means

$$I\mathfrak{m}^k = I\mathfrak{m}^{k+1} : \mathfrak{m} \quad \text{for all} \quad k \geq 0.$$

This is, by definition, tantamount to $\mathfrak{d}_3(I) = 0$, which as we have shown above is equivalent to R being regular. □

5.2. **Potential approach to a classical conjecture.** For a field K and a K-algebra R, we write as usual $\mathrm{Der}_K(R)$ for the module of K-derivations of R, i.e., the additive maps $D\colon R \to R$ that vanish on K and satisfy Leibniz rule: $D(\alpha\beta) = \alpha D(\beta) + \beta D(\alpha)$ for all $\alpha, \beta \in R$. Now assume that R is a positive-dimensional local ring which is either

$$K[x_1, \ldots, x_m]_{\mathfrak{q}}/I \quad (\mathfrak{q} \in \mathrm{Spec}\, K[x_1, \ldots, x_m]) \quad \text{or} \quad K[\![x_1, \ldots, x_m]\!]/I, \tag{4}$$

with I a proper radical ideal and $x_1, \ldots, x_m$ indeterminates over a field K of characteristic zero. In this setting, there is the following long-held classical problem.

Conjecture 5.4. (Zariski-Lipman) *Let R be as in* (4). *If* $\mathrm{Der}_K(R)$ *is free, then R is regular.*

This problem has an interesting long history, featuring in particular a strong geometric counterpart, and remains open in some cases. For details and references, see [9, Section 2] (in particular, see [9, Theorem 2.3] for a simple proof of the Zariski-Lipman conjecture in the graded case). Additionally, [16, Section 4] shows a relation between this conjecture and the $\mathfrak{m}$-full property of ideals in the (open) two-dimensional local case, which makes it somewhat natural to expect further connections.

We point out that, under the hypotheses of the conjecture, R must be a (normal) domain, and we can write

$$\mathrm{Der}_K(R) = \bigoplus_{1 \leq j \leq t} RD_j \quad \cong R^t, \quad t = \dim R,$$

for some free basis $\{D_j\}$ consisting of precisely t derivations. Conversely, if $\mathrm{Der}_K(R)$ admits a free basis $\{D_1, \ldots, D_s\}$, then necessarily $s = t$.

Remark 5.5. Let $(R, \mathfrak{m})$ be as in (4). If, as above, the R-module $\mathrm{Der}_K(R)$ is free, then our guess is that, for each minimal reduction I of $\mathfrak{m}$, there exist $\alpha_j^{(I)}, \beta_j^{(I)} \in R$, $j = 1, \ldots, t$, such that the element given by

$$\ell^{(I)} = \sum_{j=1}^{t} \beta_j^{(I)} D_j(\alpha_j^{(I)})$$

satisfies $\ell^{(I)} \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $I\mathfrak{m}^{k+1} \colon (\ell^{(I)}) = I\mathfrak{m}^k$ for all $k \geq 0$. This would confirm the validity of Conjecture 5.4, because such conditions yield $I\mathfrak{m}^k$ to be $\mathfrak{m}$-full for all $k \geq 0$, which means $\mathfrak{d}_1(I) = 0$ and hence, as we already know, $\mathfrak{d}_3(I) = 0$. Now, Theorem 5.3 ensures that R is regular.

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